Probabilistic Free Vibration Analysis of Functionally Graded Beams using Stochastic Finite Element Methods

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Abstract. In this study, we use the stochastic finite element method for vibration analysis of functionally graded (FG) Euler-Bernoulli beams considering variability in material properties. The selected FG material consists of a mix of ceramic and metal constituents. The material properties of the FG beams studied are assumed to vary smoothly over the depth according to a power law. Constituent material properties such as the Young's modulus, mass density and volume fraction index are modeled as random variables. For each simulation of these random parameters, finite element method is employed to estimate natural frequencies of FG beam. Several simulations need to be carried out for propagating overall inputs uncertainty to stochastic frequencies that are approximated as a series in an orthogonal space. The components of series will be determined based on both polynomial chaos expansion (PCE) and stochastic collocation (SC) methods. For PCE, the multivariate Hermite orthogonal functions are derived using Askey scheme. Their coefficients are estimated using both spectral projection, linear regression approaches. Standard tensor product is used to integrate the multi-dimensional integrals. In term of SC method, basis functions are Lagrange interpolation functions formed for known coefficients called collocation points. Post-analysis including reliability, sensitivity and distribution of uncertain frequencies are also studied. These results will also be compared with those of Monte Carlo Simulation.

Keywords: Functionally graded beam; Stochastic finite element method; Uncertainty propagation; Polynomial chaos expansions; Stochastic collocation.

1 Introduction

Computational models that simulate real-world physical processes are playing an ever-increasing role in engineering and physical sciences. Computational models are being used to study complex processes as large scale as the evolution of the universe and as small scale as protein folding. However, computational results almost always depend on inputs that are uncertain, rely on approximations that introduce errors, and are based on mathematical models that are imperfect representation of reality [4]. It is important to understand why predictions from physics-based models differ from observed reality and compute corrective action to bring them closer to each other. Uncertainty quantification (UQ) is one of broadly growing fields to deal with those kinds of problems. The first purpose of UQ is to quantify uncertainty in model inputs, often by specifying range or probability distribution [4]. These uncertainties will be propagated through the computational model to predict the overall uncertainty in the response. Understanding the mapping of parameters uncertainty to output uncertainty plays a fundamental role in assessment of prediction uncertainty and in gaining knowledge in model behavior.

This research will focus on uncertainty propagation. It means that the probability distribution functions of uncertain inputs are well defined. Key issue is then to propagate that uncertainty through the finite element model to obtain the statistics of the output.

Several methods have been used to model and propagate uncertainty in stochastic computational simulations. Among these, Monte Carlo Simulation (MCS) is a classical and straightforward way. However, this method requires thousands or even millions of simulation to achieve the desired result. For some computational models like finite element method that is quite costly for one simulation, it is unrealistic to apply MCS. Recently, stochastic finite element methods that based on polynomial chaos expansion (PCE) have gained considerable attention ([6],[17]). The stochastic outputs will be approximated as a series in an orthogonal space including the basis functions and their appropriate coefficients. The multivariate polynomial functions can be easily derived for different probability distribution using Askey scheme [22]. Coefficients can be estimated using spectral projection (SP) and linear regression (LR). SP approach turns out evaluating numerically multi-dimensional integral using either probabilistic or deterministic techniques [7]. Probabilistic approaches using sampling like MCS, Latin hypercube simulation (LHS) that are very costly and slowly convergent. Whereas, deterministic techniques using standard Gauss quadrature or Smolyak's quadrature are more efficient. Detailed description about LR method and their adaptive methods can be found in ([18],[2],[9],[12],[26]). A stochastic collocation (SC) method based on Lagrange interpolation functions and quadrature rule was presented in [3] for dealing with some mechanical problems like static, dynamic and spectral problems. As the number of random variables increases, the number of quadrature points in SP and SC increases exponentially cause the curse of dimensionality. Sparse grid methods were proved to be efficient for moderate multi-dimensional problems ([13],[25],[10],[27]).

Additionally, functionally graded materials (FGM) have been extensively attractive to many researchers and scientists in recent years. Those are widely employed in aerospace, micromechanics, nuclear power and health care application. A thousand of publications related to FGM have been already reported each year during the recent decade, most of all focus on stability and vibration analysis [8]. A critical review about FGM was also given in ([8],[21]) in which different type of fabrication techniques, specific applications of FGM, different theories adopted for static and dynamic response of FGM beam and plate has been discussed. However, the processing and manufacturing methods of FGM are very complex. The existing and most updated techniques cannot ensure FGM has the exactly desired characteristic ([8],[23]). Moreover, properties of material can be dependent on temperature so if FG structures work on extreme environments that may cause the significant changes in structural stiffness. All the variability in properties of materials, geometry and loading can induce uncertainty in the results. Therefore, it is important to fully understand the behavior of FG structures with randomness in material properties.

