The Complex System of Problem Solving - Providing the Conditions to Develop Proficiency.

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Abstract

Problem solving in mathematics has been a central topic of research in mathematics education, stemming mainly from Pólya’s (1945) work on problem solving heuristics (Lesh and Zawojewski, 2007). Despite the attention this topic has received, the research conducted in problem solving has not easily translated into improving school practice (Lester and Kehle, 2003). Foster et al. (2014) found that the teachers’ knowledge and ability to employ “effective strategies for teaching problem solving processes was particularly underdeveloped” (p.7). Felmer and Diaz (2016, p.289) also noted that there is little research on “teachers as problem solvers”, particularly at second level. Evidence on pre-service secondary teachers’ performance on non-routine problem solving tasks and experience with solving such problems led Felmer and Diaz (2016) to recommend that action be taken on pre-service mathematics teacher education courses.

The aim of this research was to develop a framework for teaching and assessing problem solving in mathematics and to implement this framework in the form of a teaching intervention with a sample of pre-service secondary level mathematics teachers. The author developed a Framework for Teaching and Assessing Problem Solving (F-TAPS) in mathematics, consistent with findings on significant issues identified from the review of literature. The author’s F-TAPS in mathematics integrates significant knowledge and affective factors with mathematical thinking in order to develop proficiency in problem solving in mathematics through the coherent construction of meaningful knowledge. The author integrated her F-TAPS in mathematics with the Modified Moore Method using the Modified 4C-ID model to design and implement an educational intervention in problem solving. This intervention was implemented with sixteen secondary pre-service mathematics teachers. Two assessments in problem solving were developed to assess problem solving ability before and after the intervention. The mindset (growth/fixed) of the pre-service teachers who participated in this study was also evaluated before and after the intervention.

Educational Design Research methodology which incorporates a proof of concept approach underpinned this research. Pre and post-test findings showed statistically significant increases in the growth mindset and problem solving ability of the pre-service teachers who participated in this study. The intervention (and the integrated F-TAPS) had a moderate effect on the mindset of the pre-service teachers and a strong effect on the problem solving ability of the pre-service teachers. The proof of concept approach provides evidence that the author’s F-TAPS could contribute to improved teaching of problem solving in mathematics at secondary level.
Author’s Declaration

I certify that this report is entirely of my own work, other than the counsel of my supervisors and that it has not been submitted for any academic award or part thereof, at this or any other educational institution. Where use has been made of the work of other people it has been acknowledged appropriately and fully referenced.

Name: Aoife Guerin

Signature:

Date:
Dedication

To Mam, Dad, my sisters and brothers for everything (including understanding my long absence!). To Nana and Grandad for being with me in spirit and also for everything while you were here.
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1 Introduction

“For most of us mathematics, like music, needs to be expressed in physical and human interactions before its symbols can evoke the silent patterns of mathematical ideas (like musical notes), simultaneous relationships (like harmonies) and expositions of proofs (like melodies).”

(Skemp, 1976, p.288)

1.1 Motivation For Conducting This Study

The secondary level mathematics syllabus in Ireland has been revised in recent years (progressively phased in from 2010 to 2015) with the implementation of a new mathematics curriculum known locally as Project Maths. This reform consists of changes to both the instruction and assessment of mathematics. This initiative places greater emphasis on students’ understanding of mathematical concepts with an increased focus on the development of students’ problem solving skills, “explanation, justification and communication” (State Examinations Commission 2015 p.3). Initial reviews from the pilot schools who engaged with Project Maths revealed potential problems with the confidence levels among both students and teachers with regard to problem solving (NCCA, 2011). The time available for problem solving was also of concern. An interim report on Project Maths by the School of Mathematical Sciences in University College Cork revealed a lack of provision in the syllabus, of specific information on methods of application that students can use to devise and solve context-based problems (Grannell et al., 2011). A final report by Jeffes et al. (2013) revealed that both Junior and Leaving Certificate students were experiencing difficulties in:

• “reading and interpreting large amounts of information;
• displaying their work and justifying their solutions;
• completing multi-step problems;
• in both strand 1 (number) and strand 2 (algebra), students find broader, open-ended problems more difficult than questions which ask for some specific calculation to be made”.

(Jeffes et al., 2013, p.56)

Problem solving in mathematics has been a central topic of research in mathematics education, stemming mainly from Pólya’s (1945) work on problem solving heuristics (Lesh and Zawojewski, 2007). Lester and Kehle (2003) compared their research in problem solving in mathematics to Lester’s 1994 work. Also Lester (1992) in a comparison of his own 1980 work to Schoenfeld’s 1992 work revealed
little change between what are considered fundamental issues in problem solving research. Despite the attention that problem solving in mathematics has received in educational research, Lester and Kehle (2003) came to the conclusion that the research conducted in problem solving is not easily translated into improving school practice. Felmer and Diaz (2016, p.289) also noted that there is little research on “teachers as problem solver”, particularly at second level.

The difficulties experienced in schools in mathematical problem solving, the author’s own interest in this topic and the aim of the government on driving a ‘smart economy’ prompted the author to investigate the instruction, learning and assessment of mathematical problem solving at secondary level. While many problem solving models and frameworks exist, the author’s framework is unique in its integrative features.

The author chose to address the gap in the mathematics education literature by working with secondary level pre-service mathematics teachers. This aims to serve two purposes:

1. to increase the research on “teachers as problem solvers” at secondary level;

2. to aid in the translating of research in problem solving into improving school practice.

1.2 Significance of the Research

It is estimated that there will be twenty million more jobs in the future for people who are mathematical problem solvers. Unfortunately it has also been estimated that sixty percent of all new jobs in the twenty first century will require skills that only twenty percent of the current workforce have (Glenn, 2000). The Department of Education and Science is engaged in ongoing initiatives to promote the STEM (Science, Technology, Engineering and Mathematics) subjects across the entire education system in Ireland. The application of mathematics is required in the study of science, technology and engineering. Careers in these subjects are of vital important for economic growth. STEM education has a fundamental role to play in supporting and ensuring Ireland’s economic competitiveness (Alexander, 2012; STEM Education Review Group, 2016).

Pre-service secondary level teachers have demonstrated deficiencies in their understanding of mathematics (Chazan et al., 1999; Llenares and Krainer 2006; Hourigan and O’Donoghue 2013) and the teacher is acknowledged as the “single most powerful influence on student achievement” (Hattie, 2003 p.4). Attempts at improving problem solving in mathematics education require the focus of attention to be on the education of the pre-service mathematics teacher (Zimmermann, 2016), with alignment existing between the pre-service teachers’ education and students’ education (Zimmermann, 2016). The author addresses
this gap in the mathematics education literature. Part of the pre-service mathematics teachers’ education should include the solving and posing of problems that are similar to, and some more challenging than, those they will be teaching to their students (Chapman, 2015; Zimmermann, 2016). The author’s research focuses on the solving of such problems with secondary level pre-service mathematics teachers.

