Static Capillary Structures and their Stability

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Preface

Abstract

This thesis examines the static equilibrium shapes and stability of various capillary surfaces. The equilibrium shapes are accurately approximated using asymptotic series solutions in the micro-gravity limit. The stability of two of these capillary surfaces is then examined using an energy functional method and these results corroborated using a linear stability analysis. Finally, a method of improving the stability of a vertical liquid bridge is examined numerically and experimentally. The problems considered in this thesis are motivated by a stent problem described in the first chapter but, in fact, capillary phenomena are ubiquitous in science, nature and industry and the work here has wide reaching applications.
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Finally, I must thank my grandparents for their continuous and unwavering support and encouragement which made this whole time possible.
Declaration

I, the undersigned, hereby declare that this submission is entirely my own work, in my own words, and that all sources used in researching it are fully acknowledged and all quotations properly identified. It has not been submitted, in whole or in part, by me or another person, for the purpose of obtaining any other credit / grade.

Matthew Haynes
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Chapter 1

Introduction

We encounter liquid drops on a daily basis, whether it be in the form of a raindrop on a car windscreen or an air bubble in a carbonated drink. The interface between the two phases is commonly referred to as a capillary surface because capillary effects (i.e., surface tension) play an important role in determining its shape and stability.

The first theories of capillary effects originated from the study of the phenomenon whereby liquid rises up a (so-called capillary) tube placed in a bath of water. But it soon became clear that similar effects abound. In nature, prime examples of exploiting capillary forces to one’s advantage are the water boatman (corixidae, for example), which spends its life ‘running’ upon the surface of ponds and lakes using these surface forces (for example, studied by Blake [1986]), and the adhesion mechanism used by tree frogs (for example, studied by Persson [2007]). In biology and medicine, a comprehension of many respiratory diseases follows from an understanding of the flow of gas past a liquid drop or liquid bridge (for example, studied by Cavalcante et al. [2005]). In industry, capillary effects are important in coating flows (for example, studied by Wilson et al. [2002]), contact lens manufacture (for example, studied by Murphy and Lee [2017]) among other processes. The capillary problems studied in this thesis were motivated by a problem presented at a European Study Group with
A stent is a tube inserted into a passage in the body to prevent or counteract a disease-induced reduction of flow through the passage. Stents are widely used in a large number of medical applications; however, the stent we focus on here is a throat stent. This stent is made of a superalloy wire, braided in a tubular mesh configuration and covered with a silicone polymer in such a way that it is both flexible and self-expanding. The production of such a stent involves covering the tubular mesh in a highly volatile coating solution. To achieve this, the mesh is mounted on a mandrel (a very smooth polytetrafluoroethylene cylindrical fixture), as seen in figure 1.1a, and both are coated with the solution. After the coating, the stent is then cured. During this phase, the mandrel and mesh are placed in an oven at 100°C for about an hour. In this time, the solvent evaporates and polymerization occurs, causing a steady drop in the volume. It is in this phase of the manufacturing process that, on occasion, a coating defect arises whereby holes can develop in the coating in one or more of the ‘diamonds’ (defined by the superalloy wires), as seen in figure 1.1b. This leads to undesirable wastage as the defect renders the stent unusable.

The process which causes the hole-formation in the diamonds is simplified to first understand the fluid motion within the diamond by modelling the liquid region as a capillary problem.

Referring to figure 1.1b, each diamond can be seen to support a liquid bridge and it is evident that even to accurately describe the equilibrium configuration is a challenging problem. In fact, the problem is fully three dimensional with the added complication that the bounding structure of the liquid bridge is embedded in a curved surface. Further, these structures are approximately cylindrical (wires) which allows the effective slope of the free liquid surface at the contact line to vary.

Before attempting to analyse hole-formation in stents, it is natural to first consider problems with a high degree of symmetry. This thesis discusses two possible simplifications. First, we idealise the supporting structure as a horizon-
Section 1.0

(a) Image of a stent mesh mounted on a mandrel. (b) Close up of the completed stent.

Figure 1.1: Images of a stent at different stages during the production process.

tal rectangular frame, comprising four approximately cylindrical or rectangular pillars. If the frame is sufficiently long in one direction, then the cross-section through the bridge in the shorter direction, supported by the cross section of two of the pillars, can be approximated as two dimensional. An alternate simplification we study is idealising the supporting structure as a vertical cylindrical tube, where the liquid bridge is formed inside. This thesis studies primarily the equilibrium configurations of these structures and their stability.

An improved understanding of the shapes of capillary surfaces has further applications. In recent years, models of the static shapes of capillary surfaces has allowed for more accurate evaluation of liquid properties, for example, a standard method to find the surface tension of a liquid is to use it to form a liquid bridge and match the resulting shape to a theoretical shape of the liquid bridge, see [Pétré and Wozniak, 1986]. Another example of the importance of capillary surfaces in industry is the understanding of the shape of a drop on an inclined plane, which is of significance, for example, in the ink-jet printing industry. An ink-jet printer will typically place a drop of liquid ink (with high precision) on a piece of paper after which it dries. If the physics of these complicated phenomena are not well controlled the result of the printing is unsatisfactory, see for example [Singh et al., 2010] and references therein. Details of these
Chapter 1

processes shall be discussed in section 1.1.

This thesis is structured into three sections: the first examines the shape of capillary surfaces under microgravity conditions (chapter 2); the second investigates the stability threshold for two capillary configurations (chapters 3 and 4); and the third explores the possibility of extending the stability region of a vertical liquid bridge (chapter 5). While no single capillary surface is studied throughout the thesis, the geometries chosen for each section are interesting, both physically and mathematically, and deal with previously unsolved problems. A key part of this thesis is showing that the methods used for one configuration are equally applicable to more exotic problems.

In this chapter we give both a physical and mathematical introduction to the problems. We first introduce several important theoretical results and give examples of practical applications of capillary surfaces. Then we define and refine the mathematical tools which are used through the subsequent technical chapters. This technical introduction contains a summary of most of the essential tools required throughout this thesis.

Chapter 2 studies the equilibrium shape of three capillary surfaces under the constraint of micro-gravity. This chapter shall studied three physical problems. The three problems are: an axisymmetric liquid membrane; an infinite horizontal liquid bridge; and a drop on an inclined plane. The first two problems use symmetry conditions to reduce a three dimensional problem to a two dimensional problem, while the third only considers a slice of a drop. These surfaces are examined by considering a parametric formulation and then developing an asymptotic solution in the limit of vanishing Bond number. The Bond number is a dimensionless quantity measuring the importance of body forces (gravity) when compared to surface forces (surface tension). The asymptotic solutions are accompanied with exact (albeit clumsy) solutions using elliptic integrals.

Chapter 3 investigates the stability for a symmetric horizontal liquid bridge. Two analytical methods are used to obtain an eigenvalue problem where (the real part of) the eigenvalue vanishes in the case of marginal stability. The results of this work are then verified by numerically solving the linearised Navier–Stokes
equations. These methods are also applied to an axisymmetric liquid membrane trapped inside a vertical cylindrical tube. The case of inviscid liquid membranes is also considered where some key results are analytically derived.

The results of the methods derived in chapter 3 are then discussed in chapter 4. The stable regions of the Bond number, volume, and central separation distance parameter space are found for the axisymmetric liquid membrane and the infinite horizontal liquid bridge. The perturbations are also examined either side of the neutral stability curve to determine destabilising mechanisms.

Chapter 5 investigates the proposal that the stability region of a vertical liquid bridge can be increased by vibrating the upper rod (i.e., the upper support). This study uses analytical, numerical and experimental methods. A similar result has been recently observed in experiments involving a liquid column being pulled from an infinite pool. This hypothesis is tested both numerically and experimentally. The experimental results were obtained by measuring the maximal slenderness of the liquid bridge both with and without vibration of the upper rod. The numerical results are compared with the experimental result and the asymptotic neutral stability curves previously derived.

1.1 Capillary surfaces

This section introduces the history of the field, while drawing reference to more modern works. We begin with a brief discussion of the early experimental work performed by several leading scientists from the renaissance period. This is followed by a short summary of the analytical work from the 19th and early 20th centuries. Several more modern review papers and monographs are then introduced to show the breath of the field, with each monograph having its own emphasis and motivation. This initial discussion focuses on the general field of capillary surfaces and therefore more modern (and generally more specific) texts pertaining to individual capillary surfaces (e.g., a drop) are discussed in sections 1.1.1 and 1.1.2.

A capillary surface forms the interface between two mutually immiscible flu-
The first study of capillary surfaces was performed by Leonardo da Vinci, referenced in the modern republication [da Vinci 2012], when he observed the rise of liquid in a capillary tube. This was later the interest of many leading scientists, an example can be seen in the modern republication [Hooke 2003] which gives an early description of capillary action. The main result of the early studies is attributed to Jurin [1719] who showed the height attained by the liquid is inversely proportional to the square of the diameter of the tube. This remained an empirical result without a theoretical basis until Von Segner [1751] postulated the concept of surface tension.

Mathematically, significant progress started to be made when Young [1805] and Laplace [1805] (independently) derived an equation which governs the shape of a capillary surface. This equation, now almost ubiquitous within the studies of capillary surfaces, takes both names as the ‘Young-Laplace equation’.

Young considered the forces acting upon a fluid interface. In so doing he was able to argue that the force due to surface tension must balance the force generated by pressure difference across the interface. He further argued that the force due to surface tension is proportional to the curvature of the interface. Mathematically, for a surface with curvature $\kappa$ and surface tension $\sigma$, this can be written as

$$\Delta p = \sigma \kappa,$$

where $\Delta p$ is the pressure difference across the interface.

Later, Gauss [1831] proposed an argument using the idea of minimising the energy of the system and applying the calculus of variations. The modern version of this argument was first produced by Bolza [1904], who gave the equations governing the extremals and minimisers for supported capillary surfaces.

Before the work of Bolza numerous famous (and less famous) scientists worked in the field of capillary surfaces. The reviews of Minkowski [1903] and later of Bikerman [1975] collected their results, together with the history of the problem. The classical monograph by Bakker et al. [1928] provides further details on the history and important results in the field. The review paper of
Bostwick and Steen [2015] together with the monographs of Finn [1986], de Gennes et al. [2004], and Langbein [2002] now form the starting point for the modern reader. The account of de Gennes et al. [2004] has a greater focus on wetting and spreading phenomena while the account of Langbein [2002] is greatly motivated by low gravity experiments being performed in space.

Recently, scientists have been performing a large number of experiments in space, see for example [Evans et al., 2009]. The field of capillary surfaces is one such field which has seen much success from these orbital experiments, see for example [Weislogel et al., 2009]. One advantage capillary surfaces has for these experiments over other fluid based studies is the lack of waste liquid produced during one experiment. These experiments often require an understanding of the dynamical problem, and they were often extensions or validations of work found in the detailed monograph of Myshkis et al. [1987].

Since the introduction of computational methods to the study of capillary surfaces, numerical solutions of the Young-Laplace equation have become readily available, however, this has not always been the case. Indeed, many of the early methods of solving the Young–Laplace equation involved the calculation of axisymmetric and cylindrical shapes of the liquid-gas interface. The Young–Laplace equation in these situations takes the form of a non-linear ordinary differential equation. Historically, computations were done manually, however the importance of these calculations was underlined, in 1857, by the Royal Society earmarking a sum of 50 pounds sterling for completion of these calculations. These results were later published by Bashforth and Adams [1883].

Before looking at specific capillary surface problems we give a full derivation of the Young-Laplace equation. This derivation uses a force balancing method (as Young did) following the derivation of Defay et al. [1966]. However we shall later see, in chapter 3, that the Young-Laplace equation can also be recovered from a energy minimisation approach, as observed by Gauss.

To derive the Young–Laplace equation, we begin by considering a point $P$ on the surface and drawing a curve (also on the surface) at a constant distance $\rho$ from $P$. This curve forms the boundary of the region, upon which we perform
a force balance, and we look for the equilibrium condition as $\rho$ tends to zero. Through $P$ we draw the two principal curvature sections $AB$ and $CD$ on the surface. Their respective radii of curvature at $P$ are $R_1$ and $R_2$, see figure 1.2. At the point $A$, an element $\delta l$ of the boundary line is subject to a force $\sigma \delta l$. When we project this force along the normal $PN$ we obtain $\sigma \delta l \sin \phi = \sigma \frac{\rho}{R_2} \delta l$.

If we consider similar elements on the boundary line at $B, C,$ and $D$ the total surface force acting on these elements is

$$2\rho \sigma \delta l \left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

We assume this expression is independent of the choice of sections $AB$ and $CD$ so it can be integrated around the circumference. Since four orthogonal elements are considered, the integration is made over a quarter of the circle, thus we find

$$F_S = \pi \rho^2 \sigma \left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

where $F_S$ is a force due to surface curvature. The force on the surface element which results from the pressure difference over the surface, $F_p$, is proportional to both the pressure differential and the area of the surface and we find that

$$F_p = (p_1 - p_2)\pi \rho^2.$$

Equating these we obtain the aforementioned Young-Laplace equation:

$$\Delta p = \sigma \kappa, \quad (1.1.2)$$

where $\Delta p$, $\kappa$ are, respectively, the difference in pressure across the surface and the curvature of the surface (which in this case is define to be twice the mean curvature) respectively.

In the 200 years following the original derivation of the Young-Laplace equation very little was achieved in finding exact solutions. However, Anderson et al. [2006] then gave details for finding exact solutions to the Young-Laplace equation for certain geometries. Anderson et al. [2006] presented two cases as classical results, having been given by Landau and Lifshitz [1987], and these are the cases of the half plane solution and the channel solution. In these cases the
liquid lies, respectively, in \([a, \infty)\) and \([a,b]\) for \(a,b \in \mathbb{R}\). The semi infinite case results in an inverse hyperbolic trigonometric solution and the channel solution relies on elliptic integrals. While neither of these solutions is particularly elegant, de Gennes et al. [2004] gives an exact solution for a meniscus on a vertical fibre. Using polar coordinates centred at the centre of the fibre and assuming the meniscus is axisymmetric, the height of the meniscus can be written in the form \(r = r(z)\). Further, in this case the Young-Laplace equation can be integrated to yield

\[
r = R \cosh \left( \frac{z - h}{R} \right),
\]

where \(R\) is the radius of the fibre and \(h\) is the height at which the meniscus intersects the fibre. This height \(h\) can be approximated by constraining the lateral dimension to not exceed the capillary length \(l\), which gives

\[
h \approx R \ln \left( \frac{2l}{R} \right).
\]

These exact solution for the free surface in their respective geometry show that in certain circumstances exact solution to the Young–Laplace equation do exist.

More recently, further exact solutions of the Young–Laplace equation have been developed. The solutions of Landau and Lifshitz [1987] are only applicable
to the profile of a surface bounded by one or two vertical planes. However, the results have been extended by Lv and Shi [2018] to understand the wetting states of two-dimensional drops. In this thesis we shall also develop exact solutions to the Young–Laplace equation for a two-dimensional drop on an inclined plane, and we shall compare these to the results of Lv and Shi [2018].

Thus far in this introduction we have overlooked the important field of bifurcation theory and its applications to capillary surfaces. Bifurcation theory has become an important tool in the analysis of the stability of a capillary surface. Through studying relevant phase diagrams the points of neutral stability can be found. The bifurcation method of finding these neutral stability points is, in general, computationally cheaper than other analytical tools. This method was first seen to be used by Poincaré [1885], and was modernised by Maddocks [1987]. This bifurcation theoretic approach gains information from families of equilibria and thereby reduces the amount of direct computation. This advantage is magnified if the investigation seeks only where stability changes. This method has been employed in a number of modern works, see, for example, Lowry and Steen [1995] (where they studied the stability of a vertical liquid bridge) and Luzzatto-Fegiz and Williamson [2012]. We shall use bifurcation theory to find points of neutral stability and construct stability diagrams.

1.1.1 Drops

The shape of a liquid drop on a horizontal (flat and smooth) surface has very few parameters, notable the contact angle and the Bond number. One method of determining the surface tension of a liquid is to compare the experimental profile of a liquid drop to a theoretical solution with agreeing contact angle. Adamson and Gast [1967] provides a detailed description of this method, among other experimental methods. To perform this matching, accurate theoretical solutions are required, and it is for this reason, among others, that understanding the profile of a liquid drop is important.

As mentioned in the previous section, Bashforth and Adams [1883] produced
numerical tables for various capillary surfaces, including both sessile and pendant liquid drops. In the following century several authors produced extensive tables of results for numerical solutions to the Young-Laplace equation for a liquid drop (see, for example, Staicopolus [1967]). Possibly the most extensive of these tabulated examination is that of Hartland and Hartley [1976], in which various axisymmetric capillary surfaces were considered, including both the sessile and pendant drop. These results were in table form for various surface tension and contact angle.

Since the results of Hartland and Hartley [1976], most of the work in this area has been in attempting to find accurate closed form approximate solutions for the profile of the drop (with the notable exceptions of Padday and Pitt [1972] and Boucher and Kent [1978], both of whom presented extensive analysis of sessile drop profiles). The necessity of these closed form approximations was driven by the need to allow physical chemists an accurate tool to compare the experimental profile with a theoretical profile when the experimental contact angle does not match with the tables.

With this aim, Shanahan [1982] developed first order perturbation solutions (assuming the capillary length is small). Shanahan’s method involved using the calculus of variations to recover the Young-Laplace equations and by direct manipulation first order perturbation solutions are derived. However, the methods outlined change when the contact angle becomes greater than $\pi/2$. Concus and Finn [1979] laid out a parametrisation of the Young-Laplace equation using the inclination and arc-length as independent variables. O’Brien and van den Brule [1991] manipulated these equations to remove the arc-length dependency and then found asymptotic solutions in the limit of vanishing Bond number. In chapter 2 we shall use this method to find asymptotic and exact solutions for the free surfaces for three capillary surfaces.

Concus and Finn [1979] developed an arclength method of manipulating Laplace’s results. For example, for a weightless axisymmetric drop on a flat
surface the Young–Laplace equation in polar coordinates reads,

\[
\left( \frac{rh'}{\sqrt{1 + h'^2}} \right)' = -rh,
\]

(1.1.5)

where dash denotes differentiation with respect to the radial distance from the centre of the drop, \(r\), and \(z = h(r)\) describes the height of the interface above the flat surface. This equation can be manipulated into the following equations using an arclength formulation,

\[
\frac{d\phi}{ds} = -h - \frac{\sin \phi}{r}, \quad \frac{dh}{ds} = \sin \phi, \quad \frac{dr}{ds} = \cos \phi,
\]

(1.1.6)

where \(s\) is the arclength and \(\phi\) the inclination, i.e. the angle the surface makes with the horizontal. This formulation has become widely used for solving drop problems both numerically, e.g., by Pozrikidis [2012] and asymptotically, e.g., by O’Brien and van den Brule [1991].

It is important to note that two dimensional liquid drops on an inclined surface are, in fact, unstable with respect to transverse perturbations; see Benilov and Benilov [2015] and references therein. As a result two dimensional drops are never encountered in nature. However, they still attract study because of their similarity to their three dimensional counterpart, which can be stable. Additionally, more mathematical progress can be made with the two dimensional equations than with their three dimensional counterparts.

While we shall only examine liquid drops using the Young-Laplace equation, it should be noted that Hocking [1981] developed a thin film equation to describe the profile of a liquid ridge falling down an inclined surface. A vast amount of literature builds upon this equation to study the two dimensional drop or liquid ridge on an inclined surface, for example, see Diez et al. [2012], Ding and Spelt, 2008 and Hocking, 1982.

Modern industrial applications which continue to drive the research into drops include, but are not limited to: microfluids (for example by Squires and Quake, 2005); inkjet printing (see, for example, Singh et al., 2010 and references therein) and ring staining (for example by Deegan et al., 1997).
1.1.2 Liquid bridges

A liquid bridge is a mass of liquid held between two solid supports. Liquid bridges occur and play a relevant role both in nature (see for example Persson [2007]) and in many industrial applications, such as materials engineering (see for example Kumar [2015]), powder granulation (see for example Suresh et al. [2017]) and flow in porous media (see for example Dejam et al. [2014]). Also, a liquid bridge can be used to calculate the physical parameters of a liquid, see for example Pétre and Wozniak [1986]. When the solid supports, or bars, are parallel surfaces the liquid bridge takes on a relatively simple configuration. The complex dynamics of the liquid meniscus formed next to a moving contact line may significantly affect the liquid bridge’s behaviour. The problem is simplified when the solid surfaces are disks of the appropriate size, so that the contact lines anchor to their sharp edges.

Two commonly studied liquid bridges are the vertical and horizontal liquid bridge. A vertical liquid bridge has the two solid supports placed one above the other, while the horizontal liquid bridge has the solid supports placed alongside each other. As mentioned, these problems become considerably simpler when the solid surfaces are disks of a size such that the contact lines are anchored. The simplicity is caused by the contact lines becoming pinned and then the dynamics of the contact line need not be considered.

Rayleigh [1878] showed that the maximum length of a cylindrical bridge supported by two equal sized disks is equal to the disk’s circumference. This result is famously known as the Plateau–Rayleigh stability limit, and based on Rayleigh’s theory and Plateau’s earlier experiments. The Plateau–Rayleigh stability limit is based on a hypothetical weightless bridge; gravitational forces act to reduce the stability limit by deforming the liquid bridge.

Much of the early work to develop the understanding of liquid bridges followed from the work of Haines [1925] and a correctional paper by Fisher [1926]. Their work was motivated by attempting to understand the properties of cohesion in moist soil. It was assumed that a soil is made up of small spherical
particles, all with radius $a$. These particles were assumed to be tightly packed and the liquid (which wets the soil) was assumed to be small enough in quantity that it was confined to annular rings formed around the contact of two particles. Using these assumptions a calculation (using the capillary pressure) was made to find the cohesion force due to the capillary attraction. This result was verified by experimental work.

A comprehensive chronological review of research on liquid bridges performed before 1980 is given by Mehrotra and Sastry [1980]. Following this review, two major goals were targeted by scientists. Firstly, to improve the force and energy expressions associated with liquid bridges (see, for example, Pitois et al. [2000] and Darabi et al. [2010]). Secondly, to obtain a better understanding of the stability and modelling the evolution of liquid bridges (see, for example, Mazzone et al. [1986] and Shi and McCarthy [2008]). In this thesis we shall be investigating the stability.

For a vertical bridge under an axial gravitational field$^1$ held between equal disks, Slobozhanin and Perales [1993] determined the full stability region. Several extensions have been made to this work and these are well summarised by Meseguer et al. [1995].

### 1.1.3 Contact angle hysteresis

As discussed previously, in this thesis we shall only consider capillary surfaces which have pinned contact lines. This assumption is realistic on flat surfaces because of the contact angle hysteresis interval which exists due to microscopic imperfections in (or chemical heterogeneities on) the surface. Historically it was believed that for a given surface and liquid only one contact angle can exist. This contact angle is given by the Young–Dupré equation, which equates the surface tensions between the three phases. This is commonly written as

$$\cos \theta = \frac{\gamma_{SG} - \gamma_{SL}}{\gamma_{LG}},$$

(1.1.7)

$^1$That is, when gravity acts in a direction parallel to the axis of the liquid bridge
where $\theta$ is the contact angle, $\gamma_{SG}$ is the surface tension between the solid and the gas, $\gamma_{SL}$ is the surface tension between the solid and the liquid and $\gamma_{LG}$ is the surface tension between the liquid and the gas. However, when deriving this equation it is assumed that the surface is perfectly flat and clean. The surfaces we study in the thesis are not assumed to satisfy these constraints.

To model the surface heterogeneities, referring to figure 1.3, the most up-to-date theory is that when a contact line is moving with speed $U_{sl}$, the contact angle assumes at the least its maximal (advancing) value $\theta = \theta_a$ for $U_{sl} > 0$ and at the most its minimal (receding) value $\theta = \theta_r$ for $U_{sl} < 0$. When $U_{sl} = 0$, the contact angle can take a range of values with $\theta \in (\theta_r, \theta_a)$ and it is this hysteretic range which allows equilibrium solutions in the present problems for non-zero-gravity conditions. For example, let us consider a small spherical droplet resting on a horizontal surface. If liquid evaporates from the droplet, it decreases in volume and contact angle, maintaining the same contact area with the underlying surface until it begins to recede. The contact line then recedes with a constant contact angle, $\theta_r$, characteristic of the surface chemistry and topography. If liquid is added to the drop, it ‘advances’ with contact angle $\theta_a$. For more details see, for example, de Gennes et al. 2004 or McCarthy and Gao 2006.

1.2 Technical Background

This section introduces several mathematical tools which shall be used in this thesis. Each of these tools shall be introduced by a short discussion of how they shall be used in this thesis followed by a more detailed discussion of the key features.

In this thesis we shall be constructing exact and approximate solution for the shape of certain capillary surfaces. These solutions are solutions to the Young–Laplace equation. However, not all solutions are physically observable, they also need to be stable. There are multiple methods used to examine the stability of such a solution. One method is to consider the Navier–Stokes equations which
(a) Contact angle depicted as a function of the velocity of the contact line’s velocity. Notice the region of contact angles which corresponds to $U_{sl} = 0$.

(b) Schematic of the contact line showing the contact angle.

Figure 1.3: Contact angle hysteresis as the contact line velocity varies.

govern the flow in the system. When a small perturbation is added to the static shape the stability is determined by finding if this perturbation grows or shrinks. Indeed, a shape for which all (sufficiently) small perturbations decay (monotonically) is said to be *linearly stable*. Linear stability is a necessary condition for stability. An alternate method to find the stability is to examine the static energy of the system. If the energy is minimised by the solutions to the Young–Laplace equation then the configuration is stable. Further, this method provides a sufficient conditions for stability. We shall give a more detailed discussion of the necessary conditions and sufficient conditions later in this introduction.

We begin this section by defining the curvature of a three dimensional surface. The importance of understanding the curvature can be highlighted by considering the Young–Laplace equation, which balances the pressure difference across a surfaces with the curvature of that surface. From here we move on
to introduce the Navier–Stokes equations and appropriate boundary conditions. This Navier–Stokes system is then the topic of the next two parts of the introduction, in which we scale the system into dimensionless variables/parameters and then show that viscosity dissipates the energy in the system. This dissipation result is similar to a result we shall derive in chapter 3 which shows that the energy of a perturbation to the system is also dissipated by viscosity. The remainder of this section is devoted to deriving conditions for the extrema of a functional and classifying these extrema. This, again, will be used in chapter 3 to examine the equilibrium configurations which minimise the energy of the system.

1.2.1 Curvature of three dimensional surfaces

Throughout this thesis we call on results from differential geometry; most importantly an understanding of the curvature of a three dimensional surface is required. As mentioned previously this importance is seen in the Young–Laplace equation which gives the equilibrium shape of the interfaces. In the problems we shall consider in this thesis the full three dimensional curvature can be simplified using assumptions from the geometry. For example, we shall consider a liquid membrane held inside a cylindrical vertical tube and from the geometry of the supporting structure it follows that the equilibrium shape is axisymmetric. Later in this section we shall see how such an assumption simplifies the curvature.

Stoker [1989] gives a comprehensive introduction to the field of differential geometry and the arguments presented here follow from the presented ideas. For this thesis when we consider the curvature we mean twice the mean curvature, or equivalently the sum of the principal curvatures. With this motivation, in this section we find expressions for the curvature of three dimensional lines. Using symmetries of the surfaces we are then able to find the curvature of the surface.

When considering the free surface of a capillary body we shall be considering
a three dimensional space curve whose locus can by defined by a vector

\[ \mathbf{X}(t) = (x_1(t), x_2(t), x_3(t)), \]

for \( t \) in some interval which we denote by \( t \in [a, b] \) for some \( a, b \in \mathbb{R} \). We require the functions \( x_i \) for \( i = 1, 2, 3 \) to have continuous first and second derivatives. The vector \( \mathbf{X}'(t) = (x'_1(t), x'_2(t), x'_3(t)) \) is then, by definition the tangent vector of the curve placed at the point of tangency. We note that the tangent vector is invariant under transformations of coordinates, but not under parameter transformations. However, if we define the tangent line to be the line at \( \mathbf{X}(t_0) \) in the direction of \( \mathbf{X}'(t_0) \), we find the tangent line is indeed invariant to both coordinate and parameter transformations.

Before arriving at a definition of the curvature we need to define the arc-length \( s(t) \) from a fixed point \( t_0 \) as

\[ s(t) = \int_{t_0}^{t} \sqrt{\mathbf{X}' \cdot \mathbf{X}'} \, dt. \]  

(1.2.1)

Using this definition, we shall use the arc-length as a parameter, i.e., \( \mathbf{X}(s) \). And in this case it is seen from (1.2.1) that \( |\dot{\mathbf{X}}(s)| = 1 \) (here \( \cdot \) denotes differentiation with respect to \( s \)). The length of the curve traced by \( \dot{\mathbf{X}}(s) \) on a sphere as \( s \) varies over an arc of \( \mathbf{X}(s) \) is defined as the total curvature of the arc. Hence using (1.2.1) we obtain

\[ \kappa_T = \int_{s_0}^{s_1} \sqrt{\dot{\mathbf{X}} \cdot \dot{\mathbf{X}}} \, ds, \]

which gives us the following definition for the curvature \( \kappa(s) \) at a point:

\[ \kappa(s) = \sqrt{\dot{\mathbf{X}} \cdot \dot{\mathbf{X}}}, \]  

(1.2.2)

and from this definition we see that \( \kappa \) is invariant under transformations of coordinates which preserves the orientation of the axes. In table 1.1 we give the curvature for the parametrisations which will be used in the technical chapters. Here the three dimensional curvature is defined to the twice the mean curvature. In the problems we study, there is an axis of symmetry. In table 1.2 below we list these axes and the curvature in this case. The three cases listed in table 1.2 correspond to the work from sections 2.2, 2.3 and chapter 5 respectively.
### Equation of the surface

<table>
<thead>
<tr>
<th>Equation of the surface</th>
<th>Curvature</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z = f(x, y)$</td>
<td>$\frac{\partial}{\partial x} \left( \frac{\frac{\partial f}{\partial x}}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}} \right) + \frac{\partial}{\partial y} \left( \frac{\frac{\partial f}{\partial y}}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}} \right)$</td>
</tr>
<tr>
<td>$z = f(r, \theta)$</td>
<td>$\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r \frac{\partial f}{\partial r}}{\sqrt{1+\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial f}{\partial \theta}\right)^2}} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\frac{\partial f}{\partial r}}{r \sqrt{1+\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial f}{\partial \theta}\right)^2}} \right)$</td>
</tr>
<tr>
<td>$r = f(\theta, z)$</td>
<td>$-\frac{1}{f \sqrt{1+\frac{1}{r^2} \left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}} + \frac{\partial}{\partial \theta} \left( \frac{\frac{\partial f}{\partial \theta}}{f \sqrt{1+\frac{1}{r^2} \left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}} \right) + \frac{\partial}{\partial z} \left( \frac{\frac{\partial f}{\partial z}}{\sqrt{1+\frac{1}{r^2} \left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}} \right)$</td>
</tr>
</tbody>
</table>

Table 1.1: Table of curvature for different parametrisations of a surface
Table 1.2: Table of curvature under different constraints

<table>
<thead>
<tr>
<th>Equation of the surface</th>
<th>Plane/Line of symmetry</th>
<th>Curvature</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z = f(x, y) )</td>
<td>( x = 0 )</td>
<td>( \frac{d^2 f}{dx^2} \bigg/ \left[ 1 + \left( \frac{df}{dx} \right)^2 \right]^{3/2} )</td>
</tr>
<tr>
<td>( z = f(r, \theta) )</td>
<td>( r = 0 )</td>
<td>( \frac{d^2 f}{dr^2} \bigg/ \left[ 1 + \left( \frac{df}{dr} \right)^2 \right]^{3/2} + \frac{df}{dr} \bigg/ \left[ 1 + \left( \frac{df}{dr} \right)^2 \right]^{3/2} )</td>
</tr>
<tr>
<td>( r = f(\theta, z) )</td>
<td>( r = 0 )</td>
<td>( \frac{d^2 f}{dz^2} \bigg/ \left[ 1 + \left( \frac{df}{dz} \right)^2 \right]^{3/2} - \frac{1}{f} \bigg/ \left[ 1 + \left( \frac{df}{dz} \right)^2 \right]^{3/2} )</td>
</tr>
</tbody>
</table>

1.2.2 Navier–Stokes equation and free surface boundary conditions

In their most general form the Navier–Stokes equations model the flow and pressure of a fluid. Our initial description follows a similar discussion to that of Batchelor [2000]. The momentum equation in the Navier–Stokes system can be viewed as a particular form of the Cauchy momentum equation,

\[
\frac{\partial}{\partial t} (\rho u) + \nabla \cdot (\rho uu) = \nabla \cdot T + \rho g, \quad (1.2.3)
\]

where \( \rho \) is the density, \( u \) is the flow velocity, \( T \) is the Cauchy stress tensor, and \( g \) represents the body accelerations. In this thesis we consider only incompressible liquids. Hence we can write the stress tensor as

\[
T = \mu S - pI_3, \quad (1.2.4)
\]

where \( \mu \) is the dynamic viscosity, \( p \) is the pressure, and \( S \) is the rate-of-strain tensor. Applying this and noting the density, \( \rho \), is constant throughout the
liquid, we obtain
\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p = \nu \nabla \cdot \mathbf{S} - \mathbf{g}, \]  
(1.2.5)

where \( \nu \) is the kinematic viscosity defined by \( \nu = \mu / \rho \). This is the form of the momentum equation we shall use throughout this thesis. Further, since the density is constant in the liquid we find the conservation of mass equation can be written as
\[ \nabla \cdot \mathbf{u} = 0. \]  
(1.2.6)

There are two types of boundary conditions which we use in this thesis: conditions at fixed boundaries; and conditions at free surfaces. At a stationary boundary we write the boundary conditions for \( \mathbf{u} \) as
\[ \mathbf{u} \cdot \mathbf{n} = 0, \]  
(1.2.7)
\[ \nu [\mathbf{u} - (\mathbf{n} \cdot \mathbf{u}) \mathbf{n}] = 0, \]  
(1.2.8)

In all cases 1.2.7 implies no flow through the rigid surface. For a viscous fluid, where \( \mu \neq 0 \), we obtain the additional condition in 1.2.8 that there is no flow along the surface and hence \( \mathbf{u} = \mathbf{0} \). For inviscid flow, where \( \mu = 0 \), this additional condition is unnecessary since the second-order derivative from \( \nabla \cdot \mathbf{S} \) is lost from 1.2.5.

At a fluid-fluid interface, we impose free surface conditions. These can be derived by assuming that all particles on the surface remain on the surface for all time. If we define the surface implicitly by \( F(x, y, z, t) = 0 \), then this condition can be achieved by setting
\[ \frac{D F}{D t} = 0, \]  
(1.2.9)

and hence
\[ \frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F = 0. \]  
(1.2.10)

On its own, 1.2.10 is insufficient to describe the motion of a free surface. Additionally, we need a boundary condition connecting the state of stress across
the fluid-fluid interface. Consider the forces acting on the interface. Suppose \( \Gamma \) is an area of the interface and \( \gamma \) is the bounding closed curve. The total force on the surface must be zero (by the continuity of tractions) and we find that

\[
\int_{\Gamma} \mathbf{n} \cdot \mathbf{T} \, dA + \int_{\gamma} \sigma \mathbf{n} \, ds = 0, \tag{1.2.11}
\]

where \( \sigma \) is the surface tension. By applying Stokes theorem we obtain

\[
\int_{\Gamma} [\mathbf{T} - \sigma \kappa \mathbf{I}_3] \cdot \mathbf{n} \, dA = 0, \tag{1.2.12}
\]

where \( \mathbf{T} \) is the interfacial stress tensor and \( \kappa = \nabla \cdot \mathbf{n} \) is the curvature of the interface. Note, the interfacial stress tensor can be written as \( \mathbf{T} = \mu \mathbf{S} - p \mathbf{I}_3 \), where the pressure \( p \) has been scaled such that the exterior pressure is zero. For (1.2.12) to be true on all surfaces the integrand must vanish so

\[
[\mu \mathbf{S} - (\sigma \kappa \mathbf{I}_3)] \cdot \mathbf{n} = 0. \tag{1.2.13}
\]

Equations (1.2.5), (1.2.6), with boundary conditions (1.2.7), (1.2.8) on solid surfaces and (1.2.10), and (1.2.13) on free surfaces form a complete system which we shall be investigating for various geometries throughout this thesis.

### 1.2.3 Analysis of the Navier–Stokes system

In this section we shall first scale the Navier–Stokes system be become dimensionless. We shall use the dimensionless Navier–Stokes system throughout this thesis although mostly in chapter 3. Following this, we shall show that the energy of the configuration is dissipated by viscous effects. This is a classic argument that has been presented several times (see for example Gonzalez and Stewart, 2007). However, we reproduce it here because of its similarity to a result from chapter 3 that the energy of a (linear) perturbation is dissipated by viscous effects. Finally, we shall linearise the Navier–Stokes system. The linearise Navier–Stokes system shall be use in chapter 3 to examine the stability of the configurations.
1.2.3.1 Scaling of the Navier–Stokes system

The ideas introduced here and in the following section shall later be used in chapter 3 to study the stability of two capillary interface problems. Many parts of these discussions follow the ideas laid out by Doering and Gibbon [1995]. Consider the Navier–Stokes equations with free surface boundary conditions on two liquid–gas interfaces, at \( z = h_{\pm}(x,y) \), and no-slip boundary conditions on two liquid–solid interfaces, at \( x = \pm L \) (\( L \in \mathbb{R} \));

\[
\begin{align*}
\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^* + \frac{1}{\rho} \nabla^* p^* &= \nu \nabla^* \cdot \mathbf{S}^* - g \hat{z}, \\
\nabla^* \cdot \mathbf{u}^* &= 0, \\
\frac{\partial h^*_\pm}{\partial t^*} \pm \mathbf{u}^* \cdot \mathbf{n} &= 0,
\end{align*}
\]

for \( (x^*, y^*, z^*) \in \Omega \), (1.2.14)

\[
\begin{align*}
\mathbf{S} - \frac{1}{\beta} (p^* - \sigma \kappa^*) \mathbf{I}_3 \cdot \hat{n} &= 0, \\
\mathbf{u}^* &= 0, \text{ for } x^* = \pm L,
\end{align*}
\]

where the stars refer to dimensional quantities, and \( \mathbf{S}^*, \kappa^* \) are the rate of strain tensor and surface curvature respectively. The region \( \Omega \) is given by

\[
\Omega = [-L, L] \times (-\infty, \infty) \times [h_{\pm}^*, h_{\pm}^*].
\]

(1.2.17)

We introduce the following scalings

\[
x^* = L x, \quad t^* = \sqrt{\frac{\rho L^3}{\sigma}} t, \quad \mathbf{u}^* = \sqrt{\frac{\sigma}{\rho L}} \mathbf{u}, \quad p^* = \frac{\sigma}{L} p,
\]

(1.2.18)

to obtain dimensionless equations:

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \beta \nabla \cdot \mathbf{S} - B \hat{z}, \\
\nabla \cdot \mathbf{u} &= 0, \\
\frac{\partial h^*_\pm}{\partial t} \pm \mathbf{u} \cdot \mathbf{n} &= 0,
\end{align*}
\]

for \( (x, y, z) \in \Omega \), (1.2.19)

\[
\begin{align*}
\mathbf{S} - \frac{1}{\beta} (p - \kappa) \mathbf{I}_3 \cdot \hat{n} &= 0, \\
\mathbf{u} &= 0, \text{ for } x = \pm 1,
\end{align*}
\]

(1.2.20)
where $\beta, B$ are the Ohnesorge and Bond numbers respectively. The Ohnesorge number relates the viscous forces to the inertial and surface tension forces, while the Bond number relates the gravitational forces to the surface tension forces, i.e.,

$$\beta = \frac{\mu}{\sqrt{\rho \sigma L}} = \nu \sqrt{\frac{\rho}{\sigma L}}, \quad (1.2.22)$$

$$B = \frac{\rho g L^2}{\sigma}. \quad (1.2.23)$$

### 1.2.3.2 Dissipation of Energy

We begin this analysis by showing the energy of the system is dissipated by viscous effects. Later, in chapter 3, we shall see a similar result for the perturbed energy of the system. By forming the dot product of (1.2.19a) with $u$ (analogous to premultiplying), we find

$$u \cdot \partial u \partial t + (uu) \cdot \nabla u + u \cdot \nabla p = \beta (u \nabla) : S - Bw, \quad (1.2.24)$$

where $w$ is the $z$ component of $u$. However, by writing $2U = u \cdot u$, we obtain the following relationships (taking into account (1.2.19b))

$$(uu) \cdot \nabla u = \nabla \cdot (uU), \quad (1.2.25)$$

$$u \cdot \partial u \partial t = \partial U \partial t, \quad (1.2.26)$$

$$u \cdot \nabla p = \nabla \cdot (up), \quad (1.2.27)$$

$$(u \nabla) : S = \nabla \cdot (u \cdot S) - \frac{1}{2} S : S, \quad (1.2.28)$$

which can be combined with (1.2.24) to give,

$$\frac{\partial U}{\partial t} + \nabla \cdot \left[ uU - \beta u \cdot \left( S - \frac{p}{\beta} I_3 \right) \right] + Bw = -\frac{\beta}{2} S : S. \quad (1.2.29)$$

Since $U$ is representative of the kinetic energy density, this equation can be integrated over $\Omega$ to find the energy dissipation. By noting that the liquid region $\Omega$ can change in time, and enforcing boundary conditions (1.2.20) and (1.2.21) we find

$$\frac{\partial E}{\partial t} = \frac{\partial E_k}{\partial t} + \frac{\partial E_s}{\partial t} + B \frac{\partial E_p}{\partial t} = -\frac{\beta}{2} \int_\Omega S : S \, dV, \quad (1.2.30)$$
where \( E, E_k, E_s, E_p \) are the total, kinetic, surface, and potential energies of the system, respectively, and can be given by

\[
\begin{align*}
E &= E_k + E_s + BE_p, \quad (1.2.31) \\
E_k &= \int_\Omega \mathbf{u} \cdot \mathbf{u} \, dV, \quad (1.2.32) \\
E_s &= \int_{\Gamma_+} |\mathbf{n}_+| \, dS + \int_{\Gamma_-} |\mathbf{n}_-| \, dS, \quad (1.2.33) \\
\frac{\partial E_p}{\partial t} &= \int_\Omega \tilde{w} \, dV, \quad (1.2.34)
\end{align*}
\]

where \( \Gamma_\pm \) denote the upper and lower surfaces, respectively. Note that the Bond number is assumed to be constant. We shall use a method similar to the one seen here to show a similar result for the energy of a linear perturbation.

### 1.2.3.3 Linearisation of the Navier–Stokes system

As discussed earlier, the fact a solution to the Young–Laplace equation exists does not mean it can be found in the physical world. It also needs to be stable. This sections describes a process of linearising the Navier–Stokes system to obtain an eigenvalue problem which (when solved) indicates the stability.

