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Closed form solutions for an anisotropic composite beam on a two-parameter elastic foundation

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ABSTRACT

Beams resting on elastic foundations are widely used in engineering design such as railroad tracks, pipelines, bridge decks, and automobile frames. Laminated composite beams can be tailored for specific design requirements and offer a desirable design framework for beams resting on elastic foundations. Therefore, the analysis of flexural behaviour of laminated composite beams on elastic foundations is of important consequence. Exact solutions for flexural deflection of composite beams with coupling terms between stretching, shearing, bending and twisting, resting on two-parameter elastic foundations for various types of loading and boundary conditions, are presented for the first time. The proposed new formulation is based on Euler–Bernoulli beam theory having four degrees of freedom, namely bending in two principal directions, axial elongation and twist. Governing equations and boundary conditions are derived from the principle of virtual work and expressed in a compact matrix–vector form. By decoupling bending in both principal directions from twist and axial elongation, the fourth-order differential equation for bending is derived and transformed into a system of first-order differential equations. An exact solution of this system of equations is obtained using a fundamental matrix approach. Fundamental matrices for different configurations of elastic foundation are provided. The ability of the presented mathematical model in predicting flexural behaviour of beams on elastic foundations is verified numerically by comparison with results available in the literature. In addition, the deflection of anisotropic beams is analysed for different types of stacking sequences, boundary and loading conditions. The effect of elastic foundation coefficients on the flexural behaviour is also investigated and discussed.

1. Introduction

Due to their outstanding material properties, including excellent specific strength, resistance to fatigue and damage tolerance behaviour, composites are widely used in different fields of engineering. Composite beams resting on elastic foundations traditionally have found use for predicting the response of load-carrying structural elements in the preliminary engineering design of bridges, railroads, motorways, runways, pavements, moorings, and marine pipelines with respect to the different types of the soil (clay, sand, granular or pressured soils, etc.). However, in recent years, the rapid development of new advanced areas of engineering such as aerospace, automobile, biomedical, and nano-engineering intensifies the role of composite beams resting on elastic foundations in modelling of the constructions of higher and slender buildings, high-speed railways, automobile frames, some tissues, bones and vessels implants of human body and other structures. To predict and evaluate their critical characteristics, careful analysis of flexural behaviour is required. Many researchers have contributed to this field using analytical approaches. Applying Fourier series, Naik (2000) obtained an analytical solution for the static deflection of a simply supported deep composite beam on a Winkler elastic foundation subject to transverse concentrated and uniformly distributed loads. The bending and free vibration of two-dimensional simply supported functionally graded beams resting on Winkler–Pasternak foundations were investigated by Ying et al. (2008). Following the state space method, they used trigonometric series to expand the solutions of normal and shear stresses and displacements to reduce the system of governing differential equations from partial to ordinary and subsequently to obtain exact two-dimensional elasticity solutions. The Navier approach was employed by Zenkour et al. (2010) to obtain an analytical solution for the bending response of a simply supported functionally graded viscoelastic sandwich beam resting on a Pasternak elastic foundation under the action of sinusoidal loads. Bending, buckling and vibration of carbon nanotube-reinforced composite beams resting on Pasternak elastic foundations were studied by Wattanasakulpong and Ungbhakorn (2013). Using the Navier method, analytical solutions for simply supported beams under uniform and sinusoidal loads were obtained. An

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analytical solution for the free vibration of functionally graded beams resting on Winkler elastic foundations was presented by Murin et al. (2013). Rectangular cross-section and polynomial spatial variation of material properties were considered. Applying a multilayer method for the material property homogenisation and by the direct integration method, beam deflection was obtained and used for modal analysis. Static bending behaviour of functionally graded microbeams subject to axial and transverse distributed loads resting on Winkler foundations was studied by Akgöz and Civalek (2015). An analytical solution was obtained for simply supported microbeams based on the Navier procedure. Again using the Navier technique, Atmane et al. (2017) derived the closed form analytical solutions for the bending, free vibration and buckling problems of simply supported, thick functionally graded beams subject to transverse distributed loads on two-parameter elastic foundations. A similar approach, in combination with new inverse hyperbolic beam theory, was applied by Sayyad and Ghugal (2018) to study bending, buckling and free vibration responses of functionally graded higher-order beams resting on Winkler–Pasternak elastic foundations. Analytical solutions for simply supported beams under the action of transverse sinusoidal and uniform loads were obtained. The Navier solution procedure was employed by Phuong et al. (2019) to perform bending analysis of functionally graded beams containing porosity. Simply supported Timoshenko beams under the action of transverse load were assumed. Chaabane et al. (2019) used hyperbolic shear deformation theory to formulate the static and dynamic responses of simply supported functionally graded beams on two-parameter elastic foundations. The Navier method was employed to obtain analytical solutions. Analytical solution for the bending of functionally graded beams with variable thicknesses resting on two-parameter elastic foundations was presented by Li et al. (2020). An elasticity solution was obtained by the means of Fourier expansion for simply supported beams under the action of non-uniform distributed loads. Based on the Navier method, Sobhy and Zenkour (2020) obtained analytical solutions for the bending of viscoelastic nano-beams lying on viscoelastic Pasternak foundations. Simply supported boundary conditions and time harmonic transverse loading conditions were assumed. Hadji and Bernard (2020) developed Navier–type analytical solutions for the bending and free vibration of functionally graded beams on two-parameter elastic foundations. Bourada et al. (2020) applied Hamilton's principle and the Navier method to derive analytical solutions for the buckling and free vibration of simply supported Single Walled Carbon Nanotubes (SWCNT) reinforced concrete beam on elastic foundation considering uniform and variable types of SWCNT reinforcement. Same approach was applied by Bousahla et al. (2020) to study buckling and vibration of composite beams armed with single-walled carbon nanotubes resting on Winkler–Pasternak elastic foundations. An exact elasticity solution for curved composite beams consisting of an arbitrary number of orthotropic layers resting on elastic foundations subject to pure bending was presented by Shenoi and Wang (2001). Using Reddy's layerwise formulation, Afshin and Taheri-Behrooz (2015) presented a Navier–type analytical solution for the deflection of simply supported laminated beams resting on Winkler elastic foundations under the action of transverse exponential loads. A mixed Galerkin perturbation technique was employed by Li and Zhao (2015) to perform nonlinear bending analysis for shear deformable anisotropic laminated composite beams subject to initial loads resting on various types of elastic foundations. Two types of boundary conditions, simply supported and hinged, were considered. A two-step perturbation method was employed by Shen (2015) to obtain the solution for large amplitude vibration, nonlinear bending and thermal postbuckling of anisotropic laminated beams made of fibre reinforced composites resting on elastic foundations in hygrothermal environments. Later, they employed this technique to perform nonlinear large amplitude vibration analysis of simply supported graphene-reinforced composite laminated beams resting on elastic foundations in thermal environments (Shen et al., 2017b). The same approach was applied to study nonlinear bending and thermal postbuckling behaviour of graphene-reinforced composite beams in thermal environments (Shen et al., 2017a) and the nonlinear flexural behaviour of temperature-dependent functionally graded carbon nanotube-reinforced composite laminated beams with negative Poisson's ratio (Yang et al., 2020) resting on elastic foundations. A two-step perturbation technique was also applied by Babaei et al. to investigate nonlinear bending (Babaei et al., 2018), vibration (Babaei et al., 2019a) and buckling (Babaei et al., 2019b) behaviour of shallow thick circular arches made of temperature-dependent functionally graded material and resting on three-parameter elastic foundations. Simply supported boundary conditions and thermal and mechanical loads were assumed. An analytical solution based on the Hamiltonian approach was proposed by Pakar et al. (2018) to solve the problem of nonlinear vibrations of unsymmetrical laminated composite beams on nonlinear elastic foundations. Simply supported and clamped boundary conditions were considered.