Extensive research has been conducted in problem solving models, this research has primarily focused on the heuristics and metacognitive processes involved in the phases of the problem solving solving process (Dewey, 1933; Pólya, 1945; Krulik and Rudnick, 1980) and are fundamentally very similar (Carson, 2007). More recent models added in more explicit cognitive processes and metacognitive aspects to the heuristics in the phases of the problem solving models (Schoenfeld, 1982; Garofalo and Lester, 1985; Armour-Thomas and Artzt, 1992; Montague and Applegate, 1993; Erbas and Okur, 2010). The model developed by Mason et al. (1982) includes the processes of specialising and generalising and focuses on mathematical thinking. Schoenfeld (1992) pointed out that a complete coherent explanation of how the components of mathematical thinking and problem solving fit together is not available. The author’s research reveals how the various aspects fit together by synthesising the research currently available on the component parts (chapter 2) in the formulation of the Framework for Teaching and Assessing Problem Solving in Mathematics (F-TAPS). The in-depth research conducted on the cognitive processes of mathematical thinking in the theories, models and frameworks developed by various researchers (van Hiele, Sfard, Tall) facilitated this synthesis (section 2.9). A start is made by the author in the development of a rubric (adapted by Docktor and Heller, 2009) which analyses the various aspects of mathematical thinking, problem solving abilities and skills, and mathematical proficiency under each of the phases of the problem solving process (Appendix S).

The author utilises aspects of these models (the mathematical thinking process of Mason’s model and the heuristics in the phases of Pólya’s model in particular, with metacognitive aspects (Flavelle, 1979) in the F-TAPS developed. The author’s framework contributes to the research on problem solving in mathematics education by integrating these aspects of the problem solving models, together with theories of knowledge development (mathematical thinking in particular) and theories on mindset, interest development, perseverance and intuition, in an effort to develop all aspects of mathematical proficiency. The use of the modified Four Component Instructional Design (4C-ID) Model in the organisation of the intervention, combined with the use of the Modified Moore Method in the delivery of the intervention (chapter six) while adhering to the author’s developed F-TAPS are unique to this study. The description of the sequencing of the intervention classes (chapter 6) to complete problems which developed mathematical thinking and understanding in alignment with van Hiele’s model of
mathematical thought, and Tall’s framework for the development of mathematical thinking from conceptual embodiment to formal axiomatic also contributes to the research on problem solving in mathematics.

1.3 Aims and Objectives of the Research

The aim of this research is to develop a framework for teaching and assessing problem solving in mathematics. This framework needs to consider the knowledge and affective factors which have an impact on students’ ability to engage in problem solving. Theories of developing mathematical thinking need further research. Research into current methods of teaching and assessing problem solving in mathematics needs to be conducted, with consideration given to the knowledge and skills required by teachers to teach and assess problem solving in mathematics.

The objectives of this research study based on the aims are to:

- conduct a review of the current literature in order to gain an in-depth understanding of mathematical thinking and problem solving, the knowledge and affective factors that are associated with the learning and teaching of problem solving, theories of developing mathematical thinking, and current methods of teaching and assessing problem solving;

- develop a theoretical framework for teaching and assessing problem solving in mathematics which will provide the basis for the development of the assessment and teaching intervention components of this research;

- develop the assessment component, based on findings from the literature and adjusted to incorporate significant findings from the pilot test (aligned with the Project Maths syllabus);

- design an intervention to develop and evaluate pre-service teachers’ problem solving abilities in mathematics, based on the framework developed and findings from the assessment component;

- implement the intervention with a sample of pre-service mathematics teachers to evaluate the effectiveness of the framework in the teaching and assessing of problem solving;

- determine the effectiveness of the intervention in achieving its aims by analysing the gathered quantitative and qualitative data.
1.4 Research Questions

The research programme consisted of seven phases as described below. Each phase of the research programme was guided by several factors. These factors gave rise to a number of research questions. Both the research programme and the research questions which emerged are as follows:

Phase 1: Literature Review

- The attributes of a “good” problem;
- The current performance of Irish students and students in other countries in problem solving;
- The factors which influence students’ ability to problem solve;
- The impact that a teacher’s knowledge has on students learning;
- The mathematics knowledge that a teacher requires in order to be able to teach for problem solving and the current level of possession of this particular mathematical knowledge by teachers;
- The development of conceptual understanding and the development of problem solving ability;
- The natural problem solving abilities students have and the nurturing of these abilities;
- The transfer of students’ mathematical knowledge to unfamiliar problem situations;
- The role of mathematical thinking and metacognition in problem solving.

Phase 2: Formulation of Model

- What it means to be proficient in problem solving in mathematics;
- The knowledge based factors which are most significant in developing students’ proficiency in problem solving;
- The affective factors that help/hinder the development of proficiency;
- The instruction practices which facilitate problem solving proficiency in mathematics;
- The influence which assessment has on what is taught.

Phase 3: Designing the Assessment

- The importance of the design of the assessment in order to reveal conceptual understanding and problem solving abilities of the participants;
• The difficulty levels of the tasks be determined;
• Pilot-testing the assessment and making adjustments.

Phase 4: Data Analysis (Pre-test)
• Main difficulties displayed in terms of progressing through each task;
• Common misconceptions.

Phase 5: Designing the Intervention
• Applying the findings from the pre-test and using the developed framework to construct intervention;
• Activities and teaching approaches;
• Interest maintenance;
• Perseverance;
• Enjoyment.

Phase 6: Data-Analysis (During Intervention)
• Observations during the intervention in relation to gains in conceptual understanding, impediments to progression, levels of persistence, interest maintenance, and enjoyment;
• Evidence to analyse the effectiveness of the intervention.

Phase 7: Data-Analysis (Post-test)
• Transferability of the framework and the developed intervention to a mathematics class in a secondary school;
• Comparison of the findings and results from the pre and post-test in problem solving performance and in growth/fixed mind-set;
• The effectiveness of the framework employed in the intervention;
• Other variables outside of those considered in the combination of the framework developed in phase two and the findings/observations from the assessments and the intervention that could account for a change in the problem solving ability of the students.
The research question which emerged from Phases 1 and 2 is:

1. How should mathematics be taught and assessed in order to facilitate the development of learners’ problem solving proficiency in mathematics?

The research questions which emerged from Phases 3 - 5 are:

2. How can the research on the teaching and assessing of problem solving be brought into practice?

(a) Who should the researcher implement the intervention with to best facilitate the transfer of research into practice?

(b) What activities and teaching approaches should be used in the intervention?

The research questions which emerged from Phases 6 and 7 are:

3. What evidence is there to show the effectiveness of the framework employed in the intervention?

4. Is the framework and the developed intervention transferrable to a mathematics class in a secondary level school?

1.5 Research Phases

The phases of the research are shown in Figure 1. There were seven phases of the research in the completion of this first cycle of the research.
Research Phases in the development of the F-TAPS and Intervention

Began with

Informed

Phase 1
Consisted of


Review of teacher knowledge and the effect this has on student achievement.

Analysis of instructional practices in mathematics education.

Analysis of models and theories of the development of mathematical thought.

