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Variable stiffness composite beams subject to non-uniformly distributed loads: An analytical solution

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ABSTRACT

An analytical solution is obtained for the 3D static deflection of variable stiffness composite beams subject to non-uniformly distributed loads. Governing differential equations with variable coefficients, reflecting the spatially variable stiffness properties, are presented in which four degrees of freedom are fully coupled. The general analytical solution in integral form is derived and closed-form expressions obtained using series expansion approximations. The static deflection of a number of variable stiffness composite beams that can be made by fibre steering are considered with various stacking sequences. The results obtained from the proposed method are validated against numerical results from the Chebyshev collocation method and excellent agreement is observed between the two. While the proposed methodology is applicable for variable stiffness composite beams with arbitrary span-wise variation of properties, it is also an efficient approach for capturing the complicated 3D static deflection of variable stiffness composite beams subject to non-uniformly distributed loads.

1. Introduction

Variable stiffness non-uniform structures have found many applications in real-life engineering designs such as modern high aspect ratio aircraft wings, helicopter rotor blades and wind turbine rotor blades as well as other slender beam-like structures. Thus, it is important to develop appropriate models capable of capturing the complicated behaviour of variable stiffness non-uniform beams under various loading conditions, accurately and efficiently. Undoubtedly, closed-form analytical solutions offer reliability for benchmarking purposes and as preliminary design tools.

Different strategies can be employed to achieve variable stiffness properties for beams including the span-wise distribution of material properties (e.g. by gradation of material or by using fibre steering), the change of geometry (e.g. by tapering the cross-section) or by using both strategies. Consequently, tapered beams, axially or bidirectional graded beams and beams made by fibre steering are considered as variable stiffness beams.

In the framework of anisotropic non-uniform variable stiffness beams, using the state space-based differential quadrature method, Lü et al. [1] proposed a semi-analytical approach for bending and thermal deformations of functionally graded beams with exponentially varying Young's modulus along the thickness and longitudinal directions. Shahba et al. [2] obtained exact shape functions for the static, stability and free vibration analysis of axially functionally graded plane beams. Using the power series method, Nguyen et al. [3] provided closed-form expressions for the static response of tapered axially functionally graded beams. Zhao et al. [4] obtained elasticity solutions for bi-directional functionally graded beams using the symplectic approach. Based on Euler-Bernoulli theory, Pydah and Sabale [5] presented analytical solutions for the in-plane bending of statically determinate bidirectional functionally graded beams with circular cross-section. Pydah and Batra [6] found analytical solutions for the static deflection of circular bidirectional graded shear deformable beams. Sachdeva and Padhee [7] using the variational-asymptotic method, obtained closed-form analytical solutions for the nonlinear response of bidirectional functionally graded cylindrical beams. Soltani and Asgarian [8,9] provided exact stiffnesses for the static and buckling analysis of axially and bidirectional functionally graded Timoshenko beams. Introducing an auxiliary function, Huang and Ouyang [10] obtained exact solution for bending of two-directional functionally graded Timoshenko beams subject to uniformly, non-uniformly distributed and concentrated loads.

A number of works considered numerical solutions for the variable stiffness beam problem. To name just a few, Karamanlı [11] used Symmetric Smoothed Particle Hydrodynamics method (SSPH) to solve the in-plane static deflection of bidirectional functionally graded beams. Li et al. [12] considered the bending, buckling and vibration of plane axi-

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ally functionally graded beams using a generalized differential quadrature method (DQM). Xie et al. [13] solved the static and free vibration of plane Euler-Bernoulli beams using an integrated collocation method. Günay and Timarci [14] provided finite element analysis of variable stiffness thin-walled laminated composite beams with closed cross-sections made by fibre steering. Employing the Ritz technique, Ai and Weaver [15] studied the static response of sandwich beams with a combination of geometric taper and variable stiffness of the core. Based on a unified formulation, Zappino et al. [16] considered the static analysis of tapered thin-walled composite beams using finite element method. Macquart et al. [17] proposed enriched beam elements for variable stiffness beams to enhance the accuracy of static response. Using the finite element method, Rajasekaran and Bakhshi Khaniki [18] investigated the static and dynamic behaviour of non-uniform size-dependent axially functionally graded beams. Yu et al. [19] proposed the isogeometric analysis (IGA) for bending and free vibration analysis of two-directional functionally graded microbeams.

A more comprehensive bibliography, ranging from early attempts to more recent ones in the context of analytical solution of static behaviour of variable stiffness isotropic beams can be found in Masjedi and Weaver [20].

In the context of 3D fully coupled anisotropic beams, recently, Doeva et al. [21,22] presented the exact solutions for the static response of composite beams with constant stiffness properties. While Masjedi and Weaver [20] proposed a closed-form analytical solution for the 3D static response of variable stiffness beams under the action of concentrated and uniformly distributed loads, to the authors' best knowledge no closed-form analytical solution exists for the 3D static response of fully coupled variable stiffness composite beams subject to non-uniformly distributed loads. Thus, in order to address this important lack of knowledge, the main purpose of this paper is to obtain an analytical closed-form solution for the 3D static response of variable stiffness beams subject to arbitrary non-uniformly distributed loads. It is worth noting that compared to uniform loads, arbitrary non-uniformly distributed loads are more complex and mathematically involved which make the problem more demanding to be solved. The Chebyshev collocation method which has been shown to be efficient and accurate in beam problems [23–26] is also applied as an alternative approach.

In the proposed formulation all four degrees of freedom, including in-plane, out-of-plane and axial displacements, and twist are fully coupled. It is also worth mentioning that in order to keep the formulation as general as possible, the stiffness matrix entries are expressed in terms of the engineering constants. While the proposed method can be used for different types of variable stiffness composite beams, the focus of current work is on variable stiffness composite beams made by fibre steering which has been shown to be useful for engineering designs [27–35]. In order to further highlight the new contributions of the current work, a comparison between this work and available solutions in the literature are made in Tables 1 and 2.

The content of this paper is outlined as follows: In Section 2, the governing equations of a fully coupled variable stiffness composite beam are presented. In Section 3, using a direct integration technique, a general exact solution is derived. In Section 4, series expansion representation is employed in order to obtain the closed-form expressions. In Section 5, several benchmark test problems are considered and the results obtained from the analytical approach are compared against those obtained from the Chebyshev collocation method. Some conclusions and remarks are drawn in the last section.

2. Governing equations

Assume a straight beam of length ℓ for which the x coordinate is along the beam longitudinal axis and the coordinates y and z define the cross-sectional planes (Fig. 1).

The set of governing differential equations and boundary conditions with variable coefficients for a fully coupled Euler-Bernoulli composite beam can be expressed in a compact matrix form (for more details see Appendix A):

$$(-\boldsymbol{A}(x)\boldsymbol{U}' + \boldsymbol{B}(x)\boldsymbol{W}'')' = \overline{\boldsymbol{F}}, \tag{2.1a}$$

$$\left(-\boldsymbol{B}^T(x)\boldsymbol{U}' + \boldsymbol{D}(x)\boldsymbol{W}''\right)'' = \overline{\boldsymbol{Q}}. \tag{2.1b}$$

$$\boldsymbol{U} = \boldsymbol{0} \quad \text{or} \quad \boldsymbol{A}(x)\boldsymbol{U}' - \boldsymbol{B}(x)\boldsymbol{W}'' = \boldsymbol{0}, \tag{2.2a}$$

$$\boldsymbol{W} = \boldsymbol{0} \quad \text{or} \quad -\boldsymbol{B}^T(x)\boldsymbol{U}' + \boldsymbol{D}(x)\boldsymbol{W}'' = \boldsymbol{0}, \tag{2.2b}$$

$$\boldsymbol{W}' = \boldsymbol{0} \quad \text{or} \quad \left(\boldsymbol{B}^T(x)\boldsymbol{U}' - \boldsymbol{D}(x)\boldsymbol{W}''\right)' = \boldsymbol{0}. \tag{2.2c}$$

where

$$\boldsymbol{A}(x) = \begin{bmatrix} EA(x) & S_{ET}(x) \\ S_{ET}(x) & GJ(x) \end{bmatrix}, \tag{2.3a}$$

$$\boldsymbol{B}(x) = \begin{bmatrix} S_{EF}(x) & -S_{EL}(x) \\ S_{FT}(x) & -S_{LT}(x) \end{bmatrix}, \tag{2.3b}$$

$$\boldsymbol{D}(x) = \begin{bmatrix} EI_y(x) & -S_{FL}(x) \\ -S_{FL}(x) & EI_z(x) \end{bmatrix}. \tag{2.3c}$$

and $\boldsymbol{U} = [u \quad \varphi]^T$, $\boldsymbol{W} = [w \quad v]^T$, $\overline{\boldsymbol{F}} = [q_x \quad q_\varphi]^T$ and $\overline{\boldsymbol{Q}} = [q_z \quad q_y]^T$, such that u is the axial elongation, φ is twist, w and v are the out-of-plane bending and in-plane bending, respectively, $()'$ denotes the derivative with respect to x, q_x, q_y, q_z are the non-uniformly distributed loads in the x, y, z directions respectively, and q_φ is the distributed torque. Stiffness terms, $EA(x)$ is the distribution of extensional stiffness, $GJ(x)$ is the distribution of twist stiffness, $EI_y(x)$ is the distribution of out-of-plane bending stiffness, $EI_z(x)$ is the distribution of in-plane bending stiffness, $S_{ET}(x)$ is the distribution of coupling between axial elongation and twist, $S_{EF}(x)$ is the distribution of coupling between out-of-plane bending and axial elongation, $S_{EL}(x)$ is the distribution of coupling between in-plane bending and axial elongation, S_{FT} is the distribution of coupling between out-of-plane bending and twist, $S_{LT}(x)$ is the distribution of coupling between in-plane bending and twist, and $S_{FL}(x)$ is the distribution of coupling between out-of-plane and in-plane bending.