In the context of numerical evaluation of static response of beams on elastic foundations, Al-Shujairi and Mollamahmutoğlu (2018) applied the generalised Differential Quadrature Method (DQM) to obtain the numerical solution for the buckling and free vibration problem of size dependent functionally graded sandwich micro-beams resting on two-parameter elastic foundations with thermal effects with different boundary conditions. Generalised DQM was also applied by Babaee et al. (2018) to perform nonlinear thermal buckling analysis of functionally graded beams integrated with shape memory alloy layers on nonlinear elastic foundations. Mahmoudpour et al. (2018) investigated nonlinear vibration of functionally graded nano-beams resting on elastic foundations subject to a uniform temperature rise. Using the Galerkin method, governing equations were reduced to nonlinear ordinary differential equations, then the Homotopy Analysis Method (HAM) was employed to obtain a closed form analytical solution for the problem. Trabelssi et al. (2019) studied free and forced vibration of a nonlocal Timoshenko graded nano-beam on a nonlinear elastic foundation. By means of the Galerkin technique, governing partial equations were transformed into a system of nonlinear ordinary differential equations and then solved analytically using the Method of Multiple Scale for hinged–hinged and clamped–clamped beams. Alternative numerical solutions were obtained by applying the Differential Quadrature and Harmonic Quadrature Methods.

As discussed, the structural analysis of composite beams resting on elastic foundations attracts substantial attention, resulting in various analytical and numerical solutions of the problem, wherein analytical solutions have the obvious advantage of providing fast and precise results. However, the vast majority of analytical solutions available in the literature are based on series techniques and are limited to specific types of boundary or loading conditions. Moreover, according to the authors' best knowledge, there is no solution for the flexural analysis of laminated fibre-reinforced anisotropic beams resting on elastic foundations available. This deficiency is in need of remedy as such beams provide an excellent possibility of being tailored according to specific engineering design requirements. However, this advantage also poses difficulties in its structural analysis as resulting equations involve coupling terms between stretching, shearing, bending and twisting leading to mathematical complexity of the solution.

To address these limitations, the exact analytical solution for the static flexural response of fully coupled Euler–Bernoulli composite beams resting on two-parameter elastic foundations is proposed for the first time. Towards this goal, a novel approach is proposed which consists of direct integration and fundamental matrix techniques. In the context of fully coupled composite beams, it has been already shown that expressing the governing equations in matrix–vector form is an effective technique to enable direct integration of the governing differential equations (Doeva et al., 2020a; Masjedi and Weaver, 2020a,b; Doeva et al., 2020b; Masjedi et al., 2021). However, for composite beams resting on elastic foundations, since first and second-order differential terms appear in the governing equations of bending, it is not possible to obtain the solution solely based on the direct integration. Thus, integration is first used to decouple bending from twist and axial elongation, then the method of fundamental matrices is used to obtain the exact solution. While the available analytical solutions in the literature are series-based and limited to specific boundary conditions and loading scenarios, the proposed exact solution in this work can capture arbitrary boundary and loading conditions. In the proposed formulation all possible four degrees of freedom, namely axial displacement u, twist φ, out-of-plane bending w and in-plane bending v, are taken into account. It is also worth mentioning that in the proposed model engineering constants are used to express the entries of stiffness matrix allowing the behaviour of beams made of functionally graded, fibre reinforced or other types of material, to be represented. In addition, the Chebyshev Collocation Method which has been shown to be accurate and efficient for beam problems (Masjedi and Ovesy, 2015a,b; Masjedi and Maheri, 2017; Masjedi et al., 2019) is applied to verify the obtained analytical solutions.

The outline of the rest of the paper is as follows: in Section 2 the theoretical formulation and solution procedure are provided; in Section 3 the Chebyshev Collocation Method used to obtain an alternative numerical solution is briefly described; in Section 4 the mathematical model presented in the current paper is validated first by comparison with some results available in the literature for homogeneous beams on elastic foundations, before numerical examples of flexural behaviour of laminated composite beams for various types of stacking sequences and loading and boundary conditions are carried out for different values of elastic foundation moduli; finally Section 5 presents some conclusions.

2. Mathematical modelling of Euler–Bernoulli beam on a two-parameter elastic foundation

2.1. Theoretical formulation

Consider a straight composite beam resting on a two-parameter elastic foundation of length ℓ in the x-direction, width b in the y-direction and thickness h in the z-direction as depicted in Fig. 1. The origin of the coordinate system is taken as the neutral axis of the beam. The constant elastic foundation is presented by a two-parameter model which is characterised by two moduli, namely Winkler modulus k_w and shear Pasternak modulus k_p (Robinson and Adali, 2018). Physically, this model represents an idealisation of the foundation described by a sequence of mutually independent closely spaced linear springs with stiffness k_w, with ends of these springs connected to a shear layer which acts as a horizontal linkage of vertical elements with interaction parameter k_p.

Ignoring cross-sectional warping, displacement of a generic point on the cross-section consists of two parts: (1) a rigid-body displacement of the beam reference line and (2) rotation of the beam cross-section which can be expressed as (Pai, 2007; Luo, 2008)

$$U_x = u(x) + z\theta_y(x) - y\theta_z(x), \tag{1a}$$

$$U_y = v(x) - z\varphi(x), \tag{1b}$$

$$U_z = w(x) + y\varphi(x), \tag{1c}$$

where U_x, U_y and U_z are the components of displacement vector and u, v and w denote displacements of the beam reference line in x, y and z directions and φ, θ_y and θ_z are the rotations of beam cross-section about x, y and z, respectively.