Formulating the F-TAPS and the pre-intervention assessment of problem solving.

Pilot test assessment. Adjustments to assessment based on findings from pilot test. Distribution of assessment (pre-test) to participants.

Analysis of data from assessment (pre-test).

Development of the intervention and post-test.

Implementation of the intervention.

Post-test after intervention, analysis of data from post-test and evaluation of the intervention.

Phase 2
Consisted of

Phase 3
Consisted of

Phase 4
Consisted of

Phase 5
Consisted of

Phase 6
Consisted of

Phase 7
Consisted of

Figure 1: Research Phases
1.6 Research Methodology

Educational Design Research was the methodological approach in this research. This research approach can be used to address complex problems in educational practice by formulating research based solutions (van den Akker et al., 2013, Design-Based Research Collective, 2003). Educational Design Research utilises theoretical knowledge in combination with comprehension of the relationships between theory and practice in the design of interventions.\(^1\) This methodology allows for both theoretical and practical contributions to research in education to be made (van den Akker et al., 2013).

The research process of Educational Design Research varies with respect to the “frameworks and models that describe and at times guide the process” (McKenna and Reeves, 2013 p.12). However common distinguishing characteristics (McKenna and Reeves, 2013 p.12) of the phases involved in Educational Design Research include:

- **analysis and orientation phase**
  - the analysis of the literature on the learning, teaching and assessing of problem solving in mathematics education highlighted practical problems and their causes which led to the consideration of formulating a framework (based on the analysis) for the teaching and assessing of problem solving in mathematics;

- **design and development phase**
  - the theories and models of developing mathematical thought, the problem solving models for the phases of problem solving, the theories and models on metacognition, mindset, interest development, intuition and perseverance were noted as essential contributors to the development of problem solving ability in mathematics.
  - the overall goal of developing mathematical proficiency, in addition to the essential contributors in the development of problem solving skills and ability were combined in the development of the author’s F-TAPS as a possible solution;
  - the author’s F-TAPS was utilised in conjunction with analysis on problems (from literature) to develop a pre-test, the results of which informed the design and development of the teaching intervention;
  - the author’s F-TAPS was then employed to inform the design and implementation of a teaching intervention with pre-service mathematics teachers. The design principles of the Modified 4C-ID Model combined with the theoretical and practical principles of the Modified Moore Method also informed the organisation, sequencing of tasks and implementation of the intervention;

\(^1\)Intervention refers to all entities that can be designed or developed; learning processes, learning environments, teaching learning materials, products, and systems” (van den Akker et al., 2013, p.11).
evaluation and reflection phase
-the intervention was tested with a small sample of pre-service mathematics teachers. The evaluation of the intervention (based on the author’s F-TAPS) was conducted by an analysis of pre and post-tests of performance in problem solving and pre and post-tests on mindset. The intervention was also evaluated from the participants’ point of view by conducting focus groups (section 7.10.1; Appendix I; Appendix R). This phase of Education Design Research then evaluates and reflects on solutions in practice which facilitates the consideration of further refinement to improve the implementation of the solutions.

(McKenna and Reeves, 2006, cited in van den Akker et al., 2013)

Educational Design Research fits the purpose of this study as the author wanted to use the review of existing theory to formulate a framework, which is based on the integration of concepts and theories from the areas of mathematical understanding, and affective domains to inform the design of an intervention which could provide a potential solution to the problem of problem solving in educational practice. This research employed one cycle of the three phases of Education Design Research. Findings from this cycle will inform refinements of the intervention in subsequent cycles.

A mixed methods approach was employed in this research to evaluate the findings from the pre and post-tests in problem solving and mindset and to evaluate the findings from the focus groups. The intervention was evaluated through the use of the four aspects of Shapiro’s model (1987). The methodological approach, along with the methods employed in this research is discussed further in chapter 3.

1.7 Limitations of the Research

The following were limitations for this study:

- the nature of the assessment may have possibly caused a reduction in the sample size of students. The fact that the time required to complete the assessment exceeded the time of a lecture may have put some students off of participating in the research.

- the sample size of pre-service teachers who completed all three components of this research (pre-test, intervention and post-test) is quite low (n = 13). Thirty - two pre-service teachers completed the pre-test, sixteen pre-service teachers completed the intervention, and thirteen completed the post-test. The small sample size subsequently limited the choice of data analysis.

- the time to complete the intervention was limited as the pre-service teachers had their own course work to complete as part of their degree. However
problem solving interventions of six weeks’ duration have produced statistically significant results (Woodward et al., 2012).

- the number of pre-service teachers who formed a control group was limited to three and so, although the data is presented, it is limited in the contribution it makes to the research.

- the participants did not fully engage with the homework aspect of the intervention. Those that did showed higher gains in performance in problem solving (P1 in the third year group). The intervention was voluntary and the workload (in terms of their university course) of the pre-service teachers was high. Comments made in relation to this in the focus group was that if this had been a compulsory intervention, they would have set aside time for homework (section 7.10.1; Appendix I)

1.8 Terms used in this Research

Some terms which relate specifically to the Irish education system are explained in this section. The author has explained some of the terms relative to their particular meaning in the context of this thesis.

- Foundation, Ordinary and Higher Level - There are three levels at which mathematics may be studied (for both Junior Certificate and Leaving Certificate courses) in Ireland. The three levels, Foundation, Ordinary and Higher Level consist of increasing amounts of mathematical content and are of increasing in difficulty respectively. The learning outcomes for the Foundation level are distinct from the Ordinary level and Higher level outcomes. The learning outcomes at Ordinary level are a subset of the learning outcomes at Higher level. The Ordinary and Higher level examination in mathematics consists of two examination papers, both of which have two sections. Section A (questions 1-6 usually) assesses “core mathematics topics focusing on concepts and skills and Section B (questions 7-9 usually) assesses context-based applications” (NCCA, 2015 p.44).

- Initial Schools - The 24 schools who engaged in the piloting of the new Project Maths Syllabus. Strands 1 (Probability and Statistics) and 2 (Geometry and Trigonometry) were implemented in the initial schools in 2008, Strands 3 (Number) and 4 (Algebra) were implemented in 2009, and Strand 5 Functions was implemented in 2010. As the new strands were phased in, content from the previous syllabus was simultaneously phased out.

- Junior Certificate - The state examinations taken at the end of the third year in secondary school when students are approximately 15 years of age. Students complete examinations in approximately ten subjects.

- Junior Cycle - The first three years in secondary school in Ireland is called the Junior Cycle.
• Leaving Certificate - The final state examinations taken at the end of the sixth and final year in secondary school when students are approximately 18 years of age. Students complete examinations in approximately seven subjects.

• Out of Field Teachers - An out-of-field mathematics teacher is defined as a teacher who is teaching mathematics in secondary school but is not fully qualified to do so (Ingersoll, 1999; Bossé and Törner, 2015).

• Pre-service Teachers - University students who are completing a course of study in mathematics education. These students are studying and training to be mathematics teachers.

• Primary School - A school in which students receive primary education from the approximate ages of 5-12 years.