In chapter 2 we shall study the static equilibrium shape of this capillary system. In this section we assume these solutions are known and we want to know their stability. We denote the upper and lower static equilibrium interfaces by \( z = \bar{h}_\pm \). Their stability is then analysed by introducing a (small) perturbation to the system and determining whether this perturbation grows or decays. Hence, we now consider small perturbations to the static case, i.e. consider

\[
\mathbf{u} = \tilde{\mathbf{u}}, \ p = \tilde{p} + \bar{p}, \ h_\pm = \bar{h}_\pm + \tilde{h}_\pm, \quad (1.2.35)
\]

where all tilde variables are asymptotically small. Applying these we obtain the
first order system,
\[
\begin{align*}
\frac{\partial \tilde{u}}{\partial t} + \nabla \tilde{p} &= \beta \nabla \cdot \tilde{S}, \\
\nabla \cdot \tilde{u} &= 0,
\end{align*}
\]
for \((x, y, z) \in \bar{\Omega},\) \hspace{1cm} (1.2.36)
\[
\begin{align*}
\frac{\partial \tilde{h}_\pm}{\partial t} \pm \tilde{u} \cdot n &= 0, \\
\left[ \tilde{S} - \frac{1}{\beta} (\tilde{p} - \tilde{h} - \kappa) \mathbb{I}_3 \right] \cdot \hat{n} &= 0,
\end{align*}
\]
at \(z = \bar{h}_\pm,\) \hspace{1cm} (1.2.37)
\[
\tilde{u} = 0, \text{ at } x = \pm 1. \hspace{1cm} (1.2.38)
\]

Here the region \(\bar{\Omega}\) is given by
\[
\bar{\Omega} = [-1, 1] \times (-\infty, \infty) \times [\bar{h}_-, \bar{h}_+].
\] \hspace{1cm} (1.2.39)

Importantly, notice that all the time derivatives can be equated to linear functions. It follows that the solution can be formed by taking a superposition of terms proportional to \(\exp(\omega t)\), where \(\{\omega\}\) is the set of eigenvalues. These eigenvalues form part of the solution set. Further since the geometry is unchanging in the \(y\)-direction we assume the perturbations are sinusoidal in \(y\). We therefore, apply the following expansion \(\tilde{u}(x, y, z, t) = \hat{u}(x, z)e^{\omega t + iky}\), and similar to other variables, to obtain
\[
\begin{align*}
\omega \hat{u} + \nabla^\dagger\tilde{p} &= \beta \nabla^\dagger \cdot \tilde{S}, \\
\nabla^\dagger \cdot \hat{u} &= 0,
\end{align*}
\]
for \((x, y, z) \in \bar{\Omega},\) \hspace{1cm} (1.2.40)
\[
\begin{align*}
\omega \hat{h}_\pm \pm \hat{u} \cdot n &= 0, \\
\left[ \tilde{S} - \frac{1}{\beta} (\tilde{p} - \hat{h} - \hat{\kappa}_\pm) \mathbb{I}_3 \right] \cdot \hat{n} &= 0,
\end{align*}
\]
for \(z = \bar{h}_\pm,\) \hspace{1cm} (1.2.41)
\[
\hat{u} = 0, \text{ for } x = \pm 1, \hspace{1cm} (1.2.42)
\]

where
\[
\nabla^\dagger = \left( \frac{\partial}{\partial x}, ik, \frac{\partial}{\partial z} \right), \hspace{1cm} (1.2.43)
\]
\[
\hat{\kappa}_\pm = \pm \left[ \left( \frac{h'_\pm}{(1 + h'_\pm^2)^{3/2}} \right)' - \frac{k^2 h'_\pm}{\sqrt{1 + h'_\pm^2}} \right]. \hspace{1cm} (1.2.44)
\]
Importantly, when making this transformation we must understand that the velocity of the flow is given by the real part of $\hat{u} e^{\omega t + iky}$.

In section 3.2.1 we shall integrate (1.2.40) over $\bar{\Omega}$ to find a result for the dissipation of the perturbed energy. This will be similar in nature to dissipation of the time dependent energy derived in this section, as seen in (1.2.30). However, the result for the perturbed energy will lead to important details of the nature of $\omega$, especially in the inviscid case. For example, in the inviscid case, following this method we can show that eigenvalues lie on the axes of the complex plane and are in positive/negative pairs. It follows that in the inviscid case the configuration is at best neutrally stable.

The eigenvalue which determines the stability of the equilibrium configuration is $\omega$. However, the operator whose spectrum is of interest in (1.2.40) is not generally self-adjoint and it follows $\omega$ will have both real and imaginary parts. It should also be noted this problem is sufficiently similar to the Sturm–Liouville problem that it can be readily shown that there is a countably infinite set of eigenvalues. If the real parts of all the eigenvalues are all negative, then the amplitudes of all the perturbations decay in time and the system is said to be ‘linearly stable’. If the greatest real part is zero then the system is marginally stable. Finally, if any of the eigenvalues has a positive real part then the system is linearly unstable; since even the smallest perturbation growing exponentially will eventually grow large enough to make the linear scheme inapplicable.

The concept of linear stability is a relatively weak notion. Indeed, while linear stability is a necessary condition for stability it does not guarantee the system is stable to all perturbations, only linear ones. However, linear instability is, in fact, a sufficient condition for instability. This follows because if the configuration is unstable to linear perturbations it is also unstable to the set of all (allowed) perturbations since this contains the destabilising linear perturbation(s).

To establish stability against any finite amplitude perturbations, which satisfy certain conditions (as will be discussed in chapter 3), the full nonlinear equations must be analysed. However, we note that a sufficient condition for
the stability of a system is that the energy of the system is minimised. This approach uses the calculus of variations. Important details in this field are given in the following section.

1.2.4 Minimisation of functionals

As described in the previous section, the linear stability analysis gives a necessary condition for stability. To find a sufficient condition we need to examine the stability for any (volume conserving) finite amplitude perturbations. In performing this analysis we consider the energy of the configuration and in particular we examine whether the energy is minimised by the equilibrium configuration, even when the most destabilising perturbations are used. If the energy is minimised then the system will return to this state after being perturbed and is therefore stable.

In this section we introduce the criteria which must be upheld for a functional to have a minimum. The main mathematical tool used to provide these criteria come from the field of calculus of variations, see for example [Gelfand and Fomin, 2000]. This section has three parts. We start by finding conditions for the extrema of a functional. Then conditions are found to classify the extrema. Finally, we introduce an analogous method to Lagrange’s method to find an extrema of a functional subject to a constraint.

Consider a functional $I : \mathcal{C} \rightarrow \mathbb{R}$

$$I[f] = \int_{-L}^{L} \int_{-1}^{1} F(x, y, f, f_x, f_y) \, dx \, dy,$$  \hspace{1cm} (1.2.45)

where a subscript denotes differentiation. Here $\mathcal{C}$ is the set of all continuous (and sufficiently differentiable) functions defined on $[-1, 1] \times [-L, L]$, with norm

$$\|f\| = \left( \int_{-L}^{L} \int_{-1}^{1} f^2 \, dx \, dy \right)^{1/2}.$$  \hspace{1cm} (1.2.46)

Suppose we want to find a function of two variables (a surface), $f = f^*(x, y)$, such that $I[f^*] \leq I[f]$, for all $f$ such that $\|f - f^*\| < \delta$, for some $\delta > 0$, i.e., we seek $f$ which minimises $I$. To find the extrema of a function we would consider
the derivative. Hence, we now introduce the variation of a functional, which is analogous to the derivative of a function. The variation of a functional, \( \delta I \), is derived in an analogous manner to a derivative of a function and can be given as

\[
\delta I = \int_{-L}^{L} \int_{-1}^{1} \left[ \delta f \frac{\partial F}{\partial f} + \delta f_x \frac{\partial F}{\partial f_x} + \delta f_y \frac{\partial F}{\partial f_y} \right] \, dx \, dy.
\] (1.2.47)

It can be shown that a necessary condition for a minimum of a function is that the first variation vanishes, i.e.,

\[
\delta I = 0.
\] (1.2.48)

Applying this we find

\[
\int_{-L}^{L} \int_{-1}^{1} \left[ \delta f \frac{\partial F}{\partial f} + \delta f_x \frac{\partial F}{\partial f_x} + \delta f_y \frac{\partial F}{\partial f_y} \right] \, dx \, dy = 0,
\] (1.2.49)

for all \( \delta f \). Suppose \( F = 0 \) at both \( x = \pm 1 \) and \( F(y = -L) = F(y = L) \) then we can integrate by parts to obtain

\[
\int_{-L}^{L} \int_{-1}^{1} \delta f \left[ \frac{\partial F}{\partial f} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial f_x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial f_y} \right) \right] \, dx \, dy = 0,
\] (1.2.50)

which is satisfied for all \( \delta f \) if and only if

\[
\frac{\partial F}{\partial f} = \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial f_x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial f_y} \right).
\] (1.2.51)

This is the two dimensional Euler equation. Suppose instead, we have a functional of two arguments, i.e., \( J[f, g] \), we find the variation is given by

\[
\delta J = \int_{-L}^{L} \int_{-1}^{1} \left[ \delta f \frac{\partial F}{\partial f} + \delta f_x \frac{\partial F}{\partial f_x} + \delta f_y \frac{\partial F}{\partial f_y} + \delta g \frac{\partial F}{\partial g} + \delta g_x \frac{\partial F}{\partial g_x} + \delta g_y \frac{\partial F}{\partial g_y} \right] \, dx \, dy,
\] (1.2.52)

and by considering \( \delta J = 0 \) we obtain the canonical Euler equations in two dimensions

\[
\frac{\partial F}{\partial f} = \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial f_x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial f_y} \right),
\] (1.2.53)

\[
\frac{\partial F}{\partial g} = \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial g_x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial g_y} \right).
\] (1.2.54)
It is important to note that while $\delta I = 0$ is a necessary condition for a minimum, it is not sufficient. Indeed suppose $I[f]$ has a maximum for $f = f^*$, we can show $\delta I = 0$. To determine whether the function $f = f^*$ is a local maxima or minima we need to consider the second variation.

The second variation can be derived in a similar manner to the first variation. When we consider functions whose cross derivatives vanish, i.e. $\frac{\partial^2 F}{\partial x \partial y} = 0$, the second variation of the functional can be given by

$$ \delta^2 I = \frac{1}{2} \int_{-L}^{L} \int_{-1}^{1} \left[ (\delta f)^2 \frac{\partial^2 F}{\partial f^2} + (\delta f_x)^2 \frac{\partial^2 F}{\partial f_x^2} + (\delta f_y)^2 \frac{\partial^2 F}{\partial f_y^2} \right] \, dx \, dy. \quad (1.2.55) $$

Further, for all functionals of two functions considered in this thesis we can write,

$$ J[f, g] = I_1[f] + I_2[g], \quad (1.2.56) $$

and hence

$$ \delta^2 J = \delta^2 I_1 + \delta^2 I_2. \quad (1.2.57) $$

Now returning to our problem of classifying extrema in functionals we have a sufficient condition for a functional $I[f]$ to have a minimum for $f = f^*$, given the first variation vanishes at $f = f^*$, is that its second variation is strongly positive\(^2\). A functional $\phi[f]$ is said to be strongly positive if there exists some constant $k > 0$ such that

$$ \phi[f] \geq k \| f \|^2. \quad (1.2.58) $$

Extending this to $J[f, g]$, we see $J$ has a minimum for $f^*, g^*$ if

$$ \delta J[f^*, g^*] = 0, \quad (1.2.59) $$

and

$$ \delta^2 I_1[f^*] \geq k_1 \| f^* \|^2, \quad \delta^2 I_2[g^*] \geq k_2 \| g^* \|^2, \quad (1.2.60) $$

\(^2\)For proof see Gelfand and Fomin [2000]
where \( J[f, g] = I_1[f] + I_2[g] \), and \( k_1, k_2 \) are positive constants. This final condition can be rewritten as: there exist positive constants \( k, \alpha \) such that

\[
\delta^2 J \geq k \| f^* + \alpha g^* \|^2,
\]

by using the triangle inequality.

The final part of this section discusses an analogy to the method of Lagrangian multipliers. Suppose we want to find the curve \( y = y(x) \) for which the functional

\[
I[y] = \int_a^b F(x, y, y') \, dx,
\]

has an extremum, given the curve satisfies \( K[y] = c \), for

\[
K[y] = \int_a^b G(x, y, y') \, dx.
\]

Suppose \( I[y] \) has an extremum for \( y = y^* \), then if \( y^* \) is not an extremum of \( K[y] \), there exists a constant \( \lambda \) such that \( y^* \) is an extremal of the functional

\[
(I + \lambda K)[y] = \int_a^b (F + \lambda G) \, dx.
\]

Thus to find extrema of \( I \) such that \( K = c \), we solve the second order differential equation which results from solving

\[
\delta(I + \lambda K) = 0,
\]

using the boundary conditions and the constraint, \( K = c \), to fix \( \lambda \).

### 1.3 Numerical methods

Throughout this thesis solutions to non-linear ordinary differential equations are needed. In general, these are not tractable analytically and a numerical solution is required. The first numerical method we present is the shooting method which can be used to solve boundary value problems and eigenvalues problems. Later, we introduce the method of finite elements for solving a Stokes flow problem. This will be used to solve the linearised Naiver–Stokes equations to study the stability of a two-dimension horizontal liquid bridge.
1.3.1 Shooting method

The shooting method is a numerical technique used to solve boundary value problems and differential equations with a parameter as part of the solution set (e.g., eigenvalue problems). The solution is found by reducing the boundary value problem to an initial value problem with additional parameters. After solving this initial value problem a function, $F$, is defined as the difference between the computed value of the solution at the second boundary and the value of the correct solution at the second boundary. Using modern computational methods, if the solution exists, it is possible to solve the initial value problem inexpensively. The function, $F$, can then be treated as a standard non-linear function and its roots can be found using standard algorithms. The roots of the function, $F$, coincide with the values of the parameter which produce a solution which satisfies all the boundary conditions. It is important to note this solution may not be (and in most cases is not) unique. We now illustrate this technique by considering the boundary value problem

$$y'' = f(x), \quad x \in [a, b], \quad (1.3.1)$$
$$y(a) = y_a, \quad y(b) = y_b. \quad (1.3.2)$$

We define the function $Y(x; t)$ such that

$$\frac{d^2 Y(x; t)}{dx^2} = f(x), \quad Y(a; t) = y_a, \quad \frac{dY(x; t)}{dx} \bigg|_{x=a} = t. \quad (1.3.3)$$

We see that this function $Y$ satisfies the same governing equation and Dirichlet condition at $x = a$ as $y$. However $Y$ is parametrised by a Neumann condition at $x = a$. This initial value problem can be solved using a standard Runge-Kutta method. After solving this initial value problem we define

$$F(t) = Y(b; t) - y_b, \quad (1.3.4)$$

and seek roots of $F$. Indeed suppose $F(\tau) = 0$ then we see $Y(x; \tau) = y(x)$. Roots of $F(t)$ can be found using Newton’s method.
1.3.2 Finite element method

Later, in section 3.3 we shall use the method of finite elements to solve the linearised Navier–Stokes equations, which were described in section 1.2.3.3. This is done to verify the stability results obtained in sections 3.1 and 3.2.

In this section we give a brief introduction to the finite element method. To best illustrate this we use a Stokes problem as a model problem. From this Stokes system, we derive the weak form of the problem and discuss how this is transferred into a discrete setting. This example is a common problem which can be found in several textbooks and for more details see, for example, the works of Baker [1985], Glowinski [1984] and Pironneau [1989]. Importantly, the difference between the linearised Navier–Stokes and the Stokes equations here is minimal. As described at the end of this section, a simple step can be used to transform the Stokes system into the linearised Navier–Stokes system.

We consider a very-viscous liquid held in a region $\Omega$, the flow and pressure must satisfy

\begin{align}
\mu \nabla^2 u - \nabla p &= 0, \\
\nabla \cdot u &= 0,
\end{align}

with vanishing boundary conditions on the edge of the region, i.e. $u = 0$ on $\partial \Omega$. Note the flow vector and pressure must be elements of the following sets:

\begin{align}
\mathbf{u} \in U &= \left\{ \mathbf{u} \in [H^1(\Omega)]^m : \mathbf{u} = 0, \text{ on } \partial \Omega \right\}, \\
p \in P &= \left\{ p \in L^2(\Omega) \right\},
\end{align}

where $H^n$ is the Sobolev space with $n$ weak derivatives in $L^2$ norm and $m = \dim \Omega$ is the dimension of the physical space. Let $v \in U$ and $q \in P$ and form the inner product of $v$ with (1.3.5) and aggregate this with the product of $q$ and (1.3.6) to find

\[
\mu v \cdot \nabla^2 u - v \nabla p + q \nabla \cdot u = 0.
\]
Integrating this and observing $v = 0$ on $\partial \Omega$ we find

$$
\int_{\Omega} \left[ - (\nabla u) : (\nabla v) + p \nabla \cdot v + q \nabla \cdot u \right] \, dV = 0. \tag{1.3.10}
$$

It follows the flow vector, $u$, and pressure, $p$, must be a solution to the weak form problem stated as:

$$
\begin{cases}
    \text{Find } u \in U, p \in P \text{ such that:} \\
    - (\nabla u, \nabla v) + (p, \nabla \cdot v) + (q, \nabla \cdot u) = 0, \quad \forall v \in U, q \in P.
\end{cases} \tag{1.3.11}
$$

The bracket notation is shorthand for the following

$$(A, B) = \int_{\Omega} A : B \, dV, \tag{1.3.12}$$

$$(a, b) = \int_{\Omega} ab \, dV. \tag{1.3.13}$$

When $m = 2$ we can partition $\Omega$ using a Delauney triangulation (see [Delaunay, 1934]) to allow the weak form problem to be discretised into a set of algebraic equations. In order to make this transition we need to discretise the continuous solution sets $U$ and $P$, we denote the discrete solution sets $U_D$ and $P_D$. When choosing how to discretise these sets is important to ensure the Inf-Sup condition is satisfied. This is a technical point beyond the scope of this introduction, but further details can be found in [Brenner and Scott, 2002]. We shall use a Taylor–Hood element pair to discretise our solution sets, which [Girault and Raviart, 1986] showed satisfies the Inf-Sup condition for Stokes problems and hence it will satisfy the Inf-Sup condition for the linearised Navier–Stokes problem.

For a given triangulation of the domain, $T(\Omega)$, the Taylor–Hood element pair uses

$$(U_D, P_D)(T_i) = (P_k, P_{k-1})(T_i) \tag{1.3.14}$$

where $P_k$ is the set of $k$-th order polynomials defined on the triangle $T_i$. We shall use $k = 2$ and therefore will have quadratic elements for the velocity components and linear elements for the pressure (on each triangle).
The discrete weak form therefore is
\[
\begin{cases}
\text{Find } u_D \in U_D, p_D \in P_D \text{ such that:} \\
- (\nabla u_D, \nabla v_D) + (p_D, \nabla \cdot v_D) + (q_D, \nabla \cdot u_D) = 0, \forall v_D \in U_D, q_D \in P_D.
\end{cases}
\]
(1.3.15)

Further details of how these operators are discretised shall be seen in section 3.3. Finally, this equation yields a matrix equation which can be solved to find the velocity components and pressure at each point of the triangulation.

When we later consider the linearised Navier–Stokes equations we shall have an eigenvalue problem, however the method shall follow from what we have seen here. Indeed the only difference between the Stokes problem which we have considered here and the linearised Navier–Stokes problem, which we shall later consider, is the addition of a \( \omega u \) term to the global equation, for an eigenvalue \( \omega \). This term comes from the time derivative which appears in the Navier–Stokes system which is not in the Stokes system. Indeed, the steps used here shall be reproduced and following this analysis we shall need to solve a quadratic eigenvalue problem. Details of how this is solved shall be seen in section 3.3.
Chapter 2

Micro-gravity equilibrium shapes

In this chapter, we examine equilibrium shapes of capillary surfaces both through exact solutions of the Young–Laplace equation (where possible) and using a micro-gravity (small Bond number) approximation. The mathematical problem is formulated using the Young–Laplace equation while the boundary conditions are taken from the geometry of the specific configuration. In the case of the free surface contacting a flat surface (e.g., a wall) we specify a contact angle boundary condition but, as the analysis shows, it is necessary on occasion to allow for a hysteresis interval so that the value of the contact angle lies in an interval rather than being unique, as discussed in section 1.1.3.

While the contact angle is important for determining the shape of a droplet, gravity will also play a role. The importance of gravity is determined by comparing the capillary length of the liquid with the length scale of the problem. The capillary length scale of a liquid is defined by 

\[ a = \sqrt{\frac{\sigma}{\rho g}} \]

where \( \sigma \) is the surface tension, \( \rho \) is the liquid density and \( g \) is the acceleration due to gravity. For water in contact with air under Earth’s gravity, \( a \approx 2.7 \text{mm} \). When the characteristic length scale \( L \) of the configuration under consideration is small
compared to the capillary length of the liquid ($L \ll a$), the Bond number ($L/a$) is small and an asymptotic approach is appropriate. One example of this in a familiar context might be a water drop on an inclined plane, for example a drop on a car windscreen. If the length scale of the drop is smaller than 1mm, the Bond number in this situation is smaller than $1/2.7 \approx 0.37$, which may be sufficiently small for the Bond number to be used as a small parameter in an asymptotic expansion. In microgravity, small Bond numbers can be achieved for droplets which are much larger than the capillary length on the surface of the earth (2.7mm).

In this thesis, we shall investigate both the shape and the stability of a range of capillary surfaces. Before a full understanding of stability is possible, the shape of these capillary surfaces must be well understood. This chapter focuses on the shape of three configurations in the limit of microgravity. The three configurations are a two dimensional liquid drop on an inclined plane, a two dimensional horizontal liquid bridge, and an axisymmetric liquid membrane held inside a vertical cylinder.

In each case, we shall find asymptotic expansions for the shape of the free surface which are valid in the limit of small Bond number. These asymptotic expansions shall be compared with a numerical solution of the Young–Laplace equation and the expansions will be shown to be in good agreement. We shall see, in each case, that by determining the upper contact angle together with another geometric condition the solution of the Young–Laplace equation is unique. For each of the geometries we compare both the resulting shape of the interface(s) and the physical parameters of the problem, e.g., the volume, and the lower contact angle. In this section we produce first order asymptotic series and study their properties. In appendix B we show that higher order expansions can be produced in closed form and, in so doing, we extend our asymptotic solutions to the next order.

We study the two dimensional liquid drop as a starting point for understanding the methods which shall be used in the other (perhaps more interesting) cases. However, the drop is not as simple as might first appear. Notably, a
drop on an inclined plane has no inherent symmetry because of the angle of the plane on which the drop rests. When a drop on an inclined plane is in motion (or the air around the drop is moving), the associated mathematical problem has been well studied using the thin film equation developed by Hocking [1981]. One example of which uses this thin film equation is the work of Paterson et al. [2015] who studied the motion of a liquid ridge down a slope subject to a flow in the surrounding gas. However, the case of a two dimensional drop resting on, or hanging from, an inclined plane continues to be of interest, see for example the work of Pozrikidis [2012] who studied the stability of both sessile and pendant liquid drop using a linear stability analysis.

The study of the two dimensional horizontal liquid bridge is motivated from the stent problem introduced in chapter 1. Consider one of the ‘diamonds’ in figure 1.1b. We idealise this as a horizontal rectangular frame, comprising four approximately cylindrical or rectangular pillars which support the liquid bridge. If the frame is sufficiently long in one direction, the section through the bridge in the shorter direction (supported by the cross section of two of the pillars) will be approximately two dimensional. Assuming the frame is long in one direction allows us to neglect effects (or boundary layers) at the ends. In this chapter, we shall examine the much simpler problem of determining the shape of a two dimensional liquid bridge supported by two vertical walls and will obtain asymptotic and numerical solutions for the equilibrium configuration, the former in the limit of microgravity. The examination is furthered by an exact solution which agrees with both the asymptotic and numerical solutions.

The ‘diamonds’ in figure 1.1b can instead be idealised as membranes held inside a cylinder. This problem shall be investigated through an asymptotic solution, in the limit of microgravity and a numerical solution. The added algebraic difficulty of considering the problem in cylindrical polar coordinates limits the development of an exact solution for this configuration.
2.1 Two dimensional drop on an inclined plane

In this section we describe the shape of a stationary drop on an inclined plane. The study of a drop resting on a flat plan has been extensively studied; see for example Finn [1986] and O'Brien [1991] who used an arc-length formulation to write the governing equations and used the Bond number as small parameter to obtain perturbation solutions. Here we extend this method to a drop on an inclined plane.

We study the two dimensional drop on an inclined plane as a development tool. In this section we shall use methods which later will be applied to the cases which are closer to the stent problem.

We employ an arc-length formulation to reduce the problem to a system of first-order equations and avoid the possible singularities that could otherwise occur when one of the surfaces becomes vertical. We define our $x$-axis to increase down the plane with zero at the upper contact point, and we define our $z$-axis to extend into the plane, as shown in figure 2.1.

![Figure 2.1: Geometry of a drop on an inclined plane](image)

The fluid inside the drop is stationary, by assumption, and therefore the
Navier–Stokes equations give

\[
\frac{\partial p}{\partial x} = \rho g \sin \alpha, \quad (2.1.1)
\]

\[
\frac{\partial p}{\partial z} = \rho g \cos \alpha. \quad (2.1.2)
\]

Integrating these gives

\[
p = \rho g (x \sin \alpha + z \cos \alpha) + P_1, \quad (2.1.3)
\]

for some constant \( P_1 \). Let \( P = P_1 - p_{atm} \) where \( p_{atm} \) is the (constant) atmospheric pressure outside the liquid drop. Therefore the pressure difference across the interface \( z = h(x) \) is

\[
\Delta p = \rho g (x \sin \alpha + h \cos \alpha) + P. \quad (2.1.4)
\]

Applying the Young–Laplace equation (1.1.2), and using the length of the drop (i.e., the distance between the upper and lower triple contact points) as the characteristic length scale for scaling, we obtain the dimensionless equation

\[
B(x \sin \alpha + h \cos \alpha) + P = -\frac{h''}{(1 + h'^2)^{1/2}}, \quad (2.1.5)
\]

where \( B \) is the Bond number, and dash denotes differentiation with respect to \( x \). From the scalings used we see that the dimensionless surface of the drop touches the plane at \( x = 0 \) and at \( x = 1 \); hence, we have the two boundary conditions

\[
h(0) = 0, \quad h(1) = 0. \quad (2.1.6)
\]

To solve for the constant \( P \) we require a further boundary condition. There are multiple possibilities for this, but a scientifically meaningful and mathematically convenient choice is to specify a known upper contact angle, \( \theta_+ \), so that

\[
h'(0) = \tan \theta_+. \quad (2.1.7)
\]

As we will see, specifying the upper contact angle will enable us to determine the lower contact angle. Using these two, it is then possible to determine the
cross-sectional area (a proxy for the volume in this two-dimensional case) of the drop. By integrating (2.1.5) over \( x \in [0, 1] \) and defining \( \theta_- \) as the lower contact angle we obtain

\[
B \left( \frac{\sin \alpha}{2} + V \cos \alpha \right) + P = \sin \theta_+ + \sin \theta_-,
\]

where \( V \) is the (cross-sectional) volume of the liquid drop. We will use this result later to verify our asymptotic solutions. One implication of (2.1.8) is that once the angle of the surface with the Bond number and both contact angles are specified, the volume is fixed. While it might seem more natural to specify the volume rather than the contact angle, one consequence of the scaling is that the volume has been scaled with the square of the base length of the drop.

### 2.1.1 Asymptotic solution

The Bond number becomes asymptotically small in the micro-gravity limit, i.e. \( B \ll 1 \) and we introduce the arc-length, \( s \), and inclination, \( \phi \) using

\[
\frac{dx}{ds} = \cos \phi, \tag{2.1.9}
\]

\[
\frac{dh}{ds} = \sin \phi, \tag{2.1.10}
\]

and note that the curvature in this parametrisation is the rate of change of inclination with respect to the arc-length so

\[
B(x \sin \alpha + h \cos \alpha) + P = \frac{d\phi}{ds}. \tag{2.1.11}
\]

The advantage of this formulation is that it allows exact solutions to be obtained more easily and it removes problems which can occur in the standard formulation around points with infinite gradient. We can further simplify the problem by removing the dependence on the arc-length from this system. Hence we find

\[
\frac{dx}{d\phi} = \frac{\cos \phi}{B(x \sin \alpha + h \cos \alpha) + P}, \tag{2.1.12}
\]

\[
\frac{dh}{d\phi} = \frac{\sin \phi}{B(x \sin \alpha + h \cos \alpha) + P}. \tag{2.1.13}
\]
Notice in this formulation we still require three conditions: two for the integration constants and one to fix the parameter $P$. In this formulation the boundary conditions at the upper contact point become

$$h(\phi = -\theta_+) = 0, \quad x(\phi = -\theta_+) = 0. \quad (2.1.14)$$

For the final boundary condition we require the lower contact angle. We treat the lower contact angle as a further unknown and write two boundary conditions at the lower contact point, ensuring the system remains consistent. These boundary conditions are

$$h(\phi = \theta_-) = 0, \quad x(\phi = \theta_-) = 1. \quad (2.1.15)$$

For the remainder of the section $h, x$ are functions of $\phi$ (unless otherwise specified). Physical intuition leads us to believe $\theta_+ \leq \theta_- \text{ for } 0 \leq \alpha \leq \pi$, and in the micro-gravity limit we assume the difference between the angles has a linear dependence on the Bond number, i.e., $\theta_- = \theta_+ + B\Delta$. Further, we expect $\Delta$ to change signs at $\alpha = n\pi$ for $n \in \mathbb{Z}$. Applying this to the lower contact angle boundary conditions (2.1.15) and expanding in Taylor series we obtain

$$h(\theta_+) + B\Delta h'(\theta_+) = 0, \quad x(\theta_+) + B\Delta x'(\theta_+) = 1. \quad (2.1.16)$$

Assuming all variables can be expressed in a regular asymptotic expansion, we obtain the ansätze

$$x = x_0 + Bx_1 + O(B^2), \quad (2.1.17)$$
$$h = h_0 + Bh_1 + O(B^2), \quad (2.1.18)$$
$$P = P_0 + BP_1 + O(B^2). \quad (2.1.19)$$

Applying these expansions to (2.1.12) and (2.1.13) we obtain the leading order system

$$\frac{dx_0}{d\phi} P_0 = \cos \phi, \quad (2.1.20)$$
$$\frac{dh_0}{d\phi} P_0 = \sin \phi, \quad (2.1.21)$$
with boundary conditions

\[ h_0(-\theta_+) = 0, \quad x_0(-\theta_+) = 0, \quad h_0(\theta_+) = 0, \quad x_0(\theta_+) = 1. \quad (2.1.22) \]

Notice the system now appears to be over determined. This follows from our assumption that the upper and lower contact angles are identical at leading order (i.e., the zero gravity limit). Therefore should this system have a solution our assumption is indeed correct. These are integrated to yield

\[ x_0 = \frac{1}{2} \left( 1 + \frac{\sin \phi}{\sin \theta_+} \right), \quad (2.1.23) \]

\[ h_0 = \frac{1}{2} \left( \cos \theta_+ - \cos \phi \right) \left( \sin \theta_+ \right), \quad (2.1.24) \]

\[ P_0 = 2 \sin \theta_+. \quad (2.1.25) \]

An example of this solution can be seen in figure 2.2. Importantly notice that at leading order the shape is independent of the angle of the plane. Further, by seeking a circular form we can write

\[ \left( x_0 - \frac{1}{2} \right)^2 + \left( h_0 - \frac{1}{2} \cot \theta_+ \right)^2 = \frac{1}{4} \csc^2 \theta_+. \quad (2.1.26) \]

Therefore, for a given upper contact angle, the leading order solution is simply a circle centred at \((x_0, h_0) = (1, \cot \theta_+)/2\) with radius \(\csc \theta_+/2\). This result shows that \(\theta_+ = n\pi\) gives no solution. This can be explained by noting that \(\theta_+ = n\pi\) results in a drop whose interface never leaves the plane, i.e., a drop with zero volume.

At first order the system becomes

\[ \frac{dx_0}{d\phi} (x_0 \sin \alpha + h_0 \cos \alpha + P_1) + \frac{dx_1}{d\phi} P_0 = 0, \quad (2.1.27) \]

\[ \frac{dh_0}{d\phi} (x_0 \sin \alpha + h_0 \cos \alpha + P_1) + \frac{dx_1}{d\phi} P_0 = 0. \quad (2.1.28) \]

with boundary conditions

\[ h_1(-\theta_+) = 0, \quad x_1(-\theta_+) = 0, \quad h_1(\theta_+) = -\Delta h_0'(\theta_+), \quad x_1(\theta_+) = -\Delta x_0'(\theta_+). \quad (2.1.29) \]
We solve this to find

\[ x_1 = \frac{1}{16 \sin^3 \theta_+} (X_c \cos \alpha + X_s \sin \alpha + P_1 X_p), \]  

(2.1.30)

\[ h_1 = \frac{1}{16 \sin^3 \theta_+} (Z_c \cos \alpha + Z_s \sin \alpha + P_1 Z_p), \]  

(2.1.31)

\[ P_1 = \frac{1}{4 \sin^3 \theta_+} \left[ \sin \alpha \left( \cos^2 \theta_+ \sin \theta_+ + \theta_+ \cos \theta_+ - 2 \sin \theta_+ \right) \right. \]

\[ + \cos \alpha \left( \theta_+ \sin \theta_+ - \cos \theta_+ \sin^2 \theta_+ \right) \], \]  

(2.1.32)

\[ \Delta = -\frac{\sin \alpha (\cos \theta_+ \sin \theta_+ - \theta_+)}{4 \sin^3 \theta_+}, \]  

(2.1.33)

where

\[ X_c = -\cos \theta_+ \sin \theta_+ - 2 \cos \theta_+ \sin \phi + \cos \phi \sin \phi + \phi + \theta_+, \]  

(2.1.34)

\[ X_s = \cos^2 \theta_+ - 2 - 2 \sin \phi \sin \theta + \cos^2 \phi, \]  

(2.1.35)

\[ X_p = 4 \cos^2 \theta_+ - 4 \sin \theta_+ \sin \phi - 4, \]  

(2.1.36)

\[ Z_c = \cos^2 \theta_+ - 2 \cos \theta_+ \cos \phi + \cos^2 \phi, \]  

(2.1.37)

\[ Z_s = \sin \theta_+ \cos \theta_+ - 2 \sin \theta_+ \cos \phi - \sin \phi \cos \phi + \phi + \theta_+, \]  

(2.1.38)

\[ Z_p = 4 \sin \theta_+ \cos \theta_+ - 4 \sin \theta_+ \cos \phi. \]  

(2.1.39)

The leading order and first order asymptotic solutions are plotted in figure 2.2 with parameters given in the caption. These parameters could correspond to a drop of water with length \( \approx 1.207 \) mm on a 57% slope (approximately) with a 45° contact angle. We see that gravity pulls the bulk of the drop down the slope and reduces the maximum height of the drop. We now consider the volume of the drop, to find an asymptotic result which relates the volume to the contact angle and the angle of the plane. In this parametric form we can write the volume as

\[ -V = \int_0^1 h \, dx = \int_{-\theta_+}^{\theta_+} h \frac{dx}{d\phi} \, d\phi = \int_{-\theta_+ - \epsilon \Delta}^{\theta_+} h \frac{dx}{d\phi} \, d\phi. \]  

(2.1.40)
Figure 2.2: Plot of the leading (red) and first order (blue dashed) asymptotic series for \((B, \alpha, \theta_+) = (0.2, -\pi/3, \pi/4)\).

Applying our asymptotic expansions we can write

\[-V = V_0 + BV_1 + O(B^2), \quad \text{(2.1.41)}\]

\[V_0 = \int_{-\theta_+}^{\theta_+} h_0 \frac{dx_0}{d\phi} \, d\phi, \quad \text{(2.1.42)}\]

\[V_1 = \int_{-\theta_+}^{\theta_+} \left( h_1 \frac{dx_0}{d\phi} + h_0 \frac{dx_1}{d\phi} \right) \, d\phi - \Delta \left( h_0 \frac{dx_0}{d\phi} \right) \bigg|_{\phi=-\theta_+}. \quad \text{(2.1.43)}\]

Then using our solutions for \(x_0, x_1, h_0, h_1\) we find

\[V_0 = \frac{\theta_+ - \cos \theta_+ \sin \theta_+}{4 \sin^2 \theta_+}, \quad \text{(2.1.44)}\]

\[V_1 = -\frac{1}{16 \sin^6 \theta_+} \left( V_s \sin \alpha + V_c \cos \alpha \right), \quad \text{(2.1.45)}\]

where

\[V_s = \theta_+ \sin \theta_+ (\cos^2 \theta_+ + 1) - \sin^2 \theta_+ \cos \theta_+ - \theta_+^2 \cos \theta_+, \quad \text{(2.1.46)}\]

\[V_c = -\theta_+ \cos \theta_+ \sin^2 \theta_+ + 2 \sin^3 \theta_+ - \theta_+^2 \sin \theta_+. \quad \text{(2.1.47)}\]

Importantly, we mentioned earlier \(\theta_+ = 0\) gives a zero volume drop. To verify
this we take the limit of these expressions as $\theta_+ \to 0$ and observe

$$\lim_{y \to 0} \frac{y - \cos y \sin y}{\sin^2 y} = 0,$$

(2.1.48)

$$\lim_{y \to 0} \frac{y \sin y (\cos^2 y + 1) - \sin^2 y \cos y - y^2 \cos y}{\sin^6 y} = 0,$$

(2.1.49)

$$\lim_{y \to 0} \frac{-y \cos y \sin^2 y + 2 \sin^3 y - y \sin y}{\sin^6 y} = 0.$$

(2.1.50)

The first of these forces the leading order volume to zero, while the latter two imply the first order correction also vanishes. Thus we indeed find $\theta_+ = 0$ implies $V = 0$ for any $\alpha$. Returning to our volume equation, (2.1.8),

$$B \left( \frac{\sin \alpha}{2} + V \cos \alpha \right) + P = \sin \theta_+ + \sin \theta_-,$$

(2.1.51)

and using our expansion for $V$ we obtain

$$P_0 = 2 \sin \theta_+,$$

(2.1.52)

$$\frac{\sin \alpha}{2} + \cos \alpha V_0 + P_1 = \Delta \cos \theta_+,$$

(2.1.53)

which are consistent with the previous results for $P$, (2.1.25) and (2.1.32), and $\Delta$, (2.1.33). Finally, we comment on the difference between the two contact angles. We have shown (in this micro-gravity limit)

$$\theta_- - \theta_+ = B \sin \alpha \frac{\theta_+ - \cos \theta_+ \sin \theta_+}{4 \sin^3 \theta_+},$$

(2.1.54)

and we see that the difference between the angles grows with the relative effect of gravity (the Bond number) as we physically expect, and the two contact angles are identical when $\alpha = n\pi$, for $n \in \mathbb{N}$, i.e. when the plane is horizontal. Notice further that sign of the right hand side of (2.1.54) is identical to the sign of $\alpha$.

This section has used the Bond number as a small parameter and reformulated the Young–Laplace equation using the inclination as the independent variable. We have then found asymptotic solutions for the shape, volume and difference between the contact angles. In the following sections we shall see alternate methods to find the shape of the drop, and in both cases the micro-gravity requirement is relaxed. The latter of these methods employs elliptic
integrals to write an exact solution for the Young–Laplace equation. However, we begin by solving the Young–Laplace equation numerically and comparing our asymptotic solutions to these numerical solutions.

### 2.1.2 Numerical solution

In this section we produce a numerical solution of the Young–Laplace equation. We are producing the numerical solutions to verify the asymptotic solutions of section 2.1.1.

For our numerical solution we use a shooting method as outlined in section 1.3.1 to solve the following system

\[
\begin{align*}
\frac{dx}{d\phi} &= \cos \phi - B(x \sin \alpha + h \cos \alpha) + P, \\
\frac{dh}{d\phi} &= \sin \phi - B(x \sin \alpha + h \cos \alpha) + P, \\
h &= 0, \quad x = 0, \quad \text{at } \phi = -\theta_+, \\
&h = 0, \quad x = 1, \quad \text{at } \phi = \theta_-
\end{align*}
\]

where \( \theta_- \) and \( P \) are part of the solution set and \( B, \alpha \) are known. We see a sample solution in figure (2.3). This solution is then numerically integrated, using a trapezoidal method, to find the volume, \( V \), of the numerical solution. This volume, \( V \), will also be compared to the results from the asymptotic series, (2.1.41).

![Numerical x vs. Numerical h](image)

**Figure 2.3:** Numerical solution for \( B = 1/4, \theta_+ = \pi/4, \alpha = \pi/3 \)
2.1.3 Comparing the numerical solution to the asymptotic solutions

In this section, we compare the numerical solution as described in the previous section with the asymptotic series solution obtained in 2.1.1. We make this comparison based on three metrics: difference between the values obtained for the constant $P$, the difference between the values obtained for $\theta$, and the difference between the values of the volume, $V$. These values are expected to scale like $B^2$ as $B \to 0$.

To examine the quality of the asymptotic series solution, we run two series of ‘experiments’. In all these experiments we set $\theta_+ = \pi/4$. For the first set we fix $B = 1/4$ and take 50 evenly spaced points from $\alpha \in [0, \pi]$, and for the final set we fix $\alpha = \pi/3$ and take 50 evenly spaced points from $B \in [0, 1.5]$. Figures (2.4)-(2.9) show the values of $P$, $\theta$, and $V$ obtained asymptotically and numerically, where the values of $B, \alpha, \theta_+$ are specified as in the previous sentence. We can see the asymptotic solution’s error is independent of plane angle and increases with $B$ more slowly than $B^2$.

![Figure 2.4](image)

Figure 2.4: The two values of $\theta$ over a range of $\alpha$ found using the two methods: blue is numerical, and green is asymptotic.

To conclude, we used the Bond number as a small parameter in an asymptotic series and compared this series solution against a numerical solution. We saw that the asymptotic solution provides a very accurate solution even for Bond numbers as large as one, and for all values of $\alpha$. Note that here we have
only given the graph of $\alpha = \pi/3$, however other values have been tested and the same structure was seen.

\subsection*{2.1.4 Exact solution}

To find an exact solution for this problem, we observe the shape of a drop can be initially described without knowledge of the contact angle or the angle of plane, as described by O’Brien and van den Brule \cite{1991}. Instead of using the contact angle and the angle of the plane we construct a ‘complete’ solution which can be truncated to obtain a solution for a specific drop.

For a liquid with density $\rho$ and surface tension $\sigma$ subject to a gravitation field with magnitude $g$, a common form of writing the Young–Laplace equation
Figure 2.7: The two values of $\theta_-$ over a range of $B$ found using the two methods: blue is numerical, and green is asymptotic.

Figure 2.8: The two values of $V$ over a range of $B$ found using the two methods: blue is numerical, and green is asymptotic.

is

$$\rho gh + P = \sigma \left( \frac{h'}{\sqrt{1+h'^2}} \right)^', \quad (2.1.59)$$

where dash denotes differentiation with respect to $x$, and $z = h(x)$ describes the shape of the free surface. In this formulation $P$ is a constant to be determined. Physically it corresponds to the jump in pressure at the apex of the drop. In this work we consider the complete drop. The complete drop is a full solution which does not use the contact angles as boundary conditions. The physical solution is then obtained by restricting the domain of the complete drop in order that the contact angle conditions are satisfied.