Now using Eqs. (1), the strain measures of the beam can be expressed as

$$\epsilon_{xx} = \frac{\partial U_x}{\partial x} = u' + z\theta_y' - y\theta_z', \tag{2a}$$

$$\gamma_{xy} = \frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} = \left(v' - \theta_z\right) - z\varphi', \tag{2b}$$

$$\gamma_{xz} = \frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} = \left(w' + \theta_y\right) + y\varphi'. \tag{2c}$$

According to Euler–Bernoulli beam kinematics which assumes that the cross-section remains orthogonal to the beam reference axis after deformation, then

$$\theta_y = -w', \quad \theta_z = v'. \tag{3}$$

Internal work of the beam can be expressed as follows

$$\int_V W_{int}\, dV = \int_V \left(\sigma_{xx}\epsilon_{xx} + \sigma_{xy}\gamma_{xy} + \sigma_{xz}\gamma_{xz}\right) dV. \tag{4}$$

By substituting strain measures ϵ_{xx}, γ_{xy} and γ_{xz} given by Eqs. (2) into Eq. (4), the following is obtained

$$\begin{aligned}\int_V W_{int}\, dV = \int_V \Big(&\sigma_{xx}u' + z\sigma_{xx}\theta_y' - y\sigma_{xx}\theta_z' + \sigma_{xy}\left(v' - \theta_z\right)\\ &+ \sigma_{xz}\left(w' + \theta_y\right) + \left(y\sigma_{xz} - z\sigma_{xy}\right)\varphi'\Big) dV.\end{aligned} \tag{5}$$

Defining beam internal forces and moments as

$$F_x = \int_A \sigma_{xx}\, dA, \tag{6a}$$

$$M_x = \int_A \left(y\sigma_{xz} - z\sigma_{xy}\right) dA, \tag{6b}$$

$$M_y = \int_A z\sigma_{xx}\, dA, \tag{6c}$$

$$M_z = -\int_A y\sigma_{xx}\, dA, \tag{6d}$$

and considering Eq. (3), then

$$\begin{aligned}\int_0^\ell W_{int}\, dx &= \int_0^\ell \left(F_x u' + M_x\varphi' + M_y\theta_y' + M_z\theta_z'\right) dx\\ &= \int_0^\ell \left(F_x u' + M_x\varphi' - M_y w'' + M_z v''\right) dx.\end{aligned} \tag{7}$$

The principle of virtual work for a beam resting on elastic foundation is expressed as

$$\int_0^\ell \left(\delta W_{int} + \delta W_f - \delta W_{ext}\right) dx = 0, \tag{8}$$

where δW_{int}, δW_f and δW_{ext} are the variations of internal, elastic foundation and external works respectively. Variation of internal work of the beam can be derived from Eq. (7) as

$$\int_0^\ell \delta W_{int} = \int_0^\ell \delta\boldsymbol{\epsilon}^T \boldsymbol{N}\, dx = \int_0^\ell \delta\boldsymbol{\epsilon}^T \boldsymbol{S}\boldsymbol{\epsilon}\, dx \tag{9}$$

where vector of strains $\boldsymbol{\epsilon}$, vector of internal forces and moments $\boldsymbol{N}$ and stiffness matrix $\boldsymbol{S}$ can be written as

$$\begin{aligned}&\boldsymbol{\epsilon} = \begin{bmatrix}\epsilon_x\\ \kappa_x\\ \kappa_y\\ \kappa_z\end{bmatrix} = \begin{bmatrix}u'\\ \varphi'\\ -w''\\ v''\end{bmatrix}, \qquad \boldsymbol{N} = \begin{bmatrix}F_x\\ M_x\\ M_y\\ M_z\end{bmatrix},\\ &\boldsymbol{S} = \begin{bmatrix}EA & S_{ET} & S_{EF} & S_{EL}\\ S_{ET} & GJ & S_{FT} & S_{LT}\\ S_{EF} & S_{FT} & EI_y & S_{FL}\\ S_{EL} & S_{LT} & S_{FL} & EI_z\end{bmatrix},\end{aligned} \tag{10}$$

where EA is the extensional stiffness, GJ is the twist stiffness, EI_y is the out-of-plane bending stiffness, EI_z is the in-plane bending stiffness, S_{ET} is the coupling between axial elongation and twist, S_{EF} is the coupling between out-of-plane bending and axial elongation, S_{EL} is the coupling between in-plane bending and axial elongation, S_{FT} is the coupling between out-of-plane bending and twist, S_{LT} is the coupling between in-plane bending and twist, and S_{FL} is the coupling between out-of-plane and in-plane bending.

Variation of external work can be defined by

$$\int_0^\ell \delta W_{ext}\, dx = \int_0^\ell \delta\overline{\boldsymbol{U}}^T \boldsymbol{Q}\, dx, \tag{11}$$

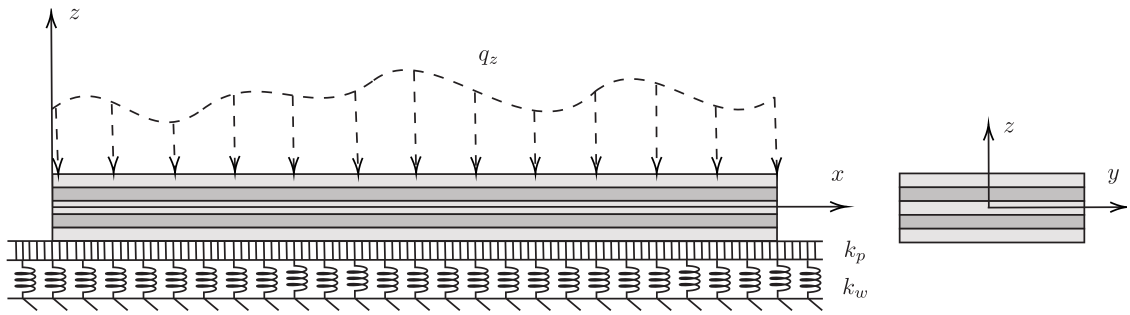


Fig. 1. Composite beam on elastic foundation.

where the vector of displacements and twist $\overline{\boldsymbol{U}}$ and vector of external forces and torque $\boldsymbol{Q}$ are written as

$$\overline{\boldsymbol{U}} = \begin{bmatrix} u \\ \varphi \\ w \\ v \end{bmatrix}, \qquad \boldsymbol{Q} = \begin{bmatrix} q_x \\ q_\varphi \\ q_z \\ q_y \end{bmatrix}. \tag{12}$$

Variation of work due to elastic foundation can be written as (Robinson and Adali, 2018)

$$\int_0^\ell \delta W_f = \int_0^\ell \left(\delta w\, k_w\, w + \delta w'\, k_p\, w'\right) dx, \tag{13}$$

Substituting Eqs. (9), (11) and (13) into Eq. (8), applying the integration by parts and collecting the coefficients of δu, $\delta\varphi$, δw, δv, $\delta w'$ and $\delta v'$, the system of governing equations can be obtained in the following form

$$\begin{aligned} &-EAu'' - S_{ET}\varphi'' + S_{EF}w''' - S_{EL}v''' = q_x \\ &-S_{ET}u'' - GJ\varphi'' + S_{FT}w''' - S_{LT}v''' = q_\varphi \\ &-S_{EF}u''' - S_{FT}\varphi''' + EI_y w^{(IV)} - S_{FL}v^{(IV)} + k_w w - k_p w'' = q_z \\ &\ \ S_{EL}u''' + S_{LT}\varphi''' - S_{FL}w^{(IV)} + EI_z v^{(IV)} = q_y. \end{aligned} \tag{14}$$

At $x = 0$ and $x = \ell$, boundary conditions can be expressed as follows

$$\begin{aligned} &u = 0 &&\text{or} && EAu' + S_{ET}\varphi' - S_{EF}w'' + S_{EL}v'' = f_x \\ &\varphi = 0 &&\text{or} && S_{ET}u' + GJ\varphi' - S_{FT}w'' + S_{LT}v'' = m_x \\ &w' = 0 &&\text{or} && -S_{EF}u' - S_{FT}\varphi' + EI_y w'' - S_{FL}v'' = -m_y \\ &v' = 0 &&\text{or} && S_{EL}u' + S_{LT}\varphi' - S_{FL}w'' + EI_z v'' = m_z \\ &w = 0 &&\text{or} && S_{EF}u'' + S_{FT}\varphi'' - EI_y w''' + S_{FL}v''' + k_p w' = f_z \\ &v = 0 &&\text{or} && -S_{EL}u'' - S_{LT}\varphi'' + S_{FL}w''' - EI_z v''' = f_y, \end{aligned} \tag{15}$$

where $()'$ denotes the derivative with respect to x; the displacements of the beam reference line in x, y and z directions are denoted as u, v and w, and φ is the twist of beam cross-section about x; q_x, q_y, q_z are functions of x representing distributed loads and q_φ is the function of x representing distributed torque; f_x, f_y, f_z are the tip loads in the x, y and z directions respectively, and m_x is the tip torque, and m_y, m_z are the tip moments about the y and z axes respectively.