A few research in uncertainty of FGM and composites have been carried out. A FEM based on direct iterative procedure combined with mean center first order perturbation technique (FOPT) was developed for nonlinear vibration and bending analysis of FG beams with random materials ([15],[16]). Statistical moments of buckling loads and frequencies of FGM plates with uncertain material properties in thermal environments were investigated using FOPT ([19],[20],[11]). Those results were compared with MCS. However, this method just estimated the mean values and covariance of outputs. The distribution functions of outputs and post-analysis like reliability and sensitivity analysis were not considered. First Order Reliability Method (FORM) and the Second Order Reliability Method (SORM) were used to investigate sensitivity of the fundamental frequency of FGM plates to material uncertainties [14]. These methods need only few times to get the results in comparison with a thousand of times required to get the results using MCS. However, each step need to derive first and second derivatives of stiffness and mass matrices symbolically that can be time consuming. Moreover, the results obtained from FORM and SORM are not very good, especially when the limit state function is high-order nonlinear. The random factor method was developed for stochastic vibration analysis of FGM beam in [23]. In this study, the randomness of the beam frequencies is explicitly expressed by random factors of constituent material parameters, and only the statistics of output were analytically computed by the statistics of random inputs. In [24], the uncertain properties of FGM were expressed as a Taylor's series using FOPT and then uncertain frequencies were computed using stochastic finite element method. Recently, Kriging metalmodel and the random sampling high dimensional model representation were used for stochastic free vibration analysis of functionally graded carbon nanotube reinforced plates. The detail of these methods can be referred in [5].

In this paper, uncertainty of FG material properties will be propagated through stochastic finite element model to estimate the uncertainty in natural frequencies of Euler-Bernoulli beam. The stochastic natural frequencies are approximated in orthogonal space using PCE and SC method. The multivariate Hermite polynomials are basis functions and both spectral projection and linear regression techniques are proposed to calculate the coefficients in PCE. Whereas, SC forms basis functions using Lagrange polynomials. Monte Carlo simulation is considered as the correct method to compare with the proposed methods. As referred to [5], the flowchart of this problem is presented in Fig. 1.

2 Theoretical Formulation

Consider a beam with the length, L, and rectangular section, $b \times h$, made of a functionally graded material (FGM) consisting of a mixture of ceramic and metal as shown in Fig. (2). The material properties of the FG beam are assumed to vary smoothly over the thickness according to a power-law form as follows:

$$P(z) = (P_c - P_m)\left(\frac{2z+h}{2h}\right)^r + P_m \tag{1}$$

where $P(z)$ represents the effective material property of interest at location z, such as the Young's modulus, E, or mass density, ρ; subscripts c and m denote the ceramic and metal phases, respectively; and r is a power-law index which is a positive scalar parameter, and $z \in [-\frac{h}{2}, \frac{h}{2}]$. All the constituent material properties and the power-law index are assumed to be random and are represented by the distributions and parameters shown in Table 1.

2.1 FG beam theory

Based on Euler-Bernoulli beam theory, the displacement field is given by the following expressions:

$$U(x,z) = u(x) - zw_{,x} \tag{2a}$$

$$W(x,z) = w(x) \tag{2b}$$

where u is the mid-plane axial displacement, w denotes the mid-plane transverse displacement of the beam, the comma indicates partial differentiation with respect to the coordinate subscript that follows.

Assuming a small deformation, strains associated with the displacement field in Eq. (2) are:

$$\epsilon_{xx}(x,z) = u_{,x} - zw_{,xx} \tag{3a}$$

$$\gamma_{xz}(x,z) = 0 \tag{3b}$$

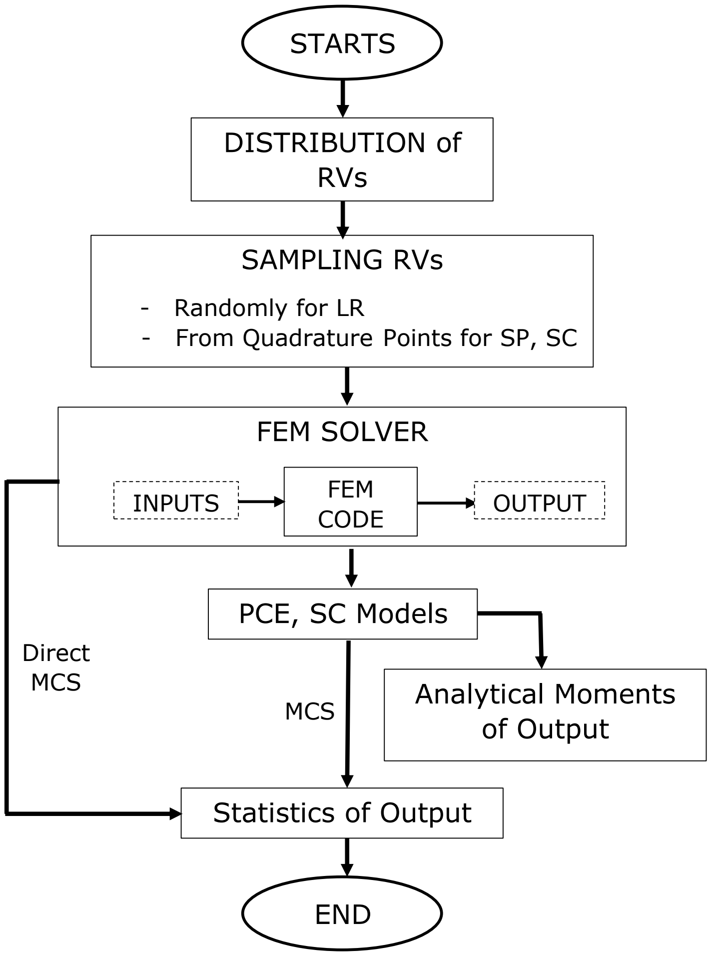

Fig. 1: Flowchart for stochastic free vibration analysis of FG beam using PCE and SC.

The normal and shear stresses are given by

$$\sigma_{xx}(x,z) = E(z)\epsilon_{xx}(x,z) \tag{4a}$$

$$\sigma_{xz}(x,z) = 0 \tag{4b}$$

2.2 Finite element model

Consider a two-node beam element for Euler-Bernoulli beam theory as shown in Fig. (3). For each element with length, l_e, Lagrange linear interpolation functions are chosen for axial displacement and Hermite cubic ones for transverse deflection. Therefore, the relationship between the displacement vector and the nodal displacement vector can be expressed as:

$$\begin{bmatrix} u & w \end{bmatrix}^T = [\mathbf{N}]\{\Delta^e\} = [\mathbf{N}]\begin{bmatrix} \Delta_1^e & \Delta_2^e & \Delta_3^e & \Delta_4^e & \Delta_5^e & \Delta_6^e \end{bmatrix}^T \tag{5}$$

where $[\mathbf{N}]$ is a shape function matrix.

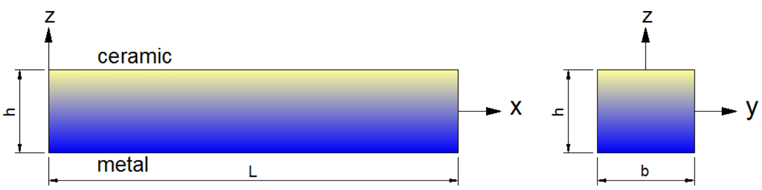

Fig. 2: Geometry of isotropic FG beam.

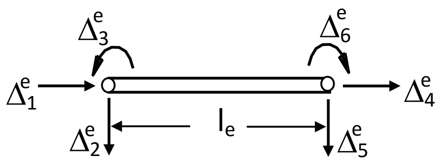

Fig. 3: A two-nodes beam element.

The strain and kinetic energy of the beam element are obtained as:

$$\begin{aligned}\mathcal{U}_e &= \frac{1}{2}\int_{V_e}(\sigma_{xx}\epsilon_{xx}+\sigma_{xz}\gamma_{xz})dV_e \\ &= \frac{1}{2}\int_0^{l_e}\Big[A_x(u_{,x})^2 - 2B_x u_{,x}w_{,xx} + D_x(w_{,xx})^2\Big]dx \end{aligned} \tag{6}$$

$$\begin{aligned}\mathcal{K}_e &= \frac{1}{2}\int_{V_e}\rho(z)(\dot{U}^2+\dot{W}^2)dV_e \\ &= \int_0^{l_e}\Big[I_0\left(\dot{u}^2+\dot{w}^2\right) - 2I_1\dot{u}\dot{w}_{,x} + I_2(\dot{w}_{,x})^2\Big]dx \end{aligned} \tag{7}$$