In the case of zero gravity the complete drop is simply circular, while when
gravity is introduced the upper half of the drop is the upper half of an ellipse. Therefore, we use the half-length of the major axis of the ellipse as the characteristic length scale, $L$. Scaling $x$ and $h$ with this characteristic length we find

$$Bh + P = \left( \frac{h'}{\sqrt{1 + h'^2}} \right)' ,$$

(2.1.60)

where $B \equiv \rho g L^2 / \sigma$ is the Bond number. Here dash denotes differentiation with respect to the scaled $x$ and $h$ has also been scaled with the length scale. To enforce this length scale we use the boundary conditions,

$$\left. \frac{dh}{dx} \right|_{x=0} = 0 ,$$

(2.1.61)

$$\left. \frac{dh}{dx} \right|_{x=1} \to -\infty .$$

(2.1.62)

In problems of this type it is common to introduce the arc-length, $s$, and inclination, $\phi$ as dependent variables, see, for example, Pozrikidis [2012] and Myshkis et al. [1987]. Writing $h$ and $x$ as functions of these variables we find

$$\frac{dx}{ds} = \cos \phi ,$$

(2.1.63)

$$\frac{dh}{ds} = \sin \phi ,$$

(2.1.64)

and hence

$$Bh + P = \frac{d\phi}{ds} .$$

(2.1.65)
The advantage of this method is instead of having a non-linear second order differential equation we now have three first order equations. This can, however, be further simplified by removing the dependence on the arc-length from the system, whence we obtain,

\[ \frac{dx}{d\phi} = \frac{\cos \phi}{Bh + P}, \]  
\[ \frac{dh}{d\phi} = \frac{\sin \phi}{Bh + P}. \]

Importantly, note the functions \( x, h \) are functions of the inclination, \( \phi \), only.

Finally the boundary conditions (2.1.61) and (2.1.62) become

\[ (x, h) = (0, 0) \text{ at } \phi = 0, \]  
\[ x = 1 \text{ at } \phi = -\frac{\pi}{2}. \]

The negative inclination follows from the graph \((x, h)\) decreasing over this interval. Finally, we recall the constant \( P \) is part of the solution set. Once this system has been solved the complete drop is found by extending the domain to \([-\pi, \pi]\), or further when necessary, as discussed later.

Equations (2.1.66), and (2.1.67) can be integrated. Here equation (2.1.66) can be integrated exactly (leading to elliptic integrals), while equation (2.1.67) can be integrated directly. Indeed, by writing \( f = Bh + P \) we find

\[ f \frac{df}{d\phi} = B \sin \phi. \]

Since \( h = 0 \) is the ‘top’ of the drop it follows \( h \leq 0 \), and in problems of this type the sign of \( P \) depends on the direction of the curvature. Here the shape is concave down everywhere, thus it follows \( P < 0 \). Therefore \( f = Bh + P < 0 \), and hence by integrating (2.1.70)

\[ f = -\sqrt{2} \sqrt{A - B \cos \phi}, \]

for some constant \( A \). By recalling \( f = Bh + P \) we obtain

\[ h = \frac{-\sqrt{2} \sqrt{A - B \cos \phi} - P}{B}. \]
and applying the boundary condition \( h(\phi = 0) = 0 \) yields

\[
P = -\sqrt{2}\sqrt{A - B}.
\]  

(2.1.73)

As expected, we observe \( h \) and \( P \) share signs. By applying (2.1.71) to (2.1.66) we find

\[
\frac{dx}{d\phi} = -\frac{\cos \phi}{\sqrt{2}\sqrt{A - B} \sin \phi}.
\]  

(2.1.74)

Integrating and enforcing \( x(0) = 0 \), we find

\[
x = -\int_0^\phi \frac{\cos \varphi}{\sqrt{2}\sqrt{A - B} \cos \varphi} \, d\varphi.
\]  

(2.1.75)

However we can show (see appendix [A])

\[
\int_0^\phi \frac{\cos \varphi}{\sqrt{2}\sqrt{A - B} \cos \varphi} \, d\varphi = \frac{1}{B} \sqrt{\frac{2}{A - B}} \left[ (B - A) E\left(\frac{\phi}{2}, \sqrt{\frac{2B}{B - A}}\right) \right.
\]

\[
\left. + AF\left(\frac{\phi}{2}, \sqrt{\frac{2B}{B - A}}\right) \right],
\]  

(2.1.76)

and therefore

\[
x = -\frac{1}{B} \sqrt{\frac{2}{A - B}} \left[ (B - A) E\left(\frac{\phi}{2}, \sqrt{\frac{2B}{B - A}}\right) \right.
\]

\[
\left. + AF\left(\frac{\phi}{2}, \sqrt{\frac{2B}{B - A}}\right) \right],
\]  

(2.1.77)

where \( E, F \) are elliptic integrals of the second and first kind, respectively, as defined in Abramowitz and Stegun [1967]. Finally, the constant \( A \) is fixed by the condition \( x(-\pi/2) = 1 \), i.e.,

\[
(B - A) E\left(-\frac{\pi}{4}, \sqrt{\frac{2B}{B - A}}\right) + AF\left(-\frac{\pi}{4}, \sqrt{\frac{2B}{B - A}}\right) = -B \sqrt{\frac{A - B}{2}}.
\]  

(2.1.78)

This equation is readily solved numerically using a trust-region dogleg method, as described by Powell [1970]. To employ this non-linear solver we write (2.1.78) in the form \( g(A, B) = 0 \). Figure 2.10 gives an example of the function \( g \).

In summary, for given Bond number, the value of \( A \) is fixed using (2.1.78) whence the analytic solution for the shape of the complete drop is given by
Figure 2.10: Plot of the function $g(A, B)$ for $B = 0.1$. The red cross shows the root of $g$. The root of this equation allows us to find the value of $A$.

By extending the domain of the functions to $[-\pi, \pi]$ we obtain the complete drop. Figure 2.11 gives three examples of the complete drop for different Bond numbers. When the Bond number is non-zero the full solution does not form a closed curve. Instead, by extending the bounding contact angles beyond $\pm \pi$, we see the solution forms ‘loops’, as seen in figure 2.13. To create a physical drop from these solutions the same method is applied and, because of the bounds of the contact angles, we find that the physical solution never contains the entire loop. Figure 2.13 gives an example of a physical drop with contact point which forces the inclination to be outside $[-\pi, \pi]$.

To find the shape of a specific drop two further conditions are needed. Here we use the upper contact angle, $\theta$, and the angle of the plane, $\alpha$. Other options include the lower contact angle and the volume of the drop. Using the contact angle and angle of the plane we can find the point on the complete drop with inclination $\phi = \alpha - \theta$. From this point we construct a straight line with gradient $\tan \alpha$. The solution is completed when the complete drop intersects this line.
again. Importantly, if one of the contact angles forces an inclination outside \([-\pi, \pi]\) we simply extend the domain of the complete solution to include the required contact angle. Further, we note that to ensure \(\theta > 0\) we must have \(\phi < \alpha\). Figure 2.12 shows some typical solutions.

![Diagram](image)

(a) \((\theta, \alpha) = (0.5, 0)\). (b) \((\theta, \alpha) = (1.4, 0.1)\). (c) \((\theta, \alpha) = (0.8, 1.3)\).

Figure 2.12: Examples of three drop solutions for \(B = 2\).

Using this method it is possible to create drops which could be physically unrealistic, e.g., figure 2.14. To complete a study into drops on an inclined plane the stability of such drop would need to be studied. However, in this work the drop is only used as a model problem to develop some of the tools which shall be used on the two dimensional horizontal liquid bridge and the cylindrical liquid membrane.

In the micro-gravity limit asymptotic solutions can be found from the exact solutions of (2.1.72) – (2.1.77). These asymptotic solutions provide, where accurate, a much faster (and attractive) means of finding the shape of a liquid
2.1.4.1 Asymptotic solution

One potential difficulty associated with using the exact solutions derived in the previous section is the need to solve (2.1.78), a transcendental equation for $A$. When the Bond number is small, however, we can use asymptotic techniques to simplify (2.1.78) and obtain approximate solutions for the surface of the drop.

We seek an asymptotic expansion solution in the limit of vanishing Bond number. We propose a regular asymptotic expansion, $A = A_0 + B A_1 + B^2 A_2 + \mathcal{O}(B^3)$ and substitute into (2.1.78). Using Taylor series expansions where appropriate, this yields

$$\sqrt{\frac{1}{2A_0} + \frac{B}{4} (\pi - 4A_1) + B^2 \left(1 - A_2 - \frac{3\pi^2}{32}\right)} + \mathcal{O}(B^3) = 1.$$  \hspace{1cm} (2.1.79)

Solving this at each order of $B$ we obtain

$$A = \frac{1}{2} + B \frac{\pi}{4} + B^2 \left(1 - \frac{3\pi^2}{32}\right) + \mathcal{O}(B^3).$$  \hspace{1cm} (2.1.80)
Applying this to (2.1.72), and expanding about $B = 0$, we find

\[
\begin{align*}
    h &= -2 \sin^2 \left( \frac{\phi}{2} \right) - B \left[ 1 - \frac{\pi}{2} + \cos \phi \right] \sin^2 \left( \frac{\phi}{2} \right) \\
    &\quad - \frac{1}{16} B^2 \left[ -8 - 12\pi + 6\pi^2 - 4(3\pi - 4) \cos \phi + 8 \cos(2\phi) \right] \sin^2 \left( \frac{\phi}{2} \right) + \mathcal{O}(B^3).
\end{align*}
\]

(2.1.81)

Similarly, applying (2.1.80) to (2.1.77) we obtain

\[
\begin{align*}
    x &= -\sin \phi + \left( \frac{\phi}{2} + \frac{1}{4} \pi \sin \phi - \frac{1}{4} \sin(2\phi) \right) B \\
    &\quad + \left[ \frac{3\pi \phi}{8} - \frac{\sin \phi}{8} - \frac{3}{16} \pi^2 \sin \phi + \frac{3}{16} \pi \sin(2\phi) - \frac{1}{8} \sin(3\phi) \right] B^2 + \mathcal{O}(B^3).
\end{align*}
\]

(2.1.82)

Figure 2.15 shows the numerical and asymptotic solutions for three small Bond numbers. We see that in the limit of vanishing Bond number the two solutions match. However, as the Bond number reaches order 0.1 a noticeable error starts to occur in the asymptotic solution. Further, by using (2.1.81) and (2.1.82) we see (2.1.66) and (2.1.67) are satisfied at each order of Bond number up to $\mathcal{O}(B^3)$. 
2.1.4.2 Comparison to Lv and Shi [2018]

In a recent paper by Lv and Shi [2018] an analytical solution of the Young–Laplace equation of two–dimensional drops on inclined planes were presented. These results were also tested against experimental and numerical results.

The solutions are listed using a different scaling to the ones we used, however once the appropriate transformations are made the solutions can be written in closed form as

\[ h = -\frac{\sqrt{2}}{B} \left( \sqrt{A - B \cos \phi} + \sqrt{A - B} \right), \]  
\[ x = -\frac{1}{\sqrt{2}} \int_0^\phi \frac{\cos \varphi \, d\varphi}{\sqrt{A - B \cos \varphi}}. \]  

In particular notice the agreement to the solutions presented in (2.1.72) and (2.1.77). Where the integral expression they have written here is equivalent to the elliptic integrals of (2.1.77).

They then go on to discuss two limiting cases. The first of these is the case of a very small droplet with a spherical shape and they conclude in this limit \( A \to \infty \). Secondly, they considered when the width of the drop becomes large and found that \( A \to 1 \) as the width of the drop tends to infinite. Importantly, both of these results agree with the solutions we presented in (2.1.72) and (2.1.77). Lv and Shi [2018] also tested these solutions against numerical simulations and experimental profiles and found a good agreement.
2.1.5 Quasi-static evaporation from the drop

The theory developed in the previous sections can also be used to analyse what happens when the volume of a drop is decreased before being increased. This is important because of a need to understand how the drop behaves while undergoing evaporation however the solutions here do not give the full picture as the drop’s contact lines depin since we only consider static drops. To develop the ideas where we have called (heavily) upon the ideas laid out in section 1.1.3.

In this section we shall discuss the quasi-static evolution of a two dimensional drop on an inclined plane as volume is reduced and later increased. The quasi-static evolution when the angle of plane is changed was discussed by Paterson et al. [2015].

Suppose $0 \leq \alpha \leq \pi$, then $\theta^+ \leq \theta^-$, and further suppose we start with a stable drop with $\theta^+$ and $\theta^-$ possibly taking on different values. As volume is reduced quasi-statically, for example by evaporation, both angles will decrease until the upper contact angle reaches the receding angle. Then both angles remain (almost) constant as the upper contact line recedes. This motion will continue until we have no more liquid, i.e., $V = 0$, or, in the case of a hydrophobic surface, until the two contact points merge. If at some point the volume stops decreasing and we increase the volume without disturbing the interface, for example through an orifice under the drop, both angles will increase until the lower angle reaches the advancing angle when the lower contact line will become depinned and will advance down the slope. The net result of increasing and decreasing the volume of the drop is a movement down the slope.

2.2 Infinite horizontal liquid bridge

As discussed earlier the two dimensional horizontal liquid bridge is studied because of its similarity to the shape of one of the ‘diamonds’ on the stent. The two dimensional horizontal liquid bridge can be thought to extent infinity into and out of the page, thus giving the name infinite horizontal liquid bridge.
In this section we consider an infinite horizontal liquid bridge held between two identical infinite vertical plates. We define our axes so that $\hat{x}$ is normal to the walls, $\hat{z}$ is opposite to the direction of the gravitational field, $\hat{y}$ to be perpendicular to both $\hat{x}$ and $\hat{z}$, as detailed in figure 2.16. If we denote the upper and lower free surfaces by $z = h_\pm(x, y)$, respectively, then the liquid is held in the region

$$(x, y, z) \in [0, 2L] \times (-\infty, \infty) \times [h_-, h_+],$$

(2.2.1)

where $L$ is the half distance between the walls. We notice that the geometry is unchanging as we move along the $y$-axis so we assume the surfaces also are unchanged. As such we consider the upper and lower free surfaces to be given by $z = h_\pm(x)$ and consider a cross section of the geometry given in (2.2.1). Once this simplification has been made the problem becomes two-dimensional, and this was the subject of [Haynes et al., 2016]. The remainder of this section follows the work of that paper.

![Figure 2.16: Schematic of a cross section of the geometry of the bridge. The bridge extends infinitely into and out of the page.](image)

Using the Young–Laplace equation we see the pressure at the interface
\( z = h_{\pm}(x) \) can be given by

\[
p = \mp \sigma \frac{d^2 h_{\pm}}{dx^2} \left[ 1 + \left( \frac{dh_{\pm}}{dx} \right)^2 \right]^{\frac{3}{2}},
\]  

(2.2.2)

However when considering the static case we must have

\[
p = -\rho g z + P.
\]  

(2.2.3)

Equating these and nondimensionalising using \( L \) as the length scale for the problem, we obtain the dimensionless system

\[
-B h_{\pm} + P = \mp \frac{d^2 h_{\pm}}{dx^2} \left[ 1 + \left( \frac{dh_{\pm}}{dx} \right)^2 \right]^{\frac{3}{2}},
\]  

(2.2.4)

where \( B \) is the Bond and \( P \) is the dimensionless pressure difference across the upper surface, defined as previously. By examination of (2.2.4) we require a total of five boundary conditions; four to fix the constants of integration and a further condition to fix \( P \). Referring to figure 2.16 and defining \( x = 0 \) to be at the left wall, we obtain the dimensionless boundary conditions

\[
\frac{dh_{\pm}}{dx} = \mp \cot \theta_{\pm}, \quad \text{at } x = 0, \quad (2.2.5)
\]

\[
\frac{dh_{\pm}}{dx} = 0, \quad \text{at } x = 1, \quad (2.2.6)
\]

\[
h_{\pm} = 0, \quad \text{at } x = 1. \quad (2.2.7)
\]

Note that (2.2.4) to (2.2.7) represent the Young–Laplace equation, the known contact angle boundary condition, the symmetry condition, and a \( z \)-axis defining condition, respectively. As discussed in the previous section, the Bond number \( B \) is asymptotically small while under micro-gravity. Thus we use it as the small parameter in an asymptotic series solution.

In the previous section we derived a set of equations with \( x, h_{\pm} \) as dependent variables, depending of the inclination \( \phi \). The same transformation can be
applied here to yield
\[
\frac{dx_\pm}{d\phi} = \frac{\cos \phi}{Bh_\pm + P},
\]
\[
\frac{dh_\pm}{d\phi} = \pm \frac{\sin \phi}{Bh_\pm + P},
\]
(2.2.8, 2.2.9)

with boundary conditions,
\[
x_\pm(0) = 1, \ x_\pm(\alpha_\pm) = 0, \ h_+(0) = 0,
\]
(2.2.10)

where \(\alpha_\pm\) are the surface inclinations at the wall, which are related to the contact angles \(\theta_\pm\) via \(\alpha_\pm = \theta_\pm - \frac{\pi}{2}\).

### 2.2.1 Asymptotic solutions

In the limit of vanishing Bond number we seek solutions of the form
\[
x_\pm = x^{(0)}_\pm + Bx^{(1)}_\pm + \mathcal{O}(B^2),
\]
\[
h_\pm = h^{(0)}_\pm + Bh^{(1)}_\pm + \mathcal{O}(B^2).
\]
(2.2.11, 2.2.12)

Further, we employ the ansatz that the upper contact angle is only slightly different to the lower angle, that is \(\alpha_- = \alpha_+ + Bc\) where \(c\) is a constant to be determined. This ansatz is later verified by balancing forces acting on the liquid. With this we obtain the boundary condition
\[
x_-(\alpha_+ + Bc) = 0.
\]
(2.2.13)

Expanding in a Taylor series around \(\phi = \alpha_+\) we obtain
\[
x_-(\alpha_+) + Bc \frac{dx_-}{d\phi} \bigg|_{\phi=\alpha_+} + \frac{B^2 c^2}{2} \frac{d^2 x_-}{d\phi^2} \bigg|_{\phi=\alpha_+} + \mathcal{O}(B^3) = 0,
\]
(2.2.14)

and therefore
\[
x_-(\alpha_+) \sim -Bc \frac{dx_-}{d\phi} \bigg|_{\phi=\alpha_+} - \frac{B^2 c^2}{2} \frac{d^2 x_-}{d\phi^2} \bigg|_{\phi=\alpha_+}.
\]
(2.2.15)

To determine \(c\) we require a further condition. This could be taken in the form of a volume condition or some other geometric condition. Here we use
that condition that the height of the bridge at its centre is known. We use
this because we believe it to be the easiest to measure accurately for someone
conducting an experiment. Mathematically, this condition can be written as
\( h_-(\phi = 0) = -\Lambda \) where \( \Lambda \) is the central separation distance.

Considering the \( O(1) \) terms we find

\[
\frac{dh^{(0)}_{\pm}}{d\phi} + \sin \phi = 0, \quad (2.2.16)
\]

\[
\frac{dx^{(0)}_{\pm}}{d\phi} - \cos \phi = 0, \quad (2.2.17)
\]

with boundary conditions

\[
h_+^{(0)} = 0, \quad h_-^{(0)} = -\Lambda, \quad x_+^{(0)} = 1, \quad \text{for } \phi = 0,
\]

\[
x_-^{(0)} = 0, \quad \text{for } \phi = \alpha_+ . \quad (2.2.19)
\]

Note \( c \) does not enter the leading order system and thus we use only 5 bound-
ary conditions. We shall see the 6th boundary condition we omitted here,
\( x_-^{(0)}(\alpha_+) = 0 \), is automatically satisfied. Solving (2.2.16) - (2.2.19) we obtain

\[
h_+^{(0)} = \frac{\cos \phi - 1}{\sin \alpha_+}, \quad (2.2.20)
\]

\[
h_-^{(0)} = \frac{1 - \cos \phi}{\sin \alpha_+} - \Lambda, \quad (2.2.21)
\]

\[
x_+^{(0)} = \frac{\sin \alpha_+ - \sin \phi}{\sin \alpha_+}, \quad (2.2.22)
\]

\[
x_-^{(0)} = \frac{\sin \alpha_+ - \sin \phi}{\sin \alpha_+}, \quad (2.2.23)
\]

\[
P^{(0)} = - \sin \alpha. \quad (2.2.24)
\]

By writing these solutions as

\[
(x_+ - 1)^2 + (h_+ + \csc \alpha_+)^2 = \csc^2 \alpha_+, \quad (2.2.25)
\]

\[
(x_- - 1)^2 + (h_- - \csc \alpha_+ + \Lambda)^2 = \csc^2 \alpha_+, \quad (2.2.26)
\]

we find the leading order solutions are simply circles of radius \(|\csc \alpha_+|\). Note
for these circles \( \csc \alpha_+ < 0 \) while the contact angle is in \((0, \pi/2)\). Consider now
terms of $O(B)$; from these, (2.2.8) and (2.2.9) yield

\[
\frac{dh^{(0)}_\pm}{d\phi}(h^{(0)}_\pm + P^{(1)}) + \frac{dh^{(1)}_\pm}{d\phi} P^{(0)} = 0, \tag{2.2.27}
\]

\[
\frac{dx^{(0)}_\pm}{d\phi}(h^{(0)}_\pm + P^{(1)}) + \frac{dx^{(1)}_\pm}{d\phi} P^{(0)} = 0, \tag{2.2.28}
\]

with the boundary conditions:

\[
h^{(1)}_\pm(0) = 0, \quad x^{(1)}_\pm(0) = 0, \tag{2.2.29}
\]

\[
x^{(1)}_+(\alpha_+) = 0, \quad x^{(1)}_-(\alpha_+) = -c \left. \frac{dx^{(0)}_\pm}{d\phi} \right|_{\phi=\alpha_+}. \tag{2.2.30}
\]

To fix $c$ we use the condition $x^{(1)}_-(0) = 0$. We can solve this system to obtain the solutions,

\[
h^{(1)}_+ = \frac{2P_1 \sin \alpha_+ + \cos \phi - 1}{2 \sin^3 \alpha_+} (\cos \phi - 1), \tag{2.2.31}
\]

\[
x^{(1)}_+ = -\frac{2P_1 \sin \alpha_+ \sin \phi + \cos \phi \sin \phi - 2 \sin \phi + \phi}{2 \sin^3 \alpha_+}, \tag{2.2.32}
\]

\[
h^{(1)}_- = \frac{2 \sin \alpha_+ (P_1 - \Lambda) + 1 - \cos \phi}{2 \sin^3 \alpha_+} (1 - \cos \phi), \tag{2.2.33}
\]

\[
x^{(1)}_- = -\frac{2 \sin \alpha_+ \sin \phi (\Lambda - P_1) + \cos \phi \sin \phi - 2 \sin \phi + \phi}{2 \sin^3 \alpha_+}, \tag{2.2.34}
\]

\[
P^{(1)} = -\frac{\sin \alpha_+ \cos \alpha_+ + \alpha_+ - 2 \sin \alpha_+}{2 \sin^2 \alpha_+}, \tag{2.2.35}
\]

\[
c = \frac{\alpha_+ + \sin \alpha_+ (\cos \alpha_+ - 2) + \Lambda \sin^2 \alpha_+}{\sin^2 \alpha_+ \cos \alpha_+}. \tag{2.2.36}
\]

We say the bridge is in its critical state when the upper and lower interfaces meet at the centre, that is $\Lambda = 0$. In the next section we shall consider the volume of a bridge, and in particular the volume of critical bridges. To find an expression for the volume we consider a force balance on the bridge, i.e., the weight of the bridge must be supported by the surface tension forces.

### 2.2.2 Force balance and volume

We note that the asymptotic solutions indicate that, at leading order, the upper and lower angles $\theta_+, \theta_-$ must be equal. This is expected since at leading order,
gravity plays no role, and the liquid bridge upper and lower surfaces are both circular arcs. We now verify the ansatz \( \theta_+ < \theta_- \) by considering an exact force balance on the liquid bridge. The liquid half-volume is found by integrating over the half-bridge whence we obtain

\[
V = \int_0^1 \int_{h_-}^{h_+} dz \, dx = \int_0^1 (h_+ - h_-) \, dx. \tag{2.2.37}
\]

However applying equation (2.2.4), and boundary condition (2.2.5) we obtain

\[
BV = \int_0^1 \left\{ \frac{d^2h_+}{dx^2} \left[ 1 + \left( \frac{dh_+}{dx} \right)^2 \right]^{\frac{3}{2}} \right\} dx + \int_0^1 \left\{ \frac{d^2h_-}{dx^2} \left[ 1 + \left( \frac{dh_-}{dx} \right)^2 \right]^{\frac{3}{2}} \right\} dx, \tag{2.2.38}
\]

\[
= \left[ \frac{dh_+}{dx} \right]_0^1 + \left[ \frac{dh_-}{dx} \right]_0^1 = \cos \theta_+ - \cos \theta_- . \tag{2.2.39}
\]

Here we have exploited the symmetry about \( x = 1 \) so that \( V \) is the half volume of the bridge. The imposes an equilibrium relation between the upper and lower contact angles. This can also be interpreted as a statement that the weight of the bridge is supported by the upward component of the surface tension at the edges. The vertical walls exert a reaction force equal to the tangential component of the surface tension force at the lines of contact. Specifically, the half-weight of the liquid is \( \rho g V \) per unit length in the in-plane direction, while the net surface tension force in the vertical direction per unit length is \( \sigma (\cos \theta_+ - \cos \theta_-) \). Hence, in order to get a net upward force, the upper and lower angle must be unequal, with \( \theta_+ < \theta_- \), this difference being accounted for by the occurrence of contact angle hysteresis.

We now verify the solutions given earlier by finding \( c \) using the exact result (2.2.41). Thus we substitute

\[
\theta_- = \theta_+ + cB + \mathcal{O}(B^2), \tag{2.2.41}
\]
into (2.2.40) to obtain
\[ c = \frac{V^{(0)}}{\sin \theta_+} = \frac{V^{(0)}}{\cos \alpha_+}. \] (2.2.42)

Now we return to (2.2.37) which we rewrite as
\[ V = \int_{\alpha_+}^{0} h_+ \frac{dx_+}{d\phi} d\phi - \int_{\alpha_-}^{0} h_- \frac{dx_-}{d\phi} d\phi. \] (2.2.43)

We expand this in an asymptotic series, using the previous expansions, to obtain
\[ V^{(0)} = \int_{\alpha_+}^{0} \left( h_+^{(0)} \frac{dx_+^{(0)}}{d\phi} - h_-^{(0)} \frac{dx_-^{(0)}}{d\phi} \right) d\phi, \] (2.2.44)
\[ V^{(1)} = \int_{\alpha_+}^{0} \left( h_+^{(1)} \frac{dx_+^{(1)}}{d\phi} + h_-^{(1)} \frac{dx_-^{(1)}}{d\phi} - h_+^{(0)} \frac{dx_+^{(0)}}{d\phi} - h_-^{(0)} \frac{dx_-^{(0)}}{d\phi} \right) d\phi \]
\[ + c^{(0)} \left( h_-^{(0)} \frac{dx_-^{(0)}}{d\phi} \right) \bigg|_{\phi=\alpha_+}. \] (2.2.45)

Therefore by applying our results for \( x_+^{(0)}, h_+^{(0)}, x_-^{(1)}, h_-^{(1)} \) we find
\[ V^{(0)} \sin^2 \alpha_+ = \Lambda \sin^2 \alpha_+ + \cos \alpha_+ \sin \alpha_+ - 2 \sin \alpha_+ + \alpha_+, \] (2.2.46)
\[ CV^{(1)} = \sin^2 \alpha_+ (\cos \alpha_+ - 2)(\cos^3 \alpha_+ - 4 \cos^2 \alpha_+ - \cos \alpha_+ + 2) \]
\[ + 2\Lambda \sin^3 \alpha_+ (\cos^3 \alpha_+ - 3 \cos^2 \alpha_+ - \cos \alpha_+ + 2) \]
\[ + \alpha_+^2 (\cos^2 \alpha_+ + \cos(2\alpha_+)) - \Lambda^2 \sin^6 \alpha_+ \]
\[ + 2\alpha_+ \sin \alpha_+ (2 \cos^3 \alpha_+ - 5 \cos^2 \alpha_+ - \cos \alpha_+ + 2), \]

where \( C = -(2 \cos^3 \alpha_+ \sin^5 \alpha_+). \) Importantly, we observe that applying these to (2.2.42) and solving the resulting equation for \( c \), we are able to recover (2.2.36), as expected.

Using equations (2.2.46) and (2.2.47) we can construct a first order asymptotic series approximation for the limiting volume of the liquid bridge (i.e., the smallest possible bridge which occurs when the upper and lower surfaces just touch in the centre), namely
\[ V_{\text{crit}} = V^{(0)} \bigg|_{\Lambda=0} + B V^{(1)} \bigg|_{\Lambda=0} + O(B^2), \] (2.2.48)
which we shall compare with the value given by numerical computation in section 2.2.5.

### 2.2.3 Exact solution

As with the drop problem studied in section 2.1.4, the Young–Laplace equation shall be integrated by employing elliptic integrals to produce exact solutions for the shape of the liquid bridge. In particular, these solutions are valid for all Bond numbers, unlike the asymptotic results of the previous section. Recall the parametric equations were given by

\[
\begin{align*}
\frac{dx_\pm}{d\phi} &= \cos \phi \frac{B h_\pm + P}{B h_\pm + P}, \\
\frac{d h_\pm}{d\phi} &= \pm \sin \phi \frac{B h_\pm + P}{B h_\pm + P},
\end{align*}
\tag{2.2.49}
\tag{2.2.50}
\]

with boundary conditions;

\[
\begin{align*}
x_\pm(0) &= 1, \\ x_+(\alpha_+) &= 0, \\ h_+(0) &= 0, \\ h_-(0) &= -\Lambda.
\end{align*}
\tag{2.2.51}
\]

We can simplify (2.2.50) by writing \( f_\pm = B h_\pm + P \), to obtain

\[
f_\pm \frac{df_\pm}{d\phi} = \pm B \sin \phi.
\tag{2.2.52}
\]

This equation is then integrated to obtain

\[
\begin{align*}
f_+ &= \pm \sqrt{2A_+ - 2B \cos \phi}, \\
f_- &= \pm \sqrt{2A_- + 2B \cos \phi},
\end{align*}
\tag{2.2.53}
\tag{2.2.54}
\]

for constants of integration \( A_\pm \). Before commenting on the signs of \( f_+ \) and \( f_- \) we recall the definition of the constant \( P \), i.e., the dimensionless pressure difference across the upper free surface. Therefore if the upper surface is convex/concave then \( P \) is positive/negative. Further since we enforce the condition \( h_+(0) = 0 \) it follows that \( h_+ \) is positive/negative when it is convex/concave. Thus \( h_+ \) and \( P \) share sign. Now, recall \( f_+ = B h_+ + P \). It follows \( f_+ > 0 \) forces both \( P > 0 \) and \( h_+ \geq 0 \), while \( f_+ < 0 \) implies \( P < 0 \) and \( h_+ \leq 0 \) (in both cases equality is only achieved at \( x = 0 \)). In the case of \( f_+ = 0 \) we immediately see \( h_+ = -\frac{P}{B} \).
which with \( h_+(0) = 0 \) forces both \( P = 0 \) and \( h_+ = 0 \). Finally we note that for \( \theta_+ \in [0, \pi/2) \) the condition at the wall of the bridge forces \( h_+ \) to be decreasing at the wall. For \( h_+ \) to be decreasing at the wall and \( h_+(0) = 0 \) we must have \( h_+ \geq 0 \). The reverse argument follows for \( \theta_+ \in (\pi/2, \pi) \). Therefore the sign of \( f_+ \) can be immediately determined from \( \theta_+ \) using

\[
\text{sign}(f_+) = \begin{cases} 
1 & \text{for } 0 \leq \theta_+ < \frac{\pi}{2}, \\
0 & \text{for } \theta_+ = \frac{\pi}{2}, \\
-1 & \text{for } \frac{\pi}{2} < \theta_+ \leq \pi.
\end{cases} \tag{2.2.55}
\]

Later we will determine a condition for the sign of \( f_- \) based on the Bond number, \( B \), the centre separation distance, \( \Lambda \), and the upper contact angle, \( \theta_+ \). We begin this analysis with the assumption that both \( f_\pm \) do not change sign. Later this assumption shall be dropped and a more general solution set shall be derived.

To allow a concise form of the solution we define the parameters \( r_\pm \) as the signs of \( f_\pm \), respectively. The upper parameter \( r_+ \) can be written explicitly in terms of \( \theta_+ \) by (2.2.55). During the following analysis conditions are found to determine \( r_- \) from the Bond number and central separation distance. Using these definitions we find equations (2.2.56) and (2.2.57) become

\[
h_+ = \frac{r_+ \sqrt{2\sqrt{A_+ - B \cos \phi} - P}}{B}, \tag{2.2.56}
\]
\[
h_- = \frac{r_- \sqrt{2\sqrt{A_- + B \cos \phi} - P}}{B}. \tag{2.2.57}
\]

Applying the condition \( h_+(0) = 0 \) we obtain

\[
P = r_+ \sqrt{2\sqrt{A_+ - B}} \tag{2.2.58}
\]

which agrees with our earlier argument that \( P \) and \( f_+ \) must share signs. Continuing, we apply \( h_-(0) = -\Lambda \) to find

\[
\frac{\sqrt{2}}{B} \left( r_- \sqrt{A_- + B} - r_+ \sqrt{A_+ - B} \right) = -\Lambda. \tag{2.2.59}
\]

Therefore we observe

\[
A_- = A_+ + \frac{B}{2} \left( A^2B - 2r_+ \sqrt{2\Lambda \sqrt{A_+ - B} - 4} \right). \tag{2.2.60}
\]
However, substitution of (2.2.60) into $h_-(0) = -\Lambda$ reveals this solution is only valid if

$$r_+ \left( r_+ \sqrt{A_+ - B} - \frac{BA}{\sqrt{2}} \right) > 0,$$

(2.2.61)

or equivalently

$$r_+ (P - BA) > 0.$$  

(2.2.62)

Importantly, we find if $\Lambda < P/B$ then $r_+ > 0$ or if $\Lambda > P/B$ then $r_+ < 0$. Further, we see when $P < 0$ we must have $r_+ < 0$ since both $\Lambda$ and $B$ must be positive. When $P < 0$ we must have $\alpha_+ > 0$ and therefore $r_+ < 0$ agrees with the force balance equation, (2.2.40), since $r_+ < 0$ corresponds to a lower surface which is convex and therefore has $\alpha_+ > 0$ as required.

To complete the solution we return to the equations for $x_\pm$ in (2.2.49), which can now be written as

$$\frac{dx_+}{d\phi} = \frac{r_+ \cos \phi}{\sqrt{2} \sqrt{A_+ - B \cos \phi}},$$

(2.2.63)

$$\frac{dx_-}{d\phi} = \frac{r_- \cos \phi}{\sqrt{2} \sqrt{A_- + B \cos \phi}},$$

(2.2.64)

with the condition

$$x_{\pm}(0) = 1.$$  

(2.2.65)

We integrate (2.2.63) and (2.2.64) to find,

$$x_+(\phi) = 1 + \frac{r_+}{\sqrt{2}A_+} \int_0^\phi \frac{\cos \theta \, d\theta}{\sqrt{1 - \frac{B}{A_+} \cos \theta}},$$

(2.2.66)

$$x_-(\phi) = 1 + \frac{r_-}{\sqrt{2}A_-} \int_0^\phi \frac{\cos \theta \, d\theta}{\sqrt{1 + \frac{B}{A_-} \cos \theta}}.$$  

(2.2.67)

However, we can show\(^1\)

$$\int_0^\phi \frac{\cos \theta \, d\theta}{\sqrt{1 + \kappa \cos \theta}} = \frac{2}{\kappa \sqrt{1 + \kappa}} \left[ (1 + \kappa)E \left( \frac{\phi}{2}, \sqrt{\frac{2\kappa}{1 + \kappa}} \right) - F \left( \frac{\phi}{2}, \sqrt{\frac{2\kappa}{1 + \kappa}} \right) \right],$$

(2.2.68)

\(^1\)See appendix A
where $F(\phi, k)$ and $E(\phi, k)$ are elliptic integrals of the first and second type respectively, as defined in [Abramowitz and Stegun 1967]. Importantly, this result is only valid for $\kappa > -1$. Indeed, if $\kappa = -1$, the respective surface is flat and this solution form is not required, and, further, if $\kappa < -1$ the integrand is then complex. Upon setting $\kappa = \mp \frac{B}{A}$ in (2.2.68) we obtain

$$x_+(\phi) = 1 + \frac{r_+ \sqrt{2} \sqrt{A_+ - B}}{B \sqrt{A_+ - B}} \left[ (B - A_+) E \left( \frac{\phi}{2}, \sqrt{-\frac{2B}{A_+ - B}} \right) + A_+ F \left( \frac{\phi}{2}, \sqrt{-\frac{2B}{A_+ - B}} \right) \right],$$

(2.2.69)

$$x_-(\phi) = 1 + \frac{-r_- \sqrt{2} \sqrt{A_- + B}}{B \sqrt{A_- + B}} \left[ (B + A_-) E \left( \frac{\phi}{2}, \sqrt{-\frac{2B}{A_- + B}} \right) - A_- F \left( \frac{\phi}{2}, \sqrt{-\frac{2B}{A_- + B}} \right) \right].$$

(2.2.70)

Finally to determine $A_+$ we set $x_+(\alpha_+) = 0$, which gives

$$(B - A_+) E \left( \frac{\alpha_+}{2}, \sqrt{-\frac{2B}{A_+ - B}} \right) + A_+ F \left( \frac{\alpha_+}{2}, \sqrt{-\frac{2B}{A_+ - B}} \right) = 0,$$

(2.2.71)

which can be solved numerically, or asymptotically for small Bond number, to obtain $A_+$. In summary, the full solution system is

$$h_+ = \frac{r_+ \sqrt{2} \sqrt{A_+ - B} \cos \phi - P}{B},$$

(2.2.72)

$$h_- = \frac{r_- \sqrt{2} \sqrt{A_- + B} \cos \phi - P}{B},$$

(2.2.73)

$$P = r_+ \sqrt{2} \sqrt{A_+ - B}$$

(2.2.74)

$$x_+ = 1 + \frac{r_+ \sqrt{2} \sqrt{A_+ - B}}{B \sqrt{A_+ - B}} \left[ (B - A_+) E \left( \frac{\phi}{2}, \sqrt{-\frac{2B}{A_+ - B}} \right) + A_+ F \left( \frac{\phi}{2}, \sqrt{-\frac{2B}{A_+ - B}} \right) \right],$$

(2.2.75)

$$x_- = 1 + \frac{-r_- \sqrt{2} \sqrt{A_- + B}}{B \sqrt{A_- + B}} \left[ (B + A_-) E \left( \frac{\phi}{2}, \sqrt{-\frac{2B}{A_- + B}} \right) - A_- F \left( \frac{\phi}{2}, \sqrt{-\frac{2B}{A_- + B}} \right) \right],$$

(2.2.76)

where the constants of integration, $A_+$ and $A_-$, are solutions to

$$0 = (B - A_+) E \left( \frac{\alpha_+}{2}, \sqrt{-\frac{2B}{A_+ - B}} \right) + A_+ F \left( \frac{\alpha_+}{2}, \sqrt{-\frac{2B}{A_+ - B}} \right),$$

(2.2.77)

$$A_- = A_+ - \frac{B}{2} \left( A^2 B - 2r_+ \sqrt{2A} \sqrt{A_+ - B} - 4 \right),$$

(2.2.78)
with
\[
\begin{align*}
    r_+ &= \begin{cases} 
        1 & \text{for } -\frac{\pi}{2} \leq \alpha_+ < 0, \\
        0 & \text{for } \alpha_+ = 0, \\
        -1 & \text{for } 0 < \alpha_+ \leq \frac{\pi}{2}.
    \end{cases} \\
    r_- &= \begin{cases} 
        1 & \text{for } \Lambda < \frac{P}{B}, \\
        0 & \text{for } \Lambda = \frac{P}{B}, \\
        -1 & \text{for } \Lambda > \frac{P}{B}.
    \end{cases}
\end{align*}
\]

(2.2.79) (2.2.80)

We see \( r_+ \) is immediately determined from \( \alpha_+ \), and, importantly, \( r_- \) is fully determined by \( \Lambda \) and the upper surface. For given \( B \) and \( \alpha_+ \), we can determine \( A_+ \) using (2.2.77) and therefore \( P \) without any information of the lower surface. To solve (2.2.77) numerically, and therefore find \( P \), we could use Newton’s method. Instead we use a trust-region dogleg method, as described by [Powell 1970], to avoid any complications with a singular Jacobian. To employ this non-linear solver we rewrite (2.2.77) into the form \( g_1(A_+, B) = 0 \). Figure 2.17 plots the function \( g_1 \) against \( A_+ \) for \( (B, \alpha_+) = (0.1, -0.5) \). \( A_+ = B \) is in fact a solution to (2.2.77), since

\[
\lim_{A_+ \to B} (B - A_+)E\left(\frac{\alpha_+}{2}, \sqrt{\frac{-2B}{A_+ - B}}\right) = 0.
\]

(2.2.81)

However, equation (2.2.77) is only valid if \( A_+ \neq B \), because in its derivation we multiplied \( x_+(\alpha_+) = -1 \) by \( A_+ - B \). Therefore we consider the root marked in figure 2.17 with a red cross.

The solutions (2.2.72)-(2.2.80) are valid while neither \( f_+ \) or \( f_- \) have a root, i.e., change sign. At low Bond number and central separation distance of order one (or less), the functions \( f_+, f_- \) do not have any roots, and therefore the aforementioned solutions can be applied for asymptotic solutions in the limit of vanishing Bond number, which we covered in section 2.2.3.2. It should be noted however that the asymptotic solutions fail when \( \Lambda \sim P/B \), since \( h_- \sim 1/B \).
Figure 2.17: Example of the function $g_1(A_+)$ for $(B, \alpha_+) = (0.1, -0.5)$. Roots of $g_1$ satisfy equation (2.2.77).

becomes large in this limit. From the definitions of $f_{\pm}$, i.e.,

$$f_{\pm} = B h_{\pm} + P,$$  \hspace{1cm} (2.2.82)

we can apply the Young–Laplace equation to equate these to the positive and negative curvatures of the upper and lower surfaces, respectively, i.e., $f_{\pm} = \pm \kappa_{\pm}$ where $\kappa_{\pm}$ are the curvatures of the upper and lower surfaces, respectively. Therefore roots of $f_{\pm}$ correspond to inflection points in the surfaces, when viewed in the $(x, h_{\pm})$ plane. When these roots are considered in the $(h_{\pm}, \phi)$ and $(x_{\pm}, \phi)$ planes they correspond to turning points, see figure 2.18. The following section discusses the solutions when there is an inflection point.

### 2.2.3.1 Inflection points

We first suppose that the upper surface has a point, $\psi$, such that $f_+(\psi) = 0$. This point will satisfy $A_+ = B \cos \psi$, and thus we find $A_+ < B$. However, from (2.2.68) we recall that $\kappa = -B/A_+$ and therefore $A_+ < B$ would imply $\kappa < -1$. 
(a) Plot of the lower solutions, \((h_-, x_-)\), as functions of \(\phi\), in black and red respectively. Importantly, both functions have the same turning point, at \(\phi \approx 0.3112\), as shown by the blue dashed line.

Figure 2.18: Example of the solutions when an inflection point exists, here we use \((B, \alpha_+, \Lambda) = (3, -0.1, 0.2)\).

Since (2.2.68) involves the integral

\[
\int_0^\phi \frac{\cos \theta \, d\theta}{\sqrt{1 + \kappa \cos \theta}}
\]

which will be complex if \(\kappa < -1\), we conclude that an inflection point on the upper surface is impossible. While inflection points are impossible on the upper surface they may exist on the lower surface.