For convenience, Eqs. (14) and (15) can be written in a compact matrix–vector format as follows

$$-\boldsymbol{A}\boldsymbol{U}'' + \boldsymbol{B}\boldsymbol{W}''' = \boldsymbol{Q_x}, \tag{16a}$$

$$-\boldsymbol{B}^T\boldsymbol{U}''' + \boldsymbol{D}\boldsymbol{W}^{(IV)} + \boldsymbol{K_w}\boldsymbol{W} - \boldsymbol{K_p}\boldsymbol{W}'' = \boldsymbol{Q_z}, \tag{16b}$$

$$\boldsymbol{A}\boldsymbol{U}' - \boldsymbol{B}\boldsymbol{W}'' = \boldsymbol{F_x}, \tag{17a}$$

$$-\boldsymbol{B}^T\boldsymbol{U}' + \boldsymbol{D}\boldsymbol{W}'' = \boldsymbol{M_y}, \tag{17b}$$

$$\boldsymbol{B}^T\boldsymbol{U}'' - \boldsymbol{D}\boldsymbol{W}''' + \boldsymbol{K_p}\boldsymbol{W}' = \boldsymbol{F_z}, \tag{17c}$$

where

$$\boldsymbol{U} = \begin{bmatrix} u \\ \varphi \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} w \\ v \end{bmatrix}, \tag{18}$$

$$\begin{aligned} &\boldsymbol{Q_x} = \begin{bmatrix} q_x \\ q_\varphi \end{bmatrix}, \quad \boldsymbol{Q_z} = \begin{bmatrix} q_z \\ q_y \end{bmatrix}, \quad \boldsymbol{F_x} = \begin{bmatrix} f_x \\ m_x \end{bmatrix}, \\ &\boldsymbol{M_y} = \begin{bmatrix} -m_y \\ m_z \end{bmatrix}, \quad \boldsymbol{F_z} = \begin{bmatrix} f_z \\ f_y \end{bmatrix}, \end{aligned} \tag{19}$$

$$\begin{aligned} &\boldsymbol{A} = \begin{bmatrix} EA & S_{ET} \\ S_{ET} & GJ \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} S_{EF} & -S_{EL} \\ S_{FT} & -S_{LT} \end{bmatrix}, \\ &\boldsymbol{D} = \begin{bmatrix} EI_y & -S_{FL} \\ -S_{FL} & EI_z \end{bmatrix}, \end{aligned} \tag{20}$$

$$\boldsymbol{K_w} = \begin{bmatrix} k_w & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{K_p} = \begin{bmatrix} k_p & 0 \\ 0 & 0 \end{bmatrix}. \tag{21}$$

2.2. Solution procedure

To derive the exact solution of Eqs. (16), (16a) should be rearranged and differentiated so the third derivative of vector $\boldsymbol{U}$ is obtained in the following form

$$\boldsymbol{U}''' = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{W}^{(IV)} - \boldsymbol{A}^{-1}\boldsymbol{Q}_{\boldsymbol{x}}'. \tag{22}$$

Substituting Eq. (22) into Eq. (16b), and rearranging it, the expression for the fourth derivative of $\boldsymbol{W}$ can be obtained

$$\begin{aligned} \boldsymbol{W}^{IV} = {} & (\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{K_p}\boldsymbol{W}'' - (\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{K_w}\boldsymbol{W} + \\ & (\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{Q_z} - (\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{Q}_{\boldsymbol{x}}'. \end{aligned} \tag{23}$$

Introducing

$$\begin{aligned} &\boldsymbol{B_1} = (\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{K_p}, \\ &\boldsymbol{B_2} = (\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{K_w}, \\ &\boldsymbol{b} = (\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{Q_z} - (\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{Q}_{\boldsymbol{x}}', \end{aligned}$$

Eq. (23) can be written as

$$\boldsymbol{W}^{IV} = \boldsymbol{B_1}\boldsymbol{W}'' - \boldsymbol{B_2}\boldsymbol{W} + \boldsymbol{b}. \tag{24}$$

This fourth-order expression can be transformed into a system of first-order linear equations by introducing 2×1 vectors of variables

$$\boldsymbol{x_1} = \boldsymbol{W}, \quad \boldsymbol{x_2} = \boldsymbol{W}', \quad \boldsymbol{x_3} = \boldsymbol{W}'', \quad \boldsymbol{x_4} = \boldsymbol{W}'''. \tag{25}$$

Then it follows that

$$\boldsymbol{x}_1' = \boldsymbol{W}' = \boldsymbol{x_2}, \tag{26a}$$

$$\boldsymbol{x}_2' = \boldsymbol{W}'' = \boldsymbol{x_3}, \tag{26b}$$

$$\boldsymbol{x}_3' = \boldsymbol{W}''' = \boldsymbol{x_4}, \tag{26c}$$

$$x_4' = W^{IV} = B_1 x_3 - B_2 x_1 + b. \quad (26d)$$

Eqs. (26) can be expressed as an eight-dimensional first-order system of linear differential equations

$$x' = Mx + f, \quad (27)$$

where the vector of unknowns x, companion matrix M and non-homogeneous term f are defined as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad M = \begin{bmatrix} 0_{2\times2} & I & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & I & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & I \\ -B_2 & 0_{2\times2} & B_1 & 0_{2\times2} \end{bmatrix}, \quad f = \begin{bmatrix} 0_{2\times1} \\ 0_{2\times1} \\ 0_{2\times1} \\ b \end{bmatrix}, \quad (28)$$

with $0_{2\times2}$ denoting 2×2 zero matrix, I denoting 2×2 identity matrix and $0_{2\times1}$ denoting 2×1 zero vector.

The general solution of Eq. (27) is

$$x = \Phi(x)C + \Phi(x)\int \Phi(s)^{-1} f(s)\mathrm{d}s, \quad (29)$$

where $\Phi(x)$ is the fundamental matrix of the solution and $C = [C_1 \;\; C_2 \;\; C_3 \;\; C_4]^T$ is a vector of arbitrary 2×1 constant vectors to be uniquely determined.

It is worth noting that computing the fundamental matrix becomes challenging as the size of companion matrix increases. However, for the reader's convenience specific expressions for $\Phi(x)$ corresponding to possible configurations of M for particular problems are presented in the following.