where the differentiation with respect to the time t is denoted by the dot-superscript convention; $(A_x,\ B_x,\ D_x)$ are the stiffnesses of FG beams given by:

$$(A_x, B_x, D_x) = \int_{-h/2}^{h/2}\left(1, z, z^2\right) b dz \tag{8}$$

and I_0, I_1, I_2 are the inertia coefficients defined by:

$$(I_0, I_1, I_2) = \int_{-h/2}^{h/2}\rho(z)(1, z, z^2) b dz \tag{9}$$

Substituting Eq. (5) into Eqs. (6), (7) and applying Hamilton's principle leads to the following finite element model for free vibration of the beam element:

$$[\mathbf{K}_e]\{\mathit{\Delta}_e\} + [\mathbf{M}_e]\{\ddot{\mathit{\Delta}}_e\} = 0 \tag{10}$$

where the element stiffness matrix $\mathbf{K}_e$ and the mass matrix $\mathbf{M}_e$ are given by:

$$[\mathbf{K}_e] = \int_0^{l_e}\Big(\{\mathbf{B}_1\}^T A_x\{\mathbf{B}_1\} - \{\mathbf{B}_3\}^T B_x\{\mathbf{B}_1\} - \{\mathbf{B}_1\}^T B_x\{\mathbf{B}_3\} + \{\mathbf{B}_3\}^T D_x\{\mathbf{B}_3\}\Big)dx \tag{11}$$

$$[\mathbf{M}_e] = \int_0^{l_e} \left(\{\mathbf{N}_1\}^T I_0 \{\mathbf{N}_1\} + \{\mathbf{N}_2\}^T I_0 \{\mathbf{N}_2\} - \{\mathbf{N}_1\}^T I_1 \{\mathbf{B}_2\} - \{\mathbf{B}_2\}^T I_1 \{\mathbf{N}_1\} + \{\mathbf{B}_2\}^T I_2 \{\mathbf{B}_2\} \right) dx \quad (12)$$

where

$$\{\mathbf{N}_1\} = \left[1 - \frac{x}{l_e} \quad 0 \quad 0 \quad \frac{x}{l_e} \quad 0 \quad 0 \right] \quad (13a)$$

$$\{\mathbf{N}_2\} = \left[0 \quad 1 - 3\left(\frac{x}{l_e}\right)^2 + 2\left(\frac{x}{l_e}\right)^3 \quad x - 2\frac{x^2}{l_e} + \frac{x^3}{l_e^2} \quad 0 \quad 3\left(\frac{x}{l_e}\right)^2 - 2\left(\frac{x}{l_e}\right)^3 \quad -\frac{x^2}{l_e} + \frac{x^3}{l_e^2} \right] \quad (13b)$$

$$\{\mathbf{B}_1\} = \frac{d\{\mathbf{N_1}\}}{dx} \quad (13c)$$

$$\{\mathbf{B}_2\} = \frac{d\{\mathbf{N_2}\}}{dx} \quad (13d)$$

$$\{\mathbf{B}_3\} = \frac{d^2\{\mathbf{N_2}\}}{dx^2} \quad (13e)$$

After assembling, applying boundary conditions and setting $\Delta = \Delta_0 e^{-i\omega t}$, the global governing equation for free vibration analysis is obtained as follows:

$$\left[[\mathbf{K}] - \omega^2 [\mathbf{M}] \right] \{\Delta\} = 0 \quad (14)$$

where ω is the natural frequency of the FG beam.

3 Stochastic spectral methods for uncertainty propagation

For uncertainty computation, the random model response can be approximated as a series in an orthogonal space. This series has two components: the basis functions and their appropriate coefficients:

$$u(x, t, \mathbf{q}) = \sum_{i=0}^{\infty} \alpha_i (x, t) \Psi_i (\mathbf{q}) \quad (15)$$

where $\mathbf{q}$ is a vector of random variables; $\{\Psi_i\}$ are the multivariate orthogonal functions and $\{\alpha_i\}$ are coefficients.

There are two approaches depending on the ways to determine its components: polynomial chaos expansion (PCE) and stochastic collocation (SC). PCE estimates the coefficients in a suitable set of bases using either projection approach or linear regression, whereas, SC forms the interpolation polynomials for the known coefficients called collocation points [1]. Herein, multivariate Hermite polynomial and Lagrange interpolation are using as the basis functions for PCE and SC, respectively.

