Geometric-Material analogy for multiscale modelling of twisted plates

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Abstract

It is well known that the macroscopic behavior of many engineering materials is strongly affected by the role of underlying microstructure. Currently though, mathematical expressions linking behavior of large scale structures to the geometry of their microscopic structure are largely lacking. In this respect, establishing quantitative links across different material lengthscales may offer new pathways for engineering design. In the present work an analogy between cross sectional geometrical properties, representing macrostructure, and a material length parameter, representing microstructure, is presented. The analogy is established through the study of a thin plate subject to axial loading undergoing finite displacements from two alternative perspectives. First, we consider a thin elastic plate with a pretwist about the loading axis where a warping term is introduced accounting for the out-of-plane deformation of the cross section. The coupled governing differential equations and the corresponding coupled boundary conditions are explicitly derived employing a classical structural mechanics approach utilising an energy variational statement. Secondly, an axially loaded thin flat plate (i.e. with no pretwist) is studied with strain gradient elasticity theory incorporating only one material length parameter representing the microstructure, in addition to the two classical Lamé stiffness constants. The ensuing analogy emerges by comparison of the governing equations of the two formulations which shows a mathematical expression can be identified, which incorporates both geometric and material length variables, that formalises the link between microscale and macroscale. This mathematical expression, which constitutes the kernel of the proposed multiscale approach, admits a twofold interpretation depending on the assumed independent variable. On the one hand, the proposed multiscale modelling approach suggests that a plate with complex global geometry can be substituted by a structurally - equivalent, flat plate with constitutive relations given by a non - local, strain gradient theory. On the other hand, the material length parameter can be interpreted on a physical basis because for the first time it has been identified as a known function of geometrical features of the structure through simple algebraic relationships for various cross sectional profiles.

Keywords: *warping deformation, pretwist, strain gradient elasticity, variational principle, Helmholtz equation*

1 Introduction

It is well known that twisted structural elements can be found in many practical engineering applications including, rotor machinery components (i.e. rotor blades, helicopter blades, wind turbines), impellers, drilling devices used

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in offshore structures and yarns in structural textiles (i.e. body armour vests). The initial stress - free state of these elements is characterised by a distinguishing feature, i.e. the initial twist, which serves various design aspects beneficially, e.g. aerodynamic performance. Inherently, initial twist in structures is associated with increased geometrical complexity, as well as, with geometric coupling between different types of loading which makes structural analysis a challenging task. Therefore, understanding the mechanical behaviour of these structural elements is crucially important in providing new pathways in engineering design.

The macroscopic mechanical response of many engineering materials is strongly affected by various interactions occurring at the scale of their microstructure. The inadequacy of classical continuum mechanics to capture such behavior necessitates the use of a more general continuum based theoretical framework capable of providing a thorough understanding and a quantification of the links across different lengthscales. Homogenisation techniques are commonly used to describe internal interactions of the microstructure utilising a continuum approach. Although generally useful, in many cases these techniques fail to adequately address microstructural features of interest and also fail to directly link them with the macrostructure.

In this respect, the motivation of the present work is to provide a new multi-scale modelling approach for structures by identifying and quantifying the interplay between two important structural design variables, i.e. geometry and material properties. Therefore, the proposed approach serves a dual target. On the one hand, it establishes an analogy that allows a direct mapping of the macroscopic geometry of the structure to the material properties of an equivalent, yet simpler, geometric structure based on non - local, strain gradient theory. In addition, the proposed analogy represents a strong - type representation because it is based on differential equations enabling geometrical variations of macrostructure to be modelled as material variations of the microstructure. On the other hand, our formulation provides a physical meaning of the micro-structural length parameters, that are currently found by empirical means in non - local strain gradient theories, by identifying them quantitatively as expressions of geometrical features of the macrostructure.

2 Background

Classical thin plate theory has been used extensively for the detailed analysis of demanding structural components over a wide range of engineering length scales spanning micro-scale to the macro-scale. At the microlevel, the structural performance of devices such as capacitive micro-sensors and micro-actuators, [1],[2], thin film/substrate components [3] have been studied within the framework of classical thin plate theory providing satisfying results. Also, at the macrolevel, structures including wing skins, flaps and vertical fins of aircraft have been modelled as thin flat plates also providing good agreement with experimental results [4]. The modelling of these structures is based upon the Kirchhoff - Love hypothesis whereby the transverse shearing stresses are neglected usually without great loss of accuracy and the in-plane displacements can be thought of as linear functions of the through thickness coordinate.

2.1 Structural mechanics for pretwisted plates

Many research studies exist in the literature regarding the analysis of thin pretwisted plates. Green [5], [6] was the first who studied the problem of a twisted strip in the framework of nonlinear plate theory investigating various instabilities. He showed that if a constant longitudinal force is applied at twisted ends of the strip there is a critical value of pretwist where the strip buckles to nodal lines perpendicular to the axis of twist due to developing compressive forces at the edges of the strip. This result demonstrates a buckling phenomenon due to geometric coupling between tension and torsion. Maunder and Reissner [7] obtained an explicit solution to the problem of pure elastic bending of a pretwisted rectangular plate within the framework of linear theory of thin elastic shells.

Note, that all of the above mentioned works are based on a purely structural mechanics perspective and the role of microstructure is not modelled and for isotropic, homogeneous materials matches with experimental data well [8]. For other materials, with well-described inhomogeneities, microstructural continuum theories may become more appropriate models and are introduced in the next section.

In this work, the displacement field of classical plate theory is modified in order to account for the effects of cross sectional warpage due to pretwist and the ensuing twisting curvature of the structural component. Note, by considering relatively small angles of pretwist then thin twisted plates can be visualised as thin shallow shells with twist curvature.

2.2 The role of underlying microstructure in plates - Size effects

The fundamental assumption in Classical Plate Theory is that one dimension of the structure is small compared with other characteristic dimensions of the structure. This assumption is beneficial in the structural analysis of plates because it allows the reduction of a three-dimensional structure to a two-dimensional one that is more easily solved mathematically. However, it is well known that when the dimensions of a structural component become comparable to the size of its material microstructure, so-called *size effects* are observed which cannot be described by a classical structural approach. It has been proven experimentally that in many engineering materials the underlying microstructure may affect the overall mechanical response of the component at the macroscopic level, [9], [10], [11]. Hence, microstructure constitutes an essential feature of the material that in some cases may lead to enhanced mechanical properties. It becomes evident, that the modelling of materials with microstructure is of great importance despite the fact that it may become a challenging and delicate task due to induced mathematical complexities.

To this end, the effects of microstructure on the overall mechanical response of a material can be described by two approaches

1. taking as a starting point some details of the microscale and then using a homogenization technique to describe the macroscale
2. taking as a starting point the macroscale and utilising some average procedures to account for the microscale.

The first approach is established on the idea that the features of interest of the microstructure are established a priori (e.g. the preferred directions of a fabric) within a representative volume element of the structure and then adopting a subsequent homogenization scheme to identify mechanical properties at larger lengthscales. This philosophy is the so-called *bottom up approach* where a remarkable volume of research articles exist in the literature [12], [13], [14], [15], [16], [17], [18]. The advantage of this approach is that the microscopic parameters are related directly, in a physically intuitive way, with the macroscopic features of the material. However, the simplifying assumptions regarding the underlying microstructure do not necessarily represent the actual behavior of the structure at the macroscale.

According to the second approach, the analysis starts at the macroscopic level and the role of microstructure is involved through additional intrinsic parameters (e.g. material length parameters) entering the formulation either through the constitutive equations or the equations of motion of the elastic medium [21],[22], [23], [24], [25], [26]. This philosophy is the so called *top down approach* which actually attempts to describe the way that microstructure manifests itself at the macroscopic level through simple models. The main drawback of this approach is that the intrinsic parameters used to describe the microstructure are not related in a physical way with the macroscopic characteristics of the structure.

Note, that the present work proposes a twofold analogy between macrostructural geometric properties and microstructural material length parameters which is based on a comparison of analogous differential equations and ensuing algebraic relationships. That means both pathways of multiscale modelling are satisfied in a strong mathematical form in contrast to the majority of studies available in the literature where the primary target is one-sided, i.e. to link microstructural parameters with the macrostructural properties, based on weak form energy statements. A key advantage of this new approach is therefore the point-wise homogenization without need of a representative volume element.