When the modulus of Winkler foundation k_w is 0, B_2 becomes a null matrix. In this case, the companion matrix M has two distinct real eigenvalues λ_3 and $\lambda_4 \neq 0$ and zero eigenvalue of multiplicity six, i.e. $\lambda_i = 0$, $i = 1, 2, 5, 6, 7, 8$. There are four corresponding real linearly independent eigenvectors v_1, v_3, v_4 and v_5. Defining vectors

$$v_2 = [0\;0\;1\;0\;0\;0\;0\;0]^T,$$
$$v_6 = [0\;0\;0\;1\;0\;0\;0\;0]^T,$$
$$v_7 = [0\;0\;0\;0\;0\;1\;0\;0]^T,$$
$$v_8 = [0\;0\;0\;0\;0\;0\;0\;1]^T,$$

the fundamental matrix for the particular case can be written as

$$\Phi(x) = [v_1 \;\; (v_1 x + v_2) \;\; v_3 e^{\lambda_3 x} \;\; v_4 e^{\lambda_4 x} \;\; v_5 \;\; (v_5 x + v_6) \;\; (v_5 x^2 + v_6 x + v_7) \;\; (v_5 x^3 + v_6 x^2 + v_7 x + v_8)].$$

When the modulus of Winkler foundation k_w is non-zero, the companion matrix M has four distinct eigenvalues $\lambda_i \neq 0$, $i = 1, \ldots, 4$ and zero eigenvalue of multiplicity four, i.e. $\lambda_j = 0$, $j = 5, \ldots, 8$. There are four corresponding linearly independent eigenvectors v_i, $i = 1, \ldots, 4$. Defining vectors

$$v_5 = [0\;1\;0\;0\;0\;0\;0\;0]^T,$$
$$v_6 = [0\;0\;0\;1\;0\;0\;0\;0]^T,$$
$$v_7 = [0\;0\;0\;0\;0\;1\;0\;0]^T,$$
$$v_8 = [0\;0\;0\;0\;0\;0\;0\;1]^T,$$

fundamental matrix for the particular case can be written as

$$\Phi(x) = [v_1 e^{\lambda_1 x} \;\; v_2 e^{\lambda_2 x} \;\; v_3 e^{\lambda_3 x} \;\; v_4 e^{\lambda_4 x} \;\; v_5 \;\; (v_5 x + v_6) \;\; (v_5 x^2 + v_6 x + v_7) \;\; (v_5 x^3 + v_6 x^2 + v_7 x + v_8)].$$

Recall that $W = x_1$. Substituting it into rearranged Eq. (16a) and integrating the obtained expression twice, the exact solution for the vector U can be obtained in the following form

$$U = A^{-1}BW' - \iint A^{-1}Q_x \mathrm{d}x + C_5 x + C_6. \quad (30)$$

The solution contains vectors of unknowns C_5 and C_6 which can be determined once the boundary conditions are known.

3. Chebyshev collocation method

The computationally efficient and accurate Chebyshev Collocation Method (CCM) is used herein to provide an alternative numerical solution for Eqs. (14).

According to CCM, an approximate solutions u, v, w and φ are presented in the truncated Chebyshev series form

$$u = \sum_{i=0}^{N} a_i T_i(x), \quad \varphi = \sum_{i=0}^{N} b_i T_i(x), \quad w = \sum_{i=0}^{N} c_i T_i(x), \quad v = \sum_{i=0}^{N} d_i T_i(x), \quad (31)$$

where Chebyshev polynomials $T_i(x)$ are used as trial functions, and $i = 1, \ldots, N$ is the degree of the Chebyshev polynomial. Substituting Eqs. (31) into the governing Eqs. (14) and minimising the residuals at the Chebyshev collocation points, the system of linear equations is obtained. To have a well-posed system of $4\times(N+1)$ equations in combination with the boundary conditions presented by Eqs. (15), the equations for u and φ associated with the residuals at the first and the last Chebyshev points are eliminated. Regarding w and v, equations associated with the first two and the last two Chebyshev points are eliminated. The system of linear equations obtained can be solved for unknown coefficients a_i, b_i, c_i and d_i. For more details of CCM see Masjedi and Ovesy (2015a). It is noted that for the current problem $N = 16$ was sufficient to obtain converged results.

4. Numerical results

4.1. Isotropic beam under distributed load

In this subsection the accuracy of the proposed mathematical model in predicting the deflection of beams resting on two-parameter elastic foundations is verified. Since numerical results for the bending of fully coupled composite beams are not available in the literature, static flexural analysis of homogeneous isotropic beam is performed and results are compared with analytical and numerical results existing in the literature. Values of mid-span deflection of uniformly loaded beam obtained from the present exact solution, analytical solution based on Green's functions introduced by Wang et al. (1998) and the DQM solution presented by Chen et al. (2004), are presented in Tables 1 and 2 for clamped–clamped and simply supported–simply supported boundary conditions, respectively. For convenience the mid-span deflection and foundation parameters are normalised as follows

$$\overline{w} = \frac{wEI_y}{q\ell^4}, \quad \overline{k}_w = \frac{k_w \ell^4}{EI_y}, \quad \overline{k}_p = \frac{k_p \ell^2}{EI_y}, \quad (32)$$

where EI_y is the flexural rigidity and the length-to-thickness ratio $\ell/h = 120$. This investigation has been performed for three different types of elastic foundation parameters, namely $\overline{k}_w = 0$ and $\overline{k}_p = 0$ (i.e. no elastic foundation); $\overline{k}_w = 0$ and $\overline{k}_p \neq 0$ and $\overline{k}_w$ and $\overline{k}_p$ both non-zero. All three sets of results obtained from different theories for different values of elastic foundation parameters are in close agreement.

4.2. Laminated composite beam under distributed load

As part of validation studies, several examples are performed regarding flexural behaviour of composite laminated beams resting on two-parameter elastic foundations. In particular, a uniform laminated beam with rectangular cross-section with properties listed in Table 3 is considered.

The analysis is performed for normalised deflections defined as

$$\overline{u} = u\frac{bh}{q_z\ell^2}E_{22}, \quad \overline{\varphi} = \varphi\frac{bh^3}{q_z\ell^3}G_{12}, \quad \overline{w} = w\frac{bh^3}{q_z\ell^4}E_{22}. \quad (33)$$

The two following examples illustrate our new solutions.

Table 1
Mid-span deflection $\overline{w} \times 10^{-2}$ of uniformly loaded beam clamped at both ends.

Foundation parameter		Present (Exact)	Present (CCM)	DQM (Chen et al., 2004)	Analytical (Wang et al., 1998)
$\overline{k}_w$	$\overline{k}_p$				
0	0	0.260417	0.260417	0.26064	0.2616
	10	0.208454	0.208454	0.20862	0.2095
	25	0.160687	0.160687	0.16081	0.1617
10	0	0.255256	0.255256	0.25547	0.2565
	10	0.205116	0.205116	0.20528	0.2062
	25	0.158681	0.158681	0.15880	0.1597
100	0	0.216547	0.216547	0.21670	0.2174
	10	0.179229	0.179229	0.17935	0.1800
	25	0.142633	0.142633	0.14273	0.1435

Table 2
Mid-span deflection $\overline{w} \times 10^{-2}$ of uniformly loaded beam simply supported at both ends.