3.1 Polynomial Chaos Expansion

For convenience, in mathematical computation, the above series Eq. (15) can be truncated using a finite number of terms. If the number of random variable is d and the qualified order of polynomial is p, the number of full polynomial terms N is the permutation of p and $d + p$, determined as follows:

$$N = \frac{(d+p)!}{d!p!} \quad (16)$$

Equation (15) becomes

$$u(x, t, \mathbf{q}) = \sum_{i=0}^{N} \alpha_i (x, t) \Psi_i (\mathbf{q}) + \epsilon \quad (17)$$

Herein, multivariate Hermite polynomials are used as the basis functions and their coefficients should be determined so that the error or residual ϵ is minimized.

Spectral Projection approach The residual minimum requires that it must be orthogonal with the projection of response in selected space or the inner product of the residual and each basis function is zero. From Eq. (17), taking the inner product of both side with respect to Ψ_j and enforcing orthogonality yields:

$$< u, \Psi_j >= \sum_{i=0}^{N} \alpha_i < \Psi_i, \Psi_j > \tag{18}$$

Because Ψ_j are mutually orthogonal, Eq. (18) becomes

$$\alpha_i = \frac{< u, \Psi_i >}{< \Psi_i, \Psi_i >} \tag{19}$$

Theoretically, all coefficients can be obtained by solving Eq. (19), however the random response u is unknown. Moreover, each inner product involves a multidimensional integral evaluated numerically using either probabilistic techniques (sampling) or deterministic techniques (quadrature rules, sparse grid approaches). This paper will use Gauss Hermite quadrature for solving $\{\alpha_i\}$. The normalization factor $< \Psi_i, \Psi_i >$ can be analytical estimated.

For 1-D integral:

$$\alpha_i \propto < u, \psi_i >= \int u(q)\, \psi_i(q)\, \rho_Q(q)\, dq = \sum_{j=1}^{N_{gp}} w_j u(q_j)\, \psi_i(q_j) \tag{20}$$

where N_{gp} is the number of quadrature points, w_j and q_j are the set of weights and quadrature points, respectively.

In case of d-dimension, it can be expressed using simple tensor products "probabilistic" Gauss-Hermite as follows

$$\alpha_i \propto < u, \Psi_i >= \sum_{j_1=1}^{N_{gp}^1} \cdots \sum_{j_d=1}^{N_{gp}^d} \left(w_{j_1}^1 \otimes \cdots \otimes w_{j_d}^d\right) u\left(q_{j_1}^1, \ldots, q_{j_d}^d\right) \Psi_i\left(q_{j_1}^1, \ldots, q_{j_d}^d\right) \tag{21}$$

If the order of integral function is q, the minimum number of Gauss point for each random variable is $N_{gp} = \frac{2p+1}{2}$ and the total number of terms to estimate coefficients is $d^{N_{gp}}$. Hence, if the model response $u\left(q_{j_1}^1 \ldots q_{j_d}^d\right)$ is obtained from finite element solution, we need to solve $d^{N_{gp}}$ deterministic problems. Clearly, this method is quite expensive for multidimensional and higher-order problems.

Linear regression approach Let $\Re = \{\mathbf{q}^1, \ldots, \mathbf{q}^{Ns}\}$ be a set of Ns $(Ns > N)$ realizations of input random vector, and $U = \{u^1, \ldots, u^{Ns}\}$ be corresponding output evaluations $\left(u^i = u\left(x, t, \mathbf{q}^i\right), i = 1, \ldots, Ns\right)$.The vector of residuals can be estimated from Eq. (17) in the compact form:

$$\Upsilon = U - \alpha^T \Psi \tag{22}$$

where Ψ is the matrix whose elements are given by $\Psi_{ij} = \Psi_j(\mathbf{q}^i)$, $i = 1, \ldots, Ns$; $j = 1, \ldots, N$. The coefficients α are estimated by minimizing the L_2-norm (least-square regression) of the residual followed as

$$\alpha = Arg\ \ min \parallel U - \alpha^T \Psi \parallel_2^2 \tag{23}$$

Solving Eq. (23), the coefficients are given by

$$\alpha = \left(\Psi^T \Psi\right)^{-1} \Psi^T U \tag{24}$$

3.2 Stochastic Collocation

In Stochastic collocation method, instead of using polynomials orthogonal with respect to probability density function as above, Lagrange interpolation functions are used. These functions are formed from Gauss quadrature points called collocation points.