• Project Maths - Post primary mathematical syllabus which was implemented in phases in secondary schools in Ireland from 2010 to 2015. Project Maths was implemented in 24 secondary level schools initially in 2008 and based on feedback from these schools the syllabus was implemented in phases in all schools from September 2010. The changes involved in the implementation of Project Maths compared to the old syllabus involve different content and a change in the emphasis of skills developed. At Leaving Certificate level, there is an increase in the amount of statistics and probability, removal of vectors and matrices (these were studied on the old syllabus) and changes made to the content of functions and calculus. Strands 1 (Probability and Statistics) and 2 (Geometry and Trigonometry) were implemented in 2010, Strands 3 (Number) and 4 (Algebra) were implemented in 2011, and Strand 5 Functions was implemented in 2012. As the new strands were phased in, content from the previous syllabus was simultaneously phased out. There is an “increased emphasis on problem solving as well as explanation, justification and communication of work” (SEC, 2015, p.4).

• Resource Teachers - These are learning support teachers who provide extra supplementary teaching to students who require extra help. Resource teachers work with staff members, parents and professionals to implement effective teaching strategies to meet the student’s particular needs.

• Secondary School - A school in which students receive secondary education from the approximate ages of 12-18 years. Students enter secondary school after completing primary school. A secondary school is also referred to as a post-primary school.

• State Examinations - Examinations for second-level education in Ireland which are prepared by the Irish State Examinations Commission. All students in the country receive the same examination.
1.9 Outline of Chapters

Chapter 1 provided a brief introduction to the research. The author’s motivation for conducting the research and the significance of this research to the field of mathematics education was discussed. The research questions arising from the aims and objectives of this study were presented and a brief description of the methodologies employed to answer these questions was provided. The author also gave an explanation of the terms utilised in this study, concluding with this outline of the chapters of the thesis.

Chapter 2 is a review of the current literature on mathematical thinking and problem solving in mathematics. The teaching, learning and assessing of problem solving is discussed. The factors which affect students’ ability to engage in problem solving are investigated and the current knowledge and performance of both students and teachers are analysed. The implications of a teacher’s knowledge on student achievement in mathematics is highlighted, with consideration given to the current knowledge of pre-service teachers and the instruction they currently receive as part of their teacher education programme. Models and theories of mathematical thinking are analysed and the chapter concludes with a comparison between these models and theories in relation to the development of the author’s F-TAPS.

Chapter 3 outlines the way in which the research in this study was undertaken. Descriptions of the research paradigms which form the foundation of this research, along with descriptions of the theoretical framework and the methods of data collection and analysis are included in this chapter. The philosophical assumptions underlying this research, along with the corresponding appropriate research methods employed are discussed. Explanations for the choice of data collection and analysis are also provided.

Chapter 4 provides a detailed description of the Framework for Teaching and Assessing Problem Solving (F-TAPS) which was developed by the author. The theory underlying the assessment component developed on the basis of the F-TAPS is also given, with the presentation of the inter-rater reliability.

Chapter 5 provides an analysis of the findings from the pre-test assessments. The analysis of the data provided information on the current performance of the pre-service teachers in problem solving in mathematics. Common difficulties and misconceptions were identified.

Chapter 6 provides a detailed description of the intervention developed, the decisions made in relation to the time, organisation and implementation of the intervention are discussed.

Chapter 7 provides an analysis of the findings from the post-test assessments. The findings of the assessments after the intervention (post-tests) are discussed.
in this chapter. These findings are discussed relative to the pre-test findings for the pre-service teachers who participated in all three phases of this research study (pre-test, intervention and post-test phases). The findings for each phase (reading and understanding, planning and solving, and solution and checking) of the problem solving process are discussed. The overall findings are also discussed and the findings of the post mindset questionnaire is discussed relative to the pre-test findings on mindset.

Chapter 8 discusses the contribution of this study to mathematics education research. A summary of the research is given and conclusions are presented. Recommendations and possible future work are discussed.
2 Literature Review

2.1 Introduction

This chapter provides a review of the literature that is relevant to the research of this thesis. A literature review serves the purpose of uncovering existing knowledge relating to the research interest and question(s) of the researcher. The review of the literature facilitates the identification of the research topic to be studied by the researcher by narrowing the focus of the research interest to a specific topic (Machi and McEvoy, 2016). The research topic then serves to frame the literature review conducted by the researcher. The comprehensive review, analysis and synthesis of the existing knowledge relating to the research topic facilitates the identification of unresolved problems and/or questions which remain to be answered. These unresolved problems provide a point from which to build the study.

This literature review discusses problem solving in mathematics at secondary level. The author first looked at the performance of students in problem solving in mathematics. The deficiencies which were found to exist in students’ problem solving performance prompted the author to look at the possible factors impacting their performance. One of the factors identified was teacher knowledge. After reading about teacher knowledge and the initial reports on Project Maths, the author noticed common factors between the aspects of the maths syllabus that the teachers found challenging and the deficiencies which were evident in students’ work. This led to the author considering working with pre-service secondary mathematics teachers. To begin this chapter, an overview of the issues pertaining to problem solving in mathematics at secondary level in Ireland is presented.

2.1.1 Overview of the Current Performance of Irish Students in State Examinations

The Chief Examiner’s report of the 2005 higher level Leaving Certificate\textsuperscript{2} mathematics paper revealed that there continued to be weakness among higher level students, particularly in the area of problem solving, as a result of insufficient understanding of mathematical concepts and poor problem solving and decision making skills (State Examination Commission (SEC), 2005; SEC, 2000).

These inadequacies caused substantial impediments to progression when the questions progress in ways which students were not accustomed to, or necessitated more than the practised application of well prepared algorithms. While students displayed adequate procedural skills, any question that required students to demonstrate a good understanding of the concepts behind these pro-

\textsuperscript{2}The Leaving Certificate examination is the final examination in the Irish secondary school system. It requires a minimum of two years’ preparation. Mathematics is studied over a duration of 180 hours. All students (>97%) study mathematics for 13/14 years in school.
cedures caused undue difficulty (SEC, 2005). There was also a decline in the perseverance for success among students, when faced with unfamiliar problems (SEC, 2005). An increase in the number of extra questions attempted gives reason to believe that this decrease in perseverance was not as a result of the time constraint (SEC, 2005). Students seem to have insufficient capability to tackle problems that are different to those which they have been predisposed to tackle.

Feedback from the SEC on the 2011 Leaving Certificate Mathematics examination (initial schools Phase 2) again highlighted conceptual understanding as problematic at all levels. Questions requiring the application of knowledge and skills in unfamiliar situations from those in which these skills were developed also caused difficulty among students.

The most recent Chief Examiner’s report of the 2015 examinations in mathematics show a general increase in the willingness of students to engage in non-routine problems. However, this report highlighted that Junior Certificate and ordinary level Leaving Certificate students predominantly employ trial and error in their approaches to solving unfamiliar problems, and struggle if the problem necessitates the use of any substantial amount of algebra (SEC, 2015).