Foundation parameter		Present (Exact)	Present (CCM)	DQM (Chen et al., 2004)	Analytical (Wang et al., 1998)
$\overline{k}_w$	$\overline{k}_p$				
0	0	1.302083	1.302083	1.302290	1.3033
	10	0.644771	0.644771	0.644827	0.6457
	25	0.366091	0.366091	0.366111	0.3671
10	0	1.180396	1.180396	1.180567	1.1814
	10	0.613275	0.613275	0.613325	0.6141
	25	0.355649	0.355649	0.355668	0.3566
100	0	0.640020	0.640020	0.640074	0.6403
	10	0.425557	0.425557	0.425582	0.4261
	25	0.282834	0.282834	0.282846	0.2836

Table 3
Beam properties.

Width (b)	0.0023	m
Number of layers	6	
Ply thickness	0.000125	m
Beam thickness (h)	0.00075	m
Beam length (ℓ)	120 h	m
E_{11}	135.64	GPa
E_{22}	10.14	GPa
G_{12}	5.86	GPa
ν_{12}	0.29	

4.2.1. Simply supported composite beam under sinusoidal distributed load

A composite beam simply supported at both ends subject to a sinusoidal distributed pressure $q_z(x) = q_0 sin(\pi x/\ell)$ is considered. This beam rests on a two-parameter foundation with $\overline{k}_w$ and $\overline{k}_p$ taking values 0, 10 and 100, and 0, 50 or 75, respectively. Numerical results for the nondimensionalised maximum deflections for symmetric $[45_3]_s$, cross-ply $[0_3/90_3]$ and asymmetric $[60_3/30_3]$ layups are tabulated in Tables 4–6 respectively. Careful analysis of these results indicates that for the symmetric layup the effect of elastic foundation parameters is more considerable on the deflection of beam, and yet less so for the deflection of beams with asymmetric and especially the cross-ply layup. This behaviour can be explained by the different combinations of coupling terms specific for each type of the layup. However, it is more important to note that the deflection is a stronger function of variations in the Pasternak foundation rather than the Winkler foundation. This deduction is confirmed by Fig. 2 which illustrates the deformed configuration of asymmetric composite beams for different values of elastic foundation parameters. It is also observed that results from the current method and those obtained from CCM are in excellent agreement.

4.2.2. Clamped composite beam under uniformly distributed load

The remaining benchmark problem, a beam clamped at both ends under the action of uniformly distributed load resting on elastic foundation, is analysed. Tables 7–9 show the nondimensionalised maximum deflections of composite beams with symmetric, cross-ply and asymmetric stacking sequences, respectively. The exact results for beams with various values of foundation parameters $\overline{k}_w$ and $\overline{k}_p$ are compared to those obtained from CCM and excellent agreement between them is observed. As in the previous example, the effects of variations in elastic foundation parameters are more significant for the symmetric type of the layup while cross-ply beams are least affected by such changes. It is clearly shown by the data in the tables and by Fig. 3 that deflections decrease as values of foundation parameters increase. Again, the results obtained from the proposed solution are in excellent agreement with CCM solutions.

Table 4
Maximum deflection of simply supported symmetric composite beam subject to sinusoidal distributed load.

Foundation parameter		Exact			CCM		
$\overline{k}_w$	$\overline{k}_p$	$\lvert\overline{u}\rvert$	$\lvert\overline{\varphi}\rvert$	$\lvert\overline{w}\rvert$	$\lvert\overline{u}\rvert$	$\lvert\overline{\varphi}\rvert$	$\lvert\overline{w}\rvert$
0	0	–	1.089099e−02	6.653712e−02	–	1.089099e−02	6.653712e−02
	50	–	5.159309e−03	3.208836e−02	–	5.159309e−03	3.208836e−02
	75	–	4.066391e−03	2.548382e−02	–	4.066391e−03	2.548382e−02
10	0	–	1.075952e−02	6.573534e−02	–	1.075952e−02	6.573534e−02
	50	–	5.129010e−03	3.190046e−02	–	5.129010e−03	3.190046e−02
	75	–	4.047386e−03	2.536510e−02	–	4.047386e−03	2.536510e−02
100	0	–	9.704845e−03	5.930363e−02	–	9.704845e−03	5.930363e−02
	50	–	4.871482e−03	3.030344e−02	–	4.871482e−03	3.030344e−02
	75	–	3.883983e−03	2.434442e−02	–	3.883983e−03	2.434442e−02

Table 5
Maximum deflection of simply supported composite beam with cross-ply layup subject to sinusoidal distributed load.

Foundation parameter		Exact			CCM		
$\bar{k}_w$	$\bar{k}_p$	$\lvert\bar{u}\rvert$	$\lvert\bar{\varphi}\rvert$	$\lvert\bar{w}\rvert$	$\lvert\bar{u}\rvert$	$\lvert\bar{\varphi}\rvert$	$\lvert\bar{w}\rvert$
0	0	6.573075e−01	–	2.168303e−02	6.573075e−01	–	2.168303e−02
	50	4.802794e−01	–	1.608038e−02	4.802794e−01	–	1.608038e−02
	75	4.223364e−01	–	1.423876e−02	4.223364e−01	–	1.423876e−02
10	0	6.547323e−01	–	2.159845e−02	6.547323e−01	–	2.159845e−02
	50	4.788790e−01	–	1.603374e−02	4.788790e−01	–	1.603374e−02
	75	4.212444e−01	–	1.420216e−02	4.212444e−01	–	1.420216e−02
100	0	6.324286e−01	–	2.086591e−02	6.324286e−01	–	2.086591e−02
	50	4.666316e−01	–	1.562583e−02	4.666316e−01	–	1.562583e−02
	75	4.116633e−01	–	1.388095e−02	4.116633e−01	–	1.388095e−02

Table 6
Maximum deflection of simply supported asymmetric composite beam subject to sinusoidal distributed load.

Foundation parameter		Exact			CCM		
$\bar{k}_w$	$\bar{k}_p$	$\lvert\bar{u}\rvert$	$\lvert\bar{\varphi}\rvert$	$\lvert\bar{w}\rvert$	$\lvert\bar{u}\rvert$	$\lvert\bar{\varphi}\rvert$	$\lvert\bar{w}\rvert$
0	0	4.476041e−01	9.803684e−03	4.886224e−02	4.476041e−01	9.803684e−03	4.886224e−02
	50	2.458331e−01	5.384379e−03	2.734382e−02	2.458331e−01	5.384379e−03	2.734382e−02
	75	1.997703e−01	4.375486e−03	2.240416e−02	1.997703e−01	4.375486e−03	2.240416e−02
10	0	4.436353e−01	9.716758e−03	4.843009e−02	4.436353e−01	9.716758e−03	4.843009e−02
	50	2.446052e−01	5.357485e−03	2.720775e−02	2.446052e−01	5.357485e−03	2.720775e−02
	75	1.989512e−01	4.357544e−03	2.231266e−02	1.989512e−01	4.357544e−03	2.231266e−02
100	0	4.108418e−01	8.998495e−03	4.485929e−02	4.108418e−01	8.998495e−03	4.485929e−02
	50	2.340802e−01	5.126962e−03	2.604139e−02	2.340802e−01	5.126962e−03	2.604139e−02
	75	1.918689e−01	4.202424e−03	2.152165e−02	1.918689e−01	4.202424e−03	2.152165e−02

Table 7
Maximum deflection of clamped symmetric composite beam subject to uniformly distributed load.