The derivation of closed-form analytical solution of the system of differential Eqs. (2.1) with variable coefficients is presented in the next section.

3. General solution

In contrast to the uniformly distributed loads case considered in [20] (which are constant along the beam reference line), distributed loads are now considered to be arbitrary functions of x and as a result while performing the integrations, loading related terms are kept in integral form. To obtain the exact general solution of Eqs. (2.1), Eq. (2.1a) is integrated and rearranged. The first derivative of the vector $\boldsymbol{U}$ can be expressed as:

$$\boldsymbol{U}' = \boldsymbol{A}^{-1}(x)\boldsymbol{B}(x)\boldsymbol{W}'' - \boldsymbol{A}^{-1}(x)\overline{\boldsymbol{F}}^{i} - \boldsymbol{A}^{-1}(x)\boldsymbol{C}_1. \tag{3.1}$$

Substituting Eq. (3.1) into Eq. (2.1b) and rearranging it, we obtain:

$$\left(\left(\boldsymbol{D}(x) - \boldsymbol{B}^T(x)\boldsymbol{A}^{-1}(x)\boldsymbol{B}(x)\right)\boldsymbol{W}'' + \boldsymbol{B}^T(x)\boldsymbol{A}^{-1}(x)\overline{\boldsymbol{F}}^{i} + \boldsymbol{B}^T(x)\boldsymbol{A}^{-1}(x)\boldsymbol{C}_1\right)'' = \overline{\boldsymbol{Q}}. \tag{3.2}$$

By integrating Eq. (3.2) twice, the following expression is obtained

$$\left(\boldsymbol{D}(x) - \boldsymbol{B}^T(x)\boldsymbol{A}^{-1}(x)\boldsymbol{B}(x)\right)\boldsymbol{W}'' + \boldsymbol{B}^T(x)\boldsymbol{A}^{-1}(x)\overline{\boldsymbol{F}}^{i} + \boldsymbol{B}^T(x)\boldsymbol{A}^{-1}(x)\boldsymbol{C}_1 = \overline{\boldsymbol{Q}}^{ii} + x\boldsymbol{C}_2 + \boldsymbol{C}_3, \tag{3.3}$$

Rearranging Eq. (3.3), the second derivative of the vector $\boldsymbol{W}$ is found as

$$\boldsymbol{W}'' = \boldsymbol{K}_D(x)\left(\overline{\boldsymbol{Q}}^{ii} + x\boldsymbol{C}_2 + \boldsymbol{C}_3\right) + \boldsymbol{K}_B^T(x)\left(\overline{\boldsymbol{F}}^{i} + \boldsymbol{C}_1\right), \tag{3.4}$$

where,

Table 1
Comparison of available analytical solutions for variable stiffness beams.

	Theory				Kinematics		Cross-Section		Coupling Terms		
Reference	**EBT**	**TBT**	**HOT**	**ET**	**2D**	**3D**	**SO**	**TW**	**BT**	**AT**	**BA**
[1]	–	–	–	✓	✓	–	✓	–	–	–	✓
[2]	✓	–	–	–	✓	–	✓	–	–	–	–
[3]	✓	–	–	–	✓	–	✓	–	–	–	✓
[4]	–	–	–	✓	✓	–	✓	–	–	–	✓
[5]	✓	–	–	–	✓	–	✓	–	–	–	✓
[6]	–	–	✓	–	✓	–	✓	–	–	–	✓
[7]	–	–	–	✓	–	✓	✓*	✓*	–	–	–
[8,9]	–	✓	–	–	✓	–	✓	–	–	–	–
[10]	–	✓	–	–	✓	–	✓	–	–	–	–
Present	✓	–	–	–	–	✓	✓	✓	✓	✓	✓

EBT: Euler-Bernoulli Beam Theory **TBT**: Timoshenko Beam Theory.
HOT: Higher Order Beam Theory **ET**: 2D/3D Elasticity Theory.
SO: Solid Cross-Section **TW**: Thin-Walled Cross-Section.
BT: Bend-Twist **AT**: Axial Elongation-Twist **BA**: Bend-Axial Elongation.
* Closed-form solutions are provided just for circular cross-sections.

Table 2
Comparison of available closed-from solutions for fully coupled composite beams.

	Theory		Stiffness		Loading Type			Solution	
Reference	**EBT**	**TBT**	**Constant**	**Variable**	**UD**	**NUD**	**CT**	**Series**	**Exact**
[20]	✓	–	–	✓	✓	–	✓	✓	–
[21]	✓	–	✓	–	–	✓	–	–	✓
[22]	–	✓	✓	–	✓	–	✓	–	✓
Present	✓	–	–	✓	–	✓	–	✓	–

EBT: Euler-Bernoulli Beam Theory **TBT**: Timoshenko Beam Theory.
UD: Uniformly Distributed Load **NUD**: Non-Uniformly Distributed Load **CT**: Concentrated/Tip Load.

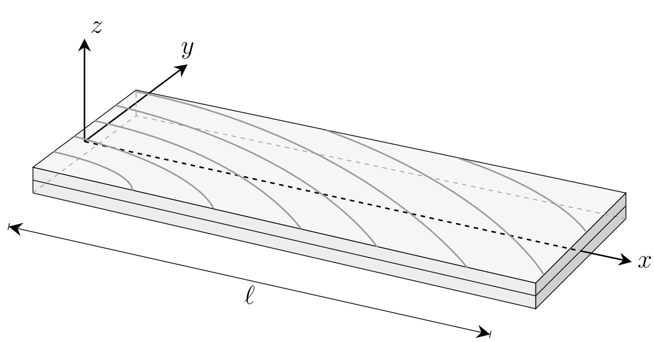


Fig. 1. Coordinate system for the composite beam.

$$\overline{\boldsymbol{F}}^i = \int \overline{\boldsymbol{F}}\,dx = \begin{bmatrix} q_x^i \\ q_\varphi^i \end{bmatrix}$$

$$\bar{\boldsymbol{Q}}^{ii} = \iint \bar{\boldsymbol{Q}}\,dx\,dx = \int \begin{bmatrix} q_z^i \\ q_y^i \end{bmatrix} dx = \begin{bmatrix} q_z^{ii} \\ q_y^{ii} \end{bmatrix}$$

Table 3 shows the analytical expressions of q_n, q_n^i and q_n^{ii} where $n = x, y, z, \varphi$, for a number of non-uniformly distributed loads commonly used in engineering problems. While the proposed integration procedure is valid for non-uniform loads distributed on the full span of the beam, it is possible to further extend this approach to cases in which a constant load is applied to a section of the beam span by employing singularity functions. Successive integration of Eq. (3.4), the general solution for the vector $\boldsymbol{W}$ can be obtained

$$\boldsymbol{W}' = \int \boldsymbol{K_D}(x)\overline{\boldsymbol{Q}}^{ii}\,dx + \int x\boldsymbol{K_D}(x)\,dx\boldsymbol{C}_2 + \int \boldsymbol{K_D}(x)\,dx\boldsymbol{C}_3 + \int \boldsymbol{K}_{\boldsymbol{B}}^T(x)\overline{\boldsymbol{F}}^i\,dx + \int \boldsymbol{K}_{\boldsymbol{B}}^T(x)\,dx\boldsymbol{C}_1 + \boldsymbol{C}_4, \tag{3.5}$$

$$\boldsymbol{W} = \iint \boldsymbol{K_D}(x)\overline{\boldsymbol{Q}}^{ii}\,dx\,dx + \iint x\boldsymbol{K_D}(x)\,dx\,dx\boldsymbol{C}_2 + \iint \boldsymbol{K_D}(x)\,dx\,dx\boldsymbol{C}_3 + \iint \boldsymbol{K}_{\boldsymbol{B}}^T(x)\overline{\boldsymbol{F}}^i\,dx\,dx + \iint \boldsymbol{K}_{\boldsymbol{B}}^T(x)\,dx\,dx\boldsymbol{C}_1 + \boldsymbol{C}_4 x + \boldsymbol{C}_5. \tag{3.6}$$

where $\boldsymbol{C}_i$, $i = 1, 2, \ldots, 5$, are the vectors of unknown integrating constants to be determined.