Here the upper surface is unchanged from the original analysis, and is given by (2.2.72), (2.2.74), (2.2.75), (2.2.77). When there exists an inflection point on the lower surface, the functions \(h_-\) and \(x_-\) are multivalued in \(\phi\), as depicted in figure 2.18. Note that we assume that only one inflection point exists in each half of the bridge. To overcome this we denote \(h_-^{(c)}, x_-^{(c)}, A_-^{(c)}\) as the central part of the solutions, and \(h_-^{(b)}, x_-^{(b)}, A_-^{(b)}\) as the boundary part of the solutions.
These lower solutions have the conditions:

\[ h^{(c)}_-(0) = -\Lambda, \]  
\[ x^{(c)}_-(0) = 1, \]  
\[ h^{(c)}_-(\psi) = h^{(b)}_-(\psi), \]  
\[ x^{(c)}_-(\psi) = x^{(b)}_-(\psi), \]

where \( \psi \) is the inclination at the inflection point, which satisfies \( f_-'(\psi) = 0 \). The first two of these conditions are the same as was used in the original analysis; the latter two are used to match the boundary and central solutions.

Since the sign of \( f_- \) changes at \( \phi = \psi \) we have to be careful with our definition of \( r_- \). We choose \( r_- \) to be the sign of \( f_- \) in the central region. We now proceed to solve for \( f_- \). We begin by noting \( f_-'(\psi) = 0 \) implies we must have

\[ \cos \psi = -\frac{A^{(c)}_-}{B}, \]  

or similarly,

\[ \cos \psi = -\frac{A^{(b)}_-}{B}. \]  

By equating these we find \( A^{(b)}_- = A^{(c)}_- \), and therefore we return to the notation \( A_- \) for both. Continuing, the centre part of the lower surface must have

\[ h^{(c)}_- = \frac{\sqrt{2}}{B} \left( r_- \sqrt{A_- + B \cos \phi} - \frac{P}{\sqrt{2}} \right). \]

Therefore, applying \( h^{(c)}_-(0) = -\Lambda \) yields

\[ A_- = A_+ + \frac{B}{2} \left( BA^2 - r_+ \Lambda 2\sqrt{2} \sqrt{A_+ - B} - 4 \right), \]

as previously. We substitute this into \( h^{(c)}_-(0) = -\Lambda \) and we observe this is only valid when either

\[ \Lambda < \frac{P}{B} \text{ and } r_- > 0, \text{ or } \Lambda > \frac{P}{B} \text{ and } r_- < 0. \]
This allows us to form the same definition for $r_-$ as in the original analysis, i.e.,

\[
r_- = \begin{cases} 
1 & \text{for } \Lambda < \frac{P}{B}, \\
0 & \text{for } \Lambda = \frac{P}{B}, \\
-1 & \text{for } \Lambda > \frac{P}{B}.
\end{cases}
\] (2.2.93)

Since $f_-$ changes sign at $\psi$ we must have

\[
h_\psi^{(b)} = \frac{\sqrt{2}}{B} \left( -r_- \sqrt{A_- + B \cos \phi - \frac{P}{\sqrt{2}}} \right),
\] (2.2.94)

which also satisfies $h_\psi^{(c)}(\psi) = h_\psi^{(b)}(\psi)$. Continuing, the solutions for $x_-$ are again similar to those previously found:

\[
x_-^{(c)} = 1 + r_- \frac{\sqrt{2}}{B \sqrt{A_- + B}} \left[ (B + A_-) E \left( \frac{\phi}{2}, \sqrt{\frac{2B}{A_- + B}} \right) - A_- F \left( \frac{\phi}{2}, \sqrt{\frac{2B}{A_- + B}} \right) \right],
\] (2.2.95)

\[
x_-^{(b)} = 2X_\psi - r_- \frac{\sqrt{2}}{B \sqrt{A_- + B}} \left[ (B + A_-) E \left( \frac{\phi}{2}, \sqrt{\frac{2B}{A_- + B}} \right) - A_- F \left( \frac{\phi}{2}, \sqrt{\frac{2B}{A_- + B}} \right) \right],
\] (2.2.96)

where $X_\psi = x_-^{(c)}(\psi)$. In summary, the solution set in the case where there is an
inflation point takes the form

\[ h_+ = \frac{r_+ \sqrt{2}}{B} \left( \sqrt{A_+ - B \cos \phi - B} \right) \] (2.2.97)

\[ x_+ = 1 + r_+ \frac{\sqrt{2}}{B \sqrt{A_+ - B}} \left[ (B - A_+) E \left( \frac{\phi}{2}, \sqrt{\frac{-2B}{A_+ - B}} \right) + A_+ F \left( \frac{\phi}{2}, \sqrt{\frac{-2B}{A_+ - B}} \right) \right], \] (2.2.98)

\[ P = r_+ \sqrt{2A_+ - 2B}, \] (2.2.99)

\[ h_{c} = \frac{\sqrt{2}}{B} \left( r_- \sqrt{A_- + B \cos \phi - \frac{P}{\sqrt{2}}} \right), \] (2.2.100)

\[ h_{b} = \frac{\sqrt{2}}{B} \left( -r_- \sqrt{A_- + B \cos \phi - \frac{P}{\sqrt{2}}} \right), \] (2.2.101)

\[ x_{c} = 1 + r_- \frac{\sqrt{2}}{B \sqrt{A_- + B}} \left[ (B + A_-) E \left( \frac{\phi}{2}, \sqrt{\frac{2B}{A_- + B}} \right) - A_- F \left( \frac{\phi}{2}, \sqrt{\frac{2B}{A_- + B}} \right) \right], \] (2.2.102)

\[ x_{b} = 2X_{\psi} - r_- \frac{\sqrt{2}}{B \sqrt{A_- + B}} \left[ (B + A_-) E \left( \frac{\phi}{2}, \sqrt{\frac{2B}{A_- + B}} \right) - A_- F \left( \frac{\phi}{2}, \sqrt{\frac{2B}{A_- + B}} \right) \right]. \] (2.2.103)

The constants \( A_+, A_-, \psi, X_\psi \) are fixed using the constraints

\[ x_+(\alpha_+) = 0, \] (2.2.104)

\[ A_- = A_+ - B \left( 2 + r_+ A \sqrt{2} \left( \sqrt{A_+ - B} \right) \right), \] (2.2.105)

\[ \cos \psi = -\frac{A_-}{B}, \] (2.2.106)

\[ X_\psi = x_{c}^{(c)}(\psi) \] (2.2.107)

with

\[ r_- = \begin{cases} 
1 & \text{for } \Lambda < \frac{P}{B}, \\
0 & \text{for } \Lambda = \frac{P}{B}, \\
-1 & \text{for } \Lambda > \frac{P}{B}.
\end{cases} \] (2.2.108)

From these solutions we infer an inflection point in the lower surface exists if

\[ |x_{c}^{(c)}(\psi)| < 1, \] where \( \psi = \cos^{-1}(-A_-/B) \) (if such a \( \psi \) exists). Therefore we
obtain the following two conditions for the existence of an inflection point on the lower surface:

\[ |A_-| < B, \]  

\[ \left| (B + A_-)E \left( \frac{1}{2} \cos^{-1} \left( -\frac{A_-}{B} \right), \sqrt{\frac{2B}{A_- + B}} \right) \right| \left| -A_- F \left( \frac{1}{2} \cos^{-1} \left( -\frac{A_-}{B} \right), \sqrt{\frac{2B}{A_- + B}} \right) \right| < B \sqrt{\frac{A_- + B}{2}}, \]  

(2.2.110)

where \( A_- \) can be quickly determined by

\[ A_- = A_+ - B \left( 2 + r_+ \Lambda \sqrt{2} \sqrt{A_+ - B} - \frac{BA_+^2}{2} \right). \]  

(2.2.111)

The first of the conditions ensures all terms in the second are real, and the second is the expanded version of \( |x_{\pm}^{(c)}(\psi)| < 1 \). Figure 2.19 gives examples of inflection points, one with \( r_- > 0 \) and the other with \( r_- < 0 \).

![Graphs showing inflection points](image)

(a) Plot of the solutions for an example with \( r_- = 1 \). Here the central separation distance is \( \Lambda = 0.02 \).

(b) Plot of the solutions for an example with \( r_- = -1 \). Here the central separation distance is \( \Lambda = 0.08 \).

Figure 2.19: Examples of both types of inflection points. For these bridges we have \( (B, \alpha_+) = (3, -0.2) \). Again the red crosses denote the inflection points.
2.2.3.2 Asymptotic Solutions

In section 2.2.1 we obtained asymptotic solutions for $x$ and $h$ that are valid in the microgravity limit where $B \to 0$. In the previous section we obtained solutions which are valid for all Bond numbers, however these are expressed in terms of elliptic integrals. By considering the behaviour of this exact solution in the limit of vanishing bond number we can obtain a further microgravity approximation of the shape of a liquid bridge, which should be equivalent to the previous asymptotic solutions. Importantly in this limit, we see $r_-$ (as defined in (2.2.80)) must equal $\frac{1}{2}$.

From this we recall from our earlier discussion that $r_+ = 1$. In this limit we use the conditions $x_+(\alpha_+) = -1$ and equation (2.2.60) to determine $A_\pm$. We write these constants as regular expansions in $B$ so that

$$A_\pm = A_\pm^{(0)} + A_\pm^{(1)} B + A_\pm^{(2)} B^2 + O(B^3). \quad (2.2.112)$$

We proceed by substituting the expansion for $A_+$ into the boundary condition $x_+(\alpha_+) = -1$ and expanding in a Taylor series around $B = 0$ to find

$$x_+(\alpha_+) = \frac{\sin \alpha_+}{\sqrt{2 A_+^{(0)}}} + B 2 \alpha_+ - 4 A_+^{(1)} \sin \alpha_+ + \sin(2 \alpha_+) \quad \frac{8 \sqrt{2} \left( A_+^{(0)} \right)^2}{2}$$

$$- B^2 \frac{12 A_+^{(1)} - 9 \sin \alpha_+ - 12 \left( A_+^{(1)} \right)^2 \sin \alpha_+}{32 \sqrt{2} \left( A_+^{(0)} \right)^2}$$

$$- B^2 \frac{16 A_+^{(0)} A_+^{(2)} \sin \alpha_+ + 6 A_+^{(1)} \sin(2 \alpha_+) - \sin(3 \alpha_+)}{32 \sqrt{2}} + O(B^3). \quad (2.2.113)$$

To now apply $x_+(\alpha_+) = 1$ we must have all non-leading order terms vanishing and the leading order approximation of $x_+(\alpha_+)$ is set to 1. Applying this we

\footnote{This follows from enforcing order 1 (or less) central separation distance.}
find

\[ A^{(0)}_+ = \frac{1}{2} \sin^2 \alpha_+ , \]  
\[ A^{(1)}_+ = \frac{1}{2} (\cos \alpha_+ + \alpha_+ + \csc \alpha_+) , \]  
\[ A^{(2)}_+ = -\frac{1}{8} \left[ -1 + 9 \csc^2 \alpha_+ - 3 \alpha_+ (\alpha_+ + \sin 2\alpha_+) \csc^4 \alpha_+ \right] . \]  
\[ (2.2.114) \]
\[ (2.2.115) \]
\[ (2.2.116) \]

These results are substituted into (2.2.60) to obtain

\[ A^{(0)}_- = \frac{1}{2} \sin^2 \alpha_+ , \]  
\[ A^{(1)}_- = \frac{1}{2} (2\Lambda + \alpha_+ \csc \alpha_+ + \cos \alpha_+ - 4) , \]  
\[ A^{(2)}_- = \frac{\Lambda}{2} \left( \Lambda + \cot \alpha_+ - 2 \csc \alpha_+ + \alpha_+ \csc^2 \alpha_+ \right) \]
\[ + \frac{1}{8} \left( -1 + 9 \csc \alpha_+ - 3 \alpha_+^2 \csc^4 \alpha_+ - 6 \alpha_+ \cos \alpha_+ \csc^3 \alpha_+ \right) . \]  
\[ (2.2.117) \]
\[ (2.2.118) \]
\[ (2.2.119) \]

Then using these results in (2.2.72) - (2.2.80), we find asymptotic approximations for \( h_\pm, x_\pm \) and \( P \), namely

\[ P = P^{(0)} + B P^{(1)} + B^2 P^{(2)} + O(B^3) , \]  
\[ h_\pm = h^{(0)}_\pm + h^{(1)}_\pm B + h^{(2)}_\pm B^2 + O(B^3) , \]  
\[ x_\pm = x^{(0)}_\pm + x^{(1)}_\pm B + x^{(2)}_\pm B^2 + O(B^3) , \]  
\[ (2.2.120) \]
\[ (2.2.121) \]
\[ (2.2.122) \]

where

\[ P^{(0)} = -\sin \alpha_+ , \]  
\[ P^{(1)} = -\frac{1}{2} \left( \cot \alpha_+ - 2 \csc \alpha_+ + \alpha_+ \csc^2 \alpha_+ \right) , \]  
\[ P^{(2)} = \frac{1}{8} \left[ (5 - 4 \cos \alpha_+) \csc^3 \alpha_+ - 9 \csc^2 \alpha_+ + 8 \alpha_+ \cot \alpha_+ \csc^3 \alpha_+ - 4 \csc^4 \alpha_+ + 4 \alpha_+^2 \csc^5 \alpha_+ \right] . \]  
\[ (2.2.123) \]
\[ (2.2.124) \]
\[ (2.2.125) \]
\[ h^{(0)}_+ = \frac{\cos \phi - 1}{\sin \alpha_{+}}, \quad (2.2.126) \]
\[ h^{(1)}_+ = \frac{2 P^{(1)} \sin \alpha_{+} + \cos \phi - 1}{2 \sin^3 \alpha_{+}} (\cos \phi - 1), \quad (2.2.127) \]
\[ h^{(0)}_- = \frac{1 - \cos \phi}{\sin \alpha_{+}} - \Lambda, \quad (2.2.128) \]
\[ h^{(1)}_- = \frac{2 \sin \alpha_{+} (P^{(1)} - \Lambda) + 1 - \cos \phi}{2 \sin^3 \alpha_{+}} (1 - \cos \phi), \quad (2.2.129) \]
\[ x^{(0)}_+ = 1 - \frac{\sin \phi}{\sin \alpha_{+}}, \quad (2.2.130) \]
\[ x^{(1)}_+ = \frac{2 P^{(1)} \sin \alpha_{+} \sin \phi + \cos \phi \sin \phi - 2 \sin \phi + \phi}{2 \sin^3 \alpha_{+}}, \quad (2.2.131) \]
\[ x^{(0)}_- = 1 - \frac{\sin \phi}{\sin \alpha_{+}}, \quad (2.2.132) \]
\[ x^{(1)}_- = -\frac{2 \sin \alpha_{+} \sin \phi (\Lambda - P^{(1)}) + \cos \phi \sin \phi - 2 \sin \phi + \phi}{2 \sin^3 \alpha_{+}}, \quad (2.2.133) \]

and
\[ h^{(2)}_+ = \sqrt{2A^{(0)}_+} \left[ \frac{(A^{(1)}_+ - \cos \phi)A^{(2)}_+}{4 \left(A^{(0)}_+\right)^2} - \frac{(A^{(1)}_+ - \cos \phi)^3}{16 \left(A^{(0)}_+\right)^3} \right], \quad (2.2.134) \]
\[ h^{(2)}_- = \sqrt{2A^{(0)}_-} \left[ -\frac{(A^{(1)}_- - \cos \phi)A^{(2)}_-}{4 \left(A^{(0)}_-\right)^2} + \frac{(A^{(1)}_- - \cos \phi)^3}{16 \left(A^{(0)}_-\right)^3} \right], \quad (2.2.135) \]
\[ x^{(2)}_+ = -\frac{12A^{(1)}_+ \phi}{32\sqrt{2} \left(A^{(0)}_+\right)^{\frac{3}{2}}} - 9 \sin \phi - 12 \left(A^{(1)}_+\right)^2 \sin \phi \]
\[ - \frac{16A^{(0)}_+ A^{(2)}_+ \sin \phi + 6A^{(1)}_+ \sin 2\phi - \sin 3\phi}{32\sqrt{2}}, \quad (2.2.136) \]
\[ x^{(2)}_- = -\frac{-12A^{(1)}_- \phi}{32\sqrt{2} \left(A^{(0)}_-\right)^{\frac{3}{2}}} - 9 \sin \phi - 12 \left(A^{(1)}_-\right)^2 \sin \phi \]
\[ - \frac{16A^{(0)}_- A^{(2)}_- \sin \phi + 6A^{(1)}_- \sin 2\phi - \sin 3\phi}{32\sqrt{2}}, \quad (2.2.137) \]

Observe the agreement between (2.2.123) - (2.2.133) and the asymptotic solution found earlier in (2.2.20) - (2.2.24) and (2.2.31) - (2.2.36). This shows the expected agreement between the two approaches.
2.2.4 Numerical solution

To verify the asymptotic and exact solutions, we can use the formulation (2.2.4) or (2.2.8) and (2.2.9). Using the former we note that we can obtain numerical solutions if we know the upper contact angle and one of the following three; lower contact angle, volume, distance between the two interfaces half way between the walls.

If we know the lower contact angle then we can numerically solve the system directly without having to introduce further parameters and constraints, since here we have two second order ordinary differential equations and one arbitrary constant \( P \) with five boundary conditions. If we know the volume of the bridge (here we really mean half volume, as in section 2.2.2) then we can find the lower contact angle from equation (2.2.40) and proceed in the same way. If instead we know the distance between the two interfaces at \( x = 1 \), then we can use this as a sixth boundary condition and set volume as a second free constant that will fix the lower contact angle. In practice it may be experimentally easiest to measure the centre distance between the two interfaces so we consider this case for the remainder of this work.

2.2.5 Comparison of results

Using these numerical solutions we can now examine the validity of the asymptotic series results. Figure 2.22 shows the asymptotic series solution plotted along with the numerical solution for \( \theta_+ = \pi/4 \) and \( B = 0.5 \) in the critical case when \( \Lambda = 0 \). In table 2.1 we see that the asymptotic series solution is in very good agreement with the numerical solution.

In figure 2.23 we see the two values of \( P \) compared to \( B \) and in figure 2.24 we see that the contact angle does not influence the error of the asymptotic solution.

In figure 2.24 we see \( P \) changes sign at \( \theta_+ = \pi/2 \). This is physically expected since \( P \) is the pressure difference across the upper interface. From the Young–Laplace this pressure difference is equated to the curvature of the interface,
which is positive for concave up interfaces and negative for concave down interfaces. Importantly note that in this micro-gravity approximation there are no inflection points in the upper interface so the upper surface is concave up for $0 \leq \theta_+ < \frac{\pi}{2}$, and concave down for $\frac{\pi}{2} < \theta_+ \leq \pi$. This agrees with our estimate of $P$.

### 2.2.6 Approximation of maximal central separation distance

In chapters 3 and 4, we shall study the stability region of an infinite horizontal liquid bridge. To do this we need to understand the existence region, especially in the micro-gravity limit. Later, the phase space of $(B, V, \Lambda)$ shall be used for the study. Here we begin by showing (in a few examples) that the liquid bridge can be equivalently defined using different conditions. Using other of these alternate conditions, we derive an expression for the maximal central separation distance for a liquid bridge of fixed volume.
To determine the maximum central separation distance we transform the system. Instead of using Neumann conditions at the walls of the bridge we enforce Dirichlet boundary conditions. This allows us to force the upper and lower interfaces to meet at the walls. For simplicity we use this point as the
Section 2.2

<table>
<thead>
<tr>
<th>( B )</th>
<th>Volume given by (2.2.48)</th>
<th>Numerical volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1703</td>
<td>0.1703</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1670</td>
<td>0.1672</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1638</td>
<td>0.1642</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1605</td>
<td>0.1612</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1572</td>
<td>0.1584</td>
</tr>
<tr>
<td>1</td>
<td>0.1408</td>
<td>0.1454</td>
</tr>
<tr>
<td>2</td>
<td>0.1081</td>
<td>0.1242</td>
</tr>
</tbody>
</table>

Table 2.1: Table comparing the asymptotic and numerical limiting volumes of the liquid bridge for \( \theta_+ = \pi/3 \) and Bond numbers up to 1.

This then allows us to examine the maximum value of \( \Lambda \) for a given volume. Note, finally, at higher Bond numbers this method could over-estimate the maximum value of \( \Lambda \) because here the two interfaces may contact away from the walls. The Dirichlet system takes the form

\[
Bh_+ - P = \frac{d}{dx} \left( \frac{1}{\sqrt{\Delta_+}} \frac{dh_+}{dx} \right), \tag{2.2.138}
\]

\[
Bh_- - P = -\frac{d}{dx} \left( \frac{1}{\sqrt{\Delta_-}} \frac{dh_-}{dx} \right), \tag{2.2.139}
\]

\[
h_+ = H_+ \text{ at } x = \pm 1, \tag{2.2.140}
\]

\[
h_- = H_- \text{ at } x = \pm 1, \tag{2.2.141}
\]

\[
V = \int_{-1}^{1} (h_+ - h_-) \, dx, \tag{2.2.142}
\]

where

\[
\Delta_\pm = 1 + \left( \frac{dh_\pm}{dx} \right)^2. \tag{2.2.143}
\]

This system is equivalent to (2.2.4)-(2.2.7) which we previously studied. Figure 2.25 and table 2.2 provide evidence these two systems are equivalent. Indeed, when the Bond number, upper contact angle and centre separation distance is
chosen the values of $H_{\pm}$ and $V$ can be calculated. If we then solve the Young–Laplace system with these values as known contact lines’ positions and volume, the values of $\theta_+$ and $\Lambda$ can be recovered using

\[
h_+^{'}(1) = \cot \theta_+, \tag{2.2.144}
\]
\[
\Lambda = h_+(0) - h_-(0). \tag{2.2.145}
\]

This method can be used to show that the solutions found using any combination of these conditions can be equivalent. However, note that when $H_+ \neq H_-$, defining the liquid bridge by its contact lines’ positions does not yield a unique solution. This shall be seen later in section 3.4.

It follows a liquid bridge can be defined by the Bond number, $B$, and any two of the four following parameters: the contact lines positions, $H_{\pm}$, the contact angles, $\theta_{\pm}$, the volume, $V$, or the central separation distance $\Lambda$. After defining the configuration using the Bond number and two parameters from the list, the other parameters can be found as part of the solution. Indeed, geometric conditions could be calculated directly by $H_{\pm} = h_{\pm}(1)$, $\cot \theta_{\pm} = \mp h_{\pm}^{'}(1)$ and
Figure 2.24: The two values of $P$ plotted over a range contact angles $\theta_+$, for $B = 0.1$, $\Lambda = 0.1$. Blue dots are numerical and green dashes are asymptotic.

$$\Lambda = h_+(0) - h_-(0)$$
while the volume could be calculated from

$$V = \int_{-1}^{1} (h_+ - h_-) \, dx. \tag{2.2.146}$$

We shall later see, in section 3.4 that defining the bridge in terms of the contact lines' positions gives multiple solutions for the system, except in the case of maximal $\Lambda$ when only one solution exists.

Figure 2.25: Graphs showing the shape of the bridge for different setups. Any two of the parameters in table 2.2 could have been chosen with the Bond number to result in the graph. The setups are given in table 2.2.
Using Dirichlet boundary conditions, i.e., known contact line position, and a volume condition we can find an expression for the maximum central separation distance, $\Lambda_{\text{max}}$, for a given volume. To do this we set $H_+$ and $H_-$ to zero in (3.1.24)–(3.1.28) and solve for $h_+$ and $h_-$, and compute $\Lambda = h_+(0) - h_-(0)$ to find $\Lambda_{\text{max}}$. Further, we find in the limit of vanishing Bond number that the maximum central separation distance can be given in parametric form by

$$\Lambda_{\text{max}} \to 2 \left( t - \sqrt{t^2 - 1} \right), \quad \text{as } B \to 0,$$

where $t$ is found by solving

$$V \to 2t^2 \csc^{-1} t - 2\sqrt{t^2 - 1}, \quad \text{as } B \to 0,$$

and thus we can give $\Lambda_{\text{max}}$ implicitly as

$$V \to \frac{(\Lambda_{\text{max}}^2 + 4)^2}{8\Lambda_{\text{max}}^2} \sin^{-1} \left( \frac{4\Lambda_{\text{max}}^2}{4 + \Lambda_{\text{max}}} \right) + \frac{\Lambda_{\text{max}}^2 - 4}{2\Lambda_{\text{max}}}, \quad \text{as } B \to 0.$$

Figure 2.26a shows both the numerical maximum central separation distance and the maximum central separation distance predicted by (2.2.149). We see this result predicts the maximum central separation distance with good accuracy. To compute the maximum error for a given Bond number, we find the errors of all the points for the range of volumes used in figure 2.26a. The maximum of these values is then used. Figure 2.26b shows that (2.2.149) offers a very good approximation of the maximal aspect ratio, for Bond numbers as large as unity. We shall later see, in chapter 4, that the vanishing Bond number approximation gives a good estimate for the maximal central separation distance even when the Bond number is order one. This is especially true for lower

<table>
<thead>
<tr>
<th>Graph</th>
<th>B</th>
<th>$(H_+, H_-)$</th>
<th>$(\theta_+, \theta_-)$</th>
<th>$\Lambda$</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2.25a)</td>
<td>0.1</td>
<td>(0.1611, -0.1771)</td>
<td>(1.2486, 1.2618)</td>
<td>0.02</td>
<td>0.25</td>
</tr>
<tr>
<td>(2.25b)</td>
<td>0.25</td>
<td>(0.2567, -0.3045)</td>
<td>(1.0459, 1.1852)</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>(2.25c)</td>
<td>2</td>
<td>(0.0434, -0.0577)</td>
<td>(1.4700, -1.3700)</td>
<td>0.18</td>
<td>0.3</td>
</tr>
</tbody>
</table>
volumes since the maximal central separation distance remains nearly constant for a larger range of Bond numbers. This discussion shall be reintroduced and emphasised in chapter 4.

Figure 2.26: Comparison of the analytic result (2.2.149) with the numerically found result.

2.2.7 Quasi-static evaporation

Following the same framework discussed in section 2.2.7, we can used the solutions obtained in this section to consider the shape of the liquid bridge as the liquid is slowly removed, e.g., by evaporation. In particular, we assume that the time scale for the evaporation is much slower than that for the free surface equilibration so that a quasi-static approximation is appropriate. Consider a liquid bridge held between two identical substrates with hysteresis interval $(\theta_r, \theta_a) = (\pi/4, \pi/2)$, as defined in section 1.1.3. While this is a large hysteresis interval, it is not unreasonable for a rough surface to have an interval this large.
We take the liquid bridge to have lower contact angle close to $\theta_a = \pi/2$. Both contact angles must be inside the hysteresis interval and satisfying (2.2.40). Therefore the triple points are ‘pinned’. When the volume is reduced we find both contact angles decrease (see figure 2.27), while still satisfying (2.2.40). As we continue to remove volume both contact angles will continue to decrease until $\theta_+ = \theta_r$. At this point if the volume continues to be reduced the upper triple point becomes ‘unpinned’ and the configuration is no longer static. The upper triple point will travel down the substrate until either the volume stops decreasing, or the critical bridge is reached. If volume is removed from the critical bridge, the bridge can no longer be supported.

If we now allow liquid vapour, present in the atmosphere, to (slowly) condense into the bridge then we expect both triple points to remain ‘pinned’ as both contact angles increase, until $\theta_- = \theta_a$. At this point further addition of liquid will cause the lower triple point to move downward. The net effect is that initially decreasing and then increasing, the liquid volume would cause the liquid bridge to move down the substrate; the initial bridge location can never be recovered in this way.

As we then reduce the volume of the bridge, both contact angles will decrease. We recall $\theta_+ < \theta_-$. This trend will continue until the upper contact angle reaches the receding value. At this point the upper interface will start to recede while the lower interface will remain pinned.

If we continue removing liquid, the bridge will reach a critical situation with $b = 0$, where any further evaporation will cause the bridge to rupture. This critical volume was discussed in section 2.2.2. The black dot-dash line in figure 2.27 shows this critical case. However if, instead, the volume stops decreasing before this critical case is reached the upper contact angle will return to a value inside the hysteresis interval.

We further comment that this process is fundamentally hysteretic. If we consider figure (2.21), a critical bridge, and allow liquid vapour, present in the atmosphere, to (slowly) condense into the bridge then we expect the upper contact angle to remain fixed while the lower increases until it reaches the advancing
angle. At this point, further addition of liquid will cause the lower contact line to move downwards. The net effect is that initially decreasing and then increasing the liquid volume would cause the bridge to move down the wall; the initial bridge location can never be recovered in this way.

Figure 2.27: Evolution of a liquid bridge from a starting position (red) as volume is removed, until the critical (black dot-dash) bridge is reached. The intermediate (blue dash) state shows where the upper contact angle has reached the receding value $\theta_r$.

### 2.3 Cylindrical liquid membrane

The study of the cylindrical liquid membrane is motivated by approximating one of the ‘diamonds’ in the stent as a cylinder. This section, on the whole, follows a similar route to that of the previous section on infinite horizontal liquid bridges. Due to the similarity of the problem, most of the workings are omitted leaving only the important results and highlighting how they are changed from
their counterparts in the previous section.

Unlike the previous sections, due to the cylindrical form of the Young–Laplace equation, a direct integration approach is not possible to produce exact solutions. This highlights the importance of the asymptotic approaches used in the previous sections. Here the asymptotic results shall be compared with numerical solutions only.

In this section we consider a liquid membrane enclosed by a vertical cylinder with radius $R$. This radius shall be used as the characteristic length scale, so the wall of the cylinder is at $r = 1$ for dimensionless $r$. In dimensionless variables the liquid region is given by

$$(r, \theta, z) \in [0, 1] \times [0, 2\pi) \times [h_-, h_+],$$  \hspace{1cm} (2.3.1)

where $h_{\pm}$ are the dimensionless heights of the upper and lower interface respectively. In the previous section the full geometry is found by extending the cross section infinitely into and out of the page, while in the section we rotate the same cross section around its centre.

Using a similar parametrisation to the previous sections, we are able to write the Young–Laplace equation in the following form:

$$\frac{dr_{\pm}}{d\phi} = \frac{r_{\pm} \cos \phi}{B h_{\pm} r_{\pm} - r_{\pm} P - \sin \phi},$$  \hspace{1cm} (2.3.2)

$$\frac{dh_{\pm}}{d\phi} = \pm \frac{r_{\pm} \sin \phi}{B h_{\pm} r_{\pm} - r_{\pm} P - \sin \phi},$$  \hspace{1cm} (2.3.3)

where $r_{\pm}, h_{\pm}$ are functions of the inclination $\phi$. Applying this transformation to the boundary conditions we obtain

$$r_{\pm}(0) = 0,$$  \hspace{1cm} (2.3.4)

$$h_{+}(0) = 0,$$  \hspace{1cm} (2.3.5)

$$r_{\pm}(\alpha_{\pm}) = 1,$$  \hspace{1cm} (2.3.6)

where $\alpha_{\pm}$ is the inclination at the contact point, which we can relate $\alpha_{\pm}$ to $\theta_{\pm}$ with the relation $\alpha_{\pm} = \theta_{\pm} - \frac{\pi}{2}$. 
2.3.1 Asymptotic solution

By using analogous assumptions to the section on horizontal liquid bridges we are able to find the following first order solutions for the shape of the liquid membrane

\[
\begin{align*}
    r_\pm^{(0)} &= \frac{\sin \phi}{\sin \alpha_+}, \\
    h_+^{(0)} &= \frac{1 - \cos \phi}{\sin \alpha_+}, \\
    h_-^{(0)} &= \frac{\cos \phi - 1}{\sin \alpha_+} - \Lambda, \\
    p^{(0)} &= -2 \sin \alpha_+.
\end{align*}
\] (2.3.7)

Then at the next order we find

\[
\begin{align*}
    r_+^{(1)} &= \frac{1}{4 \sin^3 \alpha} \left( \frac{1 + 3 \sin \alpha}{3 \sin \phi} - \frac{p^{(1)} \sin \alpha \cos 2\phi + \frac{1}{3} \cos 3\phi - \cos 2\phi + \cos \phi}{\sin \phi} \right), \\
    h_+^{(1)} &= \frac{1}{A} \left( 8(\cos \alpha + 1) \ln(\csc \phi - \cot \phi) - 8(\cos \alpha + 1) \ln(\sin \phi) \right) \\
    &+ \frac{1}{A} \left( (-12 - 4 \cos(2\alpha) - 8 \cos(\alpha)) \cos(\phi) + 4(1 + \cos \alpha) \cos(2\phi) + K \right), \\
    p^{(1)} &= \frac{1}{\sin^3(\alpha)} \left( \frac{2}{3} \cos^3 \alpha - \cos^2 \alpha + \frac{1}{3} \right), \\
    r_-^{(1)} &= \frac{1}{\sin^3 \alpha \sin \phi} \left( \frac{1}{3} \cos^3 \phi - \frac{1}{2} (1 + \Lambda \sin \alpha) \cos^2 \phi + \frac{p^{(1)} \sin^2 \alpha \cos \phi}{2} + \frac{1}{6} + \frac{\Lambda \sin \alpha}{2} + p^{(1)} \sin^2 \alpha \right), \\
    h_-^{(1)} &= \frac{1}{\sin^3 \phi} \left\{ -\cos^2 \phi + \frac{1 + \Lambda \sin \alpha}{2} \left( \cos \phi + \ln \left( \cot \frac{\phi}{2} \right) + \phi + \cot \phi \right) \right. \\
    &+ \left. \left. p^{(1)} \sin^2 \alpha \left( \cos \phi + \csc \phi - \ln(\sin \phi) \right) \right. \right. \\
    &+ \left. \left. \frac{1}{6} (\cos(2\phi) - 3) \csc \phi + \frac{\cos 2\phi}{12} + \frac{\ln(\sin \phi)}{3} \right. \right. \\
    &+ \left. \left. \left( \frac{1}{6} + \frac{\Lambda \sin \alpha}{2} + p^{(1)} \sin^2 \alpha \right) \left( \ln \left( \tan \frac{\phi}{2} \right) - \cot \phi \right) \right. \right. \\
    &- \left( \frac{1}{12} + p^{(1)} \sin^2 \alpha \right) \right\},
\end{align*}
\] (2.3.11, 2.3.12, 2.3.13, 2.3.14, 2.3.15)
\[
\Delta = -\frac{1}{\cos \alpha \sin^3 \alpha} \left( \frac{1}{3} \cos^3 \alpha - \frac{1}{2} (1 + \Lambda \sin \alpha) \cos^2 \alpha \right.
\]
\[
+ P^{(1)}(1 - \cos \alpha) \sin^2 \alpha + \frac{1}{6} + \frac{\Lambda \sin \alpha}{2} \right),
\]
where \( A = 3 \sin(4\alpha) + 6 \sin(3\alpha) - 6 \sin(2\alpha) - 18 \sin \alpha \) and \( K = 8 \cos \alpha \ln(2) + 8 \ln(2) + 4 \cos 2\alpha + 4 \cos \alpha + 8 \). Again, at leading order, we recover that the shape of the two interfaces are sections of circles. This is expected because at leading order the Young–Laplace equation states that both the curvatures are constant.

These asymptotic solutions shall be tested against a numerical solution. In the next section we give details of a numerical difficulty arising from the polar formulation, and give details on overcoming this difficulty.

### 2.3.2 Numerical solution

Returning to the equation
\[
\pm \left( \frac{h_\pm''(r)}{(1 + h_\pm'(r)^2)^{\frac{3}{2}}} + \frac{h_\pm'(r)}{r \sqrt{1 + h_\pm'(r)^2}} \right) = -P + Bh_\pm(r)
\]
we see that the boundary conditions applied at \( r = 0 \) will cause a singularity in our numerical solution. So with this in mind we look for approximate solutions valid for \( r \in [0, \delta) \) where \( \delta \ll 1 \) to start off our numerical solution. To do this we consider a Taylor series expansion of \( h_\pm(r) \) around \( r = 0 \).

Since \( h_\pm(r) \) is even and has the property \( h_\pm(0) = 0 \) the Taylor series expansion around \( r = 0 \) must be of the form
\[
h_\pm(r) = \frac{d^2h_\pm}{dr^2} \bigg|_{r=0} r^2 + O(r^4)
\]
(2.3.18)

Note that \( \frac{d^2h}{dr^2}|_{r=0} \) is a constant which we label \( H_\pm \). Applying this to (2.3.17) gives
\[
H_\pm(H_\pm^2 r^2 + 2) = \pm \frac{1}{2} (H_\pm^2 r^2 + 1)^{\frac{3}{2}} (B H_\pm r^2 - 2P)
\]
(2.3.19)

We can expand \((1 + H_\pm^2 r^2)^{\frac{3}{2}}\) around \( r = 0 \) as
\[
(1 + H_\pm^2 r^2)^{\frac{3}{2}} = 1 + \frac{3H_\pm^2 r^2}{2} + O(r^4)
\]
(2.3.20)
So we have

\[ H_{\pm}(H_{\pm}^2r^2 + 2) = \pm \frac{1}{2} \left( 1 + \frac{3}{2} r^2 H_{\pm}^2 \right) \left( B H_{\pm} r^2 - 2P \right). \]  

(2.3.21)

If we now look only at the leading order terms, i.e., terms of order \( r^0 \), we find

\[ 2H_{\pm} = \mp P \Rightarrow H_{\pm} = \mp \frac{P}{2} \]  

(2.3.22)

Therefore we take \( h_+(r) = -\frac{Pr^2}{4} \) and \( h_-(r) = -\Lambda + \frac{Pr^2}{4} \) as our solution on \([0, \delta]\). Using this we can construct boundary conditions for \( h_\pm(\delta) \) namely

\[ h_+(\delta) = -\frac{P\delta^2}{4}, \quad h_-(\delta) = -\Lambda + \frac{P\delta^2}{4} \quad \text{and} \quad \left. \frac{dh_\pm}{dr} \right|_{r=\delta} = \mp \frac{P\delta}{2} \]  

(2.3.23)

Using these boundary conditions together with \( \left. \frac{dh_\pm}{dr} \right|_{r=1} = \pm \cot \theta \) we can numerically solve (2.3.17) in the region \([\delta, 1]\) without encountering a singularity. In this method we need to use an initial estimate for the value of \( P \), so here we use the value given from the asymptotic approximation. If the returned value for \( P \) differs much from this value we then iterate the solution process with this value as the initial guess. This is a similar method to the shooting method described in section 1.3.1.

2.3.3 Comparison of results

Here we compare the two methods by examining the results they yield for \( \theta_- \) and \( P \). In table 2.3 we give a list of these values for various Bond numbers while keeping the upper contact angle and central separation distance constant.

As we can see the results all match to the given precision for \( B = 0.01 \). In figure 2.28 we see the two values of \( P \) compared to \( B \) and in figure 2.29 we see that the contact angle does not influence the error of the asymptotic solution for \( P \).

2.3.4 Maximal central separation distance

In this section we examine the critical configuration of maximal central separation distance in the micro-gravity limit. This critical configuration is studied as
Table 2.3: Table listing the numerical and asymptotic values of $P$ and $\theta$ for various Bond numbers with $\theta_+ = \pi/3$ and $\Lambda = 0$.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$P_{\text{asymptotic}}$</th>
<th>$P_{\text{numerical}}$</th>
<th>$\theta_{\text{asymptotic}}$</th>
<th>$\theta_{\text{numerical}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>-0.9987</td>
<td>-0.9987</td>
<td>1.049</td>
<td>1.049</td>
</tr>
<tr>
<td>0.1</td>
<td>-0.9870</td>
<td>-0.9870</td>
<td>1.062</td>
<td>1.062</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.9738</td>
<td>-0.9743</td>
<td>1.077</td>
<td>1.072</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.9346</td>
<td>-0.9375</td>
<td>1.123</td>
<td>1.116</td>
</tr>
<tr>
<td>1</td>
<td>-0.8692</td>
<td>-0.8804</td>
<td>1.198</td>
<td>1.175</td>
</tr>
<tr>
<td>2</td>
<td>-0.7385</td>
<td>-0.7798</td>
<td>1.349</td>
<td>1.190</td>
</tr>
</tbody>
</table>

it provides a bounding curve for the region of existence in $(B,V,\Lambda)$ parameter space. More details on finding this curve away from the microgravity limit will be seen in chapter 4.

As described in section 2.2.6 we can equivalently use Dirichlet boundary conditions to find the shape of the liquid membrane. In similarity to the two dimensional case, using Dirichlet boundary conditions does not provide a unique solution except in the case of maximal central separation distance. By setting $h_{\pm}(1) = 0$ we can determine the maximal value of $\Lambda = h_+(0) - h_-(0)$ for a given volume. We find, in the limit of vanishing Bond number, the maximum central separation distance can be given by

$$\Lambda_{\text{max}} \to 2(t - \sqrt{t^2 - 1}), \text{ as } B \to 0,$$

(2.3.24) where $t$ is found by solving

$$V = \frac{2}{3} \left[ t^3 - (t^2 - 1)\frac{2}{3} \right] - \sqrt{t^2 - 1}.$$

(2.3.25)

Thus we find $\Lambda_{\text{max}}$ is a solution to,

$$V = \frac{\Lambda_{\text{max}}}{48} (\Lambda_{\text{max}}^2 + 12),$$

(2.3.26) in the microgravity limit. In chapter 4 we shall see this limit provides a good
Section 2.4

0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
-1
-0.98
-0.96
-0.94
-0.92
-0.9
-0.88
-0.86
B
P

Figure 2.28: The two values of $P$ plotted over a range Bond numbers for $\theta_+ = \pi/3$, $\Lambda = 0.1$. Blue dots are numerical and green dashes are asymptotic.

Figure 2.28: The two values of $P$ plotted over a range Bond numbers for $\theta_+ = \pi/3$, $\Lambda = 0.1$. Blue dots are numerical and green dashes are asymptotic.

approximation even for order one Bond numbers. And further, this limit gives a better approximation for lower volumes.

2.4 Summary

This chapter has investigated solutions of the Young–Laplace equation for three different configurations. The Young–Laplace equation in all three cases was written in a parametric form, based on the inclination of the surface. While the configurations studied here share similar mathematical expressions they come from physically different backgrounds. Further, for the two two-dimensional configurations studied here, exact solutions have been found and have been verified, using the asymptotic and numerical results.

The solutions were found using three different mathematical techniques. In all three problems we were able to use an asymptotic series solution to find accurate solutions in the microgravity limit. Also, in all three problems numerical solutions were obtained using a shooting method, as described in section 1.3.1. Finally, in the two-dimensional problems we have found exact solutions using
Figure 2.29: The two values of $P$ plotted over a range upper contact angles $B = 0.1, \Lambda = 0.1$. Blue dots are numerical and green dashes are asymptotic.

elliptic integrals. These exact solutions were shown to match the numerical solutions and the behaviours of the exact solutions at small Bond numbers were found to be identical to the asymptotic solutions.

We were able to find an asymptotic approximation for the maximal central separation for the two-dimensional horizontal liquid bridge, and further showed this asymptotic approximation gives low errors for Bond numbers even as large as unity. However past unity the errors become noticeable. In all three problems, we further explored the quasi-static evolution of the shape as volume was removed and later reintroduced, and found that this was fundamentally hysteretic. The contact lines remain pinned as the volume is decreased until the upper contact angle reaches the receding contact angle. Further decreasing the volume will result in the upper contact line receding until the minimum volume is achieved. In the case of a liquid drop the minimum volume is simply zero. After decreasing the volume to a point where the upper contact line has moved, by reintroducing the removed liquid the shape will be identical however, it shall be displaced in the effective direction of the gravitational field.