In 1-D problem, after truncating Eq. (15) becomes

$$u\left(x,t,q\right)=\sum_{i=0}^{N_p}\alpha_i\left(x,t\right)L_i\left(q\right) \tag{25}$$

where N_p is the number of collocation point, $L_i(q)$ is the Lagrange interpolation function formed from a set of N_p probabilistic Gauss Hermite quadrature points $\{q_k, k=1,\ldots,N_p\}$ as follows:

$$L_i(q)=\prod_{j=1}^{N_p}\frac{q-q_j}{q_i-q_j}, j\neq i \tag{26}$$

Lagrange interpolation function satisfies $L_i(q_k)=\delta_{ik}$, so the coefficients $\alpha_i\left(x,t\right)$ in Eq. (26) are the solutions $u\left(x,t,q\right)$ at the collocation point q_i:

$$\alpha_i\left(x,t\right)=u\left(x,t,q_i\right) \tag{27}$$

And Eq. (25) becomes

$$u\left(x,t,q\right)=\sum_{i=0}^{N_{gp}}u\left(x,t,q_i\right)L_i\left(q\right) \tag{28}$$

When it comes to multi-dimensional problem, standard tensor products can be used to transfer 1-D interpolation functions to multi-dimension ones. Let $\{q^i_{j_i}, j_i=1,\ldots,N^i_{gp}, i=1,\ldots,d\}$ be the set of collocation points of i^{th} random variable, the full-tensor product interpolation of d-dimensional function can be expressed:

$$u\left(\mathbf{q}\right)=\sum_{j_1=1}^{N^1_{gp}}\cdots\sum_{j_d=1}^{N^d_{gp}}u\left(q^1_{j_1}\ldots q^d_{j_d}\right)\left(L^1_{j_1}\otimes\cdots\otimes L^d_{j_d}\right) \tag{29}$$

When dealing with probabilistic problems using PCE or SC, one need to estimate corresponding outputs $u\left(x,t,\mathbf{q}\right)$ at each realization of random inputs $\mathbf{q}_i$. In this paper, frequencies of FG beams for each set of certain inputs will be estimated by solving ω in Eq. (14) as presented in Section 2.

4 Numerical Examples

In this study, the material properties are random, but they are assumed to be independent of the geometry of FG beams. The detailed properties are summarized in Table 1. The numerical results are carried out for three boundary conditions, including clamped-free (C-F), simply supported (S-S) and clamped-clamped (C-C). The statistical moments of frequencies and sensitivity indices of parameters are analytically estimated based on the coefficients of PCE and SC as mentioned in [1]. Herein, MCS's results with 10000 simulations are considered as the references to compare with those of PCE and SC.

In order to verify the convergence of the proposed methods, Table 2 summaries the mean μ and standard variation σ of first three natural frequencies of FG beams with different

Table 1: Distributions and parameters of material properties.

Parameters	$E_c \times 10^6 (Mpa)$	$E_m \times 10^6 (Mpa)$	$\rho_c (kg/m^3)$	$\rho_m (kg/m^3)$	r
mean (μ)	151	70	3000	2707	–
COV (%)	10	10	10	10	10
Distribution	Normal	Normal	Normal	Normal	Lognormal

orders of PCE in which a number of quadrature points for each random variables $Ngp = 3$ and the number of simulation for LR $Ns = 150$. Those results are recalculated for different quadrature points chosen in SP and SC and presented in Table 3. It can be seen that if the polynomial order $p = 4$ and Gauss quadrature point $Ngp = 3$, the results are convergent and three proposed methods are excellent in agreement. This statement is further proved by comparing the present first two moments of frequencies with results obtained from MCS for different mean value of power index r. It is noted that direct MCS needs 10000 FEM solvers whereas LR, SP and SC need only 150, 243 and 243, respectively. As framework presented in Fig.1, the computational cost of FEM solver is dominant compared with other steps, so the alternative methods are by far cheaper than direct MCS. It seems that LR is cheapest, but it may have largest uncertainty due to the uncertainty in simulations for estimating the coefficients. As expected, for the same value of μ_r, the response's variance increases as the mean value increases. C-C beam has the most variance and the least variance belongs to C-F beam. Interestingly, the coefficient of variance (σ/μ) for all three beams are similar, just around 5.2%, 5.7% and 6% for $\mu_r = 2$, $\mu_r = 5$ and $\mu_r = 10$, respectively. It is clear that the variation of output is smaller than that of inputs. Additionally, as μ_r increases, the mean value of frequencies decreases but the variance has the opposite tendency. It is logical because the increase in μ_r leads to the decrease in the mean of material properties and the increase in variance of r.

