A strain-displacement variational formulation for laminated composite beams based on the modified couple stress theory

1. Introduction

Mixed energy methods, e.g. Hellinger-Reissner (HR) method, can include stresses and displacements as unknowns in the formulation and can be made to satisfy equilibrium conditions as well as minimise total energy. The HR method accurately captures the full 3D stress field of general beams and plates [1].

The present study aims to:
- develop strain-displacement (SD) mixed energy method for stress analysis of laminated beams.
- include couple-stress in the SD formulation to accurately predict 3D stresses of highly heterogeneous laminates.
- implement DQM method to solve the governing equations subject to different boundary conditions.

2. Mathematical formulation

Consider a laminated beam with the length and the rectangular cross-section being L and b x t. The beam is constituted of N homogeneous layers and fibre angle (in stacking sequence) \( g^k \) in \( k \)th layer is referred to x-axis. Equivalent single layer (ESL) theory is used.

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Displacement profile (Third-order zigzag):
- Axial strain:
  \[ \epsilon_s^{(3)} = \frac{\partial u_s}{\partial x} = \frac{1}{2} \left( \epsilon_x^{(0)} - \epsilon_y^{(0)} \right) \]
- Micro-rotation
- Axial stress:
  \[ \sigma_s^{(3)} = \frac{\partial \phi_s}{\partial x} \]
- Micro-curvature
- Stress resultant (by strain vector):
  \[ P = \int_{0}^{L} \left( \epsilon_s^{(3)} \bar{E}_s + \epsilon_y^{(3)} \bar{E}_y \right) \bar{t} \, dx = S_c \]
- Coupled stress resultant (by micro-curvature)
- Transverse shear & normal stresses
- Equilibrium in Modified couple stress theory (MCST)
- Integration through thickness (ESL)
- Lagrange multipliers (corresponding to displacements)

Variational principle is performed with respect to Strain and Displacement functionals for Governing and Boundary equations (SD MCS model).
A reduced strain-displacement (SD) model without couple stress is obtained by removing the micro-curvature vector.

3. Results

Material properties and laminate configuration:

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<th>Laminate</th>
<th>Number of layers</th>
<th>Material properties</th>
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Example 1: Simply supported beam under sinusoidal load, the span-to-thickness ratio L/t=8. Laminates A (symmetric) and B (non-symmetric).

Example 2: Clamped beam under uniformly distributed load, the span-to-thickness ratio L/t=10. Laminates 1 (non-symmetric) and Laminate 2 (symmetric).

4. Conclusions

- Both the SD and SD MCS models can accurately capture the flexural behaviour of various simply-supported beams including symmetric and antisymmetric laminates, as well as thick-soft core sandwich beams.
- The SD MCS model including the couple stress can predict localised stress efficiently near the clamped boundary, which is challenging for ESL models. (31x8 = 248 DOFs in SD MCS vs 96,000 C3D8R brick elements in 3D FE).
- Changing the length scale parameter, i.e. \( \tilde{G}^{(2)} \), in couple stress can result in better optimisation of system total energy.

References