In all three cases, some of the solutions we compute will not always be
physically realistic. One example of this is that when we increase the central separation distance beyond the maximum found in sections 2.2.6 and 2.3.4, the lower interface goes above the upper interface for sections of the domain. It follows that when finding the region where bridges can exist, further constraints beyond those considered in this chapter must be considered. Given the previous example, one such constraint is that the upper surface is above the lower surface throughout the domain. However, even with this constraint it does not guarantee that all the solutions can be physically viewed in experiments, because in experiments only stable configurations can be observed. The following two chapters study the stability of capillary surfaces. Chapter 3 develops methods for determining the stability of a given configuration, while chapter 4 uses these methods to find stable regions in the Bond number, volume, central separation distance phase space, for the two dimensional horizontal liquid bridge and the cylindrical membrane studied in the this chapter.
Chapter 3

Stability analysis I: Methods

Having found the equilibrium shape of a liquid membrane/bridge in the previous chapter, we are now interested in the stability of these membranes/bridges. There are several methods that can be used to examine the stability of capillary surfaces. In this thesis, we focus our stability analysis on two capillary surfaces, namely the infinite horizontal liquid bridge (as discussed in section 2.2) and the cylindrical liquid membrane (as discussed in section 2.3). This chapter contains a detailed description of the mathematical methods use to analyse the capillary surfaces; the results of this analysis are given in chapter 4.

One of the most commonly used methods for analysing a capillary surface is linear stability analysis. In this thesis we use linear stability analysis but also consider energy based methods and bifurcation theory methods. The energy method seeks to minimise the energy functional and in so doing we obtain a sufficient condition for the capillary surface to the stable. The principle advantage of the energy based method is that it provides a sufficient condition for stability rather than the necessary condition that the linear stability analysis provides. The bifurcation method provides a computationally inexpensive
method for finding the neutral stable systems.

The applications of these methods to a vertical liquid bridge are well documented, for example by Slobozhanin and Perales [1993]. Indeed, one famous result in this field is the Plateau–Rayleigh stability limit, first observed by Plateau [1873] and then analytically corroborated by Rayleigh [1878]. The analysis of Rayleigh [1878] began by considering small perturbations to the interface of a liquid jet. Then, by applying Lagrange’s method to the kinetic and potential energies, the dispersion relation is shown to have the form

\[ \omega^2 = \frac{\sigma}{\rho g} k R_0 \frac{I_1(kR_0)}{I_0(kR_0)} \left(1 - k^2 R_0^2\right), \]  

(3.0.1)

where \(\sigma\) is the surface tension, \(\rho\) is the density, \(\omega\) is the growth rate, \(k\) is the wave number, \(R_0\) is the initial radius of the jet, and \(I_n\) the modified Bessel function of the first type of order \(n\). The result implies that if \(0 < kR_0 < 1\) then \(\omega^2 > 0\) and the jet is unstable to disturbances whose wavelength exceed the circumference of the cylinder.

Importantly, the Plateau–Rayleigh stability limit is independent of the velocity of the falling liquid; as a result, this result can be applied to a vertical liquid bridge. In the case of a vertical liquid bridge, we see that if the length of the bridge is greater than the circumference of the supporting rods, i.e., \(L > 2\pi R\), the bridge is unstable. In dimensionless terms this result states that the slenderness of the liquid bridge must be less than \(\pi\) for a stable bridge to exist, where the slenderness is defined as the ratio of the length to the diameter of the liquid bridge.

This chapter begins in section 3.1 with an analysis of the energy functionals of the capillary surfaces of interest. We find that the equilibrium configurations studied in the previous chapter are extrema of the energy functional associated with the Navier–Stokes equation. This, on its own, does not guarantee stability. The stability of the equilibrium configurations is then analysed by considering the second variation of the functional. Following the results from section 1.2.4, the equilibrium configurations are stable if the second variation is
strongly positive, i.e.,
\[ \delta^2 E[f^*, g^*] \geq k_1 \|f^*\|^2 + k_2 \|g^*\|^2, \]  
(3.0.2)
for positive constants \(k_1\) and \(k_2\). This condition shall be reduced to an eigenvalue problem for eigenvalues \(\lambda\) and \(\mu\). We shall see that the stability or instability of the configuration can be determined from the sign of one of the eigenvalues, \(\lambda\).

The stability of the equilibrium configurations is then analysed using linear stability. In section 1.2.3.3 the Navier–Stokes equations were linearised around the static equilibrium configuration, yielding equations (1.2.40)-(1.2.44). We shall analyse these equations in the specific case of a horizontal infinite liquid bridge and a cylindrical liquid membrane. We find that linear perturbations are dissipated by viscous effects but will not dissipate without viscosity. It follows that if these equilibrium surfaces are constructed with inviscid liquid they can be, at best, marginally stable. The viscous configurations are then considered and it is shown that the marginally stable systems satisfy the same equations as the marginally stable systems predicted by the energy functional analysis.

While the fact that the two methods predict the same marginal stability curve might seem a trivial result, it is in fact rather more surprising. The concept of a system being linearly stable is a rather weak result, since it only implies that the system is stable to small perturbations. The non-linear stability, which is determined by the energy functional analysis, ensures the system is stable to all finite perturbations which conserve both the volume and the position of the contact lines.

To compute the perturbations and corresponding eigenvalues from the linearised Navier–Stokes system a finite element scheme is used. In the third section of this chapter we write the linear Navier–Stokes system in weak form. This weak form system is then discretised using quadratic basis functions for the velocity components and linear basis functions for the pressure. This discrete system is then written as a (complex) quadratic eigenvalue problem. The final parts of this section test the numerical scheme and give details of the limitations of this method.
The final section of this chapter, section 3.4, uses a bifurcation method to determine where the stability changes in a given phase space. These results are tested using the energy functional analysis to determine the point of marginal stability. Hence it is shown that the marginal stability curve truly separated the stable region from the unstable region.

3.1 Energy functional stability analysis

In this section, we consider the energy as a functional of the equilibrium surfaces. To then analyse the stability of the equilibrium surface we shall use results from section 1.2.4. This is performed first for an infinite horizontal liquid bridge and then a cylindrical liquid membrane. The equilibrium surfaces of these configurations were studied in sections 2.2 and 2.3 respectively.

For both configurations we shall begin by defining a functional to describe the energy of the system. We shall see that the energy functional \( E[h_+, h_-] \) can be written in the form

\[
E[h_+, h_-] = I_1[h_+] + I_2[h_-],
\]

for non-linear functionals \( I_1 \) and \( I_2 \). It is then shown that the equilibrium surfaces studied in the previous chapter occur at extrema of the energy functional. Determining this extrema is key to understanding the stability of the equilibrium surface; indeed if the energy functional is minimized by the equilibrium surface then that surface is stable. To determine the extrema we use a result from section 1.2.4 which states that a functional which satisfies \( (3.1.1) \) is minimised on the plane \((h_+, h_-) = (h^*_+, h^*_-)\) if the second variation of both \( I_1 \) and \( I_2 \) are strongly positive here, i.e., there exists positive constants \( k_1, k_2 \) such that,

\[
\delta^2 E[h^*_+, h^*_-] \geq k_1 \|h^*_+ + k_2 h^*_-\|^2,
\]

given the first variation also vanishes on the plane.
3.1.1 Infinite horizontal liquid bridge

Consider an infinite liquid bridge held between two identical vertical walls, separated by a distance $2R$, as seen in figure 3.1. By defining the upper and lower surfaces as $z = h_+(x, y)$ and $z = h_-(x, y)$, respectively, we define the region occupied by the liquid by

$$(x, y, z) \in [-R, R] \times (-\infty, \infty) \times [h_-, h_+].$$

Figure 3.1: The left image is a cross-section of the geometry, and on this image we define the $x, z$ axes to be horizontal and vertical respectively, with the $y$ axis into the page. The right image is a top view of the geometry which is to be extended infinitely both upwards and downwards. In the right image gravity acts into the page.

The static energy of the liquid bridge is the sum of the gravitational potential energy, $P$, and the interfacial energies associated with the gas/solid, liquid/solid, and the liquid/gas interfaces, $S_{gs}$, $S_{ls}$, $S_{lg}$ respectively. However, in this study, we assume the variations in the surface position are sufficiently small to ensure the position of the contact lines remain pinned\textsuperscript{1}. In this case the surface energies of the gas/solid and liquid/solid regions will be constant and can be excluded from the analysis.

To avoid the potential energy and interfacial energy diverging in the case

\textsuperscript{1}This assumption is not unreasonable, since in a relaxed state we can assume that the contact angle is away from the recede and advance angles. Therefore the perturbation would have to be so large as to force this angle to change up to approximately $25^\circ$ to force the angle past the recede or advance angle. See de Gennes et al., 2004 and references therein.
of an infinite bridge, we first consider a bridge which extends over \( y \in (-L, L) \) with periodic conditions at \( \pm L \), i.e., \( h_\pm(x, -L) = h_\pm(x, L) \) and the same for all their derivatives. Later we shall take the limit \( L \to \infty \).

The gravitational potential energy is given by

\[
P_L = \rho g \int_{-L}^{L} \int_{-R}^{R} \frac{1}{2} (h_+^2 - h_-^2) \, dx \, dy,
\]

where \( \rho, g \) are the density of the liquid and the gravitational acceleration, respectively. Note both \( \rho \) and \( g \) are assumed constant in the liquid. The energy of an interface is the product of the surface tension, \( \gamma \), and the area of the interface. Therefore the gas/liquid interfacial energy is

\[
S_L = \gamma \int_{-L}^{L} \int_{-R}^{R} \left( \sqrt{\Delta_+} + \sqrt{\Delta_-} \right) \, dx \, dy,
\]

where

\[
\Delta_\pm = 1 + \left( \frac{\partial h_\pm}{\partial x} \right)^2 + \left( \frac{\partial h_\pm}{\partial y} \right)^2.
\]

Using the half width of the bridge, \( R \), as a characteristic length scale for \((x, y, z)\), i.e., \((x, y, h) = R(x^*, y^*, h^*)\) and \( \gamma R^2 \) as a characteristic energy scale, i.e., \( E = \gamma R^2 E \), we find

\[
P_L = \rho g R^4 \int_{-L^*}^{L^*} \int_{-1}^{1} \frac{1}{2} (h_+^2 - h_-^2) \, dx^* \, dy^*,
\]

\[
S_L = \gamma R^2 \int_{-L^*}^{L^*} \left( \sqrt{\Delta^*_+} + \sqrt{\Delta^*_-} \right) \, dx \, dy,
\]

\[
E_L = P_L + S_L.
\]

Hence, dividing by \( \gamma R^2 \) we obtain the dimensionless static energy

\[
E_L^* = B P_L^* + S_L^*,
\]

where \( B \equiv \frac{\rho g L^2}{\gamma} \) is the Bond number and

\[
P_L^* = \int_{-L^*}^{L^*} \int_{-1}^{1} \frac{1}{2} (h_+^2 - h_-^2) \, dx^* \, dy^*,
\]

\[
S_L^* = \int_{-L^*}^{L^*} \int_{-1}^{1} \left( \sqrt{\Delta^*_+} + \sqrt{\Delta^*_-} \right) \, dx^* \, dy^*.
\]
For the remainder of this section all variables are dimensionless, and shall be written without the star. When we perturb the energy we must ensure two parameters remain fixed. Firstly, we are perturbing the boundaries of the bridge without adding or removing liquid from it, i.e., the volume must remain fixed. Secondly, while the perturbations are small we expect the contact line to remain pinned, as mentioned earlier.

While employing this method we must ensure that the contact lines remain pinned. To enforce this condition we must enforce that the perturbations to the free surfaces, \( \delta h_\pm \), vanish at both \( x = -1 \) and \( x = 1 \). The volume consideration is maintained by implementing the method of Lagrange multipliers to find the local extrema of the static energy.

By taking the variation of (3.1.10) we find
\[
\delta E_L = B \delta P_L + \delta S_L. \tag{3.1.13}
\]

We can analyse the \( \delta P_L \) and \( \delta S_L \) terms separately. From (3.1.11) and (3.1.12), we see
\[
\delta P_L = \int_{-L}^{L} \int_{-1}^{1} (h_+ \delta h_+ - h_- \delta h_-) \, dx \, dy, \tag{3.1.14}
\]
\[
\delta S_L = \int_{-L}^{L} \int_{-1}^{1} \left( \frac{\partial h_+}{\partial x} \frac{\partial \delta h_+}{\partial x} + \frac{\partial h_+}{\partial y} \frac{\partial \delta h_+}{\partial y} + \frac{\partial h_-}{\partial x} \frac{\partial \delta h_-}{\partial x} + \frac{\partial h_-}{\partial y} \frac{\partial \delta h_-}{\partial y} \right) \, dx \, dy. \tag{3.1.15}
\]

The periodic conditions at \( y = \pm L \) imply that
\[
\left( \frac{1}{\sqrt{\Delta_\pm}} \frac{\partial h_\pm}{\partial x} \delta h_\pm \right)_{y=-L} = \left( \frac{1}{\sqrt{\Delta_\pm}} \frac{\partial h_\pm}{\partial x} \delta h_\pm \right)_{y=L}. \tag{3.1.16}
\]

Using this along with the fact that \( \delta h_\pm = 0 \) at both \( x = 1 \) and \( x = -1 \), we can integrate (3.1.15) by parts, to obtain
\[
\delta S_L = - \int_{-L}^{L} \int_{-1}^{1} \left[ \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{\Delta_+}} \frac{\partial h_+}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\sqrt{\Delta_+}} \frac{\partial h_+}{\partial y} \right) \right] \delta h_+ \, dx \, dy
- \int_{-L}^{L} \int_{-1}^{1} \left[ \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{\Delta_-}} \frac{\partial h_-}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\sqrt{\Delta_-}} \frac{\partial h_-}{\partial y} \right) \right] \delta h_- \, dx \, dy. \tag{3.1.17}
\]
To find the equilibrium shape of the free surfaces of the infinite bridge we consider

$$\lim_{L \to \infty} \delta (E_L - \alpha V_L) = 0,$$  \hspace{1cm} (3.1.18)

where $\alpha$ is a Lagrange multiplier and $V_L$ is the volume of the liquid bridge, i.e.,

$$V_L = \int_{-L}^{L} \int_{-1}^{1} (h_+ - h_-) \, dx \, dy.$$  \hspace{1cm} (3.1.19)

However we note that

$$\delta V_L = \int_{-L}^{L} \int_{-1}^{1} (\delta h_+ - \delta h_-) \, dx \, dy.$$  \hspace{1cm} (3.1.20)

To ensure (3.1.18) is satisfied for all $\delta h_\pm$, the coefficients of $\delta h_+$ and $\delta h_-$ must vanish. Therefore we find

$$B h_+ - \alpha = \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{\Delta_+}} \frac{\partial h_+}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\sqrt{\Delta_+}} \frac{\partial h_+}{\partial y} \right),$$  \hspace{1cm} (3.1.21)

$$B h_- - \alpha = -\frac{\partial}{\partial x} \left( \frac{1}{\sqrt{\Delta_-}} \frac{\partial h_-}{\partial x} \right) - \frac{\partial}{\partial y} \left( \frac{1}{\sqrt{\Delta_-}} \frac{\partial h_-}{\partial y} \right).$$  \hspace{1cm} (3.1.22)

When solving (3.1.21), (3.1.22) we must have a total of five constraints to accompany the periodic conditions at $y = \pm L$. The first constraint is a known volume condition, i.e.,

$$V_L = \int_{-L}^{L} \int_{-1}^{1} (h_+ - h_-) \, dx \, dy.$$  \hspace{1cm} (3.1.23)

As discussed earlier, we imposed that $\delta h_\pm = 0$ on both $x = 1$ and $x = -1$ to ensure that the contact line remains pinned. This condition makes most sense when accompanied with Dirichlet boundary conditions for $h_\pm$. Hence we write $h_+ (\pm 1, y) = H_+ (y)$ and $h_- (\pm 1, y) = H_- (y)$, for known functions $H_\pm$. However from the geometry of the system, when $L \to \infty$, we expect no change in the shape as we progress through the bridge in the $y$-direction, hence we set derivatives with respect to $y$ to zero. Therefore we obtain the system for the
equilibrium shape of the liquid bridge

\[ B h_+ - \alpha = \frac{d}{dx} \left( \frac{1}{\sqrt{\Delta_+}} \frac{dh_+}{dx} \right), \quad (3.1.24) \]

\[ B h_- - \alpha = -\frac{d}{dx} \left( \frac{1}{\sqrt{\Delta_-}} \frac{dh_-}{dx} \right), \quad (3.1.25) \]

\[ h_+ = H_+ \text{ at } x = \pm 1, \quad (3.1.26) \]

\[ h_- = H_- \text{ at } x = \pm 1, \quad (3.1.27) \]

\[ V = \int_{-1}^{1} (h_+ - h_-) \, dx. \quad (3.1.28) \]

Note, in particular, that \( H_{\pm} \) are now constants. This system is equivalent to \( (2.2.4)-(2.2.7) \) if the Lagrangian multiplier for the volume is equal to the dimensionless pressure difference across the upper surface. Then, to determine the stability of these equilibrium shape we consider the second variation of the static energy. Recall

\[ \delta E_L = \int_{-L}^{L} \int_{-1}^{1} \left\{ B (h_+ \delta h_+ - h_- \delta h_-) + \left[ \frac{1}{\sqrt{\Delta_+}} \frac{\partial h_+}{\partial x} \frac{\partial \delta h_+}{\partial x} + \frac{1}{\sqrt{\Delta_+}} \frac{\partial h_+}{\partial y} \frac{\partial \delta h_+}{\partial y} \right] + \left[ \frac{1}{\sqrt{\Delta_-}} \frac{\partial h_-}{\partial x} \frac{\partial \delta h_-}{\partial x} + \frac{1}{\sqrt{\Delta_-}} \frac{\partial h_-}{\partial y} \frac{\partial \delta h_-}{\partial y} \right] \right\} \, dx \, dy. \quad (3.1.29) \]

When taking the variation of this it is important to keep the \( \frac{\partial h}{\partial y} \) terms until the end, since these shall show how the perturbation changes with \( y \). Therefore we take the variation and then enforcing the shape is unchanging in the \( y \)-direction, i.e., \( \frac{\partial h}{\partial y} = 0 \), to find

\[ \delta^2 E_L = \int_{-L}^{L} \int_{-1}^{1} \left\{ B (\delta h_+)^2 - B (\delta h_-)^2 + \frac{1}{\Delta_+^2} \left[ \left( \frac{\partial \delta h_+}{\partial x} \right)^2 + (1 + h_+^2) \left( \frac{\partial \delta h_+}{\partial y} \right)^2 \right] + \frac{1}{\Delta_-^2} \left[ \left( \frac{\partial \delta h_-}{\partial x} \right)^2 + (1 + h_-^2) \left( \frac{\partial \delta h_-}{\partial y} \right)^2 \right] \right\} \, dx \, dy, \quad (3.1.30) \]

where, \( \Delta_{\pm} = 1 + h_{\pm}^2 \).
and \( h_\pm \) is a function of \( x \) only, hence dash (\( \bullet \)) denotes differentiation with respect to \( x \). Note that if there is any change of the perturbation in the \( y \) direction, there is an increase in \( \delta^2 E \). Therefore the most destabilising perturbation, i.e. the one with the lowest value of \( \delta^2 E \), is one which is independent of \( y \). This can been seen in (3.1.30) since the terms involving \( \partial \delta h_\pm / \partial y \) are non-negative. Therefore, we have shown that \( \delta^2 E_L \) will be strongly positive if the same functional without the dependence of \( y \) is considered. Hence we consider

\[
E = \int_{-1}^{1} \left[ B \left( f_+^2 - f_-^2 \right) + \frac{f_+^{\prime 2}}{\Delta_+^2} + \frac{f_-^{\prime 2}}{\Delta_-^2} \right] \, dx, \tag{3.1.31}
\]

\( f_\pm = 0 \) at both \( x = -1 \), and \( x = 1 \), \( \tag{3.1.32} \)

where

\[
f_\pm(x) = \delta h_\pm(x, \cdot). \tag{3.1.33}
\]

Importantly notice,

\[
E \leq \frac{\delta^2 E_L}{L}. \tag{3.1.34}
\]

To find the sign of \( E \) we observe that \( E \) is parabolic with one global stationary point, therefore we consider this point and find the sign of the functional here. We shall show \( E \) cannot be bound above, and it follows that the stationary point is a minima. Therefore if we show this minimum is positive then the second variation of the static energy is positive. When we consider the extrema of \( E \) we examine only fixed-volume normalised\( ^2 \)perturbations. Rather than consider all possible perturbations, we observe that perturbations of the form

\[
f_+ = f_- = A \cos \left( \frac{k \pi x}{2} \right), \tag{3.1.35}\]

satisfy the boundary conditions and fixed-volume conditions for \( k = 1, 3, 5, \ldots \). The normalisation condition takes the form

\[
\int_{-1}^{1} (f_+^2 + f_-^2) \, dx = a^2, \tag{3.1.36}\]

\(^2\)By normalised we mean that the sum of the squares of the \( L^2 \) norms of the upper and lower perturbations takes a given (positive) value. See equation (3.1.36)
and to enforce this condition we set $A = a/\sqrt{2}$. For these perturbations we have
\[ \mathcal{E} = \frac{2^2}{8} \int_{-1}^{1} \left( \Delta_{+}^{-\frac{3}{2}} + \Delta_{-}^{-\frac{3}{2}} \right) \sin^2 \left( \frac{k\pi x}{2} \right) \, dx. \] (3.1.37)

To approximate this integral note $\sin^2 \left( \frac{k\pi x}{2} \right)$ is a function which does not change sign. Therefore we can apply the mean value theorem for definite integrals to find
\[ \int_{-1}^{1} \left( \Delta_{+}^{-\frac{3}{2}} + \Delta_{-}^{-\frac{3}{2}} \right) \sin^2 \left( \frac{k\pi x}{2} \right) \, dx = \left( \Delta_{+}^{-\frac{3}{2}} + \Delta_{-}^{-\frac{3}{2}} \right) (\xi) \int_{-1}^{1} \sin^2 \left( \frac{k\pi x}{2} \right) \, dx \] (3.1.38)
\[ = \left( \Delta_{+}^{-\frac{3}{2}} + \Delta_{-}^{-\frac{3}{2}} \right) (\xi), \] (3.1.39)
for some $\xi \in [-1, 1]$. To simplify the remaining algebra we define
\[ c = \left( \Delta_{+}^{-\frac{3}{2}} + \Delta_{-}^{-\frac{3}{2}} \right) (\xi). \] (3.1.40)

From (3.1.6) it follows
\[ 0 < c < 1. \] (3.1.41)

Returning back to $\mathcal{E}$, we find that
\[ \mathcal{E} = \frac{c a^2 k^2 \pi^2}{8}. \] (3.1.42)

Importantly, $c$ is determined by the equilibrium configuration of the system and is therefore independent of the perturbation $f_{\pm}$. It follows that for a given equilibrium configuration the functional $\mathcal{E} \to \infty$ as $k \to \pm\infty$, and hence $\mathcal{E}$ cannot be bound above. As discussed early, this result implies that the single stationary point of $\mathcal{E}$ must be a minimum. Further, if this minimum occurs at a positive value of $\mathcal{E}$ then the second variation of the energy is strongly positive and it follows the configuration is stable. To determine the sign of $\mathcal{E}$ at the stationary point we consider
\[ \delta \left( \mathcal{E} - \mu \int_{-1}^{1} (f_+ - f_-) \, dx - \lambda \left[ \int_{-1}^{1} (f_+^2 + f_-^2) \, dx - a^2 \right] \right) = 0, \] (3.1.43)
where $\lambda, \mu$ are analogous to Lagrange multipliers (to enforce the fixed-volume and normalised perturbations conditions), and $\alpha$ is a normalisation parameter.

This gives

$$\int_{-1}^{1} \left[ Bf_+ \delta f_+ + \frac{1}{\Delta_+^2} \frac{df_+}{dx} \frac{d\delta f_+}{dx} - Bf_- \delta f_- + \frac{1}{\Delta_-^2} \frac{df_-}{dx} \frac{d\delta f_-}{dx} \right. $$

$$- \frac{\mu}{2} (\delta f_+ - \delta f_-) - \lambda (f_+ \delta f_+ + f_- \delta f_-) \left. \right] dx = 0. \quad (3.1.44)$$

Integrating by parts, noting $\delta f_\pm = 0$ at both $x = -1$ and $x = 1$ (since $f_\pm$ is fixed at these points), we obtain

$$\int_{-1}^{1} \frac{1}{\Delta_\pm^2} \frac{df_\pm}{dx} \frac{d\delta f_\pm}{dx} dx = - \int_{-1}^{1} \frac{d}{dx} \left( \frac{1}{\Delta_\pm^2} \frac{df_\pm}{dx} \right) \delta f_\pm dx. \quad (3.1.45)$$

Therefore, requiring coefficients of $\delta f_+$ and $\delta f_-$ vanish gives

$$Bf_+ - \left( \frac{1}{\Delta_+^2} f_+' \right)' - \frac{\mu}{2} - \lambda f_+ = 0, \quad (3.1.46)$$

$$-Bf_- - \left( \frac{1}{\Delta_-^2} f_- \right)' + \frac{\mu}{2} - \lambda f_- = 0, \quad (3.1.47)$$

with

$$f_\pm = 0, \quad \text{at } x = 1 \quad \text{and } x = -1. \quad (3.1.48)$$

Since the problem involves two Lagrange multipliers, which are part of the solution set, we need to enforce two extra conditions. These conditions are that the perturbations are normalised and that they do not alter the volume of the bridge, i.e.,

$$\int_{-1}^{1} (f_+ - f_-) dx = 0, \quad (3.1.49)$$

$$\int_{-1}^{1} (f_+^2 + f_-^2) dx = a^2. \quad (3.1.50)$$

This system allows us to solve for $f_+, f_-$ and then determine whether $\mathcal{E} > 0$ at its stationary value. However if we multiply (3.1.46) and (3.1.47) by $f_+$ and $f_-$
respectively, then sum the result and integrate over $-1 \leq x \leq 1$ we find

$$\mathcal{E} = \lambda a^2,$$  \hspace{1cm} (3.1.51)

at its minimum. Further, since $a$ is user chosen we set $a = 1$ to find the stationary value of $\mathcal{E}$ coincides with $\lambda$. Recall, the functional $\mathcal{E}$ is parabolic in nature with one global minimum, which allows us to conclude that if the value of $\lambda$ is positive the energy functional, $E$, is minimized. Therefore the stability of the liquid bridge can be viewed as an eigenvalue value problem for $\lambda$, given by (3.1.46)–(3.1.50). We shall use a shooting method to solve (3.1.46)–(3.1.50) which allows us to find $\lambda$ and thus determine the stability of a liquid bridge.

In summary, if we find that $\lambda > 0$ in the solution to the eigenvalue problem, then we have shown that the energy functional for the infinite horizontal liquid bridge is minimised by the static configuration studied in chapter 2. Since, throughout this section we have shown

$$\delta^2 E_L \geq L\mathcal{E} \geq L\lambda (\|f_+\| + \|f_-\|),$$  \hspace{1cm} (3.1.52)

and because $L$ is positive, we have shown that if $\lambda > 0$ then the second variation is strongly positive. Note, here we used the norm

$$\|f_\pm\|^2 = \int_{-1}^{1} f_\pm^2 \, dx,$$  \hspace{1cm} (3.1.53)

and therefore

$$\|f_+\|^2 + \|f_-\|^2 = \int_{-1}^{1} (f_+^2 + f_-^2) \, dx = a^2.$$  \hspace{1cm} (3.1.54)

Importantly, by showing the second variation of the energy functional is minimised by the static configuration we have found that $\lambda > 0$ is a sufficient condition for the infinite liquid bridge being stable.

3.1.1.1 Test case

To verify the validity of any numerical algorithm we wish to employ to solve (3.1.46)–(3.1.50) we need a known solution for comparison. We notice that if we choose $(B, V, \Lambda) = (0, 1, 1/2)$ we find the bridge takes a rectangular form with
$h_+(x) \equiv 0$ and $h_-(x) \equiv -0.5$. Importantly in this case $h'_\pm = 0$, and hence both $\Delta_\pm = 1$. Therefore (3.1.46)–(3.1.50) become,

$$f''_\pm + \lambda f_\pm + \frac{\mu}{2} = 0, \quad \text{with } f_\pm(\pm1) = 0,$$

(3.1.55)

$$f''_\pm + \lambda f_\mp - \frac{\mu}{2} = 0, \quad \text{with } f_\mp(\pm1) = 0,$$

(3.1.56)

$$\int_{-1}^{1} (f_+ - f_-) \, dx = 0,$$

(3.1.57)

$$\int_{-1}^{1} (f_+^2 + f_-^2) \, dx = 1.$$

(3.1.58)

To ensure a solution exists we must have $\lambda > 0$. Then solving (3.1.55)–(3.1.58) leaves

$$f_\pm = \pm \frac{\mu}{2} \left( \frac{\cos (\sqrt{\lambda}x) - \cos \sqrt{\lambda}}{\lambda \cos \sqrt{\lambda}} \right),
\begin{cases}
\text{when } \mu \neq 0, \\
\sqrt{\lambda} = \tan \sqrt{\lambda},
\end{cases}
\tag{3.1.59}$$

$$f_\pm = \frac{(-1)^n + 1}{4} \sin (\sqrt{\lambda}x) - \frac{(-1)^{n-1} + 1}{4} \cos (\sqrt{\lambda}x),
\begin{cases}
\text{when } \mu = 0, \\
\lambda = \left(\frac{n\pi}{2}\right)^2, \text{ for } n \in \mathbb{N},
\end{cases}
\tag{3.1.60}$$

### 3.1.1.2 Numerical solutions of (3.1.46)–(3.1.50)

We solve (3.1.46)–(3.1.50) using a shooting method, as described in section 1.3.1. In so doing we introduce initial estimate parameters $a_\pm, \lambda_0, \mu_0$ and solve (3.1.46), (3.1.47) with

$$f_\pm(-1) = 0, \quad f'_\pm(-1) = a_\pm, \quad \lambda = \lambda_0, \quad \mu = \mu_0$$

(3.1.61)

using a standard initial value problem solver. The values of $a_\pm, \lambda_0, \mu_0$ are then fixed using a non-linear solver such that conditions (3.1.48), (3.1.49), (3.1.50), are held. Figure 3.2 shows a comparison of the leading 5 eigenvalues predicted analytically and numerically.

In the remaining part of this section we briefly discuss properties of the eigenvalues and eigenfunctions of (3.1.46)–(3.1.50). As discussed these solutions are
obtained using a shooting method as described in section 1.3.1. For a given equilibrium configuration the eigenvalue problem can be seen to have a countable infinite number of eigenvalues $\lambda$, and the set of eigenvalues forms an increasing set. Figure 3.3 plots these eigenvalues and figure 3.4 plots the corresponding eigenfunctions of the leading three eigenvalues, for $(B, \Lambda, V) = (3, 0.5, 0.3)$. We see that after the leading eigenfunction, the following eigenfunctions come in pairs. In the pairs, one of the two eigenfunctions has $f_+$ almost zero throughout and the other has $f_-$ almost zero throughout. All of the paired eigenfunctions contain a number of roots which can be related to the corresponding eigennumber. In the case given, the second and third eigenfunctions have one root, the fourth and fifth have two roots, and the pattern continues in this way.

In chapter 4 we shall be interested in where the system changes from being stable to unstable in a given phase space. Since the stability depends on the sign of $\lambda$, we shall be interested in writing $\lambda$ as a function of one of the Bond number, the volume or the central separation distance, while keeping the others fixed. Figure 3.5 gives as example of this for fixed Bond number and volume.

\textsuperscript{3}This can be proved in a similar way to the analogous result of the Sturm–Liouville eigenvalue problem, see for example Gerlach [2009].

Figure 3.2: A comparison of the analytical (blue crosses) and numerical (red circles) eigenvalues for the test case, $B = 0, V = 1, \Lambda = 0.5$. 
Figure 3.3: The leading seven eigenvalues for \((B, V, \Lambda) = (3, 0.5, 0.3)\).

Figure 3.4: The eigenfunctions of the leading three eigenvalues for \((B, V, \Lambda) = (3, 0.5, 0.3)\).

3.1.2 Cylindrical liquid membrane

The case of the cylindrical liquid membrane is rather similar to the infinite horizontal liquid bridge. As such the details of the working are omitted where the working follows a straightforward path from the previously described work. However, the key results of the work shall be highlighted with how they differ from the analogous result in the infinite horizontal case. As previously we begin by determining the energy functional of the system.

Consider a liquid bridge trapped inside a cylinder of radius \(R\), see figure...
3.6 By labelling the upper and lower surfaces as \( z = h_+(r, \theta) \) and \( z = h_-(r, \theta) \) respectively we see the liquid region can be given by

\[
(r, \theta, z) \in [0, R] \times [0, 2\pi] \times [h_-, h_+].
\] (3.1.62)

The static energy of the liquid membrane is the sum of the gravitational potential energy, \( P \), and the interfacial energies associated with the gas/solid, liquid/solid, and the liquid/gas interfaces, \( S_{gs}, S_{ls}, S_{lg} \), respectively. As previously, we assume the variations in the surface position are sufficiently small to pin the position of the contact line. As such the surfaces energies of the gas/solid and liquid/solid regions will be constant, and are excluded from the analysis. The gravitational potential energy can be given by

\[
P = \rho g \int_0^{2\pi} \int_0^R \frac{1}{2} \left( h_+^2 - h_-^2 \right) r \, dr \, d\theta,
\] (3.1.63)

where \( \rho \) and \( g \) are the density of the liquid and the gravitational acceleration, respectively. Note that both \( \rho \) and \( g \) are assumed to be constant throughout the liquid region. The interfacial energy is the product of the surface tension,
\( z = h^+ \)  
\( z = h^- \)

\( 2R \)

Figure 3.6: The left image is a cross-section of the geometry, and on this image we define the \( r, z \) axes to be horizontal and vertical respectively. The right image is a top view of the geometry. In the right image gravity acts into the page.

\( \gamma \), and the area of the interface. Therefore the gas/liquid interfacial energy is

\[
S = \gamma \int_0^{2\pi} \int_0^R \left( \sqrt{\Gamma^+} + \sqrt{\Gamma^-} \right) r \, dr \, d\theta, \tag{3.1.64}
\]

where

\[
\Gamma_{\pm} = 1 + \left( \frac{\partial h_{\pm}}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial h_{\pm}}{\partial \theta} \right)^2. \tag{3.1.65}
\]

By using the radius of the membrane, \( R \), as a characteristic length scale for both \( r \) and \( h_{\pm} \), i.e., \( r = R r^* \) and \( h_{\pm} = h^*_{\pm} \), and \( \gamma R^2 \) as the characteristic energy scale, we find

\[
P = \rho g R^4 \int_0^{2\pi} \int_0^R \frac{1}{2} \left( h^*_{\pm}^2 - h^*_{\pm}^2 \right) r^* \, dr^* \, d\theta, \tag{3.1.66}
\]

\[
S = \gamma R^2 \int_0^{2\pi} \left( \sqrt{\Gamma^*_{\pm}} + \sqrt{\Gamma^*_{\pm}} \right) r^* \, dr^* \, d\theta, \tag{3.1.67}
\]

\[
\gamma R^2 E^* = P + S. \tag{3.1.68}
\]

Hence we obtain the dimensionless free energy

\[
E^* = B I_p^* + S^*, \tag{3.1.69}
\]

where \( B \equiv \frac{\rho g R^2}{\gamma} \) is the Bond number and

\[
I_p^* = \int_0^{2\pi} \int_0^R \frac{1}{2} \left( h^*_{\pm}^2 - h^*_{\pm}^2 \right) r^* \, dr^* \, d\theta, \tag{3.1.70}
\]

\[
S = \int_0^{2\pi} \int_0^R \left( \sqrt{\Gamma^*_{\pm}} + \sqrt{\Gamma^*_{\pm}} \right) r^* \, dr^* \, d\theta. \tag{3.1.71}
\]
For the remainder of this section all variables are dimensionless, and are written without their star. The energy here is similar to the energy found in the previous section, however it is in cylindrical polar coordinates and as such has a symmetry condition at $r = 0$ rather than the conditions applied at $x = -1$. By considering the first variation of (3.1.69) while ensuring the volume is conserved we find

$$B h_+ - \beta = \frac{1}{r} \frac{d}{dr} \left( \frac{r}{\sqrt{\Gamma}} \frac{d h_+}{dr} \right),$$

(3.1.72)

$$B h_- - \beta = -\frac{1}{r} \frac{d}{dr} \left( \frac{r}{\sqrt{\Gamma}} \frac{d h_-}{dr} \right),$$

(3.1.73)

$$h_\pm = H_\pm \text{ at } r = 1,$$

(3.1.74)

$$\frac{d h_\pm}{dr} = 0 \text{ at } r = 0,$$

(3.1.75)

$$V = \int_0^1 (h_+ - h_-) r \, dr.$$

(3.1.76)

This system is equivalent to the system studied in section 2.3. Continuing in the same way as the previous section we are able to show the second variation satisfies

$$\delta^2 E = \int_0^{2\pi} \int_0^1 \left\{ B \left[ (\delta h_+)^2 - (\delta h_-)^2 \right] + \frac{1}{\Gamma^2} \left[ \left( \frac{\partial \delta h_+}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial \delta h_+}{\partial \theta} \right)^2 \right] + \frac{1}{\Gamma^2} \left[ \left( \frac{\partial \delta h_-}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial \delta h_-}{\partial \theta} \right)^2 \right] \right\} r \, dr \, d\theta,$$

(3.1.77)

It is noted here that if there is a non-axisymmetric part to the perturbation then $\delta^2 E$ is greater than the case where $\delta^2 E$ is independent of $\theta$. Therefore the most destabilising perturbation, i.e., the one with the lowest value of $\delta^2 E$, is axisymmetric. Therefore, we have shown that $\delta^2 E_L$ will be strongly positive if the same functional without the dependence of $\theta$ is considered. Hence we consider

$$E = \int_0^1 \left[ B (f^2_+ - f^2_-) + \frac{f^4_+}{\Gamma^2} + \frac{f^4_-}{\Gamma^2} \right] r \, dr,$$

(3.1.78)
with

\[ f_{\pm} = 0 \text{ at } r = 1, \quad (3.1.79) \]
\[ f'_{\pm} = 0 \text{ at } r = 0, \quad (3.1.80) \]

where

\[ f_{\pm}(r) = \delta h_{\pm}(r, \cdot). \quad (3.1.81) \]

Note that it can be shown

\[ \mathcal{E} \leq \frac{\delta^2 E}{\pi}. \quad (3.1.82) \]

To find the sign of \( \mathcal{E} \) we observe that \( \mathcal{E} \) is parabolic with one global stationary point, therefore we consider this point and find the sign of the functional here. When we consider the extremum of \( \mathcal{E} \) we examine only fixed-volume normalised perturbations. Following a similar argument to the previous section we can show that \( \mathcal{E} \) cannot be bound above, and therefore the stationary point is a minima. By considering this extremum we are able to show the perturbations \( f_{\pm} \) must satisfy the eigenvalue problem

\[ f_{+}(B - \lambda) - \frac{\mu}{2} - \frac{1}{r} \frac{d}{dr} \left( \frac{r}{\Gamma^{+}_2} \frac{\partial f_+}{\partial r} \right) = 0, \quad (3.1.83) \]
\[ f_{-}(B + \lambda) - \frac{\mu}{2} + \frac{1}{r} \frac{d}{dr} \left( \frac{r}{\Gamma^{-}_2} \frac{\partial f_-}{\partial r} \right) = 0, \quad (3.1.84) \]
\[ f_{\pm} = 0, \quad \text{at } r = 1, \quad (3.1.85) \]
\[ f'_{\pm} = 0, \quad \text{at } r = 0, \quad (3.1.86) \]
\[ \int_{0}^{1} (f_+ - f_-) r \, dr = 0, \quad (3.1.87) \]
\[ \int_{0}^{1} (f^2_+ + f^2_-) r \, dr = b^2, \quad (3.1.88) \]
where $\Gamma_{\pm} = 1 + \frac{d h_{\pm}^2}{dr}$, with $h_{\pm}$ satisfying

\begin{align*}
Bh_+ - \beta &= \frac{1}{r} \frac{d}{dr} \left( \frac{r}{\sqrt{\Gamma_+}} \frac{dh_+}{dr} \right), \\
Bh_- - \beta &= -\frac{1}{r} \frac{d}{dr} \left( \frac{r}{\sqrt{\Gamma_-}} \frac{dh_-}{dr} \right),
\end{align*}

(3.1.89) (3.1.90)

\begin{align*}
h_{\pm} &= H_{\pm} \text{ at } r = 1, \\
V &= \int_0^1 (h_+ - h_-) r \, dr.
\end{align*}

(3.1.91) (3.1.92)

This system allows us to solve for $f_+, f_-$ and then determine whether $E > 0$ at its stationary value. However if we multiply (3.1.83) and (3.1.84) by $f_+$ and $f_-$ respectively, then sum the result and integrate over $0 \leq r \leq 1$ we find

\begin{equation}
E = \lambda b^2,
\end{equation}

(3.1.93)

at the minimum. Further, since $b$ is user chosen we set $b = 1$ to find the stationary value of $E$ coincides with $\lambda$. Recall, the functional $E$ is parabolic in nature with one global minimum. Hence we conclude that if the value of $\lambda$ is positive the functional $E$ is minimized. Therefore the stability of the liquid membrane can be viewed as an eigenvalue value problem for $\lambda$, given by (3.1.83)–(3.1.88). In the upcoming section we use a shooting method to solve (3.1.83)–(3.1.88). This method allows us to find $\lambda$ and thus determine the stability of a liquid membrane.

In summary, if we find that $\lambda > 0$ in the solution to the eigenvalue problem, then we have shown that the energy functional for the infinite horizontal liquid bridge is minimised by the static configuration studied in chapter 2. Since, throughout this section we have shown

\begin{equation}
\delta^2 E_L \geq \pi \mathcal{E} \geq \pi \lambda (\|f_+\| + \|f_-\|),
\end{equation}

(3.1.94)

it follows that if $\lambda > 0$ then the second variation is strongly positive. Note, here we used the norm

\begin{equation}
\|f_\pm\|^2 = \int_{-1}^1 f_\pm^2 r \, dr,
\end{equation}

(3.1.95)
and therefore
\[
\|f_+\|^2 + \|f_-\|^2 = \int_{-1}^{1} (f_+^2 + f_-^2) r \, dr = b^2. \quad (3.1.96)
\]

Importantly, by showing the second variation of the energy functional is minimised by the static configuration we have found that \(\lambda > 0\) is a sufficient condition for the infinite liquid bridge being stable.

### 3.1.2.1 Numerical solutions of (3.1.46)–(3.1.50)

As we did in the previous subsection, here we conclude by briefly discussing properties of the eigenvalues and eigenfunctions of (3.1.83)–(3.1.88). As discussed in the previous section these solutions are obtained using a shooting method as described in 1.3.1. For a given equilibrium configuration the eigenvalue problem can be seen to have a countable infinite number of eigenvalues \(\lambda\), and the set of eigenvalues forms an increasing set. Figure 3.7 plots these eigenvalues and figure 3.8 plots the corresponding eigenfunctions of the leading three eigenvalues, for a given equilibrium configuration. It is observed that the eigenfunctions corresponding to the \(n\)-th eigenvalue have \(n-1\) roots for \(r \in (0, 1)\).