2.3 Generalised continuum theories

2.3.1 Preliminary remarks

Generalised continuum theories fall naturally within the top down approach and their distinguishing feature from classical continuum mechanics theory is based on the fact that the interaction of material particles within the continuum cannot fully be described by a traction vector. That is, the Cauchy postulate breaks down and an additional quantity, the couple stress tensor, is required to describe the aforementioned interaction. The immediate implication of this assumption is that in generalised continuum theories the stress tensor is no longer symmetric[1]. As a general remark it should be noted that generalised continuum theories are actually homogenized formulations based upon the fact that some details of the microstructure are "smeared out". Clearly, this loss of information constitutes a compromise between the realism of the model and induced complexity.

2.3.2 Material approach

According to the vast literature devoted to the generalised continuum theories, there are two ways to generalise the classical Cauchy continuum theory, both based on the strain energy density expression. First, the same deformation metric, i.e. the deformation, is retained together with some additional higher gradients of this metric or the deformation metric is extended further by including additional degrees of freedom. In this respect, there is a large variety of generalised continuum theories available in the literature depending on exactly which deformation metric is used in the strain energy expression. According to the Cosserat brothers [21] each material particle of the body can be modelled with a triad of vectors, called directors, which rotate and translate independently of the macrocontinuum. These formulations are now referred to as micropolar theories and attempt to capture effects of the microstructure introducing the concept of micro-rotation, ϕ_i, which accounts for the microrotations of the microelements, [22], [23]. In the case where the microrotation is independent of the rotation of the continuum the theory is named as *unrestricted micropolar theory* while in the case where the two rotations are related the theory is called *restricted micropolar theory* or *couple stress theory*, [24], [25]. On the other hand, Mindlin et.al. [26]

[1] Interestingly, it has been found that non symmetric tensors emerge also in the analysis of double-curved shells on the basis of classical theoretical framework [19], [20]. Such kind of "symmetry breakage" might be a key factor for further developments in multiscale modelling.

proposed a theory by considering strain energy density as a quadratic form not only of strains but also of gradient of strains. According to this formulation eighteen material parameters are involved, with two being the classical Lamé stiffness constants. The difficulty in determining experimentally all of these new material parameters makes this formulation complex and unwieldy. Later, Mindlin [27], Mindlin and Eshel [28] proposed a simplified version of the initial formulation where for a centro-symmetric, isotropic and linear elastic material five material constants are required, with two being the classical Lamé stiffness constants.

2.3.3 Structural approach - Plates and Shells

A considerable number of research studies exist in the literature regarding the analysis of thin elastic plates and shells undergoing finite deformations within the framework of generalised continuum theories. Lazopoulos [29], employing strain gradient elasticity, studied the localised buckling of a long span nonlinear plate under uniaxial in-plane compression. Papargyri-Beskou and Beskos [30] dealt with the stability analysis of gradient elastic circular cylindrical thin shells under axial compression and provided an analytical expression of the governing equation of the problem which happens to be of tenth order. Yang et.al. [31] established a modified couple stress theory by introducing an additional condition regarding the equilibrium of moments of couples between the material particles of the continuum. Tsiatas [32], based on Yang's modified theory, developed a new Kirchhoff model for isotropic linear plates with arbitrary planform shape. Asghari [33] derived the governing equation of motion for geometrically nonlinear microplates utilising Yang's modified couple stress theory. Reddy et.al. [34] provided a thorough review and possible equivalences among strain gradient, couple stress and modified couple stress models for beams and plates.

2.4 Scope and structure of the present work

In this work, a twofold analogy between the macroscopic features of the structural component and the microstructural material length scale parameter is developed. On one hand, the analogy offers the ability to model geometrical variations at the macroscale as material variations at the microscale. On the other hand, our approach suggests direct quantitative links between material length parameter, related to microstructure, and geometrical properties, related with the macrostructure. In order to identify this connection between the two distinct scales we consider an elastic thin plate which is studied under two conceptually different approaches. First, we consider that the plate is pretwisted about the loading axis and analysis is conducted within the theoretical framework of classical structural mechanics also modelling the warping deformation of the plate's cross section. The governing equilibrium equations and the corresponding boundary conditions of the elastic pretwisted thin plate derived through a variational statement are presented in Section 3. In Section 4, the strain gradient, elastic thin plate incorporating one material length parameter related to the microstructure is addressed. In Section 5, the equivalence between the two approaches is established through a Helmholtz type homogeneous differential equation. The analysis shows that the warping term constitutes the connecting link between the two approaches. Furthermore, explicit relationships between the microstructural length and the geometric properties of various cross sections with known warping functions are presented. The paper concludes with a summary in Section 6.

3 Geometrically nonlinear thin pretwisted plate with warping effects

Consider a homogeneous and isotropic thin plate, initially flat, lying in the $x-\eta$ plane coincident with the mid-plane of the plate. A cartesian coordinate system $Ox\eta\zeta$ is attached to the plate as depicted in Fig.1(a). The plate is pretwisted around the longitudinal axis x while at the same time is subject to a tension load P_x as shown in Fig.1(b). In such cases, it is more convenient to introduce a new convected coordinate system, $Oxyz$, which describes the deformation of the cross section of the pretwisted plate, locally. Each cross section of the thin plate experiences an angle of twist $d\theta/dx=\alpha_0 x$ as does the convected coordinate system which follows the local deformation of the plate. In this respect, the two coordinate systems are related through the transformation formula

$$y=\eta cos\left(\alpha_0 x\right)+\zeta sin\left(\alpha_0 x\right) \tag{1}$$

$$z=-\eta sin\left(\alpha_0 x\right)+\zeta cos\left(\alpha_0 x\right) \tag{2}$$

where the following relations hold

$$\frac{\partial y}{\partial x}=\alpha_0 z, \qquad \frac{\partial z}{\partial x}=-\alpha_0 y, \qquad \frac{\partial f\left(y,z\right)}{\partial x}=\alpha_0\left(z\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial z}\right) \tag{3}$$

and where $f(y,z)$ is an arbitrary function. In the convected coordinate system, the components of the displacement vector $\boldsymbol{u}$ can be written in the form

$$\begin{aligned} u&=u_0\left(x,y\right)+\psi\frac{\partial\theta}{\partial x}-z\frac{\partial w}{\partial x}\\ v&=v_0\left(x,y\right)-z\frac{\partial w}{\partial y}\\ w&=w\left(x,y\right)\end{aligned} \tag{4}$$

and where $u_0\left(x,y\right), v_0\left(x,y\right)$ represent in-plane displacements of the mid-plane in x,y directions respectively, $w\left(x,y\right)$ denotes the lateral deflection of the mid-plane, $\psi\left(x,y,z\right)$ denotes the warping function of the cross section accounting for the out of plane displacement and $\partial\theta/\partial x$ denotes the angle of twist per unit length of the cross section which, in general, is not constant.

It is well known, that the two dimensional equilibrium equations and the corresponding boundary conditions for the thin plate can be derived from the general three dimensional equations of nonlinear elasticity by integrating through the plate's thickness. Whereas this technique is powerful it leads to boundary conditions which often do not emerge naturally and furthermore are not consistent with the two dimensional equations. Hence, in the present analysis we use the variational statement based on the minimum potential energy of the plate in order to derive the equilibrium equations and the corresponding boundary conditions. The principal of virtual work reads

$$\delta V^p=\delta U^p-\delta W^p=0\Rightarrow\delta U^p=\delta W^p \tag{5}$$

where δV^p is the potential energy of the elastic nonlinear thin pretwisted plate, δU^p is the strain energy density while δW^p represents the work done by the external forces. The variation of the strain energy of the pre-twisted thin plate, δU^p is given by the expression

$$\delta U^p=\frac{1}{2}\int\int\int\delta\left[\sigma_{xx}\epsilon_{xx}+\sigma_{yy}\epsilon_{yy}+\tau_{xy}\gamma_{xy}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{6}$$

whilst the work done by the external forces can be written as

$$\delta W^p = \int\int \left[X\delta u + Y\delta v + Z\delta w + R\left(\psi\frac{\partial\theta}{\partial x}\right)\right]\mathrm{dxdy} + \int \left[N_u^x\delta u\right]\mathrm{dx} + \int \left[R_\psi\delta\left(\psi\frac{\partial\theta}{\partial x}\right)\right]\mathrm{dx} + \int \left[N_w^x\delta w\right]\mathrm{dx} + \int\left[m_{xx}^x\frac{\partial\delta w}{\delta x}\right]\mathrm{dx} + \int\left[N_v^x\delta v\right]\mathrm{dx} + \int\left[N_v^y\delta v\right]\mathrm{dy} + \int\left[N_w^y\delta w\right]\mathrm{dy} + \int\left[m_{yy}^y\frac{\partial\delta w}{\delta y}\right]\mathrm{dy} + \int\left[N_v^y\delta v\right]\mathrm{dy} \tag{7}$$

where the external forces X, Y, Z which are prescribed per unit area of the mid-plane, are the components of the body forces in the x, y, z directions respectively, and R is the generalised body force ("bi-moment") related to the warping displacement of the cross section. At the boundary, R_ψ represents the boundary warping term, $N_u^x, N_w^x, N_v^x, N_v^y, N_w^y, N_v^y$ represent the components of the traction forces per unit length in the x, y directions, respectively, and m_{xx}^x, m_{yy}^y represent the moments per unit length at the boundary produced by couple-like tractions in the x, y directions, respectively.