Table 2: Convergence of first three natural frequencies of FG beams with orders of PCE, $L = 8m, b = 0.8m, h = 1m$, $\mu_r = 1$.

Boundary Conditions	Polynomial Orders	Properties	LR (Ns=150)			SP (Ngp=3)		
			λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
C-F	2	μ	15.1977	93.5163	194.6697	15.1981	93.5189	194.6712
		σ	0.7781	4.7852	10.2587	0.7786	4.7883	10.2655
	3	μ	15.1982	93.5198	194.6725	15.1981	93.5189	194.6712
		σ	0.7787	4.7894	10.2674	0.7786	4.7885	10.2658
	4	μ	15.1981	93.5192	194.6717	15.1981	93.5189	194.6712
		σ	0.7786	4.7884	10.2655	0.7792	4.7925	10.2723
C-C	2	μ	96.4148	260.0021	389.5234	96.4212	260.0193	389.5365
		σ	4.9272	13.2798	20.4963	4.9387	13.3107	20.5371
	3	μ	96.4207	260.0180	389.5361	96.4212	260.0193	389.5365
		σ	4.9362	13.3040	20.5273	4.9389	13.3111	20.5377
	4	μ	96.4211	260.0189	389.5365	96.4212	260.0193	389.5365
		σ	4.9391	13.3117	20.5375	4.9431	13.3224	20.5507
S-S	2	μ	42.5766	165.9196	195.3043	42.5768	165.9203	195.3087
		σ	2.1789	8.4757	10.3248	2.1803	8.4815	10.3252
	3	μ	42.5766	165.9199	195.3075	42.5768	165.9203	195.3087
		σ	2.1803	8.4818	10.3243	2.1804	8.4818	10.3255
	4	μ	42.5766	165.9197	195.3082	42.5768	165.9203	195.3087
		σ	2.1804	8.4818	10.3253	2.1823	8.4892	10.3322

Table 3: Convergence of first three natural frequencies of FG beams with a number of quadrature points for each random variables, $L = 8m, b = 0.8m, h = 1m$, $\mu_r = 1$.

Boundary Conditions	Quadrature Points	Properties	SP (p=4) λ_1	SP (p=4) λ_2	SP (p=4) λ_3	SC λ_1	SC λ_2	SC λ_3
C-F	2	μ	15.1980	93.5187	194.6697	15.1980	93.5182	194.6691
		σ	1.0000	6.1498	13.2011	0.7751	4.7673	10.2326
	3	μ	15.1981	93.5189	194.6712	15.1979	93.5180	194.6703
		σ	0.7786	4.7885	10.2658	0.7783	4.7864	10.2623
	4	μ	15.1981	93.5189	194.6712	15.1979	93.5180	194.6703
		σ	0.7786	4.7887	10.2661	0.7783	4.7866	10.2626
C-C	2	μ	96.4210	260.0188	389.5338	96.4205	260.0174	389.5324
		σ	6.3431	17.0952	26.4104	4.9170	13.2520	20.4715
	3	μ	96.4212	260.0193	389.5365	96.4203	260.0169	389.5347
		σ	4.9389	13.3111	20.5377	4.9368	13.3053	20.5308
	4	μ	96.4212	260.0192	389.5365	96.4203	260.0168	389.5347
		σ	4.9391	13.3118	20.5383	4.9370	13.3060	20.5313
S-S	2	μ	42.5767	165.9202	195.3073	42.5765	165.9193	195.3067
		σ	2.8002	10.8912	13.2781	2.1707	8.4430	10.2922
	3	μ	42.5768	165.9203	195.3087	42.5764	165.9187	195.3078
		σ	2.1804	8.4818	10.3255	2.1795	8.4776	10.3222
	4	μ	42.5768	165.9202	195.3087	42.5764	165.9187	195.3079
		σ	2.1805	8.4823	10.3257	2.1796	8.4781	10.3225

Fig.4 compares the probability density function (PDF) and probability of exceedance (PoE) of fundamental frequency of FG beam obtained from proposed methods with MCS as $\mu_r = 1$. The results are estimated for all three boundary conditions. The exceedance plots are presented in log-scale. Again, they are all excellent in agreement. There are certainly small discrepancies at the very small probability ($< 10^{-3}$) or large frequency regions. For C-F beam, the largest frequency of MCS is lessen than those of other methods, however for S-S and C-C beams, this trend is reversed for SP and SC.