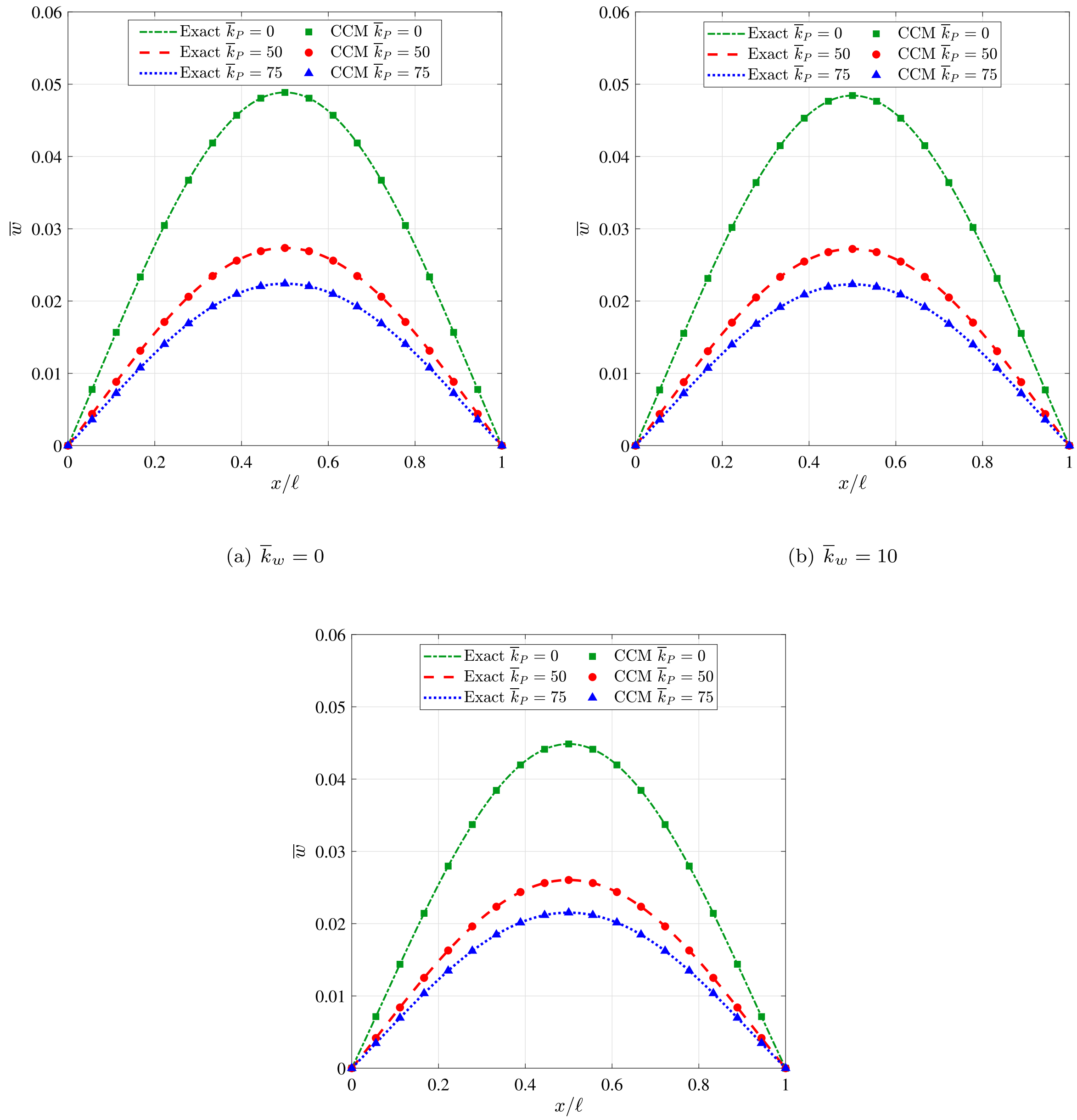
Foundation parameter		Exact			CCM		
$\bar{k}_w$	$\bar{k}_p$	$\lvert\bar{u}\rvert$	$\lvert\bar{\varphi}\rvert$	$\lvert\bar{w}\rvert$	$\lvert\bar{u}\rvert$	$\lvert\bar{\varphi}\rvert$	$\lvert\bar{w}\rvert$
0	0	–	1.286304e−02	2.157635e−02	–	1.286304e−02	2.157635e−02
	50	–	9.616726e−03	1.611227e−02	–	9.616726e−03	1.611227e−02
	75	–	8.551895e−03	1.431035e−02	–	8.551895e−03	1.431035e−02
10	0	–	1.282341e−02	2.150974e−02	–	1.282341e−02	2.150974e−02
	50	–	9.594523e−03	1.607481e−02	–	9.594523e−03	1.607481e−02
	75	–	8.534342e−03	1.428070e−02	–	8.534342e−03	1.428070e−02
100	0	–	1.247736e−02	2.092803e−02	–	1.247736e−02	2.092803e−02
	50	–	9.399205e−03	1.574528e−02	–	9.399205e−03	1.574528e−02
	75	–	8.379554e−03	1.401919e−02	–	8.379554e−03	1.401919e−02

Table 8
Maximum deflection of clamped composite beam with cross-ply layup subject to uniformly distributed load.

Foundation parameter		Exact			CCM		
$\bar{k}_w$	$\bar{k}_p$	$\lvert\bar{u}\rvert$	$\lvert\bar{\varphi}\rvert$	$\lvert\bar{w}\rvert$	$\lvert\bar{u}\rvert$	$\lvert\bar{\varphi}\rvert$	$\lvert\bar{w}\rvert$
0	0	7.763272e−01	–	9.762019e−03	7.763272e−01	–	9.762019e−03
	50	6.730439e−01	–	8.461063e−03	6.730439e−01	–	8.461063e−03
	75	6.312943e−01	–	7.933801e−03	6.312943e−01	–	7.933801e−03
10	0	7.752432e−01	–	9.748361e−03	7.752432e−01	–	9.748361e−03
	50	6.722280e−01	–	8.450763e−03	6.722280e−01	–	8.450763e−03
	75	6.305762e−01	–	7.924729e−03	6.305762e−01	–	7.924729e−03
100	0	7.656210e−01	–	9.627112e−03	7.656210e−01	–	9.627112e−03
	50	6.649725e−01	–	8.359168e−03	6.649725e−01	–	8.359168e−03
	75	6.241861e−01	–	7.843994e−03	6.241861e−01	–	7.843994e−03