Substituting into Eq.(6), the expressions of strains and stresses as given by Eqs.(A-1)-(A-6) and Eqs.(A-7)-(A-9), respectively, Eq.(5) in view of Eq.(7) after integrating through the thickness h of the plate and performing the appropriate algebra provides the four coupled equilibrium equations of the problem

$$\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{(1+\nu)}{2}\frac{\partial^2 v_0}{\partial x\partial y} + \frac{(1+\nu)}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 u_0}{\partial y^2} + \frac{(1-\nu)}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x^2} + \frac{1-\nu^2}{Eh}X = 0 \tag{8}$$

$$\frac{\partial^2 v_0}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2} + \frac{(1+\nu)}{2}\frac{\partial^2 u}{\partial x\partial y} + \frac{(1+\nu)}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 v_0}{\partial x^2} + \frac{(1-\nu)}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x\partial y} + \frac{1-\nu^2}{Eh}Y = 0 \tag{9}$$

$$\frac{\partial^2\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x^2} + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 v_0}{\partial x\partial y} + \nu\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{1-\nu^2}{Eh}R = 0 \tag{10}$$

$$-\frac{Eh^3}{12\left(1-\nu^2\right)}\nabla^4 w + \left(-R\frac{\partial w}{\partial x} + Z - Y\frac{\partial w}{\partial y}\right) = 0 \tag{11}$$

as well as the corresponding coupled boundary conditions, see Eqs.(A-10)-(A-18). In the above procedure it is assumed that the products of the initial curvatures with the induced loads are sufficiently small so as to be neglected [35], see Eqs.(A-19)-(A-21).

Note, that both coupled Eqs (8),(9) expressing the in-plane equilibrium in x, y directions, respectively, involve the warping term. Interestingly, Eq.(10) relates the warping displacement directly with the in plane displacements u_0, v_0. Eq.(11) shows that in the absence of the "bi-moment" body force R and the body force Y in y-direction the classical governing equation of equilibrium of plate is retrieved. As expected, non-classical boundary conditions emerge involving prescribed values of higher order derivatives of deflection at the two ends of the plate.

In the next section, the problem of a thin flat plate subjected to axial loading is addressed within the framework of strain gradient elasticity.

4 Simplified strain gradient theory

In this section, we investigate the mechanical response of a homogeneous and isotropic thin flat plate subject to axial loading, Fig. (2), employing the simplest form of the strain gradient theory according to Mindlin [27],[28].

Mindlin's strain gradient theory can be cast in three different but equivalent forms, namely Form I, Form II and Form III. These forms are defined using three different expressions for the strain energy density function. In Form I, the strain energy function involves only the gradients of the displacements, in Form II strain energy includes the second gradients of strain while in Form III, which is actually the couple stress theory, gradients of rotation are involved. In the present work, Form II is employed, due to simplicity, where only one material length parameter is involved in addition to the two classical Lamé stiffness constants. In this respect, the strain energy function, w, can be written in the form

$$w\left(\epsilon_{ij}, \epsilon_{ij,k}\right) = \frac{1}{2}\lambda\epsilon_{ii}\epsilon_{jj} + \mu\epsilon_{ij}\epsilon_{ij} + g^2\left(\frac{1}{2}\lambda\epsilon_{ii,k}\epsilon_{jj,k} + \mu\epsilon_{ij,k}\epsilon_{ij,k}\right) \tag{12}$$

where λ, μ are

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{13}$$

with E and ν representing Young's modulus of elasticity and Poisson's ratio respectively. The term g is the gradient coefficient with dimensions of length squared. In Fig.(2), the subscripts of the stress components follow classical continuum mechanics notation where the first subscript denotes the plane while the second subscript denotes the direction. In view of Eqs.(B-1)-(B-16) in Appendix B, under plane stress conditions, the above equilibrium equations can be expressed in terms of the in-plane displacements u_0, v_0 and the transverse deflection w, as

$$\begin{gathered}\frac{(1-\nu)}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} + \frac{(1+\nu)}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 u_0}{\partial y^2} + \frac{(1+\nu)}{2}\frac{\partial^2 v_0}{\partial x\partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial P_x}{\partial x} - \\ g^2\nabla^2\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 v_0}{\partial x\partial y} + \nu\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\right) - \frac{1-\nu}{2}g^2\nabla^2\left(\frac{\partial^2 v_0}{\partial x\partial y} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\right) = 0\end{gathered} \tag{14}$$

$$\begin{gathered}\frac{(1-\nu)}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2} + \frac{(1+\nu)}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 v_0}{\partial x^2} + \frac{(1+\nu)}{2}\frac{\partial^2 u_0}{\partial x\partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v_0}{\partial y^2} - \\ g^2\nabla^2\left[\frac{\partial^2 v_0}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 u_0}{\partial x\partial y} + \nu\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y} - \frac{1-\nu}{2}\left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial x\partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\right)\right] = 0\end{gathered} \tag{15}$$

$$-\frac{Eh^3}{12(1-\nu^2)}\nabla^4 w + g^2\frac{Eh^3}{12(1-\nu^2)}\nabla^6 w + P_x\frac{\partial^2 w}{\partial x^2} = 0 \tag{16}$$

The above three expressions constitute the set of coupled equations expressing the equilibrium equation of the gradient thin flat plate undergoing finite deformations subjected to an axial load $P_x\,[N/m]$ along the $x-$direction. Note, that in the last equation it was assumed that the products of the initial curvatures with the induced loads are sufficiently small so as to be neglected, Eqs.(A-19)-(A-21).

The gradient elastic formulation was derived based on the classical equilibrium equations of the plate and not to a variational statement as in the case of the pretwisted plate, which means generalised boundary conditions are not available. Despite this limitation the value of our analysis is not compromised.

Let us note that other sources of non-locality are possible such as stress gradients, material heterogeneity, boundary layer effects, etc. but are beyond the merit of the present work. These aspects and their implications (e.g. the loss of some local information due to homogenization procedure) to macroscopically periodic structures will be investigated in detail in future work.

In the next section the analogy between the two approaches is discussed.