The first derivative of vector $\boldsymbol{U}$ can be obtained by substituting Eq. (3.4) into Eq. (3.1). Considering Eq. (3.9) one can write

$$\boldsymbol{U}' = -\left(\boldsymbol{K_A}(x)\overline{\boldsymbol{F}}^i + \boldsymbol{K_A}(x)\boldsymbol{C}_1 + \boldsymbol{K_B}(x)\overline{\boldsymbol{Q}}^{ii} + x\boldsymbol{K_B}(x)\boldsymbol{C}_2 + \boldsymbol{K_B}(x)\boldsymbol{C}_3\right), \tag{3.7}$$

The general solution for the vector $\boldsymbol{U}$ is obtained by integrating Eq. (3.7)

Table 3
Expressions for integrals of some particular load functions (c_1 and c_2 are constants of integration).

q_n	$q_n^i = \int q_n\,dx$	$q_n^{ii} = \int q_n^i\,dx = \iint q_n\,dx\,dx$
$a\sin\left(\frac{m\pi x}{\ell}\right)$	$-\left(\frac{\ell}{m\pi}\right)a\cos\left(\frac{m\pi x}{\ell}\right) + c_1$	$-\left(\frac{\ell}{m\pi}\right)^2 a\sin\left(\frac{m\pi x}{\ell}\right) + c_1 x + c_2$
$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$	$\frac{a_n}{(n+1)}x^{n+1} + \frac{a_{n-1}}{n}x^n + \cdots + \frac{a_1}{2}x^2 + a_0 x + c_1$	$\frac{a_n}{(n+1)(n+2)}x^{n+2} + \frac{a_{n-1}}{n(n+1)}x^{n+1} + \cdots + \frac{a_1}{6}x^3 + \frac{a_0}{2}x^2 + c_1 x + c_2$

$$U = -\left(\int K_A(x)\bar{F}^i\,dx + \int K_A(x)\,dx C_1 + \int K_B(x)\bar{Q}^{ii}\,dx + \int xK_B(x)\,dx C_2 + \int K_B(x)\,dx C_3\right) + C_6, \tag{3.8}$$

where $C_i,\ i=1,2,3,6$, are the vectors of unknown coefficients to be determined. The definitions of matrices K_A, K_B, K_B^T and K_D are given as

$$K_A(x) = A^{-1}(x) + A^{-1}(x)B(x)\left[D(x) - B^T(x)A^{-1}(x)B(x)\right]^{-1}B^T(x)A^{-1}(x), \tag{3.9a}$$

$$K_B(x) = -A^{-1}(x)B(x)\left[D(x) - B^T(x)A^{-1}(x)B(x)\right]^{-1}, \tag{3.9b}$$

$$K_B^T(x) = -\left[D(x) - B^T(x)A^{-1}(x)B(x)\right]^{-1}B^T(x)A^{-1}(x), \tag{3.9c}$$

$$K_D(x) = \left[D(x) - B^T(x)A^{-1}(x)B(x)\right]^{-1}. \tag{3.9d}$$

Eqs. (3.6) and (3.8) present the general solutions for the 3D static deflection of a fully coupled variable stiffness composite beam under the action of non-uniformly distributed loads. These expressions have six vectors of unknown constants which are obtained when the boundary conditions are given. While Eqs. (3.6) and (3.8) are in integral form, the exact closed-form expressions can be derived whenever the integrands are integrable. Compared to the general solution in the case of uniformly distributed loads presented by [20], it is observed that more complexity is added to the integrands due to non-uniformly distributed loads which makes it more demanding to express the solutions in closed-form. However, in order to obtain closed-form solutions, the next section shows that a series expansion approach can be successfully employed. It is worth mentioning that the proposed approach can be applied for arbitrary distributions of stiffness properties as well as for arbitrary non-uniformly distributed loads.

4. Series expansion representation

It is not possible to guarantee exact closed-form expressions can be obtained from Eqs. (3.6) and (3.8) for arbitrary span-wise distribution of loads and stiffness properties. However, in this section a series expansion representation is used to derive the analytical closed-form expressions for arbitrary smooth distributions of loads and beam stiffness properties. Utilising a series expansion representation, the integrands in Eqs. (3.6) and (3.8) are expressed as

$$K_A(x) = R_A^0\phi(x), \quad K_A(x)\overline{F}^i = R_A^{q^i}\phi(x), \tag{4.1a}$$

$$K_B(x) = R_B^0\phi(x), \quad xK_B(x) = R_B^1\phi(x), \quad K_B(x)\overline{F}^i = R_B^{q^i}\phi(x), \quad K_B(x)\overline{Q}^{ii} = R_B^{q^{ii}}\phi(x), \tag{4.1b}$$

$$K_D(x) = R_D^0\phi(x), \quad xK_D(x) = R_D^1\phi(x), \quad K_D(x)\overline{Q}^{ii} = R_D^{q^{ii}}\phi(x), \tag{4.1c}$$

where, $R_\alpha^\beta,\ \alpha = A, B, D$ and $\beta = 0, 1, q^i, q^{ii}$ are block matrices containing the vectors of unknown weight coefficients and $\phi(x)$ is a known function of x. Introducing Eqs. (4.1) into Eqs. (3.6) and (3.8), one can write:

$$W = R_D^{q^{ii}}\iint\phi(x)\,dx\,dx + R_D^1\iint\phi(x)\,dx\,dx\,C_2 + R_D^0 \times \iint\phi(x)\,dx\,dx\,C_3 + R_B^{q^i\,T}\iint\phi(x)\,dx\,dx + R_B^{0\,T} \times \iint\phi(x)\,dx\,dx\,C_1 + xC_4 + C_5, \tag{4.2}$$

$$U = -\left(R_A^{q^i}\int\phi(x)\,dx + R_A^0\int\phi(x)\,dx\,C_1 + R_B^{q^{ii}}\int\phi(x)\,dx + R_B^1\int\phi(x)\,dx\,C_2 + R_B^0\int\phi(x)\,dx\,C_3\right) + C_6, \tag{4.3}$$

where the matrices of unknown weight coefficients are defined as:

$$R_A^0 = \begin{bmatrix} \mathbf{r}_{A_{11}}^0 & \mathbf{r}_{A_{12}}^0 \\ \mathbf{r}_{A_{21}}^0 & \mathbf{r}_{A_{22}}^0 \end{bmatrix}, \quad R_A^{q^i} = \begin{bmatrix} \mathbf{r}_{A_{11}}^{q^i} & \mathbf{r}_{A_{12}}^{q^i} \\ \mathbf{r}_{A_{21}}^{q^i} & \mathbf{r}_{A_{22}}^{q^i} \end{bmatrix}, \tag{4.4a}$$

$$R_B^0 = \begin{bmatrix} \mathbf{r}_{B_{11}}^0 & \mathbf{r}_{B_{12}}^0 \\ \mathbf{r}_{B_{21}}^0 & \mathbf{r}_{B_{22}}^0 \end{bmatrix}, \quad R_B^1 = \begin{bmatrix} \mathbf{r}_{B_{11}}^1 & \mathbf{r}_{B_{12}}^1 \\ \mathbf{r}_{B_{21}}^1 & \mathbf{r}_{B_{22}}^1 \end{bmatrix}, \quad R_B^{q^i} = \begin{bmatrix} \mathbf{r}_{B_{11}}^{q^i} & \mathbf{r}_{B_{12}}^{q^i} \\ \mathbf{r}_{B_{21}}^{q^i} & \mathbf{r}_{B_{22}}^{q^i} \end{bmatrix}, \quad R_B^{q^{ii}} = \begin{bmatrix} \mathbf{r}_{B_{11}}^{q^{ii}} & \mathbf{r}_{B_{12}}^{q^{ii}} \\ \mathbf{r}_{B_{21}}^{q^{ii}} & \mathbf{r}_{B_{22}}^{q^{ii}} \end{bmatrix}, \tag{4.4b}$$

$$R_D^0 = \begin{bmatrix} \mathbf{r}_{D_{11}}^0 & \mathbf{r}_{D_{12}}^0 \\ \mathbf{r}_{D_{21}}^0 & \mathbf{r}_{D_{22}}^0 \end{bmatrix}, \quad R_D^1 = \begin{bmatrix} \mathbf{r}_{D_{11}}^1 & \mathbf{r}_{D_{12}}^1 \\ \mathbf{r}_{D_{21}}^1 & \mathbf{r}_{D_{22}}^1 \end{bmatrix}, \quad R_D^{q^{ii}} = \begin{bmatrix} \mathbf{r}_{D_{11}}^{q^{ii}} & \mathbf{r}_{D_{12}}^{q^{ii}} \\ \mathbf{r}_{D_{21}}^{q^{ii}} & \mathbf{r}_{D_{22}}^{q^{ii}} \end{bmatrix}, \tag{4.4c}$$

where $\mathbf{r}_{\alpha_{nm}^\beta}, \alpha = A, B, D, \beta = 0, 1, q^i, q^{ii}$ and $n, m = 1, 2$ are the vectors of unknown weighting coefficients which are obtained using a standard curve fitting algorithm.