5. Conclusions

The analysis of the flexural behaviour of composite beams with coupling terms between stretching, shearing, bending and twisting, resting on two-parameter elastic foundations has been presented. The governing equations and boundary conditions were derived using the principle of virtual work under the assumptions of Euler–Bernoulli beam theory. Once the governing equations were expressed in compact matrix–vector form, the method of direct integration was used to decouple bending from axial elongation and twist, then the decoupled equations of bending in two principal direction were transformed into a system of first-order differential equations. Using a fundamental matrix approach the exact solution of the problem was obtained. The efficacy of the proposed model in predicting flexural response of composite beams on elastic foundations was verified by comparing results with those available in the literature. To demonstrate the ability of the proposed solution to predict the deflection of beams accurately with different combinations of coupling terms and different boundary and loading conditions, the flexural behaviour of fully clamped and fully simply supported composite beams subject to uniformly and



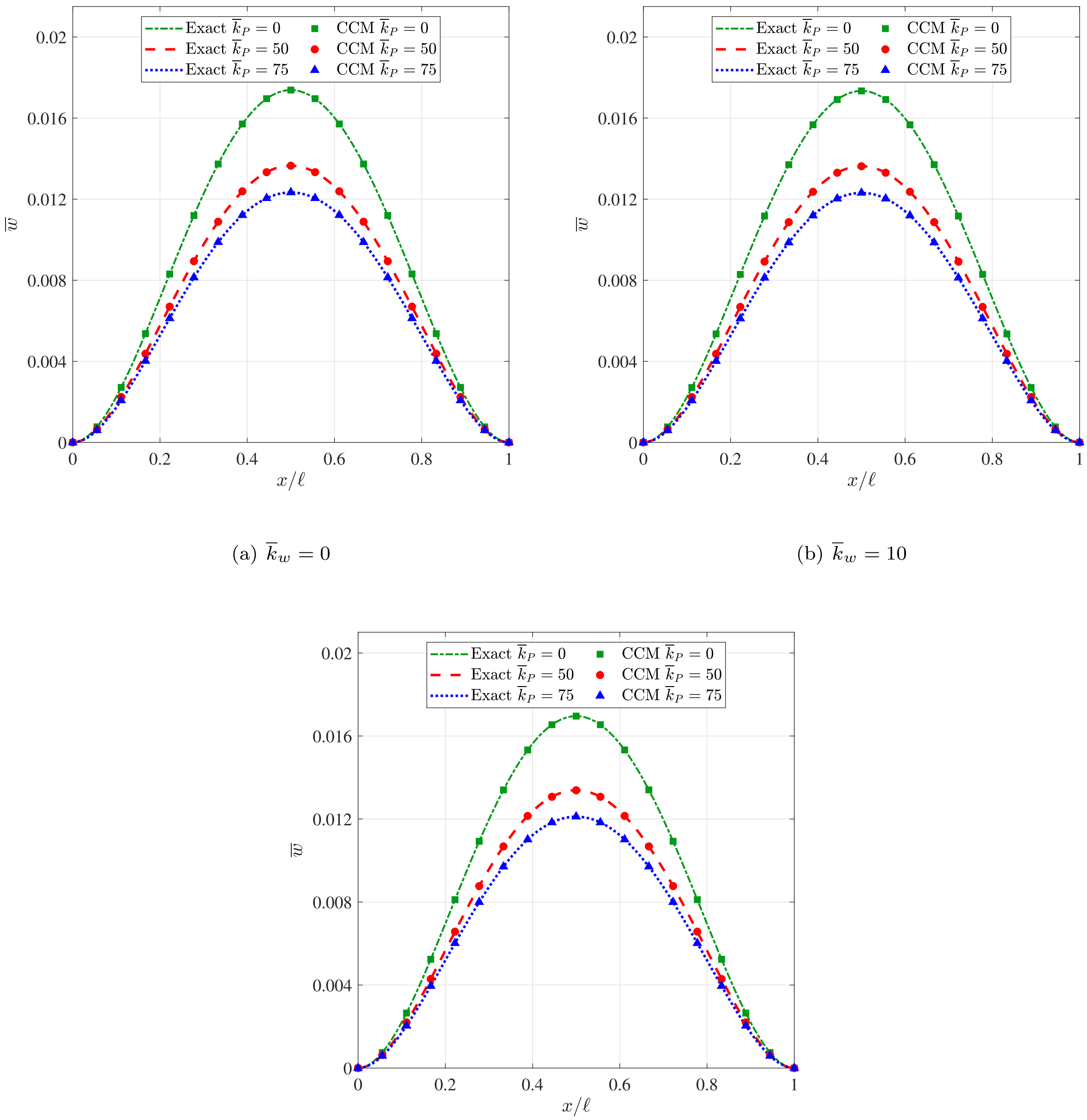
(a) $\overline{k}_w = 0$ (b) $\overline{k}_w = 10$ (c) $\overline{k}_w = 100$

Fig. 2. Normalised deflection $\overline{w}$ of simply supported asymmetric composite beam on elastic foundation subjected to sinusoidal loading.

Table 9
Maximum deflection of clamped asymmetric composite beam subject to uniformly distributed load.

Foundation parameter		Exact			CCM		
$\overline{k}_w$	$\overline{k}_p$	$\lvert\overline{u}\rvert$	$\lvert\overline{\varphi}\rvert$	$\lvert\overline{w}\rvert$	$\lvert\overline{u}\rvert$	$\lvert\overline{\varphi}\rvert$	$\lvert\overline{w}\rvert$
	0	5.286524e−01	1.157885e−02	1.738319e−02	5.286524e−01	1.157885e−02	1.738319e−02
0	50	4.154669e−01	9.099798e−03	1.365077e−02	4.154669e−01	9.099798e−03	1.365077e−02
	75	3.756769e−01	8.228295e−03	1.233280e−02	3.756769e−01	8.228295e−03	1.233280e−02
	0	5.273394e−01	1.155009e−02	1.733993e−02	5.273394e−01	1.155009e−02	1.733993e−02
10	50	4.146543e−01	9.081999e−03	1.362391e−02	4.146543e−01	9.081999e−03	1.362391e−02
	75	3.750125e−01	8.213741e−03	1.231081e−02	3.750125e−01	8.213741e−03	1.231081e−02
	0	5.158075e−01	1.129752e−02	1.695993e−02	5.158075e−01	1.129752e−02	1.695993e−02
100	50	4.074803e−01	8.924871e−03	1.338679e−02	4.074803e−01	8.924871e−03	1.338679e−02
	75	3.691362e−01	8.085036e−03	1.211635e−02	3.691362e−01	8.085036e−03	1.211635e−02

sinusoidally distributed loads were analysed for different values of elastic foundation coefficients. It was demonstrated numerically that



(a) $\overline{k}_w = 0$

(b) $\overline{k}_w = 10$

(c) $\overline{k}_w = 100$

Fig. 3. Normalised deflection $\overline{w}$ of clamped asymmetric composite beam on elastic foundation subjected to sinusoidal loading.

with increasing values of foundation stiffness, bending displacements decrease and that the effect of Pasternak stiffness parameter k_p on the displacement is greater than the Winkler stiffness parameter k_w. In addition, the considerable influence of the type of stacking sequence on the flexural behaviour of beams was discussed. In conclusion, the solutions presented within are simple and provide an efficient means for analysing the flexural behaviour of composite beams resting on elastic foundations which can also serve as benchmarks in assessing the computational accuracy and efficiency of various numerical methods.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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