5 The analogy between the classical structural theory and the gradient elastic theory

In this section we investigate the analogy between the two approaches developed in the two preceding sections. Recall that the set of Eqs.(8)-(11) expresses the equilibrium of an initially twisted thin plate subjected to axial loading taking into account the warping effects of the cross section. Assuming that body forces and the bimoment are absent ($X = Y = Z = R = 0$), solving Eq.(10) with respect to the warping term we obtain

$$\frac{\partial^2 \left(\psi \frac{\partial \theta}{\partial x}\right)}{\partial x^2} = -\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 v_0}{\partial x \partial y} + \nu \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y}\right) \tag{17}$$

Note that the right hand side expression of Eq.(17) equals the argument of the nabla operator in the second last term of Eq.(14). Substituting the last equation into Eq.(8) yields

$$\frac{\partial^2 v_0}{\partial x \partial y} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = 0, \qquad (\nu \neq 1) \tag{18}$$

Note that Eq.(18) equals the argument in the nabla operator of the last term in Eq.(14). In view of Eqs.(17),(18), Eq.(14) may be written as

$$\frac{(1-\nu)}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} + \frac{(1+\nu)}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 u_0}{\partial y^2} + \frac{(1+\nu)}{2}\frac{\partial^2 v_0}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial P_x}{\partial x} + g^2 \nabla^2 \left[\frac{\partial^2 \left(\psi \frac{\partial \theta}{\partial x}\right)}{\partial x^2}\right] = 0 \tag{19}$$

Integration of Eq.(17) with respect to x provides

$$\frac{\partial \left(\psi \frac{\partial \theta}{\partial x}\right)}{\partial x} = -\frac{\partial u_0}{\partial x} - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \nu \frac{\partial v_0}{\partial y} - \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{20}$$

which upon differentiation with respect to y yields

$$\frac{\partial^2 \left(\psi \frac{\partial \theta}{\partial x}\right)}{\partial x \partial y} = -\frac{\partial^2 u_0}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y} - \nu \frac{\partial^2 v_0}{\partial y^2} - \nu \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2} \tag{21}$$

Substituting the above equation into Eq.(9) yields

$$\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y} = -2\left(1+\nu\right)\left(\frac{\partial^2 v_0}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}\right) \tag{22}$$

Note that the left hand side of Eq.(22) is also the argument in the round parentheses of the last term in Eq.(15). Taking into account the last expression, Eq.(15) can be written as

$$\begin{aligned} &\frac{(1-\nu)}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2} + \frac{(1+\nu)}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 v_0}{\partial x^2} + \frac{(1+\nu)}{2}\frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v_0}{\partial y^2} - \\ &\nu g^2 \nabla^2 \left(\frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y} + \nu \frac{\partial^2 v_0}{\partial y^2} + \nu \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}\right) = 0 \end{aligned} \tag{23}$$

which in view of Eq.(21) finally can be written as

$$\frac{(1-\nu)}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2} + \frac{(1+\nu)}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 v_0}{\partial x^2} + \frac{(1+\nu)}{2}\frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v_0}{\partial y^2} + \nu g^2 \nabla^2 \left[\frac{\partial^2 \left(\psi \frac{\partial \theta}{\partial x}\right)}{\partial x \partial y}\right] = 0 \tag{24}$$

Note that the set of Eqs.(19),(24) links the deflection $w(x,y)$ and the in-plane displacements $u_0(x,y), v_0(x,y)$ with the material length parameter g which represents microstructure. Summarising after the above manipulations, the set of equilibrium equations for the pretwisted thin plate is given by Eqs.(8)-(11), while the equilibrium equations for the gradient elastic thin plate are given after the manipulations from Eqs.(19), (24), (16). Comparing Eq.(19) with Eq.(8) in the absence of the body forces and assuming that the axial load is constant along the $x-$axis, the following condition holds

$$\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}=g^2\nabla^2\left[\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}\right]+f_1(y) \tag{25}$$

Comparing Eq.(24) with Eq.(9) the following condition also holds

$$\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}=g^2\nabla^2\left[\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}\right]+f_2(x) \tag{26}$$

Eqs.(25),(26) imply that $f_1(y)=f_2(x)=0$ and the final condition between the internal length and the warping function takes the form

$$\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}-g^2\nabla^2\left[\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}\right]=0 \tag{27}$$

Eq.(27) is the key expression in our analysis as it substantiates our multiscale modelling approach and provides two alternative interpretations. On the one hand, it can be seen as homogeneous Helmholtz differential equation considering the term $\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}$ as primary independent variable which consists of the warping function, ψ, a quantity apparently connected with the geometry of the structure. Evidently, solving the differential equation with respect to $\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}$ a direct link between macroscopic geometrical features and microstructural material length parameter is established. That is, variations in the macrostructure may induce variations in the material microstructure that is in accordance with the top-down homogenisation approach. On the other hand, in Eq. (27) material length parameter g can be seen as the primary independent variable. In this case, assuming that the warping function of the cross section is known, Eq.(27) can be solved directly with respect to g and a direct link between material length parameter, representing the microstructure, and the dimensions of the cross section emerge. That is, variations in the microstructure impose variations in the macrostrusture which is in accordance with the bottom-up homogenization approach.

In the next two subsections both cases are discussed.

5.1 Solution of the homogeneous Helmholtz differential equation

In view of Eq.(A-1) it can be seen that the term $\frac{\partial}{\partial x}\left(\psi\frac{\partial\theta}{\partial x}\right)$ corresponds to an additional axial strain related to the warping of the cross section. If we use the notation, $\frac{\partial}{\partial x}\left(\psi\frac{\partial\theta}{\partial x}\right)=\epsilon_0^{warp}(x,y)$ then Eq.(27) becomes

$$\nabla^2\epsilon_0^{warp}-\left(\frac{1}{g^2}\right)\epsilon_0^{warp}=0 \tag{28}$$

Eq.(28) is a homogeneous differential equation of Helmholtz type and according to the method of separable variables admits a solution of the form

$$\epsilon_0^{warp}(x,y)=X(x)\,Y(y) \tag{29}$$

where the special solution $X(x)$ is a function of $x-$coordinate only and the special solution $Y(y)$ is a function of $y-$ coordinate only. Then Eq.(28) can be written in the form

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}-\left(\frac{1}{g^2}\right)=0\Rightarrow\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=-\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}+\left(\frac{1}{g^2}\right) \tag{30}$$

The last equation can only hold if both sides are equal to a positive constant, say k^2. Thus, Eq.(28) is equivalent to a pair of ordinary differential equations

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=k^2 \tag{31}$$

$$-\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}+\left(\frac{1}{g^2}\right)=k^2\Rightarrow\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=\frac{1}{g^2}-k^2 \tag{32}$$

The solution of Eq.(31) is of the form

$$X(x)=c_1cosh(kx)+c_2sinh(kx) \tag{33}$$

The solution of Eq.(32) is of the form

$$Y(y)=c_3cosh(ym)+c_4sinh(ym),\qquad m^2=\frac{1}{g^2}-k^2 \tag{34}$$

Thus, from Eq.(29), in view of Eqs. (33),(34), we conclude that the general solution of Eq.(28) is of the form

$$\epsilon_0^{warp}=\sum_{k=1}^{\infty}\left[C_1cosh(kx)\,cosh(ym)+C_2cosh(kx)\,sinh(ym)+C_3sinh(kx)\,cosh(ym)+C_4sinh(kx)\,sinh(ym)\right] \tag{35}$$

where $-\frac{1}{g}\leq k\leq\frac{1}{g}$ and C_1,C_2,C_3,C_4 are constants to be determined by appropriate boundary conditions. Defining the four constants of Eq. (35) is a difficult task because four boundary conditions, regarding the warping term, are required.

However, the above formulation may be simplified further by revisiting Eq.(27). Expanding the term of the warping strain, Eq. (27) can be written in the form

$$\nabla^2\left(\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial x}+\psi\frac{\partial^2\theta}{\partial x^2}\right)-\frac{1}{g^2}\left(\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial x}+\psi\frac{\partial^2\theta}{\partial x^2}\right)=0 \tag{36}$$

Assuming that the angle of twist along the $x-$axis is constant ($\frac{\partial\theta}{\partial x}$:constant), the last equation can be cast in the form

$$\nabla^2\left(\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial x}\right)-\frac{1}{g^2}\left(\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial x}\right)=0\Rightarrow\frac{\partial^2}{\partial x^2}\left(\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial x}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial x}\right)-\frac{1}{g^2}\left(\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial x}\right)=0 \tag{37}$$

Recalling that the warping function, to a good approximation, can be expressed in terms of the cross sectional coordinates, i.e. $\psi(x,y,z)\cong\psi(y,z)$, in view of Eq.(3) the derivative of the warping function along the $x-$ axis is independent of x. In this respect, Eq. (37) can be written as

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial x}\right)-\frac{1}{g^2}\left(\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial x}\right)=0\Rightarrow\frac{\partial^2}{\partial y^2}\left(\epsilon_{0,s}^{warp}\right)-\frac{1}{g^2}\left(\epsilon_{0,s}^{warp}\right)=0 \tag{38}$$

The term in the parenthesis, denotes the warping strain in the special case where the angle of twist per unit length is constant, as can be verified in view of Eq.(A-1).