By introducing the expression of $\phi(x)$, the closed-form solutions are derived from Eqs. (4.2) and (4.3). Using a power series representation, i.e. $\phi(x) = [1, \quad x, \quad x^2, \quad \ldots, \quad x^N]^T$ where N is the highest degree of polynomials, we obtain

$$W = R_D^{q^{ii}}\Phi^{ii}(x) + R_D^1\Phi^{ii}(x)C_2 + R_D^0\Phi^{ii}(x)C_3 + R_B^{q^i\,T}\Phi^{ii}(x) + R_B^{0\,T}\Phi^{ii}(x)C_1 + xC_4 + C_5, \tag{4.5}$$

$$U = -\left(R_A^{q^i}\Phi^i(x) + R_A^0\Phi^i(x)C_1 + R_B^{q^{ii}}\Phi^i(x) + R_B^1\Phi^i(x)C_2 + R_B^0\Phi^i(x)C_3\right) + C_6, \tag{4.6}$$

where,

$$\Phi^i(x) = \int\phi(x)\,dx = \left[x, \quad \frac{x^2}{2}, \quad \ldots, \quad \frac{x^{N+1}}{N+1}\right]^T, \tag{4.7a}$$

$$\Phi^{ii}(x) = \iint\phi(x)\,dx\,dx = \left[\frac{x^2}{2}, \quad \frac{x^3}{6}, \quad \ldots, \quad \frac{x^{N+2}}{(N+1)(N+2)}\right]^T. \tag{4.7b}$$

Eqs. (4.5) and (4.6) represent the closed-form analytical solutions for the 3D deflection of variable stiffness composite beams subject to non-uniformly distributed loads. Once boundary conditions are applied, constants of integration $C_i,\ i = 1, 2, \ldots, 6$ can be determined. It is also noted that while power series are employed in the current work, any type of appropriate series can be used in the proposed approach.

5. Numerical results

In this section a number of composite beams with constant stiffness as well as variable stiffness are presented. For all numerical examples, a slender composite beam with a rectangular cross-section is considered. Table 4 shows the material and geometric properties used for the slender beam. For comparison and benchmarking purposes, the results are given to 10 decimal places, despite the fact that in real-life problems they are not considered to be meaningful. It is also noted that the details of Chebyshev collocation method (CCM) implementation can be found in Appendix B.

Table 4
Beam properties.

Length	1	m
Width	0.01	m
Number of Layers	6	
Ply Thickness	0.000125	m
E_{11}	135.64	GPa
E_{22}	10.14	GPa
G_{12}	5.86	GPa
ν_{12}	0.29	

5.1. Constant stiffness composite beams

In order to validate the proposed approach a constant stiffness cantilever composite beam with constant fibre orientation is considered. The numerical results obtained from current approach are compared against those obtained from the exact solution presented in [21]. Different stacking sequences with different coupling terms are considered herein:

1. Symmetric $[45_3]_s$ with bend-twist coupling,
2. Antisymmetric $[45_3/-45_3]$ with extension-twist coupling,
3. Cross-ply $[0_3/90_3]$ with extension-bend coupling,
4. Unsymmetric $[60_3/30_3]$ with bend-twist, extension-twist and extension-bend coupling.

The stiffness matrices are obtained from closed-form expressions presented by Yu and Hodges [36]. Tables 5–8, show numerical results for the tip deflections of cantilever under the action of linearly distributed and sinusoidally distributed loads. For all test cases, excellent agreement is observed between different sets of results. It is worth noting that the excellent agreement observed between the exact results of [21] and those of the current work may be attributed to the fact that the proposed solutions in this work are not based on the discretisation of the unknown variables. The series representation is just used to express the final solutions in closed-form. In other words, first the problem is solved exactly (Eqs. (3.6) and (3.8)), then these exact solutions are expressed in closed-from using power series. Actually, just integrands are approximated in Eqs. (3.6) and (3.8), which does not introduce significant error in the case of sufficiently smooth functions. Thus, as expected, in the case of constant stiffness beams in which integrands are smooth and relatively simple functions, very close concordance is observed in Tables 5–8, between the analytical results proposed in this work and the exact results of [21]. Additionally, as previously mentioned, CCM has been shown to be highly accurate in beam problems, so generally very good agreement is observed between the numerical results of CCM and those of analytical solutions while in some cases CCM results deviate a little from those of analytical solution.

5.2. Variable stiffness composite beams

In this section a number of variable stiffness composite beams made by fibre steering, subject to non-uniformly distributed load and with different boundary conditions are presented. Employing the closed-form expressions given by [36], the stiffness properties distribution is obtained as a function of beam axial coordinate "x". To study the effects of different coupling terms on the static response of variable stiffness composite beams, the following stacking sequences are considered:

Table 5
Tip deformation of bend-twist coupled cantilever beam ($[45_3]_s$) under non-uniformly distributed loads.

	Exact	Analytical	CCM
Linearly distributed load ($q_z = 0.004x$ N/m)			
φ (rad)	−0.0324433326	−0.0324433326	−0.0324433326
w (m)	0.0710164499	0.0710164499	0.0710164499
Sinusoidal distributed load ($q_z = 0.001\sin(\frac{\pi x}{\ell})\,N/m$)			
φ (rad)	−0.0061416445	−0.0061416447	−0.0061414859
w (m)	0.0143036997	0.0143036997	0.0143033804

Table 6
Tip deformation of extension-twist coupled cantilever beam ($[45_3/-45_3]$) under non-uniformly distributed loads.

	Exact	Analytical	CCM
Linearly distributed load ($q_\varphi = 0.004x$ N)			
u (m)	0.0000094299	0.0000094299	0.0000094299
φ (rad)	0.0607763239	0.0607763240	0.0607763239
Sinusoidal distributed load ($q_\varphi = 0.001\sin(\frac{\pi x}{\ell})\,N$)			
u (m)	0.0000022512	0.0000022512	0.0000022512
φ (rad)	0.0145092786	0.0145092786	0.0145092789

Table 7
Tip deformation of extension-bend coupled cantilever beam ($[0_3/90_3]$) under non-uniformly distributed loads.

	Exact	Analytical	CCM
Linearly distributed load ($q_z = 0.004x$ N/m)			
u (m)	0.0000070724	0.0000070724	0.0000070724
w (m)	0.0321307287	0.0321307287	0.0321307287
Sinusoidal distributed load ($q_z = 0.001\sin(\frac{\pi x}{\ell})$ N/m)			
u (m)	0.0000013388	0.0000013388	0.0000013388
w (m)	0.0064715752	0.0064715752	0.0064714307

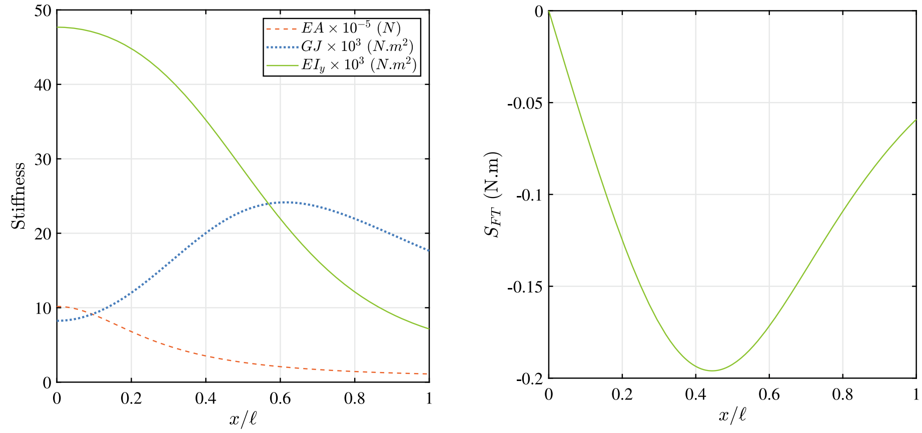
Table 8
Tip deformation of unsymmetric cantilever beam ($[60_3/30_3]$) under non-uniformly distributed loads.

	Exact	Analytical	CCM
Linearly distributed load ($q_z = 0.004x$ N/m)			
u (m)	−0.0000048161	−0.0000048161	−0.0000048161
φ (rad)	−0.0292043302	−0.0292043302	−0.0292043302
w (m)	0.0572150689	0.0572150689	0.0572150689
Sinusoidal distributed load ($q_z = 0.001\sin(\frac{\pi x}{\ell})\,N/m$)			
u (m)	−0.0000009117	−0.0000009117	−0.0000009117
φ (rad)	−0.0055284892	−0.0055284892	−0.0055283464
w (m)	0.0115239098	0.0115239098	0.0115236526

1. Symmetric $[\theta_3]_s$ with bend-twist coupling,
2. Antisymmetric $[\theta_3/-\theta_3]$ with extension-twist coupling,
3. Unsymmetric $[\theta_3/0_3]$ with bend-twist, extension-twist and extension-bend coupling.