In chapter 4 we shall be interested in where the system changes from being stable to unstable in a given phase space. We shall therefore be interested in writing \(\lambda\) as a function of one of the Bond number, the volume or the central separation distance, while keeping the others fixed. Figure 3.9 gives an example of this for fixed Bond number and volume.

### 3.2 Linear stability analysis of the Navier–Stokes equations

In this section we analyse the linearised Navier–Stokes equations derived in section 1.2.3.3 As discussed previously, a necessary condition for a capillary structure to be is stable is for the amplitude of any linear perturbation to decay in time. We begin by showing that the energy of a (linear) perturbation dissipates
due to viscous effects. We then derive the neutral stability condition for the capillary structures considered in the previous section (i.e., the infinite horizontal liquid bridge and the cylindrical liquid membrane). We find that the neutral stability curve obtained from a linear stability analysis is identical to the one obtained by analysing the energy functional. Hence, all liquid bridges/membranes one side of this curve are stable while all liquid bridges/membranes on the other are unstable.
3.2.1 Dissipation of the energy of the perturbation

This section follows a similar method to that of section 1.2.3.2 where the energy was shown to be dissipated by viscous effects. Here we aim to find a similar result for the energy of a perturbation.

The analysis in this section uses more general notation than that used previously, in section 1.2.3.3. Here we denote the region of the liquid bridge/membrane by $\Omega$, and the wetted surface of the attaching walls/cylinder by $\Gamma_1$. In this notation, the Navier–Stokes equations are applied in $\Omega$ and the no-slip boundary condition(s) on $\Gamma_1$ for both configurations.

As mentioned in section 1.2.3.3, we use a complex velocity when considering the linearised Navier–Stokes equations. The dimensionless physical velocity can be recovered by finding the real part of $ue^{\omega t + iky}$. We begin by forming the inner product of $u^*$ (where $^*$ denotes the complex conjugate) with the momentum.
equation of the linearised Navier–Stokes equations (1.2.40)–(1.2.42) to obtain
\[ \omega |u|^2 + \nabla \cdot \left[ \beta u^* \cdot \left( \frac{p}{\beta} I_3 - S \right) \right] = -\frac{\beta}{2} \|S\|^2, \quad (3.2.1) \]

where \( \|S\|^2 = S^*:S \geq 0 \). Here \( \beta \) is the Ohnesorge number, which relates the viscous forces to the inertial and surface tension forces. We shall now integrate over the volume of the bridge. The first and last terms are trivial to integrate and we observe that

\begin{align*}
\int_{\Omega} \nabla \cdot \left[ \beta u^* \cdot \left( \frac{p}{\beta} I_3 - S \right) \right] \, dV &= \oint_{\partial \Omega} -n \cdot \left[ \beta u^* \cdot \left( \frac{p}{\beta} I_3 - S \right) \right] \, dS, \\
&= \int_{z=h_+} -\beta u^* \cdot \left( \frac{p}{\beta} I - S \right) \cdot n \, dS + \int_{z=h_-} -\beta u^* \cdot \left( \frac{p}{\beta} I - S \right) \cdot n \, dS, \\
&= \omega^* \int_{z=h_+} h_+^* (Bh_+ + \kappa_+) \, dS - \omega^* \int_{z=h_-} h_-^* (Bh_- + \kappa_-) \, dS. \quad (3.2.3)
\end{align*}

Next we see, using the definition of \( \kappa_\pm \) (1.2.44) (and a similar result for the cylindrical case) and integrating by parts,

\[ \int_{z=h_\pm} h_\pm^* \kappa_\pm \, dS = \pm \int_{z=h_\pm} \left[ \frac{|h_\pm'|^2}{(1 + h_\pm^2)^2} + \frac{k^2 |h_\pm|^2}{\sqrt{1 + h_\pm^2}} \right] \, dS. \quad (3.2.5) \]

Combining these results we obtain

\[ \omega E_k + \omega^* E_s = -\beta D, \quad (3.2.6) \]
where
\[
E_k = \int_{\Omega} |u|^2 \, dV \geq 0, \tag{3.2.7}
\]
\[
E_s = \int_{z=h_+} \left[ B |h_+|^2 + \frac{|h'_+|^2}{(1 + \dot{h}_+^2)^{\frac{3}{2}}} + \frac{k^2 |h_+|^2}{\sqrt{1 + \dot{h}_+^2}} \right] dS
\]
\[+ \int_{z=h_-} \left[ -B |h_-|^2 + \frac{|h'_-|^2}{(1 + \dot{h}_-^2)^{\frac{3}{2}}} + \frac{k^2 |h_-|^2}{\sqrt{1 + \dot{h}_-^2}} \right] dS, \tag{3.2.8}
\]
\[D = \frac{1}{2} \int_{\Omega} ||S||^2 \, dV \geq 0. \tag{3.2.9}\]

Here \(E_k\) and \(E_s\) correspond to the perturbed kinetic energy and the perturbed surface energy, respectively. Taking real parts of (3.2.6), and defining the perturbed energy \(E = E_k + E_s\), we find
\[E R(\omega) = -\beta D. \tag{3.2.10}\]

Therefore for a viscous bridge (i.e., where \(\beta \neq 0\)) we see marginal stability forces \(D\) to vanish and thus \(u\) is identically zero.\(^4\) For an inviscid bridge (i.e., where \(\beta = 0\)) we see either the bridge is marginally stable or \(E = 0\). We proceed by showing that in the case of an inviscid bridge the eigenvalue \(\omega\) must lie on one of the axis of the complex plane, i.e. \(\omega^2 \in \mathbb{R}\), and further that the eigenvalues appear in positive negative pairs. After this discussion on inviscid bridges/membranes we shall return to comment on the nature of \(\omega\) for viscous bridges.

For an inviscid bridge we can take real and imaginary parts of (3.2.6) to obtain,
\[
(E_k + E_s) R(\omega) = 0, \tag{3.2.11}
\]
\[
(E_k - E_s) I(\omega) = 0. \tag{3.2.12}
\]

Suppose first \(R(\omega) = 0\) then \(I(\omega)\) is free to take any value. Instead, suppose \(R(\omega) \neq 0\) thus \(E_k = -E_s\) and hence \(2E_k I(\omega) = 0\). If \(E_k = 0\) then \(u = 0\)
which implies $\omega h_{\pm} = 0$ and since $u = 0$ we cannot also have $h_{\pm} = 0$ (because this results in no perturbation) thus we force $\omega = 0$. If $E_k \neq 0$ we must have $I(\omega) = 0$. The important result to note from all three cases is

$$\omega^2 \in \mathbb{R}.$$  \hfill (3.2.13)

Further, the governing system of an inviscid bridge/membrane can be reduced to solving the Laplace equation for a single variable with homogeneous Neumann boundary conditions on $\Gamma_1$ and mixed eigenvalue boundary conditions on $z = \bar{h}_{\pm}$. In the case of the infinite horizontal liquid bridge this becomes

$$\nabla^2 p = 0, \quad (3.2.14)$$

$$\frac{\partial p}{\partial x} = 0, \text{ on } \Gamma_1, \quad (3.2.15)$$

$$\omega^2 p - \left( \frac{\partial p}{\partial x} \bar{h}_\pm' - \frac{\partial p}{\partial z} \right) \pm \frac{\partial^2 p}{\partial x^2} \bar{h}_\pm' + 2 \bar{h}_\pm'' \frac{\partial^2 p}{\partial x^2} + \frac{\partial p}{\partial x} h_{\pm}'' - \frac{\partial^2 p}{\partial z^2} = 0, \text{ on } z = \bar{h}_\pm. \quad (3.2.16)$$

A similar eigenvalue problem can be derived in the case of the cylindrical liquid membrane. However in these eigenvalue problems only $\omega^2$ appears therefore if $\omega$ is an eigenvalue then $-\omega$ is also. Further, since $\omega^2 \in \mathbb{R}$ we see $(\omega)^2 = (\omega^*)^2$ and so we find $\omega^*$ is an eigenvalue if $\omega$ is.

Combining these results we see that an inviscid bridge is neutrally stable unless $E = 0$. When $E = 0$ there exists an eigenvalue with non-zero real part and the bridge is unstable. Although a stable eigenvalue does exist nature will always find the most unstable configuration, that is the eigenvalue with greatest real part, which is definitely positive.
3.2.2 Neutral stability equation

Recall from (3.2.10),

\[ \mathcal{E}\mathcal{R}(\omega) = -\beta D, \]

and we further commented that neutral stability occurs when \( u \) is identically zero. This result is only true if \( E \) is finite. If we assume \( u \) is identically zero and consider the boundary conditions on the surfaces (see equation (1.2.41)) we find

\[ \mathcal{I}(\omega) h_\pm = 0, \quad (3.2.18) \]

\[ p - Bh_\pm = \kappa. \quad (3.2.19) \]

The former equation here suggests that neutral stability occurs when \( \omega = 0 \), since \( h_\pm \) cannot be zero when \( u \equiv 0 \), since this results in no perturbation. The latter equation when expanded agrees with the equations for neutral stability derived in the functional analysis sections (sections 3.1.1 and 3.1.2).

Throughout this section, conditions for \( h_\pm \) at the wall(s) have been ignored. We now note that hysteresis theory forbids a small variation of a free surface from varying the contact lines’ position. Thus we have \( h_\pm = 0 \) on \( \Gamma_1 \), and in the cylindrical case we need a symmetry condition at \( r = 0 \) namely, \( h'_\pm = 0 \), where dash denotes differentiation with respect to \( r \). Also, we note that the perturbations introduced (while not mentioned at the time) are volume conserving, thus we must have

\[ \int_0^1 (h_+ - h_-) r \, dr = 0 \quad \text{for the cylindrical case,} \quad (3.2.20) \]

\[ \int_{-1}^1 (h_+ - h_-) \, dx = 0 \quad \text{for the infinite case.} \quad (3.2.21) \]

In summary the neutral stability system for the cylindrical case is (here the
Section 3.3

dash denotes differentiation with respect to $r$),

Cylindrical case ($\frac{d}{dr} = r$)

\[
\begin{align*}
\begin{cases}
    p - Bh_+ &= -\frac{1}{r} \left( \frac{rh'_+}{(1 + h'^2)^{\frac{3}{2}}} \right)' + \frac{k^2h_+}{r^2 \sqrt{1 + h'^2}} , \\
    p - Bh_- &= \frac{1}{r} \left( \frac{rh'_-}{(1 + h'^2)^{\frac{3}{2}}} \right)' - \frac{k^2h_-}{r^2 \sqrt{1 + h'^2}} , \\
    h_\pm &= 0, \text{ at } r = 1 , \\
    h'_\pm &= 0, \text{ at } r = 0 , \\
    \int_0^1 (h_+ - h_-) r \, dr &= 0.
\end{cases}
\end{align*}
\]

(3.2.22)

And in the infinite case,

Infinite case ($\frac{d}{dx} = x$)

\[
\begin{align*}
\begin{cases}
    Bh_+ - p &= \left( \frac{h'_+}{(1 + h'^2)^{\frac{3}{2}}} \right)' - \frac{k^2h_+}{\sqrt{1 + h'^2}} , \\
    Bh_- - p &= -\left( \frac{h'_-}{(1 + h'^2)^{\frac{3}{2}}} \right)' + \frac{k^2h_-}{\sqrt{1 + h'^2}} , \\
    h_\pm &= 0, \text{ at } x = 1 , \\
    h'_\pm &= 0, \text{ at } x = -1 , \\
    \int_{-1}^1 (h_+ - h_-) \, dx &= 0.
\end{cases}
\end{align*}
\]

(3.2.23)

These equations are identical to (3.1.46)-(3.1.50) and (3.1.83)-(3.1.88) respectively when we consider symmetric (or axisymmetric for the cylindrical membrane) perturbations and investigate marginally stable configuration.

We have shown the neutral stability equations predicted by an energy perturbation method match those of a linear stability analysis. Recall linear stability is only a necessary condition for stability, while stability from the energy perturbation is a sufficient condition, hence the result of the matching of their bounds is not expected, and indeed is not true for all liquid configurations.
3.3 Numerical examination using a finite element scheme

In order to verify our results from the previous two sections of analytical work we use a finite element scheme to solve the linearised Navier–Stokes equations and determine the damping factor of the system. In this section we rescale the perturbation $h_{\pm}$ by $\omega$ to allow us to see explicitly how each term in the governing equation and boundary conditions depends on $\omega$. Further, we simplify the algebra by combining two terms in the stress balance equation using $J_{\pm} = (h_{\pm} + \kappa_{\pm})$, to obtain

$$\begin{aligned}
\omega u + \nabla p &= \beta \nabla \cdot S, \\
\nabla \cdot u &= 0,
\end{aligned}$$

for $(x, z) \in [-1, 1] \times [\bar{h}_-(x), \bar{h}_+(x)] = \Omega,$

$$\begin{aligned}
u &= 0, & \text{for } x = \pm 1, \\
h_{\pm} \pm u \cdot n &= 0, & \text{for } z = \bar{h}_{\pm}(x),
\end{aligned}$$

for $z = \bar{h}_{\pm}(x), (3.3.3)$

where,

$$u = (u(x), iu(y), u(z)), \quad (3.3.4)$$

$$S = \nabla u + (\nabla u)^T, \quad (3.3.5)$$

$$\kappa_{\pm} = \mp \left[ \left( \frac{h_{\pm}'}{(1 + \bar{h}_{\pm}^2)^{1/2}} \right)' - \frac{k^2 h_{\pm}}{\sqrt{1 + \bar{h}_{\pm}^2}} \right], \quad (3.3.6)$$

$$\nabla = \left( \frac{\partial}{\partial x}, ik, \frac{\partial}{\partial z} \right), \quad (3.3.7)$$

where $k, \beta$ are known constants. Finally, $I_3$ and $n$ are the three dimensional identity tensor and outward normal vector, respectively. The functions $h_{\pm}$, defined in the previous sections as the perturbations to the free surfaces $z = \bar{h}_{\pm}(x)$, are in fact redundant from the solution as the two boundary conditions of (3.3.3) can be combined.
3.3.1 Derivation of weak form

We start by finding a weak form equation for the variations \( u, p, h \), by considering (3.3.1), (3.3.2), (3.3.3) for \((u, p)\), where \( \beta \) is a known positive constant. Let

\[
U = \left\{ u(x), u(y), u(z) \in H^1([-1, 1] \times [\bar{h}_-, \bar{h}_+]) \mid u = 0 \text{ for } x = \pm 1 \right\},
\]

(3.3.8)

\[
P = \{ p \in L^2([-1, 1] \times [h_-, h_+]) \},
\]

(3.3.9)

where \( H^n \) is the Sobolev space with \( n \) weak derivatives in \( L^2 \) norm, further, we see \( u \in U \) and \( p \in P \). Define

\[
(a, b) = \int_{-1}^{1} \int_{\bar{h}_-}^{\bar{h}_+} a \cdot b \, dz \, dx,
\]

(3.3.10)

\[
\langle \alpha, \gamma \rangle = \int_{-1}^{1} \int_{\bar{h}_-}^{\bar{h}_+} \alpha \gamma \, dz \, dx,
\]

(3.3.11)

\[
[A, B] = \int_{-1}^{1} \int_{\bar{h}_-}^{\bar{h}_+} A : B \, dz \, dx.
\]

(3.3.12)

By forming the inner product of (3.3.1a) with \( v \in U \) and multiplying (3.3.1b) by \( -q \in P \) before aggregating and integrating the results we find

\[
\omega(u, v) + (\nabla p, v) - \langle q, \nabla \cdot u \rangle = \beta(\nabla \cdot S, v), \quad \forall v \in U, q \in P;
\]

(3.3.13)

some manipulation yields

\[
\omega(u, v) - \langle q, \nabla \cdot u \rangle + \beta[S, \nabla v] = (\nabla, \beta S \cdot v - p v) + \langle p, \nabla \cdot v \rangle.
\]

(3.3.14)

We continue by applying the divergence theorem to the first term on the right-hand-side

\[
(\nabla, \beta S \cdot v - p v) = (\beta S \cdot v - p v, n)_{\partial \Omega}.
\]

(3.3.15)

where

\[
(a, b)_\gamma = \int_{\gamma} a \cdot b \, ds.
\]

(3.3.16)
We consider the sections of $\partial \Omega$ separately, first on $x = \pm 1$ we see $v = 0$ forcing these terms to also vanish. On the free surfaces, $z = \bar{h}_\pm$, we find, using (3.3.3),

$$
(\beta S \cdot v - pv, n)_{z=\bar{h}_\pm} = -\frac{1}{\omega} (v, H_\pm n)_{z=\bar{h}_\pm},
$$

(3.3.17)

where

$$
(v, H_\pm n)_{z=\bar{h}_\pm} = (v, h_\pm n)_{z=\bar{h}_\pm} = \pm k^2 \left( v, \frac{h_\pm}{\sqrt{1 + \bar{h}_\pm^2}} n \right)_{z=\bar{h}_\pm}.
$$

(3.3.18)

Integrating the second term by parts, using $v = 0$ at $x = \pm 1$, we obtain

$$
\mp \left( v, \left( \frac{h'_\pm}{(1 + \bar{h}_\pm^2)^2} \right)' n \right)_{z=\bar{h}_\pm} = \pm \left\langle \frac{h'_\pm}{(1 + \bar{h}_\pm^2)^2}, \frac{\partial (v \cdot n)}{\partial x} \right\rangle_{z=\bar{h}_\pm}.
$$

(3.3.19)

Thus, by combining these results, we find the weak form problem is

$$
\begin{cases}
\text{Find } u \in U, p \in P, \omega \in \mathbb{C} \text{ such that:} \\
\omega (u, v) - (q, \nabla \cdot u) - (p, \nabla \cdot v) + [S, \nabla v] = \frac{1}{\omega} F(u, v) \quad \forall v \in U, q \in P,
\end{cases}
$$

(3.3.20)

where, the functional $F$ is only non-zero on the boundaries $z = \bar{h}_\pm$, and is given by

$$
F = - (v, H_\pm n)_{z=\bar{h}_\pm} - (v, H_\pm n)_{z=\bar{h}_-}.
$$

(3.3.21)

### 3.3.2 Description of numerical methods

To expand on the previous section we can write the weak form problem as:

$$
\begin{cases}
\text{Find } u \in U, p \in P, \omega \in \mathbb{C} \text{ such that:} \\
\omega^2 (u, v) - \omega (\text{div } u, q) - \omega (\text{div } v, p) + \omega \beta D(u, v) = F(u, v) \quad \forall v \in U, q \in P,
\end{cases}
$$

(3.3.22)
where

\[ U = \left\{ u^{(x)}, u^{(y)}, u^{(z)} \in U[(-1, 1) \times (\tilde{h}_-, \tilde{h}_+)] \mid u = 0 \text{ for } x = \pm 1 \right\}, \quad (3.3.23) \]

\[ P = \left\{ p \in P[(-1, 1) \times (\tilde{h}_-, \tilde{h}_+)] \right\}, \quad (3.3.24) \]

\[(u, v) = \int_{-1}^{1} \int_{\tilde{h}_-}^{\tilde{h}_+} \left( u^{(x)} v^{(x)} + u^{(y)} v^{(y)} + u^{(z)} v^{(z)} \right) \, dz \, dx, \quad (3.3.25) \]

\[ (\text{div } a, b) = \int_{-1}^{1} \int_{\tilde{h}_-}^{\tilde{h}_+} \left[ 2 \left( \frac{\partial u^{(x)}}{\partial x} \frac{\partial v^{(x)}}{\partial x} + k^2 u^{(y)} v^{(y)} + \frac{\partial u^{(z)}}{\partial z} \frac{\partial v^{(z)}}{\partial z} \right) \right. \]
\[ + \left( ku^{(x)} + \frac{\partial u^{(y)}}{\partial y} \right) \left( kv^{(x)} + \frac{\partial v^{(y)}}{\partial y} \right) \]
\[ + \left( \frac{\partial u^{(y)}}{\partial z} + ku^{(z)} \right) \left( \frac{\partial v^{(y)}}{\partial z} + kv^{(z)} \right) \]
\[ + \left( \frac{\partial u^{(z)}}{\partial z} + \frac{\partial u^{(z)}}{\partial x} \right) \left( \frac{\partial v^{(z)}}{\partial z} + \frac{\partial v^{(z)}}{\partial x} \right) \right] \, dz \, dx, \quad (3.3.26) \]

\[ F = \int_{-1}^{1} \left\{ -h_+ r_+ - \frac{dh_+}{dx} \frac{dr_+}{dx} \frac{k^2 h_+ r_+}{[1 + \tilde{h}_+^2]^{3/2}} \sqrt{1 + \tilde{h}_+^2} \right. \]
\[ + \left. h_- r_- - \frac{dh_-}{dx} \frac{dr_-}{dx} \frac{k^2 h_- r_-}{[1 + \tilde{h}_-^2]^{3/2}} \sqrt{1 + \tilde{h}_-^2} \right\} \, dx, \quad (3.3.28) \]

with\footnote{The functions defined in (3.3.29) and (3.3.30) are new functions and should not be confused with any previous definitions of the same letters.}

\[ h_\pm = \left( u^{(x)} \frac{dh_\pm}{dx} - u^{(z)} \right) \bigg|_{z=\tilde{h}_\pm}, \quad (3.3.29) \]

\[ r_\pm = \left( v^{(x)} \frac{dh_\pm}{dx} - v^{(z)} \right) \bigg|_{z=\tilde{h}_\pm}. \quad (3.3.30) \]

To begin the numerical solution of this system we create a triangulation, \( T \), of our domain. A Delaunay triangulation is used, as described by Delaunay [1934].
The continuous solution sets $U$ and $P$ are discretised on the triangulation, and these discrete sets are denoted $U_D$ and $P_D$. For this problem a Taylor–Hood element pair is used to discretise the solution spaces, as described by Taylor and Hood [1973] and analysed by Girault and Raviart [1986] and Verfürth [1984]. The Taylor–Hood element pair uses
\[(U_D, P_D)(T_i) = (P_k, P_{k-1})(T_i), \tag{3.3.31}\]
where $P_k(T_i)$ is the set of $n$-th order polynomials defined on the triangle $T_i$. We use $k = 2$, and therefore have quadratic elements for velocity and linear elements for pressure (on each triangle). Therefore we write the discrete weak form as,
\[
\begin{cases}
\text{Find } u_D \in U_D, p_D \in P_D, \omega \in \mathbb{C} \text{ such that:} \\
\omega^2(u_D, v_D) - \omega(\text{div } u_D, q_D) - \omega(\text{div } v_D, p_D) + \omega \beta D(u_D, v_D) = F(u_D, v_D), \\
\forall v_D \in U_D, q_D \in P_D.
\end{cases} \tag{3.3.32}
\]

We note this leaves a quadratic eigenvalue problem, i.e., this takes the form
\[(\omega^2 A + \omega B + C) x = 0, \tag{3.3.33}\]
for some solution vector $x$ and known square matrices $A$, $B$, and $C$. The discussion of deriving (3.3.33) from (3.3.32) will follow this section, and will contain further details of $x$ and the matrices. To solve this we use a linearisation method, outlined by Tisseur and Meerbergen [2001], where we write the problem as
\[
\omega \begin{pmatrix} A & B \\ 0 & I \end{pmatrix} \begin{pmatrix} \omega x \\ x \end{pmatrix} + \begin{pmatrix} 0 & C \\ -I & 0 \end{pmatrix} \begin{pmatrix} \omega x \\ x \end{pmatrix} = 0, \tag{3.3.34}
\]
which becomes a linear eigenvalue problem for the vector $s = (\omega x, x)^T$. To solve the resulting linear eigenvalue problem we a method based on Stewart [2002].
3.3.3 Construction of matrix eigenvalue problem

In this section we only consider the discrete variable and thus the subscript $D$ has been dropped. To solve (3.3.32) we first assume that each component of $U$ has the same set of basis functions, namely $\varphi_i$ for $i = 1, 2, 3, \ldots, m$, and $P$ has basis functions $\phi_i$ for $i = 1, 2, 3, \ldots, n$. Note that each of the basis functions is zero everywhere except its corresponding node of the triangulation, where it is equal to unity. Using these we can write

\[ u = \sum_{i=1}^{m} \overline{u}_i \varphi_i \quad \text{and} \quad p = \sum_{j=1}^{n} \overline{p}_j \phi_j. \]

Where $\overline{u}_i, \overline{p}_i$ are a set of constants, and can therefore be taken from the integrand.

Let $\Omega = [-1, 1] \times [-\tilde{h}, \tilde{h}]$ and $T(\Omega)$ be a given triangulation of the geometry.

Applying the described discretisation we find, after dropping the bars

\[
\begin{align*}
(\nabla u, q) &= \left[u_i^{(x)} \int_{\Omega} \frac{\partial \varphi_i}{\partial x} \phi_j \, dV - \overline{u}_i^{(y)} \int_{\Omega} k \varphi_i \phi_j \, dV + \overline{u}_i^{(z)} \int_{\Omega} \frac{\partial \varphi_i}{\partial z} \phi_j \, dV \right] q_j, \\
(\nabla v, p) &= p_i \left[ \int_{\Omega} \varphi_i \frac{\partial \phi_j}{\partial x} \, dV - \int_{\Omega} k \varphi_i \phi_j \, dV v_j^{(y)} + \int_{\Omega} \phi_i \frac{\partial \phi_j}{\partial z} \, dV v_j^{(z)} \right], \\
(u, v) &= u_i^{(x)} \int_{\Omega} \varphi_i \phi_j \, dV v_j^{(x)} + u_i^{(y)} \int_{\Omega} \varphi_i \phi_j \, dV v_j^{(y)} + u_i^{(z)} \int_{\Omega} \varphi_i \phi_j \, dV v_j^{(z)},
\end{align*}
\]

\[
D = \left[ u_i^{(x)} \int_{\Omega} \left( 2 \frac{\partial \varphi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + k^2 \varphi_i \phi_j + \frac{\partial \varphi_i}{\partial z} \frac{\partial \phi_j}{\partial z} \right) \, dV \\
+ u_i^{(y)} \int_{\Omega} k \frac{\partial \varphi_i}{\partial x} \phi_j \, dV + u_i^{(z)} \int_{\Omega} \frac{\partial \varphi_i}{\partial z} \phi_j \, dV \right] v_j^{(x)} \\
+ \left[ u_i^{(x)} \int_{\Omega} k \varphi_i \frac{\partial \phi_j}{\partial x} \, dV + u_i^{(z)} \int_{\Omega} \frac{\partial \varphi_i}{\partial z} \phi_j \, dV \right] v_j^{(y)} \\
+ \left[ u_i^{(x)} \int_{\Omega} \frac{\partial \varphi_i}{\partial z} \phi_j \, dV + u_i^{(y)} \int_{\Omega} \varphi_i \frac{\partial \phi_j}{\partial z} \, dV \right] v_j^{(z)}.
\]
\[ F = \left( u^{(x)}_i G^{(1)}_{ij} + u^{(z)}_i G^{(3)}_{ij} \right) v^{(x)}_j + \left( u^{(x)}_i H^{(1)}_{ij} + u^{(z)}_i H^{(3)}_{ij} \right) v^{(z)}_j, \]  
(3.3.39)

\[ G^{(n)}_{ij} = G^{(n)+}_{ij} + G^{(n)-}_{ij}, \]  
(3.3.40)

\[ H^{(n)}_{ij} = H^{(n)+}_{ij} + H^{(n)-}_{ij}, \]  
(3.3.41)

where

\[ G^{(1)\pm}_{ij} = \int_{-1}^{1} \left[ \mp \phi_i \phi_j \bar{h}^\pm_0 + \frac{\bar{h}^\prime_0}{(1 + \bar{h}^2_0)^{\frac{3}{2}}} \varphi_i \varphi_j \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} \right. \]
\[ \left. - \frac{\bar{h}^\prime_0 \bar{h}^\prime\prime_0}{(1 + \bar{h}^2_0)^{\frac{1}{2}}} \left( \frac{\partial \phi_i}{\partial x} \phi_j + \varphi_i \frac{\partial \phi_j}{\partial x} \right) \right. \]
\[ \left. - \frac{k^2 \bar{h}^2_0}{1 + \bar{h}^2_0} \varphi_i \varphi_j \right] \left. \varphi_j \right|_{z=\bar{h}^\pm_0} \]  
(3.3.42)

\[ G^{(3)\pm}_{ij} = \int_{-1}^{1} \left[ \pm \phi_i \phi_j \bar{h}^\pm_0 + \frac{\bar{h}^\prime_0}{(1 + \bar{h}^2_0)^{\frac{3}{2}}} \varphi_i \varphi_j \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} \right. \]
\[ \left. + \frac{\bar{h}^\prime_0}{(1 + \bar{h}^2_0)^{\frac{3}{2}}} \varphi_i \varphi_j \right. \left. \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} \right] \left. \frac{k^2 \bar{h}^\prime_0}{1 + \bar{h}^2_0} \varphi_i \varphi_j \right|_{z=\bar{h}^\pm_0} \]  
(3.3.43)

\[ H^{(1)\pm}_{ij} = \int_{-1}^{1} \left[ \mp \phi_i \phi_j \bar{h}^\pm_0 + \frac{\bar{h}^\prime_0}{(1 + \bar{h}^2_0)^{\frac{3}{2}}} \varphi_i \varphi_j \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} \right. \]
\[ \left. + \frac{\bar{h}^\prime_0}{(1 + \bar{h}^2_0)^{\frac{3}{2}}} \varphi_i \varphi_j \right. \left. \frac{\partial \phi_j}{\partial x} \right] \left. \frac{k^2 \bar{h}^\prime_0}{1 + \bar{h}^2_0} \varphi_i \varphi_j \right|_{z=\bar{h}^\pm_0} \]  
(3.3.44)

\[ H^{(3)\pm}_{ij} = \int_{-1}^{1} \left[ \pm \phi_i \phi_j - \frac{1}{(1 + \bar{h}^2_0)^{\frac{3}{2}}} \varphi_i \varphi_j \right. \left. \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} \right] \left. \frac{k^2 \bar{h}^\prime_0}{1 + \bar{h}^2_0} \varphi_i \varphi_j \right|_{z=\bar{h}^\pm_0} \]  
(3.3.45)
with \( dV = dx \, dz \). Thus we obtain the equation

\[
\omega^2 \left( u_i^{(x)} u_i^{(y)} u_i^{(z)} p_i \right) M_{ij} \begin{pmatrix} v_j^{(x)} \\ v_j^{(y)} \\ v_j^{(z)} \\ q_j \end{pmatrix} + \omega \left( u_i^{(x)} u_i^{(y)} u_i^{(z)} p_i \right) D_{ij} \begin{pmatrix} v_j^{(x)} \\ v_j^{(y)} \\ v_j^{(z)} \\ q_j \end{pmatrix} + \left( u_i^{(x)} u_i^{(y)} u_i^{(z)} p_i \right) F_{ij} \begin{pmatrix} v_j^{(x)} \\ v_j^{(y)} \\ v_j^{(z)} \\ q_j \end{pmatrix} = 0,
\]

(3.3.46)

where \( M, D, \) and \( F \) are block matrices given by

\[
M = \begin{pmatrix} 
\int_{\Omega} \varphi_i \varphi_j \, dV & 0_{(n \times n)} & 0_{(n \times n)} & 0_{(n \times m)} \\
0_{(n \times n)} & \int_{\Omega} \varphi_i \varphi_j \, dV & 0_{(n \times n)} & 0_{(n \times m)} \\
0_{(n \times n)} & 0_{(n \times n)} & \int_{\Omega} \varphi_i \varphi_j \, dV & 0_{(n \times m)} \\
0_{(m \times n)} & 0_{(m \times n)} & 0_{(m \times n)} & 0_{(m \times m)} 
\end{pmatrix},
\]

(3.3.47)

\[
D = \begin{pmatrix} 
D^{(11)} & \int_{\Omega} k \varphi_i \left( \frac{\partial \varphi_j}{\partial x} \right) \, dV & \beta \int_{\Omega} \frac{\partial \varphi_i}{\partial z} \varphi_j \, dV & \int_{\Omega} \frac{\partial \varphi_i}{\partial x} \varphi_j \, dV \\
\beta \int_{\Omega} \frac{\partial \varphi_i}{\partial z} \frac{\partial \varphi_j}{\partial x} \, dV & D^{(22)} & \int_{\Omega} \frac{\partial \varphi_i}{\partial z} \varphi_j \, dV & \int_{\Omega} \frac{\partial \varphi_i}{\partial x} \varphi_j \, dV \\
\beta \int_{\Omega} \frac{\partial \varphi_i}{\partial z} \frac{\partial \varphi_j}{\partial x} \, dV & \beta \int_{\Omega} k \varphi_i \left( \frac{\partial \varphi_j}{\partial z} \right) \, dV & D^{(33)} & \int_{\Omega} \frac{\partial \varphi_i}{\partial z} \varphi_j \, dV \\
\int_{\Omega} \varphi_i \left( \frac{\partial \varphi_j}{\partial x} \right) \, dV & - \int_{\Omega} k \varphi_i \varphi_j \, dV & \int_{\Omega} \varphi_i \left( \frac{\partial \varphi_j}{\partial z} \right) \, dV & 0_{(m \times m)} 
\end{pmatrix},
\]

(3.3.48)

\[
F = - \begin{pmatrix} 
G^{(1)} & 0_{(n \times n)} & G^{(3)} & 0_{(n \times m)} \\
0_{(n \times n)} & 0_{(n \times n)} & 0_{(n \times n)} & 0_{(n \times m)} \\
H^{(1)} & 0_{(n \times n)} & H^{(3)} & 0_{(n \times m)} \\
0_{(m \times n)} & 0_{(m \times n)} & 0_{(m \times n)} & 0_{(m \times m)} 
\end{pmatrix},
\]

(3.3.49)
where

\[ D^{(11)} = \beta \int_{\Omega} \left( 2 \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + k^2 \phi_i \phi_j + \frac{\partial \phi_i}{\partial z} \frac{\partial \phi_j}{\partial z} \right) \, dV, \]  
\[ D^{(22)} = \beta \int_{\Omega} \left( \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + 2k^2 \phi_i \phi_j + \frac{\partial \phi_i}{\partial z} \frac{\partial \phi_j}{\partial z} \right) \, dV, \]  
\[ D^{(33)} = \beta \int_{\Omega} \left( \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + k^2 \phi_i \phi_j + 2 \frac{\partial \phi_i}{\partial z} \frac{\partial \phi_j}{\partial z} \right) \, dV, \]  

(3.3.50)

(3.3.51)

(3.3.52)

with \( m \) and \( n \) being the number of linear and quadratic nodes in the triangulation. Note that both of the bulk block matrices \( M \) and \( D \) are symmetric. In order to evaluate the surface integrals we evaluate the coefficients at all nodes on the boundary then use linear base functions to approximate the function. Thus we write

\[ G^{(1)} = -A_1^+(x_i)I_1^+ - A_2^+(x_i)I_2^+ - A_3^+(x_i)I_3^+ - A_4^+(x_i)I_4^+ - k^2A_5^+(x_i)I_5^+ \]
\[ + A_1^-(x_i)I_1^- - A_2^-(x_i)I_2^- - A_3^-(x_i)I_3^- - A_4^-(x_i)I_4^- - k^2A_5^-(x_i)I_5^-, \]  

(3.3.53)

\[ G^{(3)} = A_6^+(x_i)I_1^+ + A_2^+(x_i)I_2^+ + A_7^+(x_i)I_4^+ + k^2A_8^+(x_i)I_5^+ \]
\[ - A_6^-(x_i)I_1^- + A_2^-(x_i)I_2^- + A_7^-(x_i)I_4^- + k^2A_8^-(x_i)I_5^-, \]  

(3.3.54)

\[ H^{(1)} = A_6^+(x_i)I_1^+ + A_2^+(x_i)I_2^+ + A_7^+(x_i)I_4^+ + k^2A_8^+(x_i)I_5^+ \]
\[ - A_6^-(x_i)I_1^- + A_2^-(x_i)I_2^- + A_7^-(x_i)I_4^- + k^2A_8^-(x_i)I_5^-, \]  

(3.3.55)

\[ H^{(3)} = -(1)^I_1^+ - A_5^+(x_i)I_2^+ - k^2A_{10}^+(x_i)I_1^+ \]
\[ + (1)^I_1^- - A_5^-(x_i)I_2^- - k^2A_{10}^-(x_i)I_1^-, \]  

(3.3.56)

where \( 1_t \) is a vector of ones and where

\[ A_1^\pm = \frac{\bar{h}_e^2}{(1 + \bar{h}_e^2)^{\frac{1}{2}}}, \]
\[ A_2^\pm = \frac{\bar{h}_e^2}{(1 + \bar{h}_e^2)^{\frac{1}{2}}}, \]
\[ A_3^\pm = \frac{\bar{h}_e^2}{(1 + \bar{h}_e^2)^{\frac{1}{2}}}, \]
\[ A_4^\pm = \frac{\bar{h}_e^2}{(1 + \bar{h}_e^2)^{\frac{1}{2}}}, \]
\[ A_5^\pm = \frac{\bar{h}_e^2}{\sqrt{1 + \bar{h}_e^2}}, \]
\[ A_6^\pm = \bar{h}_e^2, \]
\[ A_7^\pm = \frac{\bar{h}_e^2}{(1 + \bar{h}_e^2)^{\frac{1}{2}}}, \]
\[ A_8^\pm = \frac{\bar{h}_e^2}{\sqrt{1 + \bar{h}_e^2}}, \]
\[ A_9^\pm = \frac{1}{\sqrt{1 + \bar{h}_e^2}}, \]
\[ A_{10}^\pm = \frac{1}{\sqrt{1 + \bar{h}_e^2}}. \]  

(3.3.57)

(3.3.58)

(3.3.59)
with

\[ I_1 = \int \varphi_i \varphi_j \bigg|_{z=h} \, dx, \quad I_2 = \int \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} \bigg|_{z=h} \, dx, \quad (3.3.60) \]

\[ I_3 = \int \left( \frac{\partial \varphi_i}{\partial x} \varphi_j + \varphi_i \frac{\partial \varphi_j}{\partial x} \right) \bigg|_{z=h} \, dx, \quad I_4 = \int \frac{\partial \varphi_i}{\partial x} \varphi_j \bigg|_{z=h} \, dx, \quad (3.3.61) \]

\[ I_5 = \int \varphi_i \frac{\partial \varphi_j}{\partial x} \bigg|_{z=h} \, dx. \quad (3.3.62) \]

We now give details on the evaluation of the terms in these matrices, which come in two categories: volume and surface integrals.

**Volume Integrals**

Let us consider, as an example

\[ I = \int_{\Omega} \varphi_i \varphi_j \, dV \quad (3.3.63) \]

and recall \( T(\Omega) \) is a triangulation of the domain. Then we can write

\[ I = \sum_{k=1}^{N} \int_{T_k} \varphi_i \varphi_j \, dV, \quad (3.3.64) \]

where \( T_k \) is the \( k \)-th triangle and \( N \) is the number of triangles in \( T \). Now let us consider one of these integrals

\[ I_k = \int_{T_k} \varphi_i \varphi_j \, dV, \quad (3.3.65) \]

and suppose \( T_k \) has vertices \((x, z) = (x_1, z_1), (x_2, z_2), (x_3, z_3)\). When using quadratic basis functions we need to define the three vertices and the three midpoints, as shown in figure 3.10. We map this triangle onto a reference triangle \( \hat{T} \) with vertices \((\xi, \zeta) = (0, 0), (1, 0), (0, 1)\) using the linear transformation

\[
\begin{pmatrix}
  x - x_1 \\
  z - z_1
\end{pmatrix} = 
\begin{pmatrix}
  x_2 - x_1 & x_3 - x_1 \\
  z_2 - z_1 & z_3 - z_1
\end{pmatrix}
\begin{pmatrix}
  \xi \\
  \zeta
\end{pmatrix}.
\]

(3.3.66)

Also recall, from the definition of \( U \) in (3.3.31), that \( \varphi_i \) are quadratic basis functions, and thus \( i \) runs from 1 to 6 and each function is zero everywhere except at one point, either at a vertex or a midpoint of an edge.
Therefore, on the reference triangle, we have

\[ \hat{\phi}_1 = (1 - 2\xi - 2\zeta)(1 - \xi - \zeta), \]  
\[ \hat{\phi}_2 = 4\xi(2\xi - 1), \]  
\[ \hat{\phi}_3 = 4\zeta(2\zeta - 1), \]  
\[ \hat{\phi}_4 = 4\xi(1 - \xi - \zeta), \]  
\[ \hat{\phi}_5 = 4\xi\zeta, \]  
\[ \hat{\phi}_6 = 4\zeta(1 - \xi - \zeta). \]

Applying this transformation to \( I_k \) we find

\[ I_k = |\text{det} A_k| \int_0^1 \int_0^{1-\xi} \hat{\phi}_i \hat{\phi}_j \, d\zeta \, d\xi, \]  
\[ \text{where } A_k \text{ is the matrix in } (3.3.66). \]  
For derivatives we need to apply the chain rule to find

\[ \frac{\partial}{\partial x} = \frac{1}{\text{det} A_k} \left[ (z_3 - z_1) \frac{\partial}{\partial \xi} - (z_2 - z_1) \frac{\partial}{\partial \zeta} \right], \]  
\[ \frac{\partial}{\partial z} = \frac{1}{\text{det} A_k} \left[ -(x_3 - x_1) \frac{\partial}{\partial \xi} + (x_3 - x_1) \frac{\partial}{\partial \zeta} \right]. \]

Finally we must repeat this procedure for \( \phi \), however almost all results are identical except the basis functions themselves on the reference triangle. In
this case we have
\[
\begin{align*}
\hat{\phi}_1 &= 1 - \xi - \zeta, \quad (3.3.76) \\
\hat{\phi}_2 &= \xi, \quad \quad \quad \quad \quad (3.3.77) \\
\hat{\phi}_3 &= \zeta. \quad \quad \quad \quad \quad (3.3.78)
\end{align*}
\]

Using these results we can evaluate all the matrices required for \( M \) and \( D \).

**Surface integrals**

For the surface integrals, we evaluate the integrals \( I_j \) and then pre-multiply by a diagonal matrix made up of the required function evaluated at each point. This is better explained by an example; let us consider
\[
A_1^+ I_1 = \bar{h}^2 \int_{-1}^{1} \varphi_i \varphi_j |_{z = \bar{h}_+} \, dx,
\]
\[
= \bar{h}^2 \sum_{k=1}^{N} \int_{T_k} \hat{\varphi}_i \hat{\varphi}_j |_{z = \bar{h}_+} \, dx,
\]
where \( T_k \) is the \( k \)-th triangle and \( N \) is the the number of triangles in \( T(\Omega) \), as before. However in the case of this surface integral, only the triangles with two points on the curve \( z = \bar{h}_+ \) have a non-zero contribution to the integral. Hence consider this integral taken over one of the triangles with two points on the curve. We write
\[
dx = \left| \frac{\partial x}{\partial \xi} \right| d\xi + \left| \frac{\partial x}{\partial \zeta} \right| d\zeta
\]
to obtain a transformation of the integral. However we note the curve \( z = \bar{h}_+ \) gets mapped to \( \zeta = 0 \), thus the integral becomes
\[
\int_{0}^{1} \hat{\varphi}_i \hat{\varphi}_j |_{\zeta = 0} \, d\xi,
\]
which we can evaluate as in the previous section. Thus to compute the value of \( A_1^+ I_1 \) we now need to compute the pre-multiplier. As part of the domain definition we know \( \bar{h}^2_+ \) at each point on the curve. Therefore we can generate a diagonal \( n \times n \) matrix, \( \mathcal{H} \), where \( n \) is the number of points in the triangulation, where the diagonal entry is zero if the corresponding point is not on the curve.
z = \tilde{h}_+ and is equal to \tilde{h}_+^2(x_i) when the i-th node of the triangulation lies on the curve z = \tilde{h}_+. Therefore,

\[ A_1^T I_1 = \mathcal{H} \sum_k \int_{T_k} \hat{\varphi}_i \hat{\varphi}_j |_{z=\tilde{h}_+} \, dx. \tag{3.3.82} \]

We can similarly compute all other surface integral terms.