Interestingly, it can be concluded that in the special case where the the angle of twist is constant, the expressions of Eqs.(28) and Eq.(38) denote that the nabla operator, ∇^2, arising from the gradient formulation, can be replaced with the operator $\frac{\partial^2}{\partial y^2}$, and consequently the differential equation, Eq.(38), expressing the analogy is simplified and admits a solution of the form

$$\epsilon_{0,s}^{warp} = c_1 e^{\frac{y}{g}} + c_2 e^{-\frac{y}{g}} \tag{39}$$

It is reasonable to assume that the warping strain along the y-axis is zero, i.e.

$$\epsilon_{0,s}^{warp}\Big|_{y=0} = 0 \Rightarrow c_1 + c_2 = 0 \tag{40}$$

Also, it can be assumed that the warping strain at the two ends of the plate, $y = \pm\frac{a}{2}$ has a prescribed value, say ϵ_1, i.e.

$$\epsilon_{0,s}^{warp}\Big|_{y=\pm a} = \epsilon_1 \Rightarrow c_1 e^{\frac{a}{2g}} + c_2 e^{\frac{-a}{2g}} = \epsilon_1 \tag{41}$$

The constants c_1, c_2 are determined by solving the system of Eqs. (40)-(41), i.e.

$$c_1 = \frac{e^{\frac{a}{2g}}}{e^{\frac{a}{g}} - 1}\epsilon_1, \qquad c_2 = -\frac{e^{\frac{a}{2g}}}{e^{\frac{a}{g}} - 1}\epsilon_1 \tag{42}$$

Consequently, the solution of Eq.(38) reads

$$\epsilon_{0,s}^{warp} = \frac{e^{\frac{a}{2g}}}{e^{\frac{a}{g}} - 1}\epsilon_1 \left(e^{\frac{y}{g}} - e^{\frac{-y}{g}}\right) \tag{43}$$

Eq.(43) is plotted in Fig. (3) for various values of ϵ_1. It can be seen that when $g \to 0$ the warping strain vanishes, $\epsilon_{0,s}^{warp} = 0$ as expected from Eq. (38), indicating that the strains in the plate are provided by Eq.(A-1) excluding the warping term, hence retrieving the classical solution. Also, Fig. (3) provides an understanding of the interplay between the warping term and the material parameter showing that for increasing values of the material length parameter, g, the contribution of the warping strain to the total strain of the plate increases as can be verified by Eq. (A-1).

5.2 Cross sections with known warping functions

In this section, Eq. (27) is solved explicitly with respect to the material length parameter g demonstrating the "bottom-up" homogenization approach . In this respect, the microstructural parameter is related with geometrical features in a direct quantitative way. By taking this approach allows us to identify the microstructural material parameter g as an explicit function of geometrical features. In order to highlight our approach rectangular, elliptical and triangular cross sections are examined. Note that the general formulation of the analogy, expressed by Eq. (27), holds for general forms of warping (e.g. secondary warping effects may be included) as well as for non-constant angles of twist, in general. In this section, secondary warping phenomena are not taken into account and the angle of twist is assumed to be constant along the loading axis as in the previous section.

5.2.1 Thin Rectangular Cross Section

Consider a thin walled rectangular cross section with dimensions $2a \times h$, Fig. (4). In this case, the warping function $\psi(x, y, z)$ can be approximated by the expression

$$\psi(x, y, z) \cong -yz \tag{44}$$

In the convected coordinate system, in view of Eq (3), we obtain

$$\epsilon_0^{warp}(x,y) = \frac{\partial}{\partial x}\left(\psi \frac{\partial \theta}{\partial x}\right) = \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial x} = \alpha_0 \frac{\partial \theta}{\partial x}\left(y^2 - z^2\right) \tag{45}$$

Substitution of the last equation into Eq. (28) provides

$$2\alpha_0 \frac{\partial \theta}{\partial x} - \frac{1}{g^2}\alpha_0 \frac{\partial \theta}{\partial x}\left(y^2 - z^2\right) = 0 \Rightarrow g^2 = \frac{1}{2}\left(y^2 - z^2\right) \tag{46}$$

By integrating Eq.(46) along the y, z directions and divide by the area of the cross section we obtain the mean value of the internal length, $\langle g \rangle$, given as

$$\langle g^2 \rangle = \frac{1}{2ah}\int_{-\frac{h}{2}}^{+\frac{h}{2}}\int_{-a}^{+a}\left(y^2 - z^2\right)\mathrm{dydz} \Rightarrow \langle g^2 \rangle = \frac{a^2}{3} - \frac{h^2}{12} \tag{47}$$

Eq. (47) is a key expression because it provides a direct relationship between the average value of microstructural length parameter and the geometrical characteristics of the cross section. To our best knowledge this relationship is presented for the first time in the literature. Note, that for the special case where $\alpha = h/2$ the mean value of the material length parameter vanishes. To further demonstrate the identification of microstrustural length parameter g in the following two subsections we consider elliptical and triangular cross sections.

5.2.2 Elliptical cross section

Consider a cross section with elliptical profile, Fig. (5), where a, h denote the semi-axis in the η, ζ directions respectively. The warping function $\psi(x, y, z)$ can be approximated by the expression

$$\psi(x,y,z) \cong -\frac{a^2 - h^2}{a^2 + h^2}yz \tag{48}$$

In the convected coordinate system, in view of Eq (3), we obtain

$$\epsilon_0^{warp}(x,y) = \frac{\partial}{\partial x}\left(\psi \frac{\partial \theta}{\partial x}\right) = \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial x} = \alpha_0 \frac{a^2 - h^2}{a^2 + h^2}\left(y^2 - z^2\right)\frac{\partial \theta}{\partial x} \tag{49}$$

Substituting Eq. (49) into Eq. (28), yields

$$2\alpha_0 \frac{a^2 - h^2}{a^2 + h^2}\frac{\partial \theta}{\partial x} - \frac{1}{g^2}\frac{a^2 - h^2}{a^2 + h^2}\left(y^2 - z^2\right)\frac{\partial \theta}{\partial x} = 0 \Rightarrow g^2 = \frac{1}{2}\left(y^2 - z^2\right) \tag{50}$$

Integrating the last equation along the y, z directions, respectively, and divide by the area of the cross section we obtain the mean value of the internal length, $\langle g \rangle$, given as

$$\langle g^2 \rangle = \frac{1}{2\pi ah}\int_{-a}^{+a}\int_{-\frac{h}{a}\sqrt{a^2-y^2}}^{\frac{h}{a}\sqrt{a^2-y^2}}\left(y^2 - z^2\right)\mathrm{dydz} \Rightarrow \langle g^2 \rangle = \frac{1}{8}\left(a^2 - h^2\right) \tag{51}$$

The last expression provides a direct relationship between the average material length parameter and the geometrical characteristics of the elliptical cross section. Note, that for the special case where $\alpha = h$ (i.e. circular cross section) the mean value of the material length parameter vanishes.

5.2.3 Equilateral triangular cross section

In this section we consider an equilateral triangular cross section as depicted in Fig. (6). The warping function $\psi(x, y, z)$ can be approximated by the expression

$$\psi(x, y, z) \cong -\frac{1}{2a}\left(z^3 - 3y^2 z\right) \tag{52}$$

In the convected coordinate system, in view of Eq (3), we obtain

$$\epsilon_0^{warp}(x, y) = \frac{\partial}{\partial x}\left(\psi \frac{\partial \theta}{\partial x}\right) = \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial x} = \frac{3\alpha_0}{2a} y\left(3z^2 - y^2\right)\frac{\partial \theta}{\partial x} \tag{53}$$

Substituting Eq.(53) into Eq. (28), yields

$$g^2 = \frac{1}{6}\left(y^2 - 3z^2\right) \tag{54}$$

Integrating the last equation along the y, z directions, respectively, and divide by the area of the cross section we obtain the mean value of the internal length $\langle g \rangle$ given as

$$\langle g^2 \rangle = \frac{\sqrt{3}}{3a^2}\int_{-\frac{a}{3}}^{\frac{2a}{3}}\int_0^{-\frac{1}{\sqrt{3}}y+\frac{2\sqrt{3}a}{9}}\left(y^2 - 3z^2\right)\mathrm{d}y\mathrm{d}z \Rightarrow \langle g^2 \rangle = \frac{a^2}{54} \tag{55}$$

Eq.(55) provides a direct relationship between the average material length parameter, g, and the geometrical characteristics of the equilateral triangular cross section, in particular the cross sectional dimension, a.