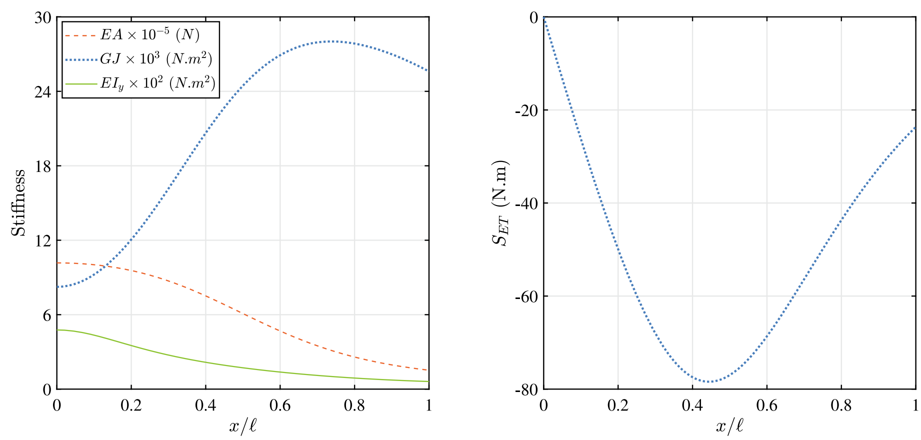
where the fibre angle θ varies linearly along the beam span as: $\theta(x) = \frac{\theta_t-\theta_r}{\ell}x + \theta_r$, where θ_r is the fibre angle at $x = 0, \theta_t$ is the fibre angle at $x = \ell, \theta_r = 0^\circ$ and $\theta_t = 45^\circ$. The distribution of stiffness properties for each lay-up is shown in Figs. 2–4.

The numerical results for the static response of variable stiffness composite beams with various boundary conditions and stacking sequences subject to non-uniformly distributed loads are presented in the following subsections. It is also worth noting that in order to construct the power series using a curve fitting approach, twenty sampling points are chosen which are distributed evenly along the beam length and the integrands are approximated accordingly which is shown to be sufficiently accurate.



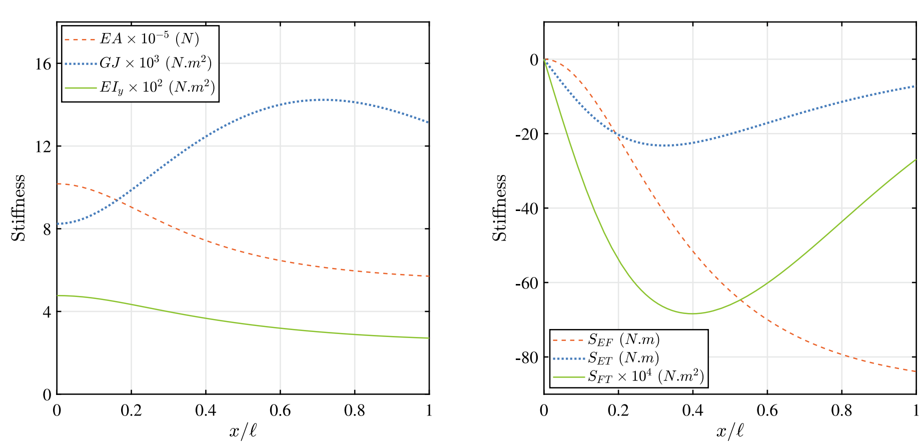
(a) (b)

Fig. 2. Distribution of stiffness properties, symmetric $[\theta_3]_s$.



(a) (b)

Fig. 3. Distribution of stiffness properties, asymmetric $[\theta_3/-\theta_3]$.



(a) (b)

Fig. 4. Distribution of stiffness properties, unsymmetric $[\theta_3/0_3]$.

Table 9
Tip deformation of bend-twist coupled cantilever beam ($[\theta_3]_s$) under non-uniformly distributed loads.

	Analytical	CCM
Linearly distributed load ($q_z = 0.01x$ N)		
φ (rad)	−0.0469687194	−0.0469687194
w (m)	0.0354972571	0.0354972571
Parabolic distributed load ($q_z = -0.01x^2 + 0.01$ N)		
φ (rad)	−0.0206670142	−0.0206670244
w (m)	0.0176764127	0.0176764198
Sinusoidal distributed load ($q_z = 0.01\sin(\frac{\pi x}{\ell})\,N$)		
φ (rad)	−0.0307536589	−0.0307536589
w (m)	0.0255147604	0.0255147604

Table 10
Tip deformation of extension-twist coupled cantilever beam ($[\theta_3/-\theta_3]$) under non-uniformly distributed loads.

	Analytical	CCM
Linearly distributed load ($q_\varphi = 0.001x\,N$)		
φ (rad)	0.0285741886	0.0285741928
u (m)	0.0000021463	0.0000021463
Parabolic distributed load ($q_\varphi = -0.001x^2 + 0.001$ N)		
φ (rad)	0.0241784665	0.0241781052
u (m)	0.0000013627	0.0000013627
Sinusoidal distributed load ($q_\varphi = 0.001\sin(\frac{\pi x}{\ell})$ N)		
φ (rad)	0.0297069622	0.0297069694
u (m)	0.0000018676	0.0000018676

Table 11
Tip deformation of unsymmetric cantilever beam ($[\theta_3/0_3]$) under non-uniformly distributed loads.

	Analytical	CCM
Linearly distributed load ($q_z = 0.01x$ N)		
φ (rad)	−0.0210453145	−0.0210453185
u (m)	−0.0000026184	−0.0000026184
w (m)	0.0256389470	0.0256389512
Parabolic distributed load ($q_z = -0.01x^2 + 0.01$ N)		
φ (rad)	−0.0095198575	−0.0095182262
u (m)	−0.0000009585	−0.0000009583
w (m)	0.0137357943	0.0137340564
Sinusoidal distributed load ($q_z = 0.01\sin(\frac{\pi x}{\ell})$ N)		
φ (rad)	−0.0141115872	−0.0141115937
u (m)	−0.0000014696	−0.0000014696
w (m)	0.0195230072	0.0195230141

5.2.1. Variable stiffness composite cantilever beam under non-uniformly distributed loads

A composite cantilever beam subject to various non-uniformly distributed load is considered. Numerical results for the tip deflections are given based on the analytical solution and the Chebyshev collocation method (CCM). Tables 9–11, show the tip deflections for different stacking sequences. For all types of distributed load there is excellent agreement between the two sets of results. Figs. 5–7 show the composite beam deformations for different stacking sequences based on the analytical solution and CCM for linearly distributed load. Deformations with zero value are not shown. It is clearly observed that the static response of variable stiffness composite beam is affected by different coupling terms.

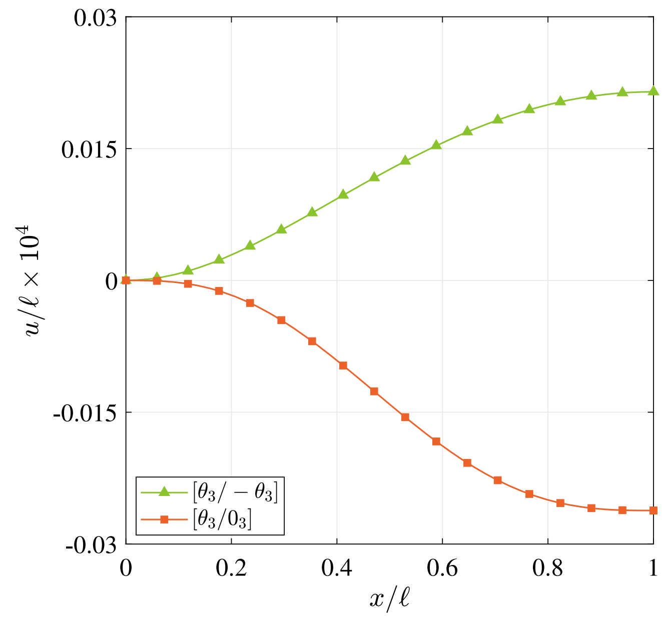


Fig. 5. Axial elongation of a cantilever beam under the action of linearly distributed load for different stacking sequences.