It was noted at Leaving Certificate ordinary level that students generally ceased in their attempts to solve the problem once any difficulty was experienced (SEC, 2015). The majority of students at ordinary Leaving Certificate level were not able to solve unfamiliar problems, even problems of an easy level (SEC, 2015). The students (ordinary leaving certificate level) were “unable to formulate and represent information in mathematical form and hence solve problems” (SEC, 2015 p.24).

The report revealed a higher capacity among students of all levels (at both Junior and Leaving Certificate) to apply knowledge to problem situations but there are many who still struggle with this objective (SEC, 2015). Similarly while there was improvement in the level of explanations and supporting work shown, students (in particular Junior Certificate higher level) are still experiencing considerable difficulties in this aspect. In general, Junior Certificate higher level students were unable to formulate and solve a problem involving simultaneous equations (question 11(c) paper 1), and experienced great difficulty in forming equations (question 14(b) paper 2). Junior Certificate higher level students experienced great difficulty when algebra featured heavily in the solution of problems (SEC, 2015).

It was noted that Leaving Certificate higher level students showed a lack of use of representation in their work and failure to consider different approaches to problems (SEC, 2015). An inability to interpret information presented diagrammatically and in text (question 7(a) paper 2) led to errors among many students. There was also a decline in their (Leaving Certificate higher level) basic algebraic manipulation ability (SEC, 2015). While there is an increase in students’ capability to apply their knowledge to an unfamiliar problem, many
higher level Leaving Certificate students are still struggling to solve problems in unfamiliar contexts (SEC, 2015).

The report of the 2015 state examinations in mathematics identified the following as causes for concern:

- the performance of higher level Leaving Certificate students in applying basic skills appropriately and accurately to mathematical situations;
- decline in basic algebraic manipulation of higher level Leaving Certificate students causing them to struggle in solving non-routine problems;
- inadequate basic competency of ordinary level Leaving Certificate students in algebra (algebraic manipulation in particular), inability to engage with unfamiliar problems and lack of perseverance;
- inadequate basic competency of ordinary level Junior Certificate students in algebra (algebraic manipulation in particular).

Some of the concerns present in the 2015 Chief Examiner’s report are similar to those highlighted in the 2005 report (difficulty with appropriate and accurate application of skills, inability to engage with unfamiliar problems and lack of perseverance). The findings of the 2005 Chief Examiner’s report had been a factor leading to a review into the mathematics curriculum at second level. This resulted in the implementation of a new curriculum, namely Project Maths aimed at improving these issues.

2.1.2 The Implementation of Project Maths

In 2005, the National Council for Curriculum and Assessment (NCCA) in Ireland conducted a review of the second-level mathematics curriculum and assessment. This review was prompted by concerns raised by the findings of several research studies and Chief Examiner’s reports (NCCA, 2012). The concerns related to the curriculum and assessment in place at that time included the following:

- more significance placed on developing procedural skills rather than developing understanding;
- high difficulty experienced with applying mathematics in real-life contexts;
- low grades achieved by ordinary level Leaving Certificate students in particular;
- small percentage of students opting to study mathematics at higher Leaving Certificate level;
- lack of deep understanding of basic mathematical concepts;
omission of certain parts of the syllabus by teachers and students due to choice available in the State examination and little integration of topics in the State examination;

predictable nature of the State examinations and the assessment of instrumental understanding\(^3\) over the assessment of relational understanding;

deficiencies in the basic understanding of concepts in arithmetic, algebra and geometry displayed by a number of third-level students, including those who studied mathematics at higher Leaving Certificate level with subsequent difficulties experienced with the mathematics in their chosen course at third-level.

(NCCA, 2006, p.7)

“At present one might describe the teaching of mathematics as a pre-occupation with the ‘how’ of the subject to the almost total neglect of the ‘why’ of the subject. It is vital that the mathematical material presented in future texts and examinations cater more to applications so that students can acquire a greater understanding and appreciation of mathematics as a problem-solving discipline”.

(NCCA, 2006, p.10)

Project Maths was implemented in 24 secondary level schools initially in 2008 and based on feedback from these schools the syllabus was implemented in phases in all schools from September 2010 (SEC, 2015). The changes involved in the implementation of Project Maths compared to the old syllabus involve different content and a change in the emphasis of skills developed (SEC, 2015). At Leaving Certificate level, there is an increase in the amount of statistics and probability, removal of vectors and matrices (these were studied on the old syllabus) and changes made to the content of functions and calculus. Strands 1 (Probability and Statistics) and 2 (Geometry and Trigonometry) were implemented in 2010, Strands 3 (Number) and 4 (Algebra) were implemented in 2011, and Strand 5 Functions was implemented in 2012. As the new strands were phased in, content from the previous syllabus was simultaneously phased out. There is an “increased emphasis on problem solving as well as explanation, justification and communication of work” (SEC, 2015, p.4).

A report conducted by the NCCA in 2012 revealed that the more traditional approach to teaching mathematics (textbook and practice, copy from board and practice) continues to be prevalent (greater than 85 percent) in schools (NCCA, 2012). The introduction of the Project Maths curriculum encourages a shift away from the traditional approach (mathematics in context, investigation/exploration of concepts and ideas, applications of mathematics and group activities (NCCA, 2012)). The traditional approach was marginally (< 1%) less

\(^3\)section 2.8
commonly used in schools that were involved in phase one of the Project Maths initiative. The present system will not change unless examination questions require thinking skills from students (NCCA, 2012).

The Leaving Certificate syllabus (2015) states that the emphasis in Project Maths is on the development of an interconnected understanding of mathematics. It is expected that as learners progress through the course, their mathematical conceptual knowledge and skills are cultivated by working with more challenging contexts and refining their problem solving approaches. This suggests an approach to the teaching of mathematics where problem solving is integral to the development of an understanding of mathematical concepts. The Junior Certificate syllabus also suggests a problem solving approach to teaching and learning mathematics, stating that problem solving should infiltrate all aspects of the teaching and learning experience.

The lack of clearly-defined methodologies to interpret, simplify, abstract, and formulate mathematical problems in context in the Project Maths syllabus may prove problematic for teachers and students (Grannell et al., 2011). It should not be left to the individual teacher to interpret what is meant by ‘more sophisticated approaches to problem solving’ (Leaving Certificate syllabus 2015, p. 7) and to discover how to develop these approaches to problem solving among students.

In the report of the impact of Project Maths on student achievement, learning and motivation (Jeffes et al., 2013), evidence of students’ work (154 pieces examined, in phase one and non-phase one schools, ranging from Junior Certificate to Leaving Certificate level) suggests that students have deficiencies in problem solving. Formulating representations to illustrate their thinking or as a means to help them to interpret and solve problems was also very scarcely presented in their work. There is almost negligible evidence, in the work reviewed, of students showing rational thinking and verification, or revealing that they have formed connections between different mathematical topics.