6 Closure

In the present study, a multiscale modelling approach for twisted plates has been presented. Comparing the governing differential equations of a twisted thin plate including warping effects with that of a flat gradient elastic thin plate an important analogy emerges based on a mathematical expression. The key characteristic of the proposed multiscale modelling approach lies in the fact that a plate with complex geometry can be represented by a structurally equivalent material based gradient plate with simpler geometry (flat plate in the current case). The proposed homogenization approach is of pointwise character and a representative volume element is not needed. Furthermore, the proposed multiscale modelling approach offers a physical interpretation of the material length parameter and highlights direct links with geometric features of the structure through explicit algebraic equations which are presented for the first time in the literature.

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Appendix A. Geometrically nonlinear thin pretwisted plate

A.1. Kinematics

The strains are assumed to be small but nonlinear and for a material point in distance z from the mid-plane can be written in terms of the displacement components as

$$\epsilon_{xx} = \epsilon^0_{xx} + \frac{\partial\left(\psi \frac{\partial\theta}{\partial x}\right)}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \tag{A-1}$$

$$\epsilon_{yy} = \epsilon^0_{yy} - z\frac{\partial^2 w}{\partial y^2} \tag{A-2}$$

$$\gamma_{xy} = \gamma^0_{xy} - z\frac{\partial^2 w}{\partial x \partial y} \tag{A-3}$$

where $\epsilon^0_{xx}, \epsilon^0_{yy}$, are the normal components of strain in x, y directions respectively, while γ^0_{xy} is the shear component of strain which according to von Kármán's hypothesis are

$$\epsilon^0_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{A-4}$$

$$\epsilon^0_{yy} = \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{A-5}$$

$$\gamma^0_{xy} = \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{A-6}$$

Note that, whereas the in plane displacements u_0, v_0 are assumed to be small, the quadratic terms $(\partial w/\partial x)^2$, $(\partial w/\partial y)^2$, $\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$, are of the same order of magnitude as $\partial u_0/\partial x, \partial v_0/\partial y, \partial v_0/\partial x, \partial u_0/\partial y$. Furthermore, according to the Kirchhoff hypothesis, the transverse shear strains are assumed to be negligible, i.e. $\epsilon_{zz} = \gamma_{xz} = \gamma_{zx} = \gamma_{yz} = \gamma_{zy} = 0$.

A.2. Constitutive law

As the strains are assumed to be small the accompanying stresses are provided by generalised Hooke's law, i.e.

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\epsilon_{xx} + \nu\epsilon_{yy}\right) = \frac{E}{1-\nu^2}\left\{\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \nu\left[\frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2}\right]\right\} \tag{A-7}$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\epsilon_{yy} + \nu\epsilon_{xx}\right) = \frac{E}{1-\nu^2}\left\{\frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2} + \nu\left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right]\right\} \tag{A-8}$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = \frac{E}{2(1+\nu)}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y}\right) \tag{A-9}$$

where σ_{xx}, σ_{yy} represent the normal Cauchy stresses along the x, y directions, respectively, τ_{xy} denotes the shear stresses, E the modulus of elasticity and ν the Poisson's ratio of the material. Strictly speaking, small strains imply that the current (deformed) configuration of the plate is the same as the initial (undeformed) configuration for equilibrium purposes and that the classical Cauchy stress measure, σ_{ij}, adequately describes the stress state of the plate in the current state.

A.3. Boundary conditions

The boundary conditions emerge from the variation of the strain energy read

$$\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}+\nu\frac{\partial v_0}{\partial y}+\frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2-\frac{1-\nu^2}{Eh}N_u^x=0 \quad or \quad \delta u=0 \tag{A-10}$$

$$\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}+\nu\frac{\partial v_0}{\partial y}+\frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2-\frac{1-\nu^2}{Eh}R_\psi=0 \quad or \quad \delta\left(\psi\frac{\partial\theta}{\partial x}\right)=0 \tag{A-11}$$

$$\begin{aligned}&\frac{\partial w}{\partial x}\left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}+\frac{\partial u_0}{\partial x}+\nu\frac{\partial v_0}{\partial y}+\frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]+\frac{(1-\nu)}{2}\frac{\partial w}{\partial y}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)-\\&\frac{h^2}{12}\frac{\partial^3 w}{\partial x^3}-\frac{\nu h^2}{12}\frac{\partial^3 w}{\partial x\partial y^2}-\frac{1-\nu^2}{Eh}N_w^x=0 \quad \text{or} \quad \delta w=0\end{aligned} \tag{A-12}$$

$$\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}+2\left(1-\nu\right)\frac{\partial^2 w}{\partial x\partial y}-\frac{12\left(1-\nu^2\right)}{Eh^3}m_{xx}^x=0 \quad \text{or} \quad \frac{\partial\delta w}{\partial x}=0 \tag{A-13}$$

$$\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-\frac{2\left(1+\nu\right)}{Eh}-\frac{2\left(1+\nu\right)}{Eh}N_v^x=0 \quad \text{or} \quad \delta v=0 \tag{A-14}$$

$$\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\nu\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}+\nu\frac{\partial u_0}{\partial x}+\frac{\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2-\frac{1-\nu^2}{Eh}N_v^y=0 \quad \text{or} \quad \delta v=0 \tag{A-15}$$

$$\begin{aligned}&\frac{\partial w}{\partial y}\left[\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\nu\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}+\frac{\partial v_0}{\partial y}+\nu\frac{\partial u_0}{\partial x}+\frac{\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]+\frac{(1-\nu)}{2}\frac{\partial w}{\partial x}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)-\\&\frac{h^2}{12}\frac{\partial^3 w}{\partial y^3}-\frac{\nu h^2}{12}\frac{\partial^3 w}{\partial x^2\partial y}-\frac{1-\nu^2}{Eh}N_w^y=0 \quad \text{or} \quad \delta w=0\end{aligned} \tag{A-16}$$

$$\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}-\frac{12\left(1-\nu^2\right)}{Eh^3}m_{yy}^y=0 \quad \text{or} \quad \frac{\partial\delta w}{\partial y}=0 \tag{A-17}$$

$$\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-\frac{2\left(1+\nu\right)}{Eh}-\frac{2\left(1+\nu\right)}{Eh}N_v^y=0 \quad \text{or} \quad \delta v=0 \tag{A-18}$$

A.4. Curvatures-induced loads products

Products of the initial curvatures with the induced loads are sufficiently small so as to be neglected, i.e.

$$\left(\frac{\partial^2 w}{\partial x^2}\int_{-h/2}^{h/2}\sigma_{xx}\mathrm{dz}\simeq 0\right)\to\frac{\partial u_0}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial v_0}{\partial y}\frac{\partial^2 w}{\partial x^2}+\frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}\simeq 0 \tag{A-19}$$

$$\left(\frac{\partial^2 w}{\partial y^2}\int_{-h/2}^{h/2}\sigma_{yy}\mathrm{dz}\simeq 0\right)\to\frac{\partial v_0}{\partial y}\frac{\partial^2 w}{\partial y^2}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial u_0}{\partial x}\frac{\partial^2 w}{\partial y^2}+\frac{\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial y^2}\frac{\partial\left(\psi\frac{\partial\theta}{\partial x}\right)}{\partial x}\simeq 0 \tag{A-20}$$

$$\left(\frac{\partial^2 w}{\partial x\partial y}\int_{-h/2}^{h/2}\tau_{xy}\mathrm{dz}\simeq 0\right)\to\frac{\partial v_0}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial u_0}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\simeq 0 \tag{A-21}$$

Appendix B. Strain gradient theory for thin flat plate.

B.1. Kinematics

Components of the classical infinitesimal strain tensor ϵ_{ij} are provided as

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{B-1}$$

B.2. Strain energy density

The strain energy U^g stems from the integration of Eq.(12) with respect to the volume Ω occupied by the plate, i.e.

$$U^g = \frac{1}{2}\int_\Omega \left(\tau_{ij}\epsilon_{ij} + \mu_{ijk}\kappa_{ijk}\right) \mathrm{d}\Omega \tag{B-2}$$

where τ_{ij} are the components of the Cauchy stress tensor μ_{ijk} are the components of the double stress tensor and κ_{ijk} are the components of the gradient of strain.