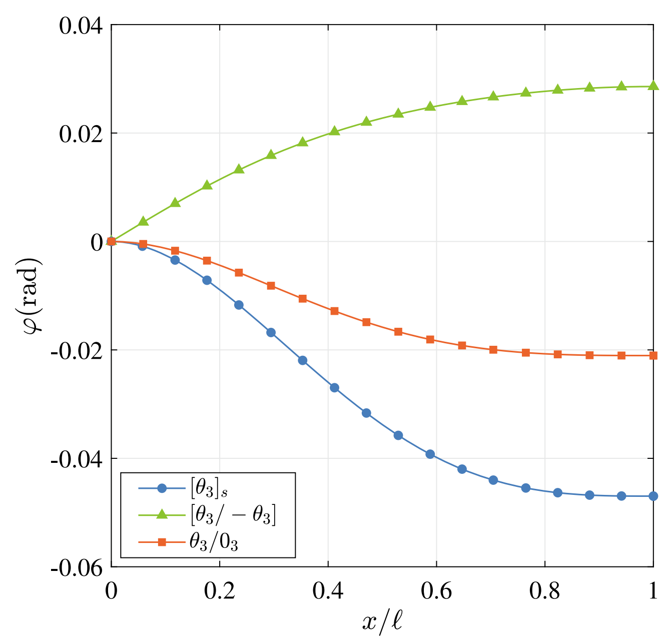


Fig. 6. Twist deformation of a cantilever beam under the action of linearly distributed load for different stacking sequences.

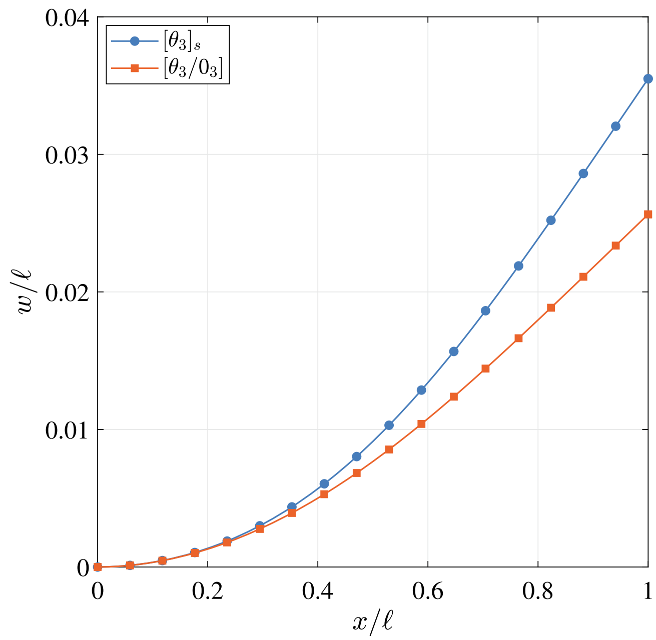


Fig. 7. Bending deformation of a cantilever beam under the action of linearly distributed load for different stacking sequences.

Table 12
Maximum deformation of bend-twist coupled simply supported beam ($[\theta_3]_s$) under non-uniformly distributed loads.

	Analytical	CCM
Linearly distributed load ($q_z = 0.1x$ N)		
	Analytical	CCM
φ (rad)	−0.0712544563	−0.0712544563
w (m)	0.0395249232	0.0395249232
Sinusoidal distributed load ($q_z = 0.1\sin(\frac{\pi x}{\ell})$ N)		
	Analytical	CCM
φ (rad)	−0.0961780046	−0.0961780046
w (m)	0.0580173717	0.0580173717

Table 13
Maximum deformation of extension-twist coupled simply supported beam ($[\theta_3/-\theta_3]$) under non-uniformly distributed loads.

	Analytical	CCM
Linearly distributed load ($q_\varphi = 0.01x$ N)		
	Analytical	CCM
φ (rad)	0.0385826815	0.0385826807
u (m)	0.0000030274	0.0000030274
Sinusoidal distributed load ($q_\varphi = 0.01\sin(\frac{\pi x}{\ell})$ N)		
	Analytical	CCM
φ (rad)	0.0662356439	0.0662356417
u (m)	0.0000043242	0.0000043242

5.2.2. Simply supported variable stiffness composite beam under non-uniformly distributed load

A variable stiffness composite beam simply supported at both ends under the action of non-uniformly distributed load is considered in this section. The values of the maximum deformations are presented in Tables 12–14 based on the analytical solution and the CCM. Excellent agreement is observed between the analytical results and those of the CCM for all cases. Figs. 8–10 show the deformed configuration of variable stiffness composite beams for different stacking sequences based on the two methods for sinusoidal non-uniformly distributed load. For the sinusoidal load case the loading and boundary conditions are symmetric. For symmetric $[\theta_3]_s$ and unsymmetric $[\theta_3/0_3]$ stacking sequences, the deformation in w occurs directly due to the applied load. However, the maximum value of w does not occur necessarily at the mid-span which is due to the fact that bending stiffness EI_y decreases from $x = 0$ to $x = \ell$. Axial elongation u and twist φ occur as a results of coupling terms. Due to the symmetric loading and boundary conditions, the internal shear force in the z direction is distributed symmetrically along the beam span with a zero value at the mid-span. It is clear that a spanwise location exists where w''' becomes zero. From the constitutive equations, w''' is observed to be proportional to φ'' and/or u''. Consequently, at a specific location the values of φ'' and/or u'' are zero, so a change of curvature in u and φ is expected. Considering the symmetric loading and boundary conditions, the values of u and φ should inevitably have two local extrema.

It is worth mentioning that the complicated behaviour of variable stiffness composite beams is highly influenced by the distribution of stiffness properties, loading and boundary conditions and it is not necessarily intuitive to interpret their static behaviour.

Table 14
Maximum deformation of unsymmetric simply supported beam ($[\theta_3/0_3]$) under non-uniformly distributed loads.

	Analytical	CCM
Linearly distributed load ($q_z = 0.1x$ N)		
	Analytical	CCM
φ (rad)	−0.0188198375	−0.0188198376
u (m)	−0.0000032672	−0.0000032672
w (m)	0.0219769883	0.0219769885
Sinusoidal distributed load ($q_z = 0.1\sin(\frac{\pi x}{\ell})$ N)		
	Analytical	CCM
φ (rad)	0.0259152902	0.0259152903
u (m)	0.0000039655	0.0000039655
w (m)	0.0339815869	0.0339815873

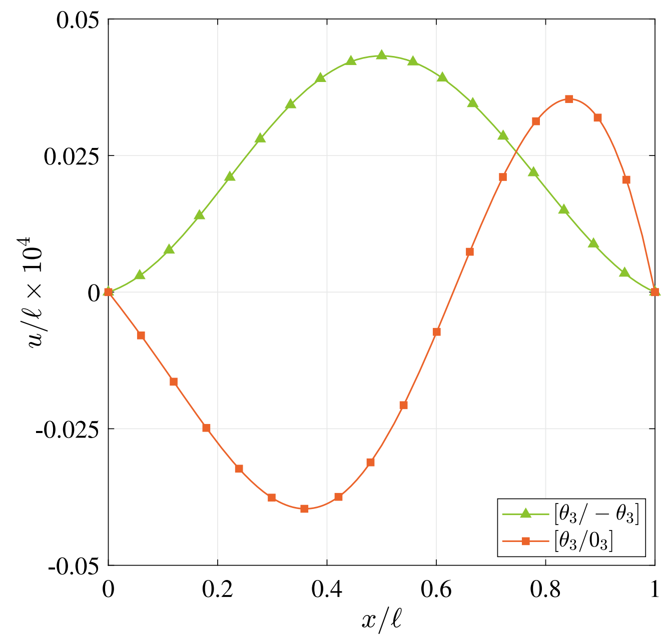


Fig. 8. Axial elongation of a simply supported beam under the action of sinusoidal distributed load for different stacking sequences.

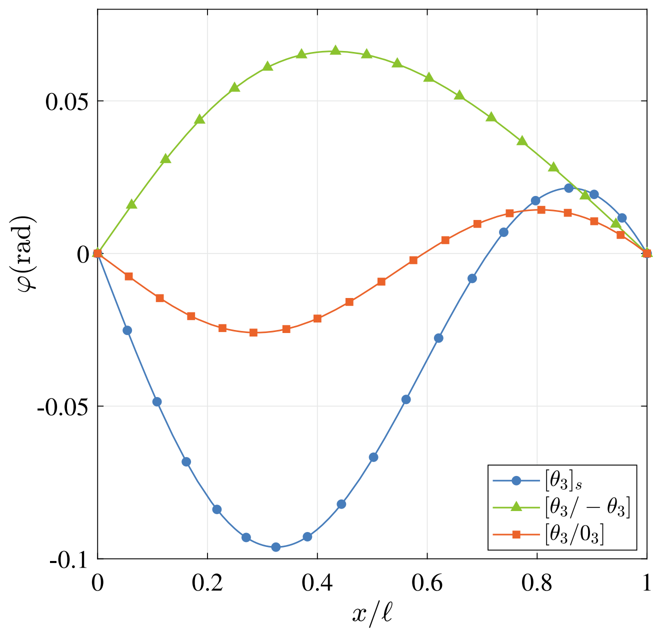


Fig. 9. Twist deformation of a simply supported beam under the action of sinusoidal distributed load for different stacking sequences.