Also revealed in this report (Jeffes et al., 2013) were concerns among teachers. One concern is the time available for teaching the new curriculum through a more problem-solving approach. A number of teachers reported that they are teaching new material using the traditional methods of drill and practice as a result of time pressure. The length of a class period was reported to be a limiting factor in engaging with new practices. Teachers feel they need more time to teach for and develop understanding. Recognising connections between strands is also proving to be a challenging task, as is formulating assessments (other than summative examinations) which reveal the level of understanding of the students.

The review of students’ work and the findings on teachers’ concerns suggest that a certain alignment exists between what teachers consider to be challeng-
ing components of the new curriculum and the deficiencies which are evident in
students’ work.

Teachers need more support to gain skills in the effective use of new teaching
methods and in facilitating improved learning experiences for their students.
Students require the regular provision of quality tasks which necessitate their
engagement with problem solving, forming connections between topics in math-
ematics and gaining skills in communicating their reasoning and justifying their
solutions both verbally and in writing (Jeffes et al., 2013).

The most recent report by the NCCA (2014) reveals that there continues to be
issues in the implementation of the Project Maths course in secondary schools.
Teachers have reported a struggle with the time demand to complete the course.
In particular, teachers (of higher level Leaving Certificate mathematics) state
that there is difficulty in the apportioning of time so that students engage with
the new approaches and emphasis on problem solving, while still allowing suffi-
cient time to ensure that the fundamental aspects of mathematical knowledge
and skills are being developed (NCCA, 2014). Teachers also stated that there
is increased pressure to focus on state examination preparation so that students
will be able to cope with the problem solving questions on the paper (NCCA,
2014).

Suggestions on possible future professional development made in this report
(NCCA, 2014) include the building of ideas for mathematical tasks which are
effective in developing thinking and problem solving skills. The report also rec-
ommends that a “reconceptualisation of mathematics teaching and learning” is
required in order to understand the changes in the approaches and emphasis
in mathematics teaching and learning involved in Project Maths (NCCA, 2014
p.10). In particular the collaboration of students with teachers and peers in de-
veloping mathematical knowledge and skills as opposed to the teacher-centred
approach of the old mathematics syllabus. There is pressure to re-design the
Leaving Certificate syllabus to make the links between topics more clear to
teachers and also to use examples to show how particular learning objectives
are being attained. The research report (NCCA, 2014) comments on the fol-
lowing aspects of the new mathematics course which have not yet been fully
implemented in schools:

• although evidence exists to show that students are engaging with the new
activities and approaches (mathematics in context, investigation/exploration
of concepts and ideas, applications of mathematics, group work to formu-
late ideas, problem solve and assess methods) in the Project Maths syllabi,
traditional approaches to teaching mathematics (text book and practice,
copy from board and practice) continues to be prevalent in schools;

• there is the possibility that teachers are focusing on the content rather
than on the processes involved in the Project Maths syllabi;
• while students are showing evidence of proficiency in mathematical procedures, there is less evidence shown of proficiency in problem solving and forming connections between topics in mathematics.

A report by Prendergast and Treacy (2017, p.12) revealed that teachers are in a state of transition and although they state that they are trying to use the new approaches to teaching and learning they also report that with some approaches (functions based approach to algebra) they are “not really sure what they are doing or supposed to be doing”. For the revised syllabus, with its emphasis on problem solving to be fully implemented in the classroom, teachers need to fully understand the changes made to both the theory and practical aspects of the syllabus (Carless, 1999).

The realisation that some students are entering third level education with an inability to “formulate and represent information in mathematical form and hence solve problems” (SEC, 2015, p.24) of even a basic level is quite concerning. The initial part of this chapter introduced the problem at the focus of this research study. To proceed from here it is necessary to clearly define what is meant by the terms “problem” and “problem solving”, since there exists various interpretations. The next section discusses both.

2.2 Problems and Problem Solving in Mathematics

2.2.1 Problems in Mathematics

Jonassen (2000) identifies two significant aspects of a mathematical problem, the existence of an unknown entity and a value (social/intellectual/personal) associated with finding this unknown (solving the problem). This is similar to Dewey’s (1933) idea of reflective thought. If an unknown is not perceived in a task, then there is no problem, i.e. there is no need for reflective thought in order to complete the task. Secondly, if one perceives an unknown entity in a task but doesn’t associate a value with determining this unknown, then one is unlikely to make an attempt to determine the unknown, i.e. to solve the problem.

Mayer (1996) describes a problem in mathematics as having to determine a way from a given state to a goal state. There is a question (given state) to which the individual (problem solver) does not know the answer (goal state) and does not know immediately how to go about obtaining this answer. Cognitive activity is required by the problem solver(s) in order to obtain the answer to a problem. A task in mathematics is a problem to an individual/group if they perceive an unknown entity in the question, associate some value with finding this unknown and by doing so, employ cognitive activities in the form of mathematical thinking, to apply the combination of existing knowledge, previous experiences and skills in order to figure out how to begin the problem (task is non-routine) and to arrive at a solution, alternatively or in addition to recognising the need to
learn new knowledge and/or skills in order to solve the problem.

The measure of a problem’s difficulty is represented by the relationship between the learner and the problem (Lester, 1994). Several researchers in the area of problem solving concur with this (Burkhardt, 2006; Schoenfeld, 1985), in that what is a problem for one individual may not be considered a problem by another and also, what is a problem to one today may not be a problem for that same person in the future. Jonassen (2000) also notes the complexity of a problem as being a variable of its difficulty. The more complex the problem is, the more inherent cognitive operations are required to solve it, and the greater the load is on working memory (Kluwe, 1995, cited in Jonassen, 2000). The way a problem is structured and presented/represented has a bearing on its complexity, these factors are often intertwined (Jonassen, 2000). Problems in mathematics differ in structure, in how they are presented and the degree of complexity involved in solving them. Mayer and Wittrock (1996) described problems as ill-defined, well-defined and routine, nonroutine. In a well-defined problem, the ‘given state’, ‘goal state’ and mathematical operators allowed are clearly specified, whereas in ill-defined problems, one or more of these states/operators are not clearly specified. Jonassen (1997) classified mathematical problems into well-structured and ill-structured problems. The differences between these are presented in Table 1. In general, ill-structured problems tend to be more complex than well-structured problems, although there exists well-structured problems which are very complex, and some ill-structured problems which are quite simple (Jonassen, 1997).
It was found in a study conducted by Schraw et al. (1995, cited in Jonassen 2000) that the thinking required to solve well-defined problems differs from the thinking employed when engaged with ill-defined problems. Good transfer of knowledge, therefore does not necessarily occur from learning to solve well-defined problems to learning to solve ill-defined problems. It is therefore important that students be exposed to both types during their learning experience of mathematics. Generally, the skills developed in engaging in ill-defined problems are capable of being transferred more than those developed in engaging in well defined problem solving, which are more likely to have limited transferability (Kirkley, 2003). Students need to become accustomed to engaging with the complexities of ill-defined problems to compete in the global job market (Lombardi, 2007).