Using the methods described here and the previous volume integral section we can now compute all the required integrals for the matrices \( M, D, \) and \( F. \)

**3.3.4 Solving the quadratic eigenvalue problem**

We want to solve the quadratic eigenvalue problem,

\[ \omega^2 M s + \omega D s + F s = 0, \tag{3.3.83} \]

where \( s = (u^{(x)} u^{(y)} u^{(z)} p)^T, \) and we write

\[ M = \begin{pmatrix} M_1 & 0_{n \times n} & 0_{n \times n} & 0_{n \times m} \\ 0_{n \times n} & M_1 & 0_{n \times n} & 0_{n \times m} \\ 0_{n \times n} & 0_{n \times n} & M_1 & 0_{n \times m} \\ 0_{m \times n} & 0_{m \times n} & 0_{m \times n} & 0_{m \times m} \end{pmatrix}, \tag{3.3.84} \]

\[ D = \begin{pmatrix} D_{11} & D_{12} & D_{13} & X \\ D_{12}^T & D_{22} & D_{23} & Y \\ D_{13}^T & D_{23}^T & D_{33} & Z \\ X^T & Y^T & Z^T & 0_{m \times m} \end{pmatrix}, \tag{3.3.85} \]

\[ F = \begin{pmatrix} F_{11} & 0_{n \times n} & F_{13} & 0_{n \times m} \\ 0_{n \times n} & 0_{n \times n} & 0_{n \times n} & 0_{n \times m} \\ F_{11} & 0_{n \times n} & F_{33} & 0_{n \times m} \\ 0_{m \times n} & 0_{m \times n} & 0_{m \times n} & 0_{m \times m} \end{pmatrix}, \tag{3.3.86} \]

with \( n \) and \( m \) being the number of nodes in the quadratic and linear mesh, respectively. We see that the matrix \( F \) is singular, so numerical zeros will be produced by any algorithm attempting to solve this system. Further, we note the block components of \( F, \) e.g., \( F_{13} \) are zero except for entries which correspond to
nodes on the free surfaces. With this in mind we reorder the vector \( s \) such that all the free surface nodes are moved to the end of the vector, i.e., \( s = (s^i, s^b) \) where \( s^i, s^b \) correspond to the interior and free surface boundary quadratic nodes, respectively. Importantly, we do not apply this transformation to the linear nodes. We must also reorder the corresponding matrices,

\[
M_1 = \begin{pmatrix} M_1^i & 0_{n[i] \times n[b]} \\ 0_{n[b] \times n[i]} & M_1^b \end{pmatrix}, \tag{3.3.87}
\]

\[
D_{ij} = \begin{pmatrix} D_{ij}^i & 0_{n[i] \times n[b]} \\ 0_{n[b] \times n[i]} & D_{ij}^b \end{pmatrix}, \tag{3.3.88}
\]

\[
X = (X^i, X^b)^T, \tag{3.3.89}
\]

\[
F_{ij} = \begin{pmatrix} 0_{n[i] \times n[i]} & 0_{n[i] \times n[b]} \\ 0_{n[b] \times n[i]} & F_{ij}^b \end{pmatrix}, \tag{3.3.90}
\]

with \( Y \) and \( Z \) following the same transformation as \( X \) and \( n[i], n[b] \) are the number of interior and boundary quadratic nodes. Therefore we find \( (3.3.83) \) is equivalent to

\[
\omega^2 M + \omega D + F = 0, \tag{3.3.91}
\]

where

\[
M = \begin{pmatrix} 0_{n[i] \times n[i]} & 0_{n[i] \times n[b]} & 0_{n[i] \times n[i]} & 0_{n[i] \times n[b]} & 0_{n[i] \times n[i]} & 0_{n[i] \times n[b]} & 0_{n[i] \times m} \\ 0_{n[b] \times n[i]} & M_1^b & 0_{n[b] \times n[i]} & 0_{n[b] \times n[b]} & 0_{n[b] \times n[i]} & 0_{n[b] \times n[b]} & 0_{n[b] \times m} \\ 0_{n[i] \times n[i]} & 0_{n[i] \times n[b]} & 0_{n[i] \times n[i]} & 0_{n[i] \times n[b]} & 0_{n[i] \times n[i]} & 0_{n[i] \times n[b]} & 0_{n[i] \times m} \\ 0_{n[b] \times n[i]} & 0_{n[b] \times n[b]} & 0_{n[b] \times n[i]} & M_1^b & 0_{n[b] \times n[i]} & 0_{n[b] \times n[b]} & 0_{n[b] \times m} \\ 0_{n[i] \times n[i]} & 0_{n[i] \times n[b]} & 0_{n[i] \times n[i]} & 0_{n[i] \times n[b]} & 0_{n[i] \times n[i]} & 0_{n[i] \times n[b]} & 0_{n[i] \times m} \\ 0_{n[b] \times n[i]} & 0_{n[b] \times n[b]} & 0_{n[b] \times n[i]} & 0_{n[b] \times n[b]} & M_1^b & 0_{n[b] \times n[i]} & 0_{n[b] \times m} \\ 0_{m \times n[i]} & 0_{m \times n[b]} & 0_{m \times n[i]} & 0_{m \times n[b]} & 0_{m \times n[i]} & 0_{m \times n[b]} & 0_{m \times m} \end{pmatrix}, \tag{3.3.92}
\]
We notice that \( \mathcal{M}, \mathcal{D}, \) and \( \mathcal{F} \) are all singular. Therefore both zeros and infinities exist as solutions for the eigenvalue \( \omega \). However if \( k = 0 \), we observe \( u^{(y)} \) is no longer part of the solution set and therefore we remove the third and fourth block rows and columns from \( \mathcal{M}, \mathcal{D}, \mathcal{F} \). In particular we notice after removing the fourth block row from \( \mathcal{F} \), that it is no longer singular, and therefore the zero solutions are removed from the set.

To solve \((3.3.91)\) we consider \( \omega(s^i, s^b) \) as a separate variable and write

\[
\omega \left( \begin{array}{cc} \mathcal{M} & \mathcal{D} \\ 0 & I \end{array} \right) \left( \begin{array}{c} \omega(s^i, s^b) \\ (s^i, s^b) \end{array} \right) + \left( \begin{array}{cc} 0 & \mathcal{F} \\ -I & 0 \end{array} \right) \left( \begin{array}{c} \omega(s^i, s^b) \\ (s^i, s^b) \end{array} \right) = 0. \tag{3.3.95} \]

By writing \( r = \left( \begin{array}{c} \omega(s^i, s^b) \\ (s^i, s^b) \end{array} \right) \) we can solve \((3.3.95)\) using a standard linear eigenvalue solver, e.g., Stewart \cite{Stewart2002}, for the eigenpair \((\omega, r)\).
3.3.5 Special case: Inviscid liquid

If the liquid is inviscid we see that the parameter $\beta$ vanishes\(^6\). In this case we redefine the space of the pressure field using the transformation $P \rightarrow \omega P$, where $P$ is the new (modified) pressure field space. Note this transformation is bijective only if $\omega \neq 0$. Using this definition and setting $\beta = 0$ we obtain the eigenvalue problem,

$$\mu (u_i^{(x)} u_i^{(y)} u_i^{(z)} p_i) M_{ij} \begin{pmatrix} v_j^{(x)} \\ v_j^{(y)} \\ v_j^{(z)} \\ q_j \\ \end{pmatrix} + (u_i^{(x)} u_i^{(y)} u_i^{(z)} p_i) F_{ij} \begin{pmatrix} v_j^{(x)} \\ v_j^{(y)} \\ v_j^{(z)} \\ q_j \\ \end{pmatrix} = 0,$$

where $F_{ij}$ is as previously defined in (3.3.49), $\mu = \omega^2$, and

$$M = \begin{pmatrix}
\int_{\Omega} \varphi_i \varphi_j \, dV & 0_{(n \times n)} & 0_{(n \times n)} & \int_{\Omega} \frac{\partial \varphi_i}{\partial z} \phi_j \, dV \\
0_{(n \times n)} & \int_{\Omega} \varphi_i \varphi_j \, dV & 0_{(n \times n)} & -\int_{\Omega} k \varphi_i \phi_j \, dV \\
0_{(n \times n)} & 0_{(n \times n)} & \int_{\Omega} \varphi_i \varphi_j \, dV & \int_{\Omega} \frac{\partial \varphi_i}{\partial z} \phi_j \, dV \\
\int_{\Omega} \phi_i \frac{\partial \varphi_j}{\partial x} \, dV & -\int_{\Omega} k \phi_i \varphi_j \, dV & \int_{\Omega} \phi_i \frac{\partial \varphi_j}{\partial z} \, dV & 0_{(m \times m)}
\end{pmatrix}. \tag{3.3.97}
$$

Therefore we have a linear eigenvalue problem, made up of matrices we have previously calculated for the viscous ($\beta > 0$) case.

3.3.6 Testing

In order verify our numerical scheme, we consider a related system in which we can solve the governing equations analytically. We set $\beta = 0$, $k = 0$, $B = 0$, and $\Lambda = 1/2$. In this case we find $\bar{h}_\pm$ are both constant, with $\bar{h}_+ = 0$ and $\bar{h}_- = 0$. \(^6\)Recall from (1.2.22) that $\beta$ is defined by $\beta = \mu/\sqrt{\rho \sigma L}$.\(^6\)
\( h_\pm = -1/2 \). Thus the system \([1.2.40]-(1.2.42)\) becomes,

\[
\begin{align*}
\omega u + \frac{\partial p}{\partial x} &= 0, \\
\omega w + \frac{\partial p}{\partial z} &= 0, \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \\
\bar{u} &= 0 \text{ on } x = \pm 1, \\
\omega p - w + \frac{\partial^2 w}{\partial x^2} &= 0 \text{ on } z = \bar{h}_\pm,
\end{align*}
\]

where \((u, v, w) = u\). This simplification transforms the system from being elliptic to now being parabolic. However, the number of boundary conditions has also been reduced, since in this system the fluid is inviscid and so the no slip condition has been relaxed to a no through flow condition. As discussed in section 1.2.2 only the \(x\) component of \(u\) is guaranteed to vanish at the walls, i.e., \(u = 0\) at \(x = \pm 1\). This system is equivalent to the single variable system;

\[
\begin{align*}
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} &= 0, \\
\frac{\partial p}{\partial x} &= 0 \text{ on } x = \pm 1, \\
\omega^2 p + \frac{\partial p}{\partial z} + \frac{\partial^3 p}{\partial z^3} &= 0 \text{ on } z = 0, \\
\omega^2 p + \frac{\partial p}{\partial z} - \frac{\partial^3 p}{\partial z^3} &= 0 \text{ on } z = -\frac{1}{2}.
\end{align*}
\]

Here we notice that this system is separable, and thus let \(p(x, z) = X(x)Z(z)\) to find,

\[
\begin{align*}
\frac{X''(x)}{X(x)} + \frac{Z''(z)}{Z(z)} &= 0, \\
X'(x = \pm 1) &= 0, \\
\omega^2 Z(z = \bar{h}_\pm) + Z'(z = \bar{h}_\pm) \pm Z'''(z = \bar{h}_\pm) &= 0.
\end{align*}
\]

Looking at \([3.3.107]\) we see both terms must be constant and of opposite sign. Thus we have,

\[
X'' + KX = 0
\]
where \( (3.3.108) \) forces \( K \geq 0 \). We see \( K = 0 \) leaves the solution \( \omega = 0 \) and \( p = \text{const.} \). Finally, we take \( t \in \mathbb{R} \setminus \{0\} \) such that \( t^2 = K \), and hence

\[
X = a \cos(tx) + b \sin(tx), \quad (3.3.111)
\]
\[
Z = c \exp(tx) + d \exp(-tx). \quad (3.3.112)
\]

Applying \( (3.3.108) \) we find two solution types for \( X \), namely

\[
X_1 = A \cos(t_1 x) \quad \text{where} \quad t_1 = n\pi, \quad \text{or} \quad , \quad (3.3.113)
\]
\[
X_2 = A \sin(t_2 x) \quad \text{where} \quad t_2 = \frac{1}{2}(2n - 1)\pi, \quad (3.3.114)
\]

where \( n \in \mathbb{N} \). Applying these values for \( t \) to \( Z \) and employing \( (3.3.109) \) we find \( Z \) has the following form

\[
Z = B \left[ \frac{t^3 + t + \omega}{t^3 + t - \omega} \exp(-tz) + \exp(tz) \right], \quad (3.3.115)
\]

where

\[
\omega^2 = \frac{-2t^3 \cosh \left( \frac{t}{2} \right) \pm t \sqrt{4t^4 - 2 + 2 \cosh t}}{2 \sinh \left( \frac{t}{2} \right)}. \quad (3.3.116)
\]

Note that for all \( k_1 \) and \( k_2 \) we find \( \omega^2 < 0 \) therefore \( i\omega \in \mathbb{R} \). In figure (3.11a) the leading eight eigenvalues are plotted, and in figure (3.11b) the leading eigenfunction is graphed.

**Testing results**

To test the code we first ensure it accurately predicts the eigenvalue and eigenfunction in the test case mentioned previously. Further we shall see that the numerical approximation converges to the analytic solution as we increase the number of mesh points. The code is then tested by allowing small changes in each of the parameters we set to 0 to run the test, and ensuring the resulting eigenvalues and eigenfunctions remain qualitatively similar.

### 3.3.7 Limitations

Apart from the limitation in the numerical scheme of the two interfaces becoming very close which reduces the number of triangles in the central region to an
insufficient amount. There is a further limitation. For all configurations several fictitious zero eigenvalues will be found. To show this, recall the weak form problem

\[
\begin{cases}
\text{Find } u \in U, p \in P, \omega \in \mathbb{C} \text{ such that:} \\
\omega^2(u, v) - \omega(\text{div } u, q) - \omega(\text{div } v, p) + \omega \beta D(u, v) = F(u, v), \quad \forall v \in U, \ q \in P.
\end{cases}
\]

(3.3.117)

In particular, notice if there exists a function, \( u^* \), such that the functional \( F(u^*, v) = 0 \) for all \( v \in U \) then \( u^* \) is an eigenfunction of this problem with corresponding eigenvalue \( \omega = 0 \). Further, we can write the functional \( F(u^*, v) \) as

\[
F(u^*, v) = -\int_{-1}^{1} \left[ J_+ (u^* \cdot n)|_{z = h_+} + J_- (u^* \cdot n)|_{z = h_-} \right] \, dx.
\]

(3.3.118)

Therefore any function \( u^* \) which satisfies \( u^* \cdot n = 0 \) on the free surfaces is an eigenfunction with corresponding eigenvalue \( \omega = 0 \). Further, notice this function does not have to satisfy any other conditions, e.g., divergence free, and will therefore not satisfy the strong form system. Such eigenvalue-eigenfunction pairs might be produced by the numerical scheme, but do not correspond to physical solutions. We avoid these by tracing the eigenvalue from a known
Section 3.4

Bifurcation analysis

In this section we seek to further validate our neutral stability results from the previous sections. We shall consider a bridge/membrane with fixed volume and Bond number, and we shall examine the relationship between the dimensionless central separation distance, $\Lambda$, and the dimensionless boundary separation distance, $\xi$. By plotting $\xi$ as a function of $\Lambda$, we find that these bifurcation diagrams contain a simple fold. For both the infinite horizontal liquid bridge and the cylindrical liquid membrane this simple fold is seen as a minimum of $\xi(\Lambda)$. The value of $\Lambda$ which corresponds to this minimum shall be seen to correspond to the neutral stability point.

It is well established that when one branch of a simple fold in a bifurcation
Chapter 3

The bifurcation diagram represents stable solutions, the other branch then represents unstable solutions (see, e.g., [Meseguer et al., 1995] and references therein). Maddocks [1987] was able to predict instability of some branches close to folds, without any knowledge of the adjacent branch. Based on this theory [Lowry and Steen, 1995] provided a detailed example for the case of a (vertical) liquid bridge. They further summarised a general method for determining the stability of a capillary surface. Indeed, in this summary they gave special attention to a constrained problem, e.g., constant volume surfaces, and showed if the bifurcation diagram has a maximum then the right branch is stable while the left is unstable. The reverse is true for a minimum.

The principle advantage of this bifurcation method over the previous methods is its ability to find points of neutral stability computationally inexpensively. Indeed, both this bifurcation and the energy minimisation method of section 3.1 seek to find roots of nonlinear functions/curves to find the neutral stability points. The energy minimisation method seeks roots of \( \lambda(\Lambda) \) for a given Bond number and volume, while the bifurcation method (we shall see) seeks roots of \( \frac{\partial \xi}{\partial \Lambda} \) for a given Bond number and volume. The derivative here is taken numerically and even so it is computationally cheaper to find \( \frac{\partial \xi}{\partial \Lambda} \) than \( \lambda \). This is because to find \( \frac{\partial \xi}{\partial \Lambda} \) at \( \Lambda = \Lambda^* \), requires us to find \( \xi \) at both \( \Lambda = \Lambda^* + \varepsilon \) and \( \Lambda = \Lambda^* - \varepsilon \), for some small \( \varepsilon \). Each of these two computations requires the solution of two coupled nonlinear ordinary differential equations subject to one constraint. While to find \( \lambda \), at \( \Lambda = \Lambda^* \) we must solve four coupled nonlinear ordinary differential equations subject to three constraints.

This section is split into two parts, one for each of the geometries studied thus far in this chapter. In each case we shall provide further details to the ideas already discussed.

### 3.4.1 Infinite horizontal liquid bridge

We look to examine the relation between the dimensionless lengths \( \Lambda \) and \( \xi \) for a given Bond number and volume, where \( \xi \) is the separation distance between the
upper and lower surfaces at the walls of the domain, i.e. \( \xi = h_+(1) - h_-(1) \). In figure 3.13 we see an example of the general shape of this diagram. Importantly, we note that \( \xi > 0 \) is a necessary condition for the existence of the configuration while it is not sufficient. Indeed, it is possible for the two interfaces to cross on the interior of the domain twice between \( x = 0 \) and \( x = 1 \) to ensure both \( \Lambda \) and \( \xi \) are positive.

![Graph](image)

Figure 3.13: A plot of \( \xi \) as a function of \( \Lambda \). The parameters used are \((B, V) = (3, 0.8)\).

We can see, in figure 3.13, there is a simple fold in the bifurcation diagram, in this example at \( \Lambda \approx 0.82 \). We therefore expect one of the branches coming from this fold to be stable and the other to be unstable. Further, we expect this fold to be located exactly at the neutral stability point. To test these hypotheses we determine the stability of each configuration on the curve using the energy functional minimisation method. Figure 3.14 shows the results of this investigation close to the fold. Importantly, we indeed observe the matching of the neutral stability limits. Further, the fact the left branch is the stable branch agrees with the work of Lowry and Steen [1995] for a fold which is a minimum.
Figure 3.14: A plot of $\xi$ as a function of $\Lambda$ for $(B, V) = (3, 0.8)$. The blue dot-dashed line represents the stable configurations and the red dashed line the unstable configurations. The black solid line shows the neutral stability value of $\Lambda$ predicted by the energy functional minimisation method.

3.4.2 Cylindrical liquid membrane

In the whole of this chapter the working for the cylindrical liquid membrane has been omitted because of its similarity to the case of the horizontal liquid bridge; this section is no different. We again examine the relation between the dimensionless lengths $\Lambda$ and $\xi$ for a given Bond number and volume. The two lengths are defined analogously to those for the liquid bridge, where $\Lambda$ is the distance between the upper and lower surfaces at the centre, while $\xi$ is the distance between the two surfaces at the boundary. Figure 3.15 shows an example of this relationship.

Similarly to the liquid bridge, there is a simple fold in the bifurcation diagram and this is interpreted as the point where stability changes. By determining the stability of each configuration on the curve using the energy functional minimisation technique (developed in section 3.1) we are able to confirm this
idea. Figure 3.16 shows the stable and unstable curves, close to the fold, for a fixed Bond number and volume. Importantly, notice that the neutral stability limit coincides with the fold.

Figure 3.15: A plot of $\xi$ as a function of $\Lambda$. The parameters used are $(B, V) = (3.5, 0.3)$.

3.5 Summary

This chapter considered the stability of two related capillary configurations. The cases considered were an infinite horizontal liquid membrane and a liquid bridge trapped inside a vertical cylinder.

The energy functional takes the functions for the free surfaces and outputs the static energy of the system. By taking the variation of this functional we obtained a system to determine the equilibrium configuration, i.e., we recovered the Young–Laplace equation. To classify the equilibrium the second variation was considered. When the equilibrium is a minimum in the energy landscape then the configuration is stable. This follows because a natural system al-
Figure 3.16: A plot of $\xi$ as a function of $\Lambda$ for $(B,V) = (3.5,0.3)$. The blue dot-dashed line represents the stable configurations and the red dashed line the unstable configurations. The black solid line shows the neutral stability value of $\Lambda$ predicted by the energy functional minimisation method.

ways tends towards its lowest energy state and any perturbation from a minima will be returned back to this minimum configuration. By examining the second variation it has been shown the most destabilising perturbations are axisymmetric for the cylindrical membrane and unchanging along the infinite horizontal bridge. By examining the most destabilising perturbation we found that if the eigenvalue $\lambda$ (as defined as solutions to equations (3.1.46)-(3.1.48) for the bridge and (3.1.83)-(3.1.88) for the membrane) is positive then the second variation of energy can be shown to satisfy $\delta^2 E \geq \lambda \left[ \|\delta h_+\|^2 + \|\delta h_-\|^2 \right]$ where $\delta h_{\pm}$ are the variations to the upper and lower surfaces. This result shows if $\lambda > 0$ the energy is at a minimum for its equilibrium configuration, and therefore is stable.

Using the Navier–Stokes equations we were able to show that the energy of any perturbation is dissipated by viscous effects. It follows that inviscid bridges/membranes can at best be neutrally stable. Indeed, it was shown that
the damping factors for the inviscid system appear in positive/negative pairs. Therefore if the system has a stabilising perturbation it must also have a destabilising one. Finally, while the system is neutrally stable, the perturbations to the free surfaces were shown to satisfy the same conditions as those predicted by the energy minimisation method.

This analysis was completed using a finite element scheme. The weak form was found by integrating the linearised Navier–Stokes equations in a similar way to the method used to show that the energy of a perturbation is dissipated by viscous effects. To discretise the weak form a Taylor–Hood element pair was used with quadratic basis functions for velocity and linear basis functions for pressure. This quadratic eigenvalue problem was solved by transforming the quadratic matrix problem into a linear matrix pencil. The results from this numerical scheme agree with the previously derived results, where the numerical scheme was accurate.

The final part of the chapter analysed bifurcation diagrams to further verify the results of the variational method. Indeed, it was shown that turning points in the $\Lambda - \xi$ bifurcation diagram occur at the same value of $\Lambda$ as the neutral stability point predicted from the variational method. Here $\xi$ and $\Lambda$ are the dimensionless distance between the upper and lower interfaces at the boundary and centre of the domain, respectively. Therefore, the neutral stability curves predicted by the two methods are in agreement.
Chapter 4

Stability analysis II: Results

This chapter uses the methods given in the previous chapter to produce stability diagrams for the infinite horizontal liquid bridge and the cylindrical liquid membrane. In both cases neutral stability curves are plotted in the Bond number and central separation distance phase space for (various) fixed volumes. However, before looking at the question of stability, the region of existence is first found. In both cases the existence region is bounded by three curves: the maximal central separation distance curve, the zero central separation distance curve and the zero central separation distance curve.

The maximal central separation distance was discussed in sections 2.2.6 and 2.3.4 for small Bond numbers. This curve shall also be referred to as the physical existence limit since it splits the region of valid solutions to the Young–Laplace equation into two regions; one where the upper surface is everywhere above the lower surface and one where the lower surface passes above the upper surface. In the following discussion a more general approach to that of sections 2.2.6 and 2.3.4 is taken. Consider the function

\[ F(B, V, \Lambda) = \min(h_+ - h_-), \]  

(4.0.1)

\[ ^1 \text{Here a valid solution is a solution to the Young–Laplace equation which is finite throughout the domain} \]
where the minimum is taken over all values of \( x \in [-1, 1] \) for the horizontal bridge and \( r \in [0, 1] \) for the cylindrical membrane. The function \( F \) gives the smallest distance between the upper and lower surfaces and it follows that \( F > 0 \) is a sufficient condition for the existence of the configuration, given that the solution to the Young–Laplace equation is finite throughout the domain. Therefore, to find the bounding curves of the existence region we consider roots of \( F \). For given Bond number and volume, \( F \) becomes a function of \( \Lambda \) only. It is clear that \( \Lambda = 0 \) corresponds to one root of this function, then as \( \Lambda \) is increased, \( F \) shall increase linearly until reaching a maximum where the smallest distance between the interfaces moves from the centre of the bridge/membrane to its extremity(ies). Thereafter \( F \) matches the function \( \xi \) as defined in section 3.4. A comparison between these functions is given in figure 4.1. Roots of \( F \) as function of \( \Lambda \) can be found, for fixed Bond number and volume, using a standard root finder.

As an aside, these graphs work to show that defining the configuration in the \((B, V, \Lambda)\) space is a better choice than the \((B, V, \xi)\) space, since in the latter case there are multiple solutions for some choices of \( \xi \).

The final curve bounding the existence region is the curve given by bridges/membranes with zero upper contact angle (this shall also be referred to as the mathematical existence limit). Note that these bridges/membranes have infinite gradient at the boundary. In section 2.2.3 we showed it was impossible for the two dimensional horizontal liquid bridge to have an inflection point of its upper surface, and this result is directly applicable to the infinite horizontal liquid bridge. We assume this result also applies to the cylindrical liquid membrane. Since the bridge/membrane cannot have an inflection point, it follows the upper surface can not have a point of infinite gradient inside the domain. Therefore the regions of existence are bounded by the curves given by configurations with zero upper contact angle.

To find \( \Lambda \) which corresponds to a zero upper contact angle we use a limiting process. This process is required to avoid any singularities which would occur from the infinite gradient which corresponds to zero contact angle. In
Figure 4.1: A comparison of $\xi$ (blue dashed line) and $F$ (red dot-dashed line) as functions of $\Lambda$ for the horizontal liquid bridge. The parameters used are $(B, V) = (3, 0.8)$. Notice that in this case the maximal value which corresponds to $F = 0$ occurs at $\Lambda \approx 0.65$

this method we define the equilibrium shape by its Bond number, volume and the upper contact angle. Numerically, it is more straightforward to define the gradient of the upper surface at the wall, rather than the contact angle. Note that zero contact angle corresponds to $h'_+ \to +\infty$ at the wall. By defining the Bond number, volume and Neumann boundary condition at the wall of the domain we can solve the Young–Laplace equation, then from this solution we have calculate the central separation distance using $\Lambda = h_+(0) - h_-(0)$. In the limiting process we track the value of $\Lambda$ as $h'_+$ becomes large (positive) at the wall. Figure 4.2 shows $\Lambda$ as the Neumann boundary condition at the wall is increased. In figure 4.2 we can see that $\Lambda$ quickly converges to a critical value. We shall use this critical value of $\Lambda$ as the maximal central separation distance for a given Bond number and volume.

The remainder of this chapter studies the stability diagrams and the most
destabilising perturbations. Firstly, we shall look at the stability space of the infinite horizontal liquid bridge. These results shall be tested using the results of finite element method, as described in section 3.3. The second part of this chapter investigates the stability space of the cylindrical liquid membrane. In both cases the stable region is found in the Bond number, volume and central separation distance phase space. Finally, we shall compare the most destabilising perturbations of the two geometries and discuss reasons for the difference in the two stability regions.

4.1 Infinite horizontal liquid bridge

The stability region must be fully contained by the existence region. As we discussed earlier the existence region is enclosed by the curves; \( \Lambda = \Lambda_{\text{max}} \), \( \Lambda = 0 \) and \( \Lambda = \Lambda^* \), where \( \Lambda_{\text{max}} \) is the maximal central separation distance and
Λ* is the central separation distance associated with zero contact angle bridge. Figure 4.3 gives an existence region with example equilibrium shapes inset in the regions where the Young–Laplace equation can be numerically solved.

Figure 4.3: Graph of the existence region for $V = 0.6$. The blue (solid) line is the curve of physical existence and the black (dotted) line is the curve of mathematical existence. Inset are example solutions of the Young–Laplace equations at $(B, \Lambda) = (1, 0.2)$ and $(3, 0.5)$.

Inside the existence region there exist both stable and unstable liquid bridges. To determine the threshold of the stable bridges we use the energy minimisation technique. Using this technique we seek $\Lambda$ such that the eigenvalue $\lambda = 0$. As discussed in section 3.1.1, the eigenvalue $\lambda$ can be interpreted as a function of $\Lambda$ for given Bond number and volume. It follows that we can use a non-linear solver to find roots of $\lambda(\Lambda)$. The root is then plotted on the $(B, \Lambda)$ phase space for given a volume, this process is repeated for various Bond numbers to form
a curve. This curve is the marginal stability curve which separates the stable and unstable regions. Figures 4.4 and 4.5 plot the stable region for different volumes and Bond numbers, respectively. In both cases the stable region is the lower-left section of the existence region.

![Graphs depicting the stable region in the Bond number – separation distance plane, for given volumes. In each case the solid line depicts the physical existence limit, the dotted line depicts the mathematical existence limit, and the dot-dash line depicts the curve of neutral stability.](image)

Figure 4.4: Graphs depicting the stable region in the Bond number – separation distance plane, for given volumes. In each case the solid line depicts the physical existence limit, the dotted line depicts the mathematical existence limit, and the dot-dash line depicts the curve of neutral stability.

The original motivation of the finite element analysis was to test the neutral stability curve. However, because of the limitation described in section 3.3.7, it is impossible to track the eigenvalue $\omega$ as it gets close to zero. It follows that it is not possible to accurately determine where the bridge changes stability using the finite element analysis produced here. However, using the bifurcation analysis it is possible to construct the same neutral stability curves as seen in
Figure 4.5: Graphs depicting the stable region in the volume – separation distance plane, for given Bond numbers. In each case the solid line depicts the physical existence limit, the dotted line depicts the mathematical existence limit, and the dot-dash line depicts the curve of neutral stability.

Importantly, this agrees with the results of the energy methods. Figure 4.6 shows the real part of eigenvalue $\omega$ along two stable curves on the phase plane. Importantly, notice that the energy methods predicts all of these curves to be stable. The real part of the eigenvalue being negative throughout these curve corroborates this result.

4.2 Cylindrical liquid membrane

The general discussion of this section follows that of the previous section. However, for this geometry there are no finite element method results with which
Figure 4.6: Graphs of the growth rate against the slenderness ratio, for volumes 0.2 (red crosses), 0.3 (blue circles), and 0.4 (green stars). The Bond number is given in the subfigure caption.

to test the results. The similarity to the previous section is sufficient to allow the stability diagrams to be presented without any further discussion on their production. Later, in section 4.3, we shall discuss the differences between the stable regions of the two geometries, and the reasons for these differences.

Figures 4.7 and 4.8 show the stable regions in the volume – central separation distance and Bond number – central separation distance planes, respectively. Looking at figure 4.7 we observe in general increasing the Bond number reduces the area of the stable region. However, interestingly, there is a section of the stable region for higher Bond number which is not stable in the case of lower Bond number. This result is in contrast to the case of a vertical liquid bridge where all bridges which are stable for a higher Bond number are also stable for all lower volumes, see for example [Slobozhanin and Perales 1993].

Figure 4.8 shows increasing the volume also increases the maximum achievable central separation distance and that this maximum occurs at a lower volume. However, by increasing the volume, the maximum Bond number is decreased and the minimum central separation distance is increased.
4.3 Comments

In this section we shall address three questions: What is the relation between the stability curves presented in this chapter and the maximal central separation distance approximations derived in chapter 2? How do the most destabilising perturbations affect the equilibrium shape? And how do the two geometries we have considered compare to each other?

In sections 2.2.6 and 2.3.4 we derived expressions for the maximal central separation distance for a given volume, in the microgravity limit. At that stage, we commented that the results have a much lower than expected error as the Bond number increases. Further the range of Bond numbers for which the approximation is accurate increases as the volume is reduced.

These microgravity approximations provide accurate results even at order one Bond number because, with reference to figures 4.3 and 4.8, the maximal separation distance curve is approximately constant for the small Bond number section of the graph (up to $B \approx 2.3$ for the cylindrical membrane with $V = 0.3$). We can also see that maximal separation curve remains almost constant until higher Bond numbers are reached for lower volume. This then implies that the
Figure 4.8: Stable regions of the Bond number – central separation distance plane. Here the blue and red curves depict the boundary of the stable region for $V = 0.35$ and $V = 0.3$, respectively.

The microgravity approximation will remain accurate for higher Bond numbers when using lower volumes. In summary, the maximal separation distance is effectively constant in the Bond number up to a critical value of the Bond number, and as a result higher order expansions (based on asymptotically small Bond number) could be significantly less accurate than the leading order approximation.

Moving onto the second question, we see, in figure 4.9, the most destabilising perturbations and the equilibrium shapes of an infinite horizontal liquid bridge for two different configurations. The perturbations of the upper surface always take the same form and they work to increase the central height of the upper interface. For nearly unstable and unstable configurations the result of the perturbation would be to decrease the upper surface’s curvature, i.e. to flatten the surface.

In a similar way the lower perturbation works to increase the height of the central section of the lower surface; however, for nearly unstable and unstable configurations there are regions near the boundaries where the perturbation drives the interface away from the upper surface.

Recall the final question: how do the two geometries we have considered
Figure 4.9: Examples of two configurations of an infinite horizontal liquid bridge with their most destabilising perturbations. Here the upper windows show the equilibrium shape and the lower window shows the corresponding most destabilising perturbations. The blue (dashed) curve corresponds to the upper surface and perturbations while the red (dot-dashed) line corresponds to the lower surface and perturbations.

(a) \((B, V, \Lambda) = (6, 0.4, 0.38)\) with \(\lambda = 0.0107\).

(b) \((B, V, \Lambda) = (2, 0.4, 0.1)\) with \(\lambda = 2.1640\).

compare to each other? Notice that a key difference between the stable regions two geometries is the nature of the maximal slenderness curve. For the infinite horizontal liquid bridge this curve appears to reach a maximum for the separation distance when it reaches the mathematical existence limit. The turning point of the cylindrical membrane is, in contrast, a maximum for the Bond number as a function of the separation distance.

By comparing figures 4.10 and 4.11 we see the cause of the aforementioned difference in turning points of the maximum slenderness curve is interior contact points between the upper and lower surfaces. In the cylindrical liquid membrane we see that the contact point is at the boundary, while for the infinite horizontal bridge the contact points can be away from the boundary of the domain.
(a) $(B, V, \Lambda) = (2.75, 0.3, 1.1798)$.  
(b) $(B, V, \Lambda) = (2.75, 0.3, 1.6464)$.

Figure 4.10: Equilibrium configurations of a cylindrical liquid membrane on the maximum slenderness curve either side of the turning point.

Figure 4.11: Equilibrium configuration for an infinite horizontal liquid bridge with $(B, V, \Lambda) = (4, 0.6, 0.5470)$. Notice, in particular, that the upper and lower surfaces meet at a point $x \in (-1, 1) \setminus \{0\}$. 
Chapter 5

Examining the effect of vibration on a liquid bridge

In chapters 3 and 4 we concentrated on two geometries where the weight of the liquid is balanced by the surface tension. In this chapter we consider a vertical liquid bridge, which is better understood (see Slobozhanin and Perales, 1993 for example). However, methods of stabilising the bridge are still to be understood. In this chapter we explore (numerically and experimentally) an idea that the stability region of a vertical liquid bridge can be extended by vibrating the supporting structure. A vertical liquid bridge is a volume of liquid held in equilibrium between two surfaces, one below and one above. Here we assume that the liquid bridge’s radius is the same as that of the two surfaces, rods or bars, and therefore remains anchored or pinned to their edges.

Previous work has examined various attempts to destabilise (and stabilise) capillary surfaces. In the case of a liquid bridge, Meseguer et al. (1995) summarises the destabilising effect of rotating the bridge, off-setting the bars, and using bars of different radii. When attempting to stabilise a liquid bridge, several procedures have been investigated. For example, Ponce-Torres et al. (2016) studied the dynamics of a surfactant-laden liquid bridge as surfactants can act
to reduce the effective Bond number. More recently [Benilov 2016] showed that vibrating the upper rod can induce a pressure field which works against the destabilising gravitational force.

This chapter builds on the work of [Benilov 2016] by first examining the equations produced in that analysis. The theory of improving the stability by vibrating the upper rod is then tested experimentally and numerically, as described in [Haynes et al. 2018].

### 5.1 Stability examination

![Schematic of the geometry of the liquid bridge.](image)

Figure 5.1: Schematic of the geometry of the liquid bridge.

The examination of [Benilov 2016] began with the Navier–Stokes equations with free surface boundary conditions on \( r = R(z) \) and no slip boundary conditions on the vibrating plate(s) which hold the bridge, see figure 5.1 for the geometry. By assuming the amplitude of the vibrations is much smaller than the length scale of the problem, a multiple scales technique is then used to derive
the following system,

$$\nabla^2 q = 0,$$  

(5.1.1)

$$q = 0, \text{ at } r = R,$$  

(5.1.2)

$$\frac{\partial q}{\partial z} = -W, \text{ at } z = \Lambda,$$  

(5.1.3)

$$\frac{\partial q}{\partial z} = 0, \text{ at } z = -\Lambda,$$  

(5.1.4)

$$\frac{\partial u}{\partial T} + u \cdot \nabla u + \nabla \left( \frac{1}{4} |\nabla q|^2 + p \right) = -B\dot{z} + \text{Oh} \nabla \cdot S,$$  

(5.1.5)

$$\nabla \cdot u = 0,$$  

(5.1.6)

$$\left[ \left( C - p - \frac{1}{2} |\nabla q|^2 \right) \mathbf{I} + \text{Oh} \mathbf{S} \right] \cdot \hat{n} = 0, \text{ at } r = R,$$  

(5.1.7)

$$\frac{\partial R}{\partial T} + u \cdot \hat{n} = 0, \text{ at } r = R,$$  

(5.1.8)

$$R = 1, \text{ at } z = \pm \Lambda.$$  

(5.1.9)

Here $q$ is the vibration induced pressure field, $W$ is the dimensionless amplitude of the upper rod’s vibration, $T$ is the long time scale, $u$ describes the liquid’s velocity, $p$ describes the liquid’s pressure, and $\Lambda$ is the slenderness of the bridge, defined by $\Lambda = L/(2R)$. Also $C$ is the curvature of the free surface $r = R(z)$, and $B$, $\text{Oh}$ are the dimensionless Bond and Ohnesorge numbers, respectively.

This system can be seen as having two parts. The first part describing the vibration induced pressure $q$ consists of (5.1.1)-(5.1.4), and the second is a modified Navier–Stokes system (5.1.5)-(5.1.9). Benilov [2016] uses two assumptions (namely that the bridge is nearly cylindrical and is subject to weak gravitational and vibration forces) to derive the following neutral stability criterion for the bridge:

$$\frac{\pi^2 V}{\Lambda^2} - 1 = \left[ \frac{9}{\sqrt{2} \pi^2} \left( B - \frac{kW^2}{8\Lambda} \right) \right]^\frac{3}{2} + \frac{3}{2} \left( \frac{\pi}{\Lambda} - 1 \right)^2,$$  

(5.1.10)

where

$$k = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{a_n a_m} \left[ \frac{\tanh(a_n \pi) \coth(a_m \pi) + 1}{1 + (a_m + a_n)^2} + \frac{\tanh(a_n \pi) \coth(a_m \pi) - 1}{(1 + (a_m - a_n)^2)} \right]$$

$$\approx 0.024794,$$  

(5.1.11)
with $a_n$ being the zeros of the Bessel function of the first kind, and
\[ V = \frac{1}{2\Lambda^2} \int_{-\Lambda}^{\Lambda} R^2 \, dz. \] (5.1.12)

Figure 5.2 shows the maximum slenderness calculated from (5.1.10) for $V = 1$ as a function of $W$ for different Bond numbers. If we consider the cylindrical approximation, i.e., $V = 1, B = O(\varepsilon)$, we find
\[ x^2 - 1 = \left[ \frac{9}{\sqrt{2} x^2} (B - W) \right]^{\frac{2}{3}} + \frac{3}{2} (x - 1)^2, \] (5.1.13)
where $x = \pi/\Lambda$ and $W = kW^2/8\pi$. If we further assume the vibrations are small we can write $W = BW^*$, and we find
\[ x = 1 + \left[ \frac{9}{4} (1 - W^*)B \right]^{\frac{2}{3}} + O(B^{\frac{4}{3}}), \] (5.1.14)
and therefore
\[ \Lambda = \pi - \pi \left[ \frac{9}{4} (1 - W^*)B \right]^{\frac{2}{3}} + O(B^{\frac{4}{3}}). \] (5.1.15)
It follows that setting $W^* = 1$ will overcome the destabilising effect of gravity and will return the stability limit to the Plateau–Rayleigh limit $\Lambda = \pi$.

Therefore the optimal value for the vibrational constant is
\[ W = \sqrt{\frac{8\pi B}{k}}. \] (5.1.16)

Further, setting $W = 0$ in (5.1.15) returns us to the case of Meseguer et al. [1995] where the asymptotic approximation for the neutral stability limit was given by
\[ \Lambda = \pi \left[ 1 - \left( \frac{9}{4} B \right)^{\frac{2}{3}} \right]. \] (5.1.17)

Equation (5.1.10) shows that the vibration of the upper bar reduces the effective Bond number, and therefore the comparison between the Bond number and $kW^2/(8\Lambda)$ determines the dominant effect of gravity versus the vibrational stabilisation. For this reason we introduce
\[ \tilde{W} = \frac{kW^2}{8}, \]
as an independent variable for our experiments. Note that in the experiments we will be using liquid bridges with slenderness $\Lambda \approx 1$. 

5.2 Experimental design

This section is devoted to describing the experiments performed to test the neutral stability limit for both a vibrated and un-vibrated bridge. Later we discuss the data and the processing we perform to produce the results, and verify our hypothesis that a liquid bridge can indeed be stabilised by vibrating the upper rod.