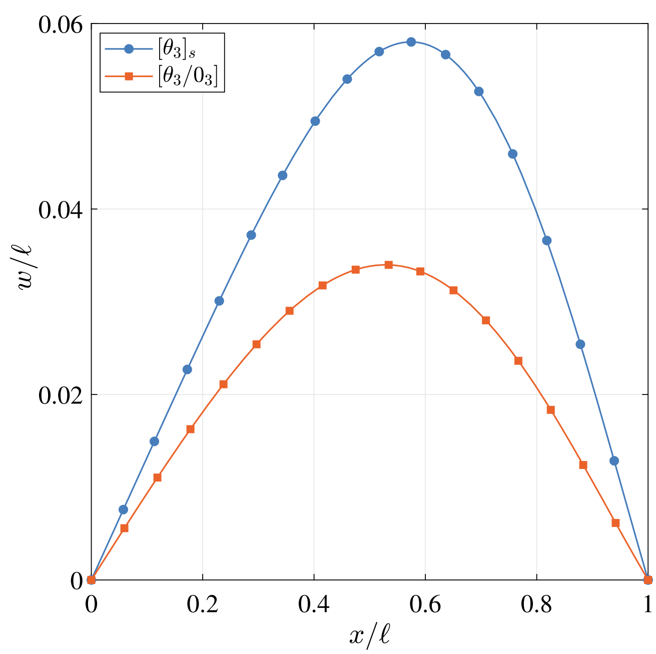


Fig. 10. Bending deformation of a simply supported beam under the action of sinusoidal distributed load for different stacking sequences.

6. Conclusions

A closed-form analytical solution is presented for the first time for the fully coupled 3D static deflection of variable stiffness composite beams subject to non-uniformly distributed loads. Based on Euler-Bernoulli theory, the governing differential equations with variable coefficients are presented and expressed in a compact matrix form where all degrees of freedom, namely: bending in two principal directions; twist and axial elongation are fully coupled. Engineering constants are used to express the entries of the stiffness matrix allowing the behaviour of the beams with arbitrary cross-sections to be considered in a unified framework. Compared to uniform loads, arbitrary non-uniformly distributed loads introduce mathematical complications to obtain closed-form solutions. However, in the current work, the general analytical solutions are derived by direct integration, and series expansion representation is employed to obtain the closed-form expressions. Additionally, in order to verify the analytical solution, the accurate and computationally efficient method of Chebyshev collocation is also applied. A slender variable stiffness beam with rectangular cross-section made by fibre steering subject to non-uniformly distributed loads with various boundary conditions is considered to obtain the numerical solution. A span-wise linearly varying fibre angles are assumed and three types of stacking sequences showing various combinations of coupling terms are considered to show the effects of anisotropy on the static deflection. The results obtained from the analytical solution and the Chebyshev collocation method are shown to be in excellent agreement. The complicated static behaviour of variable stiffness composite beams is not a trivial matter to study intuitively. However, the proposed solutions, while offering unique characteristics compared to the available solutions, such as being 3D, fully coupled and applicable to anisotropic beams with arbitrary cross-sections, present a reliable and efficient means for future investigations and engineering design studies.

CRediT authorship contribution statement

Pedram Khaneh Masjedi: Conceptualization, Methodology, Software, Writing - original draft, Writing - review & editing. **Paul M. Weaver:** Supervision, Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Derivation of governing equations

In order to be self-contained, the procedure of obtaining the governing equations and boundary conditions of a variable stiffness Euler-Bernoulli composite beam is reported herein from Masjedi and Weaver [20].

The principle of virtual work is expressed as:

$$\int_0^\ell (\delta W_{int} - \delta W_{ext})dx = 0, \tag{A.1}$$

where δW_{int} and δW_{ext} are the variations of internal and external works respectively.

Variation of internal work of the beam can be expressed as:

$$\int_0^\ell \delta W_{int} dx = \int_0^\ell \delta\varepsilon^T S \varepsilon dx, \tag{A.2}$$

where vector of strains ε and stiffness matrix S are written as:

$$\varepsilon = [u' \quad \varphi' \quad -w'' \quad v'']^T, \tag{A.3a}$$

$$S = \begin{bmatrix} EA(x) & S_{ET}(x) & S_{EF}(x) & S_{EL}(x) \\ S_{ET}(x) & GJ(x) & S_{FT}(x) & S_{LT}(x) \\ S_{EF}(x) & S_{FT}(x) & EI_y(x) & S_{FL}(x) \\ S_{EL}(x) & S_{LT}(x) & S_{FL}(x) & EI_z(x) \end{bmatrix}, \tag{A.3b}$$

Variation of external work can be expressed as:

$$\int_0^\ell \delta W_{ext} dx = \int_0^\ell \delta \overline{U}^T \overline{q} dx, \tag{A.4}$$

where displacement vector $\overline{U}$ and vector of external loads $\overline{q}$ are written as:

$$\overline{U} = [u \quad \varphi \quad w \quad v]^T, \tag{A.5a}$$

$$\overline{q} = \begin{bmatrix} q_x & q_\varphi & q_z & q_y \end{bmatrix}^T, \tag{A.5b}$$

Now Eqs. (A.2) and (A.4) can be written as:

$$\begin{aligned}\int_0^\ell \delta W_{int} dx = & \int_0^\ell \{\delta u'(EA(x)u' + S_{ET}(x)\varphi' - S_{EF}(x)w'' + S_{EL}(x)v'') \\ & + \delta\varphi'(S_{ET}(x)u' + GJ(x)\varphi' - S_{FT}(x)w'' + S_{LT}(x)v'') \\ & - \delta w''(S_{EF}(x)u' + S_{FT}(x)\varphi' - EI_y(x)w'' + S_{FL}(x)v'') \\ & + \delta v''(S_{EL}(x)u' + S_{LT}(x)\varphi' - S_{FL}(x)w'' + EI_z(x)v'')\}dx,\end{aligned} \tag{A.6}$$

and

$$\int_0^\ell \delta W_{ext} dx = \int_0^\ell \left(\delta u q_x + \delta\varphi q_\varphi + \delta w q_z + \delta v q_y\right) dx. \tag{A.7}$$

Substituting Eqs. (A.6) and (A.7) into Eq. (A.1), the set of governing differential equations with variable coefficients are obtained as:

$$\delta u : (-EA(x)u' - S_{ET}(x)\varphi' + S_{EF}(x)w'' - S_{EL}(x)v'')' = q_x \tag{A.8a}$$

$$\delta\varphi : (-S_{ET}(x)u' - GJ(x)\varphi' + S_{FT}(x)w'' - S_{LT}(x)v_{,x})' = q_\varphi \tag{A.8b}$$

$$\delta w : \left(-S_{EF}(x)u' - S_{FT}(x)\varphi' + EI_y(x)w'' - S_{FL}(x)v''\right)'' = q_z \tag{A.8c}$$

$$\delta v : (S_{EL}(x)u' + S_{LT}(x)\varphi' - S_{FL}(x)w'' + EI_z(x)v'')'' = q_y. \tag{A.8d}$$

At $x = 0$ and $x = \ell$, boundary conditions can be expressed as follows:

$$u = 0 \quad \text{or} \quad EA(x)u' + S_{ET}(x)\varphi' - S_{EF}(x)w'' + S_{EL}(x)v'' = 0 \tag{A.9a}$$

$$\varphi = 0 \quad \text{or} \quad S_{ET}(x)u' + GJ(x)\varphi' - S_{FT}(x)w'' + S_{LT}(x)v'' = 0 \tag{A.9b}$$

$$w' = 0 \quad \text{or} \quad -S_{EF}(x)u' - S_{FT}(x)\varphi' + EI_y(x)w'' - S_{FL}(x)v'' = 0 \tag{A.9c}$$

$$v' = 0 \quad \text{or} \quad S_{EL}(x)u' + S_{LT}(x)\varphi' - S_{FL}(x)w'' + EI_z(x)v'' = 0 \tag{A.9d}$$

$$w = 0 \quad \text{or} \quad (S_{EF}(x)u' + S_{FT}(x)\varphi' - EI_y(x)w'' + S_{FL}(x)v'')' = 0 \tag{A.9e}$$

$$v = 0 \quad \text{or} \quad (-S_{EL}(x)u' - S_{LT}(x)\varphi' + S_{FL}(x)w'' - EI_z(x)v'')' = 0, \tag{A.9f}$$

Using matrices (2.3), the governing Eqs. (4.8) and boundary conditions (4.9) are written in a compact matrix form:

$$\left(-A(x)\begin{bmatrix} u \\ \varphi \end{bmatrix}' + B(x)\begin{bmatrix} w \\ v \end{bmatrix}''\right)' = \begin{bmatrix} q_x \\ q_\varphi \end{bmatrix}, \tag{A.10a}$$

$$\left(-B^T(x)\begin{bmatrix} u \\ \varphi \end{bmatrix}' + D(x)\begin{bmatrix} w \\ v \end{bmatrix}''\right)'' = \begin{bmatrix} q_z \\ q_y \end{bmatrix}. \tag{A.10b}$$

$$\begin{bmatrix} u \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{A.11a}$$

$$\begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{A.11b}$$

$$\begin{bmatrix} w \\ v \end{bmatrix}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A.11c}$$

$$A(x)\begin{bmatrix} u \\ \varphi \end{bmatrix}' - B(x)\begin{bmatrix} w \\ v \end{bmatrix}'' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{A.12a}$$

$$-B^T(x)\begin{bmatrix} u \\ \varphi \end{bmatrix}' + D(x)\begin{bmatrix} w \\ v \end{bmatrix}'' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{A.12b}$$

$$\left(B^T(x)\begin{bmatrix} u \\ \varphi \end{bmatrix}' - D(x)\begin{bmatrix} w \\ v \end{bmatrix}''\right)' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A.12c}$$