Young children are naturally curious, they possess innate ability for making sense of number and have substantial intuition for solving problems (Jung et al., 2007). Many studies have revealed that students are actually better at solving problems prior to participating in mathematics classes (Boaler, 1997; Carpenter et al., 2003). Children work through a problem by using reasoning to gain an understanding of the problem and possible methods to obtain a solution in their own creative way. After many mathematics classes, in which students are often inactive learners, and are presented with numerous rules, their innate ability to problem solve seems to get ‘knocked out’ of them (Boaler, 2009, p.38). Students come to the conclusion that they need to remember all these rules and

<table>
<thead>
<tr>
<th>Well-structured Problem</th>
<th>Ill-structured Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present all elements of problem to learners</td>
<td>All elements not presented to learners</td>
</tr>
<tr>
<td>Require the application of a limited number of routine rules and principles that are organised in predictive and prescriptive ways</td>
<td>Possess multiple criteria for evaluating solutions, so there is uncertainty about which concepts, rules, and principles are necessary for the solution and how they are organised</td>
</tr>
<tr>
<td>Have knowledgeable, comprehensible solutions where the relationship between decision choices and problem states is known or probabilistic</td>
<td>Possess multiple solutions, solution paths or no solutions at all</td>
</tr>
<tr>
<td></td>
<td>Often require learners to make judgements and express personal opinions or beliefs about the problem, making it a uniquely human interpersonal activity</td>
</tr>
</tbody>
</table>

Table 1: Problem Type (Jonassen, 1997, p.68)
following the rules takes precedence over employing their common sense. In this way, mathematics may be viewed by students as a set of arbitrary rules that must be followed in order to achieve success. Resnick (1986) proposed that the procedural focus on learning mathematics may deter students from drawing upon their intuitions, while Schoenfeld (1991) also identified the perception held by students, of mathematics studied in the classroom as being separate from common sense. This narrow perception conflicts with the sense-making subject that mathematics actually is. This view of mathematics may also contribute to an inability to see connections between and within topics in mathematics (Boaler, 1997).

Students (primary level, secondary level, and pre-service teachers) claim they can complete a mathematics problem correctly without understanding what they are doing (Spangler, 1992). Students also rarely check if their solution makes sense in the context of the question. It seems that students do not necessarily use logical thought/common sense when they are engaged in solving a mathematics problem.

Students’ claims of being able to complete a mathematics problem correctly without understanding what they were doing demonstrates a misconception of the term ‘problem’. Having defined what is meant by a problem in mathematics, the next section discusses the term ‘problem solving’.

### 2.2.2 Problem Solving in Mathematics

There are various interpretations of the term ‘problem solving’. Schoenfeld (1992) describes it as learning to tackle tasks, which the learner is unaccustomed to, when the appropriate solution methods, (including those which the learner is not fully proficient in) are not known in advance by the person engaged in the task. Lesh and Zawojeski (2007) refer to the necessity of developing a more ‘productive way of thinking’ when engaged in problem solving (p.782). Problem solving is defined in the 2015 Leaving Certificate syllabus as being ‘engaged in a task for which the solution is not immediately obvious’ (p.10).

Common in all definitions is the idea of cognitively searching for a solution to a problem, when the path to the solution is unclear or completely unknown. In the 2015 Leaving Certificate Mathematics syllabus, it is stated that there are three requirements for learners, when involved in mathematical problem solving. The first is, they must understand the problem, that is, they need to make sense of what it is that is being asked of them and also what information has been presented to them, that they can make use of in their attempt to solve the problem. They need to be able to formulate a correct representation of the problem in their mind. Secondly, they must understand the mathematics they employ to solve the problem and also, the mathematics which they may learn from solving the problem. Finally, it is desired that they arrive at the correct solution. Lesh and Zawojeski’s (2007) reference to the development of a
‘more productive way of thinking’ describes the goal of engaging in mathematical problem solving. Solving a complex problem is the result of engaging in problem solving, which is the process of developing this ‘more productive way of thinking’.

Lester and Kehle (2003), cited in Lester, (2013) summarise mathematical problem solving as:

‘the coordination of previous experiences, existing knowledge, familiar representations, patterns of inference and intuition in an effort to generate new representations and related patterns of inference that resolve some unknown, which was the basis for the problem solving cognitive activity’.

(Lester 2013, p.249)

This definition includes several factors, not mentioned in the definitions included at the beginning of this section, mainly the coordination of the various factors employed in problem solving and the use of intuition in the process. This is more closely aligned with the true nature of problem solving as a set of interconnected relationships within and between several factors required for successful problem solving. This definition of problem solving, along with the previous definitions and knowledge gained from the review of the literature is summarised in Appendix A.

Problem solving is a complex human activity (Begle, 1971; Milgram, 2005), there are many variables involved in the successful engagement with mathematical problem-solving, both during the process and governing the process. These variables are attributable to the nature of the problem task, the problem solver(s) and the teacher, and interactions within and between them and the learning environment. These are variables such as: mathematical content knowledge: both procedural and conceptual, strategies, beliefs, dispositions, emotions, interest development, mindset, metacognition, expectations, intuition and logical reasoning processes (Stacey, 2005; Mayer and Wittroch, 2006). This given list is not collectively exhaustive. These factors are discussed in section 2.4.

This complex nature of mathematical problem solving, as an interconnected web of cognitive processes, influenced by several factors as mentioned, render making improvements in this area difficult (Lester, 2013; Lesh, 2005). Difficulties in engaging in mathematical problem solving may lie at any/ several point(s) in this interconnected web. Thus in attempting to make improvements in the ability of students to engage in mathematical problem solving, and to the instruction/learning and assessment of mathematical problem solving, the area of mathematical problem solving needs to be viewed as a whole.

A complex phenomenon consists of an interconnected web of components, which interact with each other in varying combinations within the phenomenon and
also interact with the phenomenon itself. These relationships between component-component, component-whole system and combinations of these relationships collectively form the complex phenomenon. These combinations of relationships constitute the complex system and how the system behaves/responds to an environment needs to be studied in the completeness of its complexity since to study it alternatively would diminish the meaning of the findings (Waldrop, 1992 in Grootenboer, 2010).

Mathematical problem solving is one such complex phenomenon. In determining a suitable framework for instruction and assessment, attention must be focused on the overall constitution of mathematical problem solving, which will also facilitate the consideration of its individual components and their relationships within and between the system. The tendency to conceptualise problem solving in a more simplistic manner has been noted as being an inhibitory factor in the effectiveness of instruction on improving students’ ability to problem solve (Lester, 2013).

There is a distinction between problem solving and solving problems. It is possible to solve some types of problems (school text book problems for which a procedure for obtaining the solution is known in advance), without being engaged in the problem solving process. Authentic problem solving is not the mechanical application of a given algorithm to a problem to arrive at a correct solution. It is the cognitive processes that one uses (both verbal and non-verbal (Miligram, 2005)), to both understand the problem and arrive at a correct solution that is the essence of problem solving. Omitting this process from problem-solving in mathematics, reduces the activity to a lesser demanding task of a question which needs to be answered, not a problem which needs to be solved. A student who has memorised procedures for solving certain mathematical problems may only be able to solve problems he has seen already, whereas a student who is a mathematical problem solver has developed the understanding to attempt to solve problems he has never encountered before.