Figure 5.3 shows the experimental set up we use to test the stability of a liquid bridge while vibrating the upper rod. A liquid bridge is formed between two parallel and coaxial disks (A, B) of equal radius, $R_0 = 1\, \text{mm}$. The lower disk has an orifice $200\, \mu\text{m}$ in diameter at its centre, which is used to feed and remove liquid using a syringe pump connected to a stepping motor. The upper disk is fixed to a piezocomposite actuator (C) connected to a power amplifier. Harmonic vibrations of the upper disk are produced with a 10 MHz function and arbitrary waveform generator is connected to the power amplifier. The frequency of the vibration is precisely controlled; however, the amplitude is the
result of electro-mechanical transfer, which is a function not only of the gain
of the power amplifier but also of the mass and rigidity of the connection to
the actuator. The lower disk is fixed to a high precision orientation system (D)
to ensure the correct alignment with the upper disk. This system is mounted
on a vertical motorised stage (E) to set the distance between the disks. The
stage was controlled by a stepper motor whose speed could be selected within
the range $0.005 - 2.6 \text{ mm/s}$.

Digital images of the liquid bridge at a resolution of $1280 \times 960$ pixels are
acquired at 30 frames per second with an exposure time of $25 \mu s$ using a CCD
camera (F). The camera is equipped with a tele-centric objective (G) provid-
ing a magnification of approximately $7.16 \mu \text{m/pixel}$. The camera can be moved
horizontally and vertically using a triaxial translation stage (H) to focus the
image. The liquid configuration is illuminated from the back with a cool white
light transmitted through an optical fibre (I). A frosted diffuser (J) is positioned
between the optical fibre and the liquid to provide a uniformly lit background.
To ensure axisymmetry of the bridge a second CCD camera (not shown in fig
5.3) is positioned with an optical axis perpendicular to the main camera. These
elements are mounted on an optical table with a pneumatic anti-vibration iso-
lation system (K) to dampen the vibrations coming from the building. Further, to
analyse the motion of the upper rod under vibration a high-speed video camera
is used in place of the primary CCD camera (F).

Figure 5.3: The experimental setup. Figure taken from (Haynes et al., 2017)
To verify the result for different Ohnesorge numbers we use two different silicone oils, whose surface tension remains largely unchanged by contamination. To measure the surface tensions of these liquid a Theoretical Image Fitting Analysis (TIFA) method is used, following Cabezas et al. [2005]. We note, also, that in these liquids both the Bond number and $\tilde{W}$ are similar, since their ratios of surface tension to density are approximately equal.

Finally, before conducting experiments we verify the movement of the upper rod. Images of the upper disk vibrating without an attached liquid bridge are acquired with a high-speed video camera. The movement is filmed without the liquid bridge to improve accuracy in the determination of the position of the disk’s edge. From the images the position of the rod is found as a function of time, and this data is then fitted to a harmonic function, see figure 5.4. The prescribed value of the vibration was found to have an error less than 0.01%. This procedure was conducted several times throughout the experimental procedure to ensure this accuracy is consistent.

Figure 5.4: Displacement of the upper disk $z_d$ in dots with the overlay $z_d = a \cos(2\pi ft + \phi)$. Figure taken from Haynes et al. [2018].

As mentioned previously, the amplitude of the vibration is not simply a
function of the electro-mechanical transfer, but also the mass connected to the actuator and the rigidity of that connection. For a given mass and connection, the amplitude has spikes of resonance. It is at one of these spikes that the experiments were conducted to increase the energy of the vibration and therefore \( W \). Because of this resonance spike, a small shift in the mass or rigidity would considerably alter the amplitude of the vibration. For this reason, the amplitude we measure from the actuator calibration cannot be used for comparison with theoretical results. Instead, we estimate \( \tilde{W} \) by fitting a numerical liquid bridge contour to the experimental one.

The focus of the procedure is to determine the liquid bridge stability limit. The disks start a short distance apart to fill the gap between them with liquid. The disks are moved apart by withdrawing the lower disk until a slender liquid bridge has formed. We now introduce, by means of injection through a hole in the lower disk, liquid to produce a bridge with near unity volume. For the experiments with vibration, the actuator is now switched on. The lower disk is now slowly pulled away from the upper disk at 3.5\( \mu \text{m/s} \), while we inject liquid at 0.04 ml/hr to keep the volume approximately constant. Images of the liquid bridge are taken throughout the quasi-static stretching process. This withdrawal is stopped when the bridge breaks and the last image of the liquid bridge before breakup is processed, using a TIFA method similar to Cabezas et al. [2005], to measure both the volume and distance between the two disks. This experimental procedure is conducted both with and without the upper disk vibration.

5.3 Processing

During the experimental procedure it was clear that keeping the Bond number constant was impossible. This is because of impurities which can be adsorbed from the air, or (in smaller amounts) from the experimental process. Ponce-Torres et al. [2016] studied this in detail showing liquid left in air quickly loses surface tension and therefore the Bond number can change rapidly over time.
Further, protection of the bridge by means of a glass cover may not be sufficient to maintain a constant Bond number. Hence the results we obtain for the maximum slenderness ($\Lambda$) depends on three dimensionless variables, namely: the volume ($V$), the Bond number ($B$), the vibration field ($\tilde{W}$). However during the experimental process the volume is kept close to unity, the experimental perturbation of the Bond number is minimised, and, finally, the vibration field is small. Hence we can write,

$$V = 1 + \delta, \quad (5.3.1)$$
$$B = \bar{B} + \varepsilon, \quad (5.3.2)$$
$$\tilde{W} \ll 1, \quad (5.3.3)$$

where $\bar{B}$ is the average value of the Bond number through the experiments. These small perturbations motivate a Taylor series analysis to reduce the degrees of freedom. To start we note that $\Lambda$ is symmetric in $W$, then we write,

$$\Lambda(1 + \delta, \bar{B}, \tilde{W}) = \Lambda(1, \bar{B}, 0) + \delta \left. \frac{\partial \Lambda}{\partial V} \right|_{(1, \bar{B}, 0)} + \frac{1}{2} \left[ \delta^2 \left. \frac{\partial^2 \Lambda}{\partial V^2} \right|_{(1, \bar{B}, 0)} + W^2 \left. \frac{\partial^2 \Lambda}{\partial W^2} \right|_{(1, \bar{B}, 0)} \right], \quad (5.3.4)$$

$$\Lambda(1 + \delta, \bar{B} + \varepsilon, \tilde{W}) = \Lambda(1, \bar{B}, 0) + \delta \left. \frac{\partial \Lambda}{\partial V} \right|_{(1, \bar{B}, 0)} + \varepsilon \left. \frac{\partial \Lambda}{\partial B} \right|_{(1, \bar{B}, 0)} + \frac{1}{2} \left[ \delta^2 \left. \frac{\partial^2 \Lambda}{\partial V^2} \right|_{(1, \bar{B}, 0)} + 2\delta \varepsilon \left. \frac{\partial^2 \Lambda}{\partial V \partial B} \right|_{(1, \bar{B}, 0)} + \varepsilon^2 \left. \frac{\partial^2 \Lambda}{\partial B^2} \right|_{(1, \bar{B}, 0)} + W^2 \left. \frac{\partial^2 \Lambda}{\partial W^2} \right|_{(1, \bar{B}, 0)} \right]. \quad (5.3.5)$$

Thus, by subtraction, we find

$$\Lambda(1 + \delta, \bar{B}, \tilde{W}) = \Lambda(1 + \delta, \bar{B} + \varepsilon, \tilde{W}) - \varepsilon \left. \frac{\partial \Lambda}{\partial B} \right|_{(1, \bar{B}, 0)} - \varepsilon \delta \left. \frac{\partial^2 \Lambda}{\partial V \partial B} \right|_{(1, \bar{B}, 0)} - \varepsilon^2 \left. \frac{\partial^2 \Lambda}{\partial B^2} \right|_{(1, \bar{B}, 0)} \quad (5.3.6)$$

This result allows us to plot the experimental results in the maximum slenderness–volume plane. The first term on the right hand side of (5.3.6) can be obtained
from the experiment, and the remaining terms will need to be calculated. Importantly, notice, in these terms there is no vibration field, so classical methods can be used to determine these quantities. We use a method which follows that of Slobozhanin and Perales [1993] and then derivatives are calculated using a finite (central) differencing method. In the following paragraphs this method of finding the stability limit is discussed.

### 5.3.1 Method of Slobozhanin and Perales

The stability curve (in the absence of vibration) is produced following the method of Slobozhanin and Perales [1993]. The equilibrium shape is found using an arclength formulation. The shape’s stability is then analysed by examining the functions $\phi_1$ and $D_1$ described in the system below, where the bridge is stable if both $\phi_1$ and $D$ only have roots at $s = 0$, where $s$ is the arclength from the base of the bridge.

The equilibrium shape $(r(s), z(s))$ is determined by solving

\[
\frac{d^2 r}{ds^2} + \frac{dz}{ds} \frac{d\beta}{ds} = 0, \tag{5.3.7}
\]

\[
\frac{d^2 z}{ds^2} - \frac{dr}{ds} \frac{d\beta}{ds} = 0, \tag{5.3.8}
\]

\[
r(0) = r_0 \equiv \sqrt{B}, \quad r(s_f) = r_0, \tag{5.3.9}
\]

\[
z(0) = 0, \quad z(s_f) = h \equiv 2\Lambda \sqrt{B}, \tag{5.3.10}
\]

\[
\int_0^{s_f} \frac{r^2}{hr_0^3} \frac{dz}{ds} ds = V, \tag{5.3.11}
\]

\[
\frac{d\beta}{ds} = -z + c - \frac{1}{r} \frac{dz}{ds}, \tag{5.3.12}
\]

\[
\beta(0) = \beta_0, \tag{5.3.13}
\]

where $V, B, \Lambda$ are the volume, Bond number, and slenderness respectively. They are assumed known, and $s_f, \beta_0, c$ are constants to be found as part of the solution. We use a shooting method to solve this system, in so doing we transform

---

1The functions $\phi_1$ and $D$ can be derived by ensuring the second variation of the energy functional has no points which are conjugate to the boundary ($s = 0$) inside the domain.
this boundary value problem into an initial value problem using the conditions

\[
\frac{dr}{ds}\bigg|_{s=0} = \cos \beta_0, \quad \frac{dz}{ds}\bigg|_{s=0} = \sin \beta_0.
\]  

(5.3.14)

The stability is then analysed by examining the functions \(\phi_1\) and \(D\), determined by,

\[
\frac{d^2\phi_1}{ds^2} + \frac{1}{r} \frac{dr}{ds} \frac{d\phi_1}{ds} + \left(a(s) + \frac{1}{r^2}\right) \phi_1 = 0,
\]  

(5.3.15)

\[
\phi_1(0) = 0, \quad \frac{d\phi_1}{ds}\bigg|_{s=0} = 1,
\]  

(5.3.16)

\[
D(s) = \phi_{01}(s) \left[ \cos \beta_1 \int_0^s r(t)\phi_{02}(t) \, dt - \frac{r^2(s) - r_0^2}{2} \right] + \left(\frac{dr}{ds} - \phi_{02}(s) \cos \beta_1\right) \int_0^s r(t)\phi_{01}(t) \, dt,
\]  

(5.3.17)

where,

\[
\frac{d^2\phi_{01}}{ds^2} + \frac{1}{r} \frac{dr}{ds} \frac{d\phi_{01}}{ds} - a(s)\phi_{01} = 0,
\]  

(5.3.18)

\[
\frac{d^2\phi_{02}}{ds^2} + \frac{1}{r} \frac{dr}{ds} \frac{d\phi_{02}}{ds} - a(s)\phi_{01} = 0,
\]  

(5.3.19)

\[
\phi_{01}(0) = 0, \quad \frac{d\phi_{01}}{ds}\bigg|_{s=0} = 1,
\]  

(5.3.20)

\[
\phi_{02}(0) = 1, \quad \frac{d\phi_{02}}{ds}\bigg|_{s=0} = 0,
\]  

(5.3.21)

with

\[
a(s) = -\frac{dr}{ds} - \left(\frac{d\beta}{ds}\right)^2 - \frac{1}{r^2} \left(\frac{dz}{ds}\right)^2.
\]  

(5.3.22)

The bridge is stable if both \(\phi_1\) and \(D\) only have roots at \(s = 0\), because this ensures the interior of the domain has no points conjugate to \(s = 0\). For a given Bond number, slenderness and volume we can now find the equilibrium shape and using the functions \(D\) and \(\phi_1\) determine the stability of the bridge, as seen in figure 5.5. The bridge is stable if neither \(\phi_1\) or \(D\) change sign in \((0, s_f]\). If the bridge is unstable, its instability can be caused by axisymmetric or non-axisymmetric perturbations and depending which function \(D(s)\) or \(\phi_1\) has the first (sign-changing) root, this destabilising perturbation will be axisymmetric or non-axisymmetric respectively.
We are working with bridges whose Bond numbers are less than 1 and Slobozhanin and Perales [1993] showed that destabilising perturbations for bridges in this region are axisymmetric. Thus to determine a bridge’s stability we investigate whether $D$ changes sign. Further since we are not interested in where the root is we can use the maximum and minimum values of $D$, $D_{\text{max}}$, $D_{\text{min}}$ respectively, and we say $D$ has a root if

$$D_{\text{max}} - \text{tol} > 0 \quad \text{and} \quad D_{\text{min}} + \text{tol} < 0,$$

(5.3.23)

where tol is used to remove numerical zeros, and is user chosen.

To find the point of neutral stability for a given Bond number and volume we take a coarse discretisation of the expected range of maximum slenderness, for example if we take $B = 0.1$ and $V = 1$ we expect the maximum slenderness to be between one and two. We then take a fine discretisation of the range around the minimum value of slenderness which is unstable. We take the smallest slenderness value which corresponds to an unstable bridge to be the neutral stability limit. We can then find the whole neutral stability curve by using this method over a range of Bond numbers and volumes, as desired. We show a neutral stability curve in figure 5.6.

The advantage of this method is that it does not require any previous knowl-
edge of solutions. Even if we are very unsure of the initial region we could take a very large region, for example [0,10], and be sure to capture the point of neutral stability. An alternative, possibly quicker, method to find the whole neutral stability curve is to use a small step size between data points and use a fine discretisation around the previous neutral stability point.

![Neutral stability curve](image)

Figure 5.6: Neutral stability curve for volumes around 1 and Bond number \( B = 0.478 \).

In particular we can determine the values required for our Taylor series formula. Taking our average experimental Bond number to be \( \bar{B} = 0.478 \) and step sizes of \( h = 0.001, k = 0.001 \), the coefficients,

\[
\Lambda_B = \frac{\Lambda(1, \bar{B} + h, 0) - \Lambda(1, \bar{B} - h, 0)}{2h} \approx -1, \tag{5.3.24}
\]

\[
\Lambda_{VB} = \frac{\Lambda(1 + k, \bar{B} + h, 0) - \Lambda(1 + k, \bar{B} - h, 0) - \Lambda(1 - k, \bar{B} + h, 0) + \Lambda(1 - k, \bar{B} - h, 0)}{4hk} \approx 0, \tag{5.3.25}
\]

\[
\Lambda_{BB} = \frac{\Lambda(1, \bar{B} + h, 0) - 2\Lambda(1, \bar{B}, 0) + \Lambda(1, \bar{B} - h, 0)}{h^2} \approx 0, \tag{5.3.26}
\]

and hence our Taylor series formula becomes

\[
\Lambda(1 + \delta, B, W) \approx \Lambda(1 + \delta, B + \varepsilon, W) + \varepsilon. \tag{5.3.27}
\]
5.4 Results

In this section we present the results found by the numerical simulations, before discussing the results of the experiments. Figure 5.7 shows both the damping factor and oscillation frequency characterizing the first axisymmetric oscillation mode as a function of liquid bridge slenderness while the rest of parameters remain fixed. Without vibration, the oscillation frequency vanishes and the damping factor curve splits into two branches (only the dominant one is plotted) at $\Lambda \approx 1.78$. The dominant branch behind the curve split decreases very sharply. The numerical method fails to calculate the equilibrium shape very close to the stability limit, so the marginal stability point must be determined by extrapolation. Interestingly, the damping factor does not monotonically decrease as the slenderness increases. On the contrary, there is a small interval of $\Lambda$ before the split of the damping factor curve where the liquid bridge stabilizes as $\Lambda$ increases. This effect has been previously observed close to the minimum volume stability limit by Ponce-Torres et al. [2016], and must be attributed to gravity because it does not appear in the analysis by Nicolás and Vega [2000] of cylindrical shapes. The liquid bridge vibration does not significantly affect the eigenfrequency for slenderness smaller than that of the split point. However, it produces an almost constant lateral displacement of the dominant branch between the split and the neutral stability point. As a consequence, liquid bridges with slenderness within the interval $1.7 \leq \Lambda \leq 1.8$ are stabilized by the upper disk vibration for $\tilde{W} = 5.24 \times 10^{-3}$. Further, we see the damping factor far from the stability limit is not significantly affected by the disk vibration and this implies that the stabilization effect cannot be anticipated from the damping of perturbations in stable shapes.

In figure 5.8 the left graph shows the free surface position for the same liquid bridge with and without the vibration-induced pressure field. The lines and symbols correspond to the experimental and numerical contours, respectively. The difference between the volumes enclosed by the two curves is smaller than 0.03%. The numerical results perfectly fit the experimental contour in the ab-
Figure 5.7: Damping factor $\gamma$ and oscillation frequency $\omega$ for the leading eigenvalue, $\lambda = -\gamma + i\omega$, for $\bar{W} = 0$ (blue solid line) $\bar{W} = 2.86 \times 10^{-3}$ (red solid line) and $5.24 \times 10^{-3}$ (black dot-dashed line). With $V = 1$, $B = 0.478$, $Oh = 0.237$.

Figure 5.7: Damping factor $\gamma$ and oscillation frequency $\omega$ for the leading eigenvalue, $\lambda = -\gamma + i\omega$, for $\bar{W} = 0$ (blue solid line) $\bar{W} = 2.86 \times 10^{-3}$ (red solid line) and $5.24 \times 10^{-3}$ (black dot-dashed line). With $V = 1$, $B = 0.478$, $Oh = 0.237$.

sence of vibration. As can be observed, the vibration-induced pressure field partially compensates for the effect of gravity on the liquid bridge equilibrium shape. The upper disk vibration reduces the liquid bridge deformation, which slightly decreases the average free surface curvature, as shown in the right graph of figure 5.8. The mean curvature of the surface is expected to be linear in $z$ since, by differentiating the Young–Laplace equation, we find

$$\frac{dC}{dz} = -B,$$

thus we use this gradient of curvature for an initial estimate for the numerical fitting of the Bond number.

After the initial disturbance from the un-vibrated, curvature the two curva-
tures remain practically parallel in the rest of the liquid bridge. The slope $B_{\text{eff}}$ of the linear function $C(z) = C^0 - B_{\text{eff}}z$, where $C(z), C^0$ are the curvature of the vibrated and un-vibrated bridges respectively, fitted to the experimental values can be regarded as the effective Bond number accounting for both gravity and the vibration-induced pressure field. Figure 5.9 shows the values of $B_{\text{eff}}$ mea-
Figure 5.8: Free surface position $r = F(z)$ and local mean curvature $\hat{C}(z)$ for $\Lambda = 1.722, V = 0.8893, B = 0.4822, \text{Oh} = 0.237$. The dash lines and open symbols correspond to experimental and numerical results for an un-vibrated bridge respectively. The solid lines and symbols are the experimental and numerical results for $\tilde{W} = 5.24 \times 10^{-3}$ respectively.

The decrease of the effective Bond number is one order of magnitude larger than the $\tilde{W}$ value.

Figure 5.10 shows the maximum slenderness of 1 and 35 cSt silicone oil liquid bridges with and without vibration and the curve of neutral stability predicted using Slobozhanin and Perales method. When the upper disk vibrates, the maximum slenderness increases around $0.03 – 0.1$ depending on the liquid bridge volume. This increase is around 0.03 for $V \approx 1$, which is consistent with the numerical results shown in figure 5.7. The two liquids follow the same trend despite the large difference between the corresponding values of the Ohnesorge number, which indicates that viscosity does not significantly effect the stability limit. This result could be anticipated from the linear stability analysis because both the frequency and damping factor become zero at marginal stability, as shown in figure 5.7 and therefore the eigenmode velocity field vanishes there. However, it should be noted that the liquid bridge was not strictly at equilib-
Figure 5.9: Experimental values of the effective Bond number $B_{\text{eff}}$ as a function of slenderness, $\Lambda$. The open and solid symbols correspond to without and with vibration ($W = 1.3$), respectively.

rium when vibration was applied due to the finite values taken by both the vibration amplitude and frequency. In fact, the dynamical effects of the upper disk vibration on the base state became more apparent as viscosity decreased. The fact that viscosity did not significantly affect the maximum slenderness in our experiments, as shown in figure 5.10, suggests that those dynamical effects do not considerably alter the liquid bridge stability.

5.5 Conclusions

In this chapter we have shown vibrating the upper rod of a liquid bridge to be a stabilising process. Importantly the vibrations are small enough in amplitude to not cause any noticeable capillary surface waves. In fact, if there were waves with an amplitude similar to the characteristic length of the bridge the process would be very different.

An important note about this stabilisation process is that it does not require the liquid to have any special properties, in contrast with stabilisation processes
Figure 5.10: Experimental data points for the maximum slenderness with average Bond number 0.4782, $\text{Oh} = 6.91 \times 10^{-3}$ (crosses) and 0.237 (circles). The red and blue points represent with and without vibration. The blue line is the maximum slenderness line predicted using the method of Slobozhanin and Perales.

using electro-magnetic fields, which utilise magnetic properties of the liquid to counteract gravitational effects by efficient alignment of electro-magnetic fields. 

[Benilov 2016] gave a description of this stabilisation method and examined the case for near cylindrical bridges, i.e., bridges with asymptotically small Bond number and modified volume asymptotically close to unity. Here we have extended this idea to show that stabilisation effect occurs even for bridges of larger Bond number, however to produce a similar increase in the stable zone one requires a much higher frequency vibration. We have seen by examining the neutral stability condition, increasing the energy of the vibration aids the stability if the centre of mass of the bridge is below its geometric centre. Further, when these two coincide we find the slenderness of the bridge is exactly $\pi$, in agreement with the Plateau–Rayleigh stability condition.

One example of an application for this method is the transfer of a liquid drop from a solid surface to another solid surface through the formation, stretching
and break up of a liquid bridge. By controlling the vibration one can exactly control the transfer ratio of the mass. This is obtained because the vibration acts in the same way gravity does, just in an opposing direction, on a liquid bridge and therefore even when the vibration is large enough to be a destabilising factor it would still be drawing mass towards the upper surface.

The analysis presented in the section could be extended to a liquid bridge not anchored to the edge of the rods, e.g., if the rods are replaced by large, flat plates. The contact lines would then unpin during the quasi-static evolution we describe in the experiment. Further, this analysis could be extended to both liquid lenses and pendant drops.
Chapter 6

Discussion

“I just sit at the typewriter and
curse a bit”

P.G. Wodehouse

This thesis has examined the static equilibrium shapes and stability of various capillary surfaces. The equilibrium shapes were accurately approximated using an asymptotic series solution in the micro-gravity limit. The stability of two of these capillary surfaces was then examined using an energy functional method and the neutral stability curves were found to agree with those found using a linear stability analysis. Finally, a method of improving the stability of a vertical liquid bridge was examined numerically and experimentally.

At the start of this thesis we described the production process of a throat stent to motivate the work which followed. Recall that during the production of a stent, a liquid-polymer is placed onto a mesh before being baked to produce a plastic coating. The liquid-polymer, before being baked, is assumed to be entirely supported by the mesh and furthermore the liquid in one ‘diamond’ is not connected to the liquid in another ‘diamond’. The liquid contained in one ‘diamond’ is then the area of study for the majority of this thesis.

As mentioned, the liquid in one ‘diamond’ is supported by the mesh which is made up of overlapping cylindrical wires. This geometry is simplified by
firstly considering a rectangular region contained within vertical walls. When one of the lengths of this rectangular region is much greater than the other this can be studied by examining the cross section, which is a two-dimensional horizontal liquid bridge. The other simplification we consider is by considering the liquid being contained within a cylinder. While the resulting structure is three-dimensional, its axisymmetry permits a similar investigation.

Both the simplifications we used have removed corners from the geometry and have replaced the curved surfaces of the cylindrical wires with flat vertical walls. These simplifications remove any complications caused by boundary layers which can appear near the corners of such domains.

For these geometries we started by constructing asymptotic series solutions in the micro-gravity limit, since the length scale of the stent can be smaller than 2mm. The study of these structures is then continued by examining their stability using a functional analysis method, a linear stability examination and finally a bifurcation method. From these, we find the stable regions in the Bond number–central separation distance and volume–central separation distance phase planes, for given volumes and Bond numbers respectively.

Future work in this area can get closer to the stent problem by either using a more physically relevant geometry or by including more physical elements, for example temperature gradients and fluid rheology. In the following paragraphs we give a few examples of these methods. In all these cases the closer we get to the full stent problem, the more challenging the problem becomes.

One very similar geometry to that of the liquid membrane trapped inside a cylinder which is closer to the stent problem is that of a liquid membrane trapped inside a torus. In similarity to the problem studied here, this geometry retains the axisymmetric property, while adding the curved nature of the supporting structure which is similar to the stent problem. Due to the similarity to the problems considered here it is expected that all three methods of studying the stability should be tractable.

Another progressive step toward the stent problem is to consider a liquid membrane trapped inside a rectangular prism, i.e., replacing the circular cross-
section of the cylindrical case studied here by a rectangular cross-section. In this case, the Young–Laplace equation becomes a partial differential equation of two variables. In general this equation will not be directly solvable in the same way that cases studied in this thesis were, and therefore to find the shape of the membrane is significantly more challenging than with the cases we have studied, and the stability will be a further complication.

The full geometry of the stent problem with the ‘criss-cross’ of cylinders could later be examined through a numerical study, building on an analysis of a rectangular prism, to complete the work in this area.

During the baking process it is expected that the liquid will not be isothermal, as was assumed in this thesis. The temperature dependence will induce a gradient of the surface tension\textsuperscript{1} which will cause a mass transfer along the surface known as the Marangoni effect. The liquid will flow from areas of low surface tension to areas with higher surface tension. The study of thermocapillary vertical liquid bridges has been a fruitful area of study, see for example the work of Wanschura et al.\textsuperscript{1995}, Schwabe\textsuperscript{1988} (and references therein) and more recently Shitomi et al.\textsuperscript{2019}. The work of Schwabe\textsuperscript{1988} gives a detailed monograph of the application of the thermocapillary liquid bridges to crystal growth in melts. Wanschura et al.\textsuperscript{1995} used numerical techniques to study the most destabilising perturbations of a thermocapillary liquid bridge and Shitomi et al.\textsuperscript{2019} again used numerical computations to understand experimental results from the International Space Station.

The changing temperature will not only cause gradients in the surface tension, but it will also cause polymerisation of the polymer-liquid. This was the primary area of interest for the initial investigation by Chapman et al.\textsuperscript{2010}, although in that study the liquid is not only supported by the mesh frame but also by a solid structure beneath the mesh. This change in geometry sufficiently stabilises the liquid structure so that rupture can only occur through insufficient volume rather than, as shown in this thesis, through a growing unstable perturbation.

\textsuperscript{1}Surface tension is a function of temperature.
Finally, as the liquid evaporates from the bulk of the liquid the contact line will not remain pinned. In particular, oscillations to the free surface, after a sufficient volume of the liquid has evaporated, will not preserve the position of the contact line. There have been numerous methods presented to model the dynamics of a contact line. Certain key methods are listed in Table 6.1. To fully understand the stability of the polymer-liquid as evaporation (and polymerisation) occurs, contact line dynamics must be incorporated to the model.

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Van der Waals / Mesoscopic precursor film</td>
<td>de Gennes and Hervet 1984</td>
</tr>
<tr>
<td>Molecular film</td>
<td>Eres et al. 2000</td>
</tr>
<tr>
<td>Navier slip</td>
<td>Huh and Scriven 1971</td>
</tr>
<tr>
<td>Nonlinear slip</td>
<td>Thompson and Troian 1997</td>
</tr>
<tr>
<td>Surface roughness</td>
<td>Hocking 1976</td>
</tr>
<tr>
<td>Shear thinning</td>
<td>Gorodtsov 1989</td>
</tr>
<tr>
<td>Diffuse interface</td>
<td>Seppecher 1996</td>
</tr>
<tr>
<td>Normal stresses</td>
<td>Boudaoud 2007</td>
</tr>
<tr>
<td>Self induced Marangoni effect</td>
<td>Shikhmurzaev 1993</td>
</tr>
<tr>
<td>Dynamic contact angle</td>
<td>Greenspan 1978</td>
</tr>
</tbody>
</table>

Table 6.1: Table listing the mechanisms proposed to overcome the singularities which occur at the contact point and their reference.

Other capillary problems can be studied using methods developed in this thesis. The remainder of this section is dedicated to describing some example of these and how methods presented in this thesis could be applied to these problem.

One problem of interest is how a drop evaporates on a flat chemically heterogeneous surface. It has recently been shown, in two dimensions, that even for smooth (i.e. differentiable) heterogeneities, the contact line can jump during evaporation, see Pradas et al. 2016. A similar result can be obtained by
studying the turning points in the volume–radius and height–radius graphs. Indeed, the stability changes as a turning point is passed on a bifurcation diagram is identical to the results we have presented here. This method should also be applicable to the three dimensional problem.

The bifurcation method should also be applicable to study how rotation of the cylinder in the cylindrical problem of sections 2.3 and 3.1.2 affects the stability. In this case the modified Young–Laplace equations to model the shape of the interfaces can be derived either directly from the Navier–Stokes equations or from a method similar to that described by Myshkis et al. [1987].

Finally, the method of vibrating a substrate could be used to improve the stability of pendant droplets. The underlying mechanic of stabilisation should prove to be similar to that of the vertical liquid bridge studied in chapter 5 where the vibration causes a pressure gradient which balances the effect of gravity.

Overall, the work of this thesis has found produced three principle results. We found accurate approximate solutions for the shape of certain capillary surfaces and tested them against numerical and exact solutions. These approximate solutions were built on the work of O’Brien and van den Brule [1991] and the exact solutions are similar to those derived by Lv and Shi [2018]. The stability of these surfaces was then studied and stability diagrams were presented using methods outlined by Benilov and Cummins [2013]. Finally, we showed a liquid bridge can be stabilised by vibrating the supporting structure, which verified the result of Benilov [2016]. The study of capillary surfaces continues to be a rich area of study and we expect many more interesting problems to arise in this area.
Appendices
Appendix A

Derivation of (2.2.68)

We consider an integral of the form

\[ \int_0^\phi \frac{\cos \theta \, d\theta}{\sqrt{1 + \kappa \cos \theta}} \]  

(A.0.1)

with \( \kappa > -1 \). Using the equality \( \cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} \) we rewrite the integral as

\[
\int_0^\phi \frac{\cos \theta \, d\theta}{\sqrt{1 + \kappa \cos \theta}} = \frac{1}{\sqrt{1 + \kappa}} \left( \int_0^\phi \frac{d\theta}{\sqrt{1 - \frac{2\kappa}{1 + \kappa} \sin^2 \frac{\theta}{2}}} - 2 \int_0^\phi \frac{\sin^2 \frac{\theta}{2} \, d\theta}{\sqrt{1 - \frac{2\kappa}{1 + \kappa} \sin^2 \frac{\theta}{2}}} \right)
\]  

(A.0.2)

substitution of \( \gamma = \frac{\theta}{2} \) yields

\[
\int_0^\phi \frac{\cos \theta \, d\theta}{\sqrt{1 + \kappa \cos \theta}} = \frac{1}{\sqrt{1 + \kappa}} \left( 2 \int_0^{\frac{\phi}{2}} \frac{d\gamma}{\sqrt{1 - \frac{2\kappa}{1 + \kappa} \sin^2 \gamma}} - 4 \int_0^{\frac{\phi}{2}} \frac{\sin^2 \gamma \, d\gamma}{\sqrt{1 - \frac{2\kappa}{1 + \kappa} \sin^2 \gamma}} \right)
\]  

(A.0.3)
Note the first integral is now in the form of an elliptic integral of the first kind $F(\phi, k) = F\left(\frac{\phi}{2}, \sqrt{\frac{2\kappa}{1+\kappa}}\right)$. It just remains to rewrite the second integral:

\[
\int_0^{\hat{\phi}} \frac{\sin^2 \gamma \, d\gamma}{\sqrt{1 - \frac{2\kappa}{1+\kappa} \sin^2 \gamma}} = -\frac{1 + \kappa}{2\kappa} \int_0^{\hat{\phi}} \frac{-\frac{2\kappa}{1+\kappa} \sin^2 \gamma \, d\gamma}{\sqrt{1 - \frac{2\kappa}{1+\kappa} \sin^2 \gamma}}
\]

\[
= \frac{1 + \kappa}{2\kappa} \left( \int_0^{\hat{\phi}} \frac{d\gamma}{\sqrt{1 - \frac{2\kappa}{1+\kappa} \sin^2 \gamma}} - \int_0^{\hat{\phi}} \frac{1 - \frac{2\kappa}{1+\kappa} \sin^2 \gamma \, d\gamma}{\sqrt{1 - \frac{2\kappa}{1+\kappa} \sin^2 \gamma}} \right)
\]

\[
= \frac{1 + \kappa}{2\kappa} \left( \int_0^{\hat{\phi}} \frac{d\gamma}{\sqrt{1 - \frac{2\kappa}{1+\kappa} \sin^2 \gamma}} - \int_0^{\hat{\phi}} \sqrt{1 - \frac{1 + \kappa}{1+\kappa} \sin^2 \gamma} \, d\gamma \right).
\]

(A.0.4)

Observe the first integral in the above expression is again $F\left(\frac{\phi}{2}, \sqrt{\frac{2\kappa}{1+\kappa}}\right)$. The second integral is an elliptic integral of the second kind $E(\phi, k) = E\left(\frac{\phi}{2}, \sqrt{\frac{2\kappa}{1+\kappa}}\right)$. Thus aggregating the expressions, we can conclude

\[
\int_0^{\phi} \frac{\cos \theta \, d\theta}{\sqrt{1 + \kappa \cos \theta}} = \frac{2}{\kappa \sqrt{1 + \kappa}} \left( (1 + \kappa)E\left(\frac{\phi}{2}, \sqrt{\frac{2\kappa}{1+\kappa}}\right) - F\left(\frac{\phi}{2}, \sqrt{\frac{2\kappa}{1+\kappa}}\right) \right),
\]

(A.0.5)

as required.
Appendix B

Second order terms for

Chapter 2

In chapter 2 we found leading and first order terms for the asymptotic series solution of the Young–Laplace equation in the three cases of a drop resting freely upon an inclined plane, an infinite horizontal liquid bridge, and a cylindrical horizontal liquid membrane. This appendix, for the first two cases, begins by writing a general iterative solution. This general iterative solution is written in terms of lower order coefficients and their integrals, while these solutions may be algebraic expensive to calculate a highly accurate series solution can be obtained in terms of the leading order solutions.

B.1 Drop on an inclined plane

Recall the shape of the drop is given, in inclination formulation, by

\[ h' [\varepsilon (x \sin \alpha + h \cos \alpha) + p] = \sin \phi, \quad (B.1.1) \]

\[ x' [\varepsilon (x \sin \alpha + h \cos \alpha) + p] = \cos \phi. \quad (B.1.2) \]
We write \( h, x, p \) as power series of \( \varepsilon \), i.e.,
\[
    h = \sum_{i=0}^{\infty} h_i(\phi)\varepsilon^i, \tag{B.1.3}
\]
\[
    x = \sum_{i=0}^{\infty} x_i(\phi)\varepsilon^i, \tag{B.1.4}
\]
\[
    p = \sum_{i=0}^{\infty} p_i\varepsilon^i, \tag{B.1.5}
\]
to obtain
\[
    (h_0'p_0 - \sin \phi) + \sum_{i=0}^{\infty} \left( \sin \alpha \sum_{k=0}^{i} h_k'x_{i-k} + \cos \alpha \sum_{k=0}^{i} h_k'h_{i-k} + \sum_{k=0}^{i+1} h_k'p_{i-k+1} \right)\varepsilon^{i+1} = 0 \tag{B.1.6}
\]
\[
    (x_0'p_0 - \cos \phi) + \sum_{i=0}^{\infty} \left( \sin \alpha \sum_{k=0}^{i} x_k'x_{i-k} + \cos \alpha \sum_{k=0}^{i} x_k'h_{i-k} + \sum_{k=0}^{i+1} x_k'p_{i-k+1} \right)\varepsilon^{i+1} = 0. \tag{B.1.7}
\]
Consider now, the \( \varepsilon^{n+1} \) term
\[
    h_{n+1}'p_0 + \sum_{k=0}^{n} (h_k'x_{n-k} \sin \alpha + h_k'h_{n-k} \cos \alpha + h_k'p_{n-k+1}) = 0, \tag{B.1.8}
\]
\[
    x_{n+1}'p_0 + \sum_{k=0}^{n} (x_k'x_{n-k} \sin \alpha + x_k'h_{n-k} \cos \alpha + x_k'p_{n-k+1}) = 0, \tag{B.1.9}
\]
we will proceed by integrating, and noting \( p_i \) are constant (with respect to \( \phi \)) for all \( i \). First, let
\[
    I_n = \int \sum_{k=0}^{n} h_k'x_{n-k} \, d\phi, \tag{B.1.10}
\]
then
\[
    h_{n+1}p_0 + I_n \sin \alpha + \sum_{k=0}^{n} h_k \left( h_{n-k} \frac{\cos \alpha}{2} + p_{n-k+1} \right) = 0, \tag{B.1.11}
\]
\[
    x_{n+1}p_0 - I_n \cos \alpha + \sum_{k=0}^{n} x_k \left( x_{n-k} \frac{\sin \alpha}{2} + h_{n-k} \cos \alpha + p_{n-k+1} \right) = 0, \tag{B.1.12}
\]
which, upon calculation of $I_n$, allows us to write $h_{n+1}$ and $x_{n+1}$ in terms of lower order terms, and the as-yet undetermined $p_{n+1}$. Indeed, when calculating $I_n$ we observe that all terms in (B.1.11) and (B.1.12) vanish at $\phi = -\theta_+$ except the term involving $I_n$, therefore the constant of integration must be chosen to force $I_n$ to vanish at $\phi = -\theta_+$ also. To evaluate $p_{n+1}$ we use the $n$-th order Taylor series expansions of $h(\theta_+ + \varepsilon \Delta) = 1$ and $x(\theta_+ + \varepsilon \Delta) = 0$, and use a similar power series for $\Delta$. Therefore setting $n = 1$ in the above solutions we find

$$h_2 = -\frac{1}{p_0} \left[ I_1 \sin \alpha + h_0 h_1 \cos \alpha + h_1 p_1 + h_0 p_2 \right], \quad (B.1.13)$$

$$x_2 = -\frac{1}{p_0} \left[ -I_1 \cos \alpha + x_0 x_1 \sin \alpha + (x_0 h_1 + x_1 h_0) \cos \alpha + x_1 p_1 + x_0 p_2 \right], \quad (B.1.14)$$

with

$$I_1 = \int (h_0' x_1 + h_1' x_0) \, d\phi, \quad (B.1.15)$$

and $p_2$ is found by solving,

$$x_0' \left( h_2 + \Delta_0 h_1' + \frac{\Delta_0^2 h_0''}{2} \right) = h_0' \left( x_2 + \Delta_0 x_1' + \frac{\Delta_0^2 x_0''}{2} \right) \text{ at } \phi = \theta_+. \quad (B.1.16)$$

### B.2 Infinite horizontal liquid bridge

Recall the equilibrium surfaces are given by

$$\begin{cases} (\varepsilon h_\pm + P) x'_\pm = \cos \phi, \\ (\varepsilon h_\pm + P) h'_\pm = \pm \sin \phi, \end{cases} \quad (B.2.1)$$

following the same method as the previous section we use power series solutions in $\varepsilon$ to obtain

$$\begin{cases} \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} x_{\pm}^{(j)} h_{\pm}^{(i-j)} \right) \varepsilon^{i+1} + \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} x_{\pm}^{(j)} P^{(i-j)} \right) \varepsilon^{i} = \cos \phi, \\ \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} h_{\pm}^{(j)} h_{\pm}^{(i-j)} \right) \varepsilon^{i+1} + \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} h_{\pm}^{(j)} P^{(i-j)} \right) \varepsilon^{i} = \sin \phi. \end{cases} \quad (B.2.2)$$
Consider the $\varepsilon_n^{n+1}$ term (for $n \neq -1$)
\[
\begin{cases}
  x_\pm^{(n+1)} P^{(0)} + \sum_{j=0}^{n} \left( x_\pm^{(j+1)} h_\pm^{(n-j)} + x_\pm^{(j+1)} P^{(n-j+1)} \right) = 0 \\
  h_\pm^{(n+1)} P^{(0)} + \sum_{j=0}^{n} \left( h_\pm^{(j+1)} h_\pm^{(n-j)} + h_\pm^{(j+1)} P^{(n-j+1)} \right) = 0
\end{cases}
\] (B.2.3)

We will proceed by integrating these, however first we define
\[
I_\pm^{(n)} = \int \sum_{j=0}^{n} x_\pm^{(j+1)} h_\pm^{(n-j)} \, d\phi
\] (B.2.4)
then we find,
\[
\begin{cases}
  x_\pm^{(n+1)} P^{(0)} + I_\pm^{(n)} + \sum_{j=0}^{n} x_\pm^{(j)} P^{(n-j+1)} = 0, \\
  h_\pm^{(n+1)} P^{(0)} + \frac{1}{2} \sum_{j=0}^{n} h_\pm^{(j)} h_\pm^{(n-j)} + \sum_{j=0}^{n} h_\pm^{(j)} P^{(n-j+1)} = 0.
\end{cases}
\] (B.2.5)

In particular we can set $n = 1$ to find the second order expression. We find,
\[
\begin{align*}
x_+^{(2)} &= -\frac{1}{P^{(0)}} \left[ I_+^{(1)} + x_+^{(0)} P^{(2)} + x_+^{(1)} P^{(1)} \right], \\
h_+^{(2)} &= -\frac{1}{P^{(0)}} \left[ \left( h_+^{(0)} + P^{(1)} \right) h_+^{(1)} + h_+^{(0)} P^{(2)} \right], \\
x_-^{(2)} &= -\frac{1}{P^{(0)}} \left[ I_-^{(1)} + x_-^{(0)} P^{(2)} + x_-^{(1)} P^{(1)} \right], \\
h_-^{(2)} &= -\frac{1}{P^{(0)}} \left[ \left( h_-^{(0)} + P^{(1)} \right) h_-^{(1)} + h_-^{(0)} P^{(2)} \right],
\end{align*}
\] (B.2.6-2.9)

where
\[
\begin{align*}
I_+^{(1)} &= \int \left[ h_+^{(0)} x_+^{(1)} + h_+^{(0)} x_+^{(0)} \right] \, d\phi, \\
I_-^{(1)} &= \int \left[ h_-^{(0)} x_-^{(1)} + h_-^{(0)} x_-^{(0)} \right] \, d\phi
\end{align*}
\] (B.2.10-11)
the constants of integration are found by forcing $x_\pm^{(2)}(0) = 0$, which yields,
\[
I_\pm^{(1)}(0) = -P^{(2)}.
\] (B.2.12)

Then $P^{(2)}$ is found by using $x_+(\alpha_+) = 0$, which gives,
\[
I_+^{(1)}(\alpha_+) = 0.
\] (B.2.13)

Indeed, these boundary conditions could have been applied in the reverse order yielding the same result.
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