Appendix B. Chebyshev collocation method

The Chebyshev collocation method (CCM) is applied as an alternative numerical solution for Eq. (4.8). For this purpose, unknown variables u, v, w and φ are discretised using the Chebyshev polynomials, and the Chebyshev points are employed as the collocation points. The Chebyshev polynomial of the first kind in x and of degree n is defined as [20]:

$$T_n(x) = \cos(n\theta) \quad when \quad x = \cos\theta, \quad -1 \leqslant x \leqslant +1, \quad n = 0, 1, \ldots \tag{B.1}$$

For an arbitrary interval $a \leqslant x \leqslant b$ the transformed Chebyshev points are:

$$x_i = \frac{1}{2}(a+b) - \frac{1}{2}(b-a)\cos\left(\frac{2i-1}{2N}\pi\right), \quad i = 1, 2, \ldots, N+1, \tag{B.2}$$

where N is the highest degree of the Chebyshev polynomials.

Using the Chebyshev polynomials, unknown variables are discretised as:

$$u = \sum_{i=0}^{N} a_i T_i(x), \quad \varphi = \sum_{i=0}^{N} b_i T_i(x), \quad w = \sum_{i=0}^{N} c_i T_i(x), \quad v = \sum_{i=0}^{N} d_i T_i(x). \tag{B.3}$$

Eqs. (B.3) are substituted into Eqs. (4.8) and the residuals at the Chebyshev points are set equal to zero. It is worth mentioning that in order to have a well-posed system of $4 \times (N+1)$ equations in conjunction with the boundary conditions (4.9), the same approach proposed by [20] is followed herein: in the case of u and φ, the equations associated with the residuals at the first and the last Chebyshev points are eliminated, whereas in the case of w and V, those equations associated with the first two and the last two Chebyshev points from Eq. (B.2) are discarded systematically. Finally, unknown coefficients a_i, b_i, c_i and d_i where $i = 0, 1, 2, \ldots, N$ are determined solving the system of linear equations.

References

[1] Lü C, Chen W, Xu R, Lim CW. Semi-analytical elasticity solutions for bi-directional functionally graded beams. Int J Solids Struct 2008;45:258–75.
[2] Shahba A, Attarnejad R, Hajilar S. A mechanical-based solution for axially functionally graded tapered euler-bernoulli beams. Mech Adv Mater Struct 2013;20:696–707.
[3] Nguyen N, Kim N, Cho I, Phung Q, Lee J. Static analysis of transversely or axially functionally graded tapered beams. Mater Res Innov 2014;18:S2-260.
[4] Zhao L, Zhu J, Wen XD. Exact analysis of bi-directional functionally graded beams with arbitrary boundary conditions via the symplectic approach. Struct Eng Mech 2016;59:101–22.
[5] Pydah A, Sabale A. Static analysis of bi-directional functionally graded curved beams. Compos Struct 2017;160:867–76.
[6] Pydah A, Batra R. Shear deformation theory using logarithmic function for thick circular beams and analytical solution for bi-directional functionally graded circular beams. Compos Struct 2017;172:45–60.
[7] Sachdeva C, Padhee SS. Analysis of bidirectionally graded cylindrical beams using variational asymptotic method. AIAA J 2019:1–13.
[8] Soltani M, Asgarian B. Finite element formulation for linear stability analysis of axially functionally graded nonprismatic timoshenko beam. Int J Struct Stab Dyn 2019;19:1950002.
[9] Soltani M, Asgarian B. Exact stiffness matrices for lateral–torsional buckling of doubly symmetric tapered beams with axially varying material properties, Iran J Sci Technol Trans Civil Eng; 2020..
[10] Huang Y, Ouyang Z-Y. Exact solution for bending analysis of two-directional functionally graded timoshenko beams. Arch Appl Mech 2020:1–19.
[11] Karamanlı A. Elastostatic analysis of two-directional functionally graded beams using various beam theories and symmetric smoothed particle hydrodynamics method. Compos Struct 2017;160:653–69.
[12] Li X, Li L, Hu Y, Ding Z, Deng W. Bending, buckling and vibration of axially functionally graded beams based on nonlocal strain gradient theory. Compos Struct 2017;165:250–65.
[13] Xie X, Zheng H, Zou X. An integrated spectral collocation approach for the static and free vibration analyses of axially functionally graded nonuniform beams. Proc Inst Mech Eng C J Mech Eng Sci 2017;231:2459–71.
[14] Günay MG, Timarci T. Static analysis of thin-walled laminated composite closed-section beams with variable stiffness. Compos Struct 2017;182:67–78.
[15] Ai Q, Weaver PM. Simplified analytical model for tapered sandwich beams using variable stiffness materials. J Sandwich Struct Mater 2017;19:3–25.

[16] Zappino E, Viglietti A, Carrera E. Analysis of tapered composite structures using a refined beam theory. Compos Struct 2018;183:42–52.
[17] Macquart T, Pirrera A, Weaver PM. Finite beam elements for variable stiffness structures. AIAA J 2018;56:3362–8.
[18] Rajasekaran S, Bakhshi Khaniki H. Finite element static and dynamic analysis of axially functionally graded nonuniform small-scale beams based on nonlocal strain gradient theory. Mech Adv Mater Struct 2019;26:1245–59.
[19] Yu T, Zhang J, Hu H, Bui TQ. A novel size-dependent quasi-3d isogeometric beam model for two-directional fg microbeams analysis. Compos Struct 2019;211:76–88.
[20] Masjedi PK, Weaver PM. Analytical solution for the fully coupled static response of variable stiffness composite beams. Appl Math Model 2020;81:16–36.
[21] Doeva O, Masjedi PK, Weaver PM. Exact solution for the deflection of composite beams under non-uniformly distributed loads. In: AIAA Scitech 2020 Forum; 2020a. p. 0245..
[22] Doeva O, Masjedi PK, Weaver PM. Static deflection of fully coupled composite timoshenko beams: an exact analytical solution. Eur J Mech-A/Solids 2020. 103975.
[23] Masjedi PK, Ovesy HR. Chebyshev collocation method for static intrinsic equations of geometrically exact beams. Int J Solids Struct 2015;54:183–91.
[24] Masjedi PK, Ovesy HR. Large deflection analysis of geometrically exact spatial beams under conservative and nonconservative loads using intrinsic equations. Acta Mech 2015;226:1689–706.
[25] Masjedi PK, Maheri A. Chebyshev collocation method for the free vibration analysis of geometrically exact beams with fully intrinsic formulation. Eur J Mech-A/Solids 2017;66:329–40.
[26] Masjedi PK, Maheri A, Weaver PM. Large deflection of functionally graded porous beams based on a geometrically exact theory with a fully intrinsic formulation. Appl Math Model 2019;76:938–57.
[27] Abdalla MM, Setoodeh S, Gürdal Z. Design of variable stiffness composite panels for maximum fundamental frequency using lamination parameters. Compos Struct 2007;81:283–91.
[28] Setoodeh S, Abdalla MM, IJsselmuiden ST, Gürdal Z. Design of variable-stiffness composite panels for maximum buckling load. Compos Struct 87; 2009:109–117..
[29] Ghiasi H, Fayazbakhsh K, Pasini D, Lessard L. Optimum stacking sequence design of composite materials part ii: variable stiffness design. Compos Struct 2010;93:1–13.
[30] Stodieck O, Cooper JE, Weaver PM, Kealy P. Improved aeroelastic tailoring using tow-steered composites. Compos Struct 2013;106:703–15.
[31] Wu Z, Weaver PM, Raju G. Postbuckling optimisation of variable angle tow composite plates. Compos Struct 2013;103:34–42.
[32] Capuzzi M, Pirrera A, Weaver P. Structural design of a novel aeroelastically tailored wind turbine blade. Thin-Walled Struct 2015;95:7–15.
[33] Albazzan MA, Harik R, Tatting BF, Gürdal Z. Efficient design optimization of nonconventional laminated composites using lamination parameters: a state of the art. Compos Struct 2019;209:362–74.
[34] Serhat G, Bediz B, Basdogan I. Unifying lamination parameters with spectral-tchebychev method for variable-stiffness composite plate design. Compos Struct 2020;112183.
[35] Thomas MA, Hallett SR, Weaver PM. Design considerations for variable stiffness, doubly curved composite plates. Compos Struct 2020;112170.
[36] Yu W, Hodges DH. Best strip-beam properties derivable from classical lamination theory. AIAA J 2008;46:1719–24.