B.3. Constitutive law

In view of Eqs.(12), (B-2) we obtain

$$\tau_{ij} = \frac{\partial U^g}{\partial \epsilon_{ij}} = \lambda\epsilon_{qq}\delta_{ij} + 2\mu\epsilon_{ij} \tag{B-3}$$

$$\mu_{ijk} = \frac{\partial U^g}{\partial \kappa_{ijk}} = g^2\left(\lambda\epsilon_{qq}\delta_{ij} + 2\mu\epsilon_{ij}\right)_{,k} = g^2\tau_{ij,k} \tag{B-4}$$

where the components of the gradient of the deformation tensor, κ_{ijk} are related to the classical infinitesimal strain tensor components through the relationship

$$\kappa_{ijk} = \epsilon_{ij,k} = \epsilon_{ij} = \frac{1}{2}\left(u_{i,jk} + u_{j,ik}\right) \tag{B-5}$$

The components of the total stress tensor, σ_{ij}, are given as the difference between the Cauchy stresses and the gradient of the double stresses, i.e.

$$\sigma_{ij} = \tau_{ij} - \mu_{ijk,k} \tag{B-6}$$

and in view of Eqs.(B-3),(B-4) may be written in the form

$$\sigma_{ij} = \lambda\epsilon_{qq}\delta_{ij} + 2\mu\epsilon_{ij} - g^2\left(\lambda\epsilon_{qq}\delta_{ij} + 2\mu\epsilon_{ij}\right)_{,kk} \tag{B-7}$$

The last equation, which is actually the constitutive equation for the material of the thin plate, can be written in plane Cartesian coordinates x, y in the form

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\epsilon_{xx} + \nu\epsilon_{yy}\right) - g^2\frac{E}{1-\nu^2}\nabla^2\left(\epsilon_{xx} + \nu\epsilon_{yy}\right) \tag{B-8}$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\epsilon_{yy} + \nu\epsilon_{xx}\right) - g^2\frac{E}{1-\nu^2}\nabla^2\left(\epsilon_{yy} + \nu\epsilon_{xx}\right) \tag{B-9}$$

$$\sigma_{xy} = \frac{E}{1+\nu}\epsilon_{xy} - g^2\frac{E}{1+\nu}\nabla^2\epsilon_{xy} \tag{B-10}$$

The geometrical relationships of the infinitesimal strains and the finite displacements of the thin plate are given through the von Kármán hypothesis as in Eqs.(A-1)-(A-6). Substituting the relations of strains in the constitutive equations as given in (B-8)-(B-10) the stress components are expressed in terms of the displacements, i.e.

$$\begin{aligned}\sigma_{xx} &= \frac{E}{1-\nu^2}\left(\epsilon_{xx}+\nu\epsilon_{yy}\right) - g^2\frac{E}{1-\nu^2}\nabla^2\left(\epsilon_{xx}+\nu\epsilon_{yy}\right)\\ &= \frac{E}{1-\nu^2}\left\{\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2}+\nu\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2}\right]\right\}-\\ &\quad g^2\frac{E}{1-\nu^2}\left\{\frac{\partial \nabla^2 u_0}{\partial x}+\nabla^2\left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]-z\nabla^2\left(\frac{\partial^2 w}{\partial x^2}\right)+\nu\left[\frac{\partial \nabla^2 v_0}{\partial y}+\nabla^2\left[\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]-z\nabla^2\left(\frac{\partial^2 w}{\partial y^2}\right)\right]\right\}\end{aligned} \tag{B-11}$$

$$\begin{aligned}\sigma_{yy} &= \frac{E}{1-\nu^2}\left(\epsilon_{yy}+\nu\epsilon_{xx}\right) - g^2\frac{E}{1-\nu^2}\nabla^2\left(\epsilon_{yy}+\nu\epsilon_{xx}\right)\\ &= \frac{E}{1-\nu^2}\left\{\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2}+\nu\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2}\right]\right\}-\\ &\quad g^2\frac{E}{1-\nu^2}\left\{\frac{\partial \nabla^2 v_0}{\partial y}+\nabla^2\left[\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]-z\nabla^2\left(\frac{\partial^2 w}{\partial y^2}\right)+\nu\left[\frac{\partial \nabla^2 u_0}{\partial x}+\nabla^2\left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]-z\nabla^2\left(\frac{\partial^2 w}{\partial x^2}\right)\right]\right\}\end{aligned} \tag{B-12}$$

$$\begin{aligned}\tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy} - g^2\frac{E}{2(1+\nu)}\nabla^2\gamma_{xy} = \frac{E}{2(1+\nu)}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-2z\frac{\partial^2 w}{\partial x\partial y}\right)-\\ &\quad g^2\frac{E}{2(1+\nu)}\left[\frac{\partial \nabla^2 v_0}{\partial x}+\frac{\partial \nabla^2 u_0}{\partial y}+\nabla^2\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)-2z\nabla^2\left(\frac{\partial^2 w}{\partial x\partial y}\right)\right]\end{aligned} \tag{B-13}$$

B.4. Resultant forces and moments

Assuming that the thickness of the thin plate, h, is uniform along both directions, the resultant in-plane forces, N_x, N_y, N_{xy}, and moments M_{xx}, M_{yy} (bending), M_{xy} (twisting), see Fig. 2 result from the through thickness integration, i.e.

$$\begin{aligned}N_{xx} &= \int_{-h/2}^{+h/2}\sigma_{xx}\mathrm{dz}+P_x = \frac{Eh}{1-\nu^2}\left\{\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\nu\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\right\}-\\ &\quad g^2\frac{Eh}{1-\nu^2}\left\{\frac{\partial \nabla^2 u_0}{\partial x}+\nabla^2\left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]+\nu\left[\frac{\partial \nabla^2 v_0}{\partial y}+\nabla^2\left[\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\right]\right\}+P_x\end{aligned} \tag{B-14}$$

$$\begin{aligned}N_{yy} &= \int_{-h/2}^{+h/2}\sigma_{yy}\mathrm{dz} = \frac{Eh}{1-\nu^2}\left\{\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\nu\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right\}-\\ &\quad g^2\frac{Eh}{1-\nu^2}\left\{\frac{\partial \nabla^2 v_0}{\partial y}+\nabla^2\left[\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]+\nu\left[\frac{\partial \nabla^2 u_0}{\partial x}+\nabla^2\left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right]\right\}\end{aligned} \tag{B-15}$$

$$N_{xy} = \int_{-h/2}^{+h/2}\tau_{xy}\mathrm{dz} = \frac{Eh}{2(1+\nu)}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)-g^2\frac{Eh}{2(1+\nu)}\left[\frac{\partial \nabla^2 v_0}{\partial x}+\frac{\partial \nabla^2 u_0}{\partial y}+\nabla^2\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right] \tag{B-16}$$

$$M_{xx} = \int_{-h/2}^{+h/2}\sigma_{xx}z\mathrm{dz} = -\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)+g^2\frac{Eh^3}{12(1-\nu^2)}\nabla^2\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right) \tag{B-17}$$

$$M_{yy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy} z \mathrm{d}z = -\frac{Eh^3}{12\left(1-\nu^2\right)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) + g^2 \frac{Eh^3}{12\left(1-\nu^2\right)}\nabla^2\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \tag{B-18}$$

$$M_{xy} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} z \mathrm{d}z = \frac{Eh^3}{12\left(1+\nu\right)}\frac{\partial^2 w}{\partial x \partial y} - g^2 \frac{Eh^3}{12\left(1+\nu\right)}\nabla^2\left(\frac{\partial^2 w}{\partial x \partial y}\right) \tag{B-19}$$

where P_x is the axial loading along the $x-$ direction. The equilibrium equations of the thin plate can be written in terms of the resultant forces and moments as

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{B-20}$$

$$\frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \tag{B-21}$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + N_{xx}\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_{yy}\frac{\partial^2 w}{\partial y^2} = 0 \tag{B-22}$$