5 Conclusion

This paper presents three different stochastic FEM for probabilistic free vibration to FG beams. All proposed methods are able efficiently propagate the uncertainty of FG material properties to natural frequencies of Euler-Bernoulli FG beams. Monte Carlo simulation is also considered as the exact method to compare with the proposed methods. The results show that all three proposed approaches are good agreement with MCS, but they require less computational cost than MCS.

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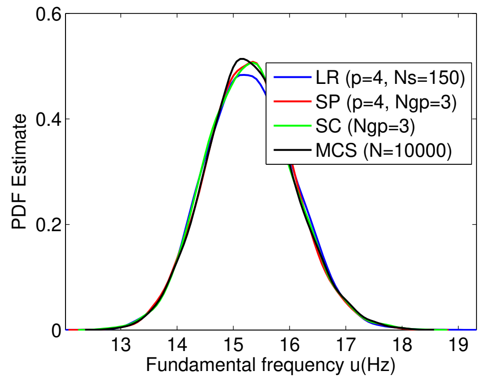
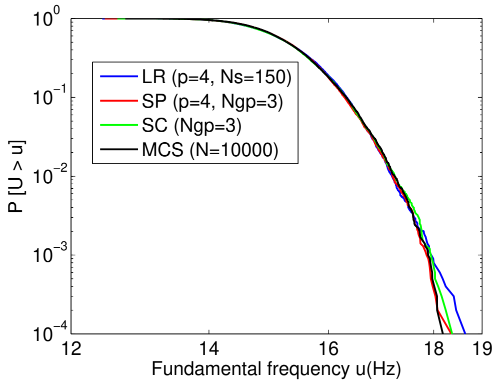
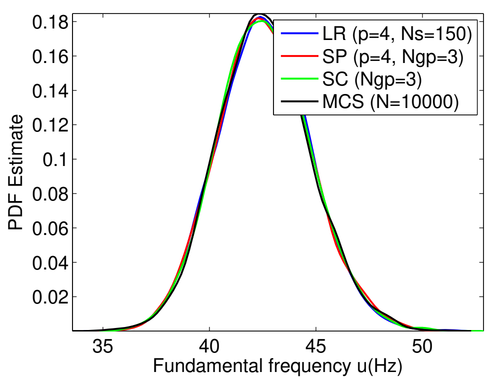


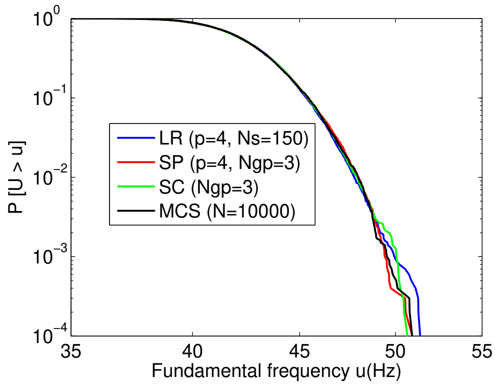
(a) PDF (C-F)



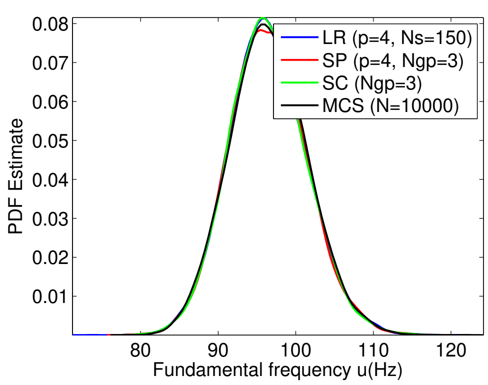
(b) PoE (C-F)



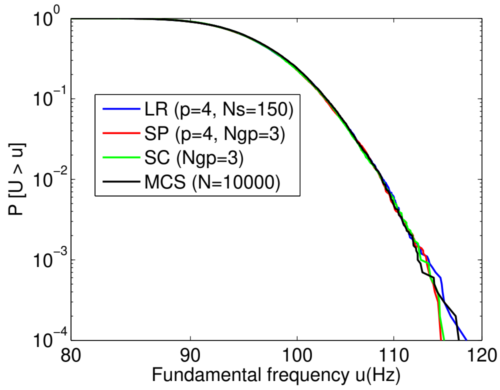
(c) PDF (S-S)



(d) PoE (S-S)



(e) PDF (C-C)



(f) PoE (C-C)

Fig. 4: Probability density function (pdf) and probability of exceedance (PoE) of fundamental frequency for C-F, S-S and C-C FG beams, $L = 8m, b = 0.8m, h = 1m$, $\mu_r = 1$.