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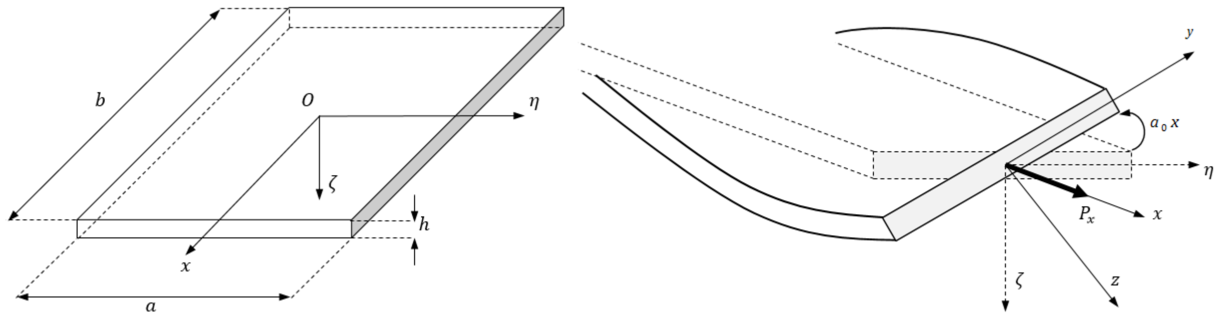

Figure 1: (a) The coordinate system $Ox\eta\zeta$ in the flat thin plate, (b) the local coordinate system $Oxyz$.

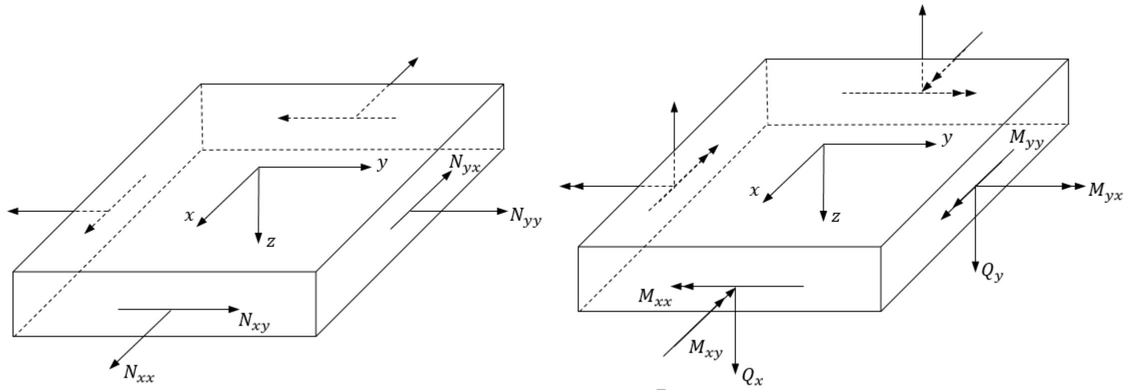

Figure 2: Resultant (a) in plane forces, (b) moments in the thin plate.

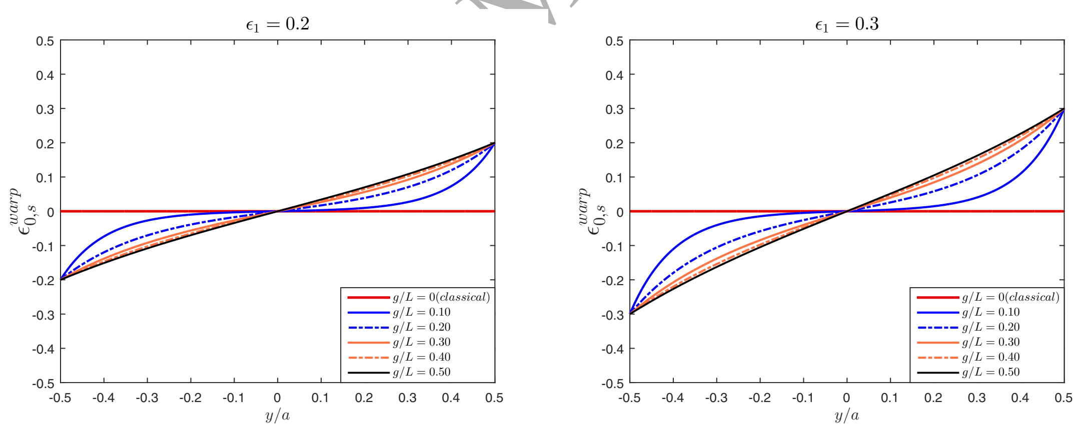

Figure 3: Variation of the warping strain $\epsilon_{0,s}^{warp}$ for (a) $\epsilon_1 = 0.20$, (b) $\epsilon_1 = 0.30$, (c) $\epsilon_1 = 0.40$, (d) $\epsilon_1 = 0.50$.

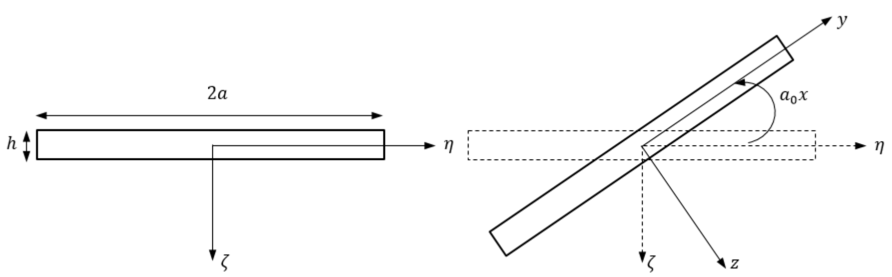

Figure 4: Rectangular cross section (a) without initial twist, (b) with initial twist $\alpha_0 x$.

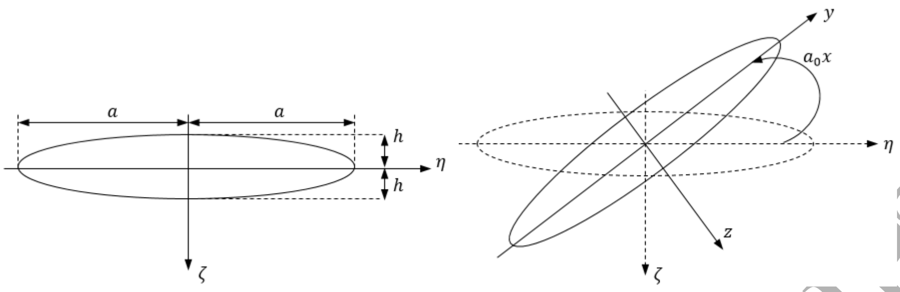

Figure 5: Elliptical cross section (a) without initial twist, (b) with initial twist $\alpha_0 x$.

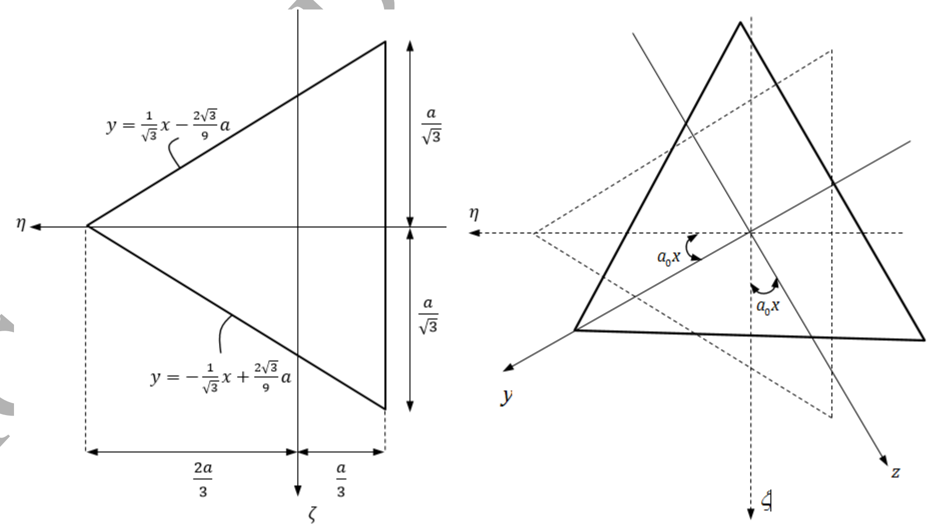

Figure 6: Equilateral triangular cross section (a) without initial twist, (b) with initial twist $\alpha_0 x$.