With the terms problem and problem solving defined, the author next looked at the performance of Irish students in problem solving at primary level, and at secondary level relative to the performance by students in other countries.

### 2.3 Performance of Students in Mathematical Problem Solving

As previously mentioned in section 2.1, the 2015 Chief Examiner’s report revealed a lack of perseverance and a general inability among ordinary level Leaving Certificate students, to apply their mathematical knowledge to solve unfamiliar problems. The ability of higher level Leaving Certificate students to apply basic skills appropriately and accurately was also noted as a point for concern. In addition, the struggle among Junior Certificate level students and Leaving Certificate ordinary level students to solve problems which necessitated
the use of any amount of algebra was substantial.

Problem solving in primary school is also of concern. Deficiencies have been noted in students’ ability to solve word problems on national assessments in mathematics at second and sixth class levels\(^4\) (Shiel and Kelly, 2001). A report on the 2009 national assessment revealed a continuing trend of weak performance in problem solving in mathematics, at second and sixth class levels (Eivers et al., 2010). Also identified in this report (Eivers et al., 2010), from the data obtained from teachers in the study, is the possible belief that teachers attach more value to speed and accuracy in mathematics than they do to the ability to problem solve. This has been identified as a problematic measure of students’ achievement (Boaler, 1997). This perception is also more likely to be accompanied with the emphasis in teaching being on procedural knowledge rather than conceptual understanding (Philipp, 2007). Recommendations given in the 2010 report on the national assessment of mathematics in Irish speaking schools\(^5\) include placing more emphasis on developing mathematical reasoning and problem solving in the senior classes in primary school.

A report of the outcome of Irish students’ achievement in the 2011 Trends in International Mathematics and Science Study (TIMMS), revealed a significant relative weakness (18 points lower than overall mathematics result) on reasoning in mathematics among ten year old pupils, while displaying a significant relative strength on knowledge of mathematics (12 points above the national mean result). The application of mathematics, while two points above the national mean mark, was not of statistical significance (Eivers and Clerkin, 2012). This highlights that although students have acquired knowledge of mathematics, and have some knowledge of when to apply this knowledge, they lack the ability to explain/justify their decisions regarding the use of their mathematical knowledge and also why/how mathematics works the way that it does. In this study, students were examined on number, geometric shapes and measures, and data display. Ireland ranked seventeenth of fifty countries. Pupils in thirteen countries performed significantly better than students in Ireland.

An established indicator of mathematical performance, is the OECD (Organisation for Economic Co-operation and Development) Programme for International Student Assessment (PISA). This programme assesses fifteen year olds in OECD countries and partner countries and economies in the areas of mathematics, problem solving, science and reading. Results in PISA assessments are considered to be a reliable way of comparing students’ performance across countries (OECD, 2014). The problem solving assessment in PISA 2012, examined the thinking processes employed in problem solving, as opposed to the ability to

\(^4\)Students in second class and sixth class at primary school level are 8 years and 12 years old on average respectively.

\(^5\)Irish is the language of instruction and communication in an Irish-speaking (or Irish-medium) school. Irish-medium schools follow the standard curriculum set by the Department of Education and Skills.
solve problems in a particular knowledge domain. This aimed to eliminate the need to use knowledge from curricular areas. Knowledge of mathematical content from the curriculum was examined in the problem solving involved in the mathematics assessment in PISA. The problems given in the problem solving assessment were novel problems for which the solutions were not obvious. The framework for assessment includes the examining of various cognitive processes involved in problem solving; exploring and understanding, representing and formulating, planning and executing and, monitoring and reflecting (OECD, 2014, p.2). Students completed the problem solving assessment on computer. This facilitated the storage of consecutive processes conducted by the students, as they solved the problem and also allowed for the exploration of virtual problem situations.

In 2003 the mean score achieved in problem solving for Ireland was 499 compared to a mean score of 500 for OECD countries (OECD, 2004). Students in Ireland achieved a mean score of 498 on the assessment of problem solving in 2012, the mean score for OECD countries was 500 (OECD, 2014, p.52). Ireland ranked seventeenth of the twenty eight OECD countries, and twenty second of the forty four countries, which participated in the assessment in 2012. Students in eighteen countries, which include thirteen OECD countries performed significantly better than students in Ireland on this assessment of problem solving (Perkins and Shiel, 2014, p.8). Irish students’ performance in problem solving is thus at OECD average level. In general, the top performers in mathematics, reading and science are also among the best achievers in problem solving. Given the PISA results obtained by Irish students in mathematics, science and reading, Irish students performed lower than expected in problem solving (Perkins and Shiel, 2014, p.15). Students in Ireland are stronger than expected in exploring and understanding problem situations and in monitoring and reflecting on their solution. Students in Ireland are weaker than expected at formulating and representing, and planning and solving problem situations.

Students in Ireland who are among the top in terms of achievement in mathematics (≈ 90th percentile) are underperforming in mathematics relative to their international peers of equal levels of achievement (significant decrease from 2003 – 2012, (OECD, 2014)). These students are below average in spotting appropriate situations where mathematics could be used in a real life context and also in formulating the problem into mathematical language (OECD, 2014).

Students from Singapore, Korea and Japan outperform students from Ireland in both the PISA problem solving assessment and the PISA mathematics assessment and have a mean performance in these assessments that is above the OECD average (top three results). Students in Singapore, Korea and Japan achieved mean scores of 562, 561 and 552 respectively on the assessment of problem solving in 2012 (OECD, 2014)). This is significantly above Ireland’s mean score of 498. In mathematics, students in Singapore and Japan achieved mean scores of 573 and 536 respectively, students in Ireland achieved a mean
score of 501.3 (OECD, 2014). The OECD average was 494 (OECD, 2014).

2.3.1 Problem Solving in Mathematics - Singapore, Korea, Japan, and the Hungarian Approach

Problem solving is the central focus of the mathematics framework underpinning the mathematics curriculum in Singapore (Ministry of Education, 2013). The mathematics framework for the curriculum also emphasizes the development of conceptual understanding, proficiency in mathematical processes and skills and gives due consideration to metacognition and attitudes (Ministry of Education, 2013). Problem solving strategies are introduced at primary school level in Singapore and pre and in-service training of teachers contribute to teachers’ high content knowledge. Fewer topics are studied at primary level but in greater depth and to mastery level (Ginsberg et al., 2005). As part of Singapore’s spiral curriculum, topics are revisited later to be studied at increasing levels of depth and difficulty. The primary syllabus in Singapore includes an overview of the aims of the syllabus across primary, secondary and pre-university levels (Ministry of Education, 2013). This overview facilitates teachers’ understanding of the syllabus in terms of how the body of knowledge and understanding that they are teaching fits in to the bigger picture. The dependence of future syllabi on previous syllabi is explicitly evident in the primary syllabus and this contributes to an understanding of the importance of building a well connected and deeply understood foundation.

Mathematical processes (reasoning, communicating, making connections, applications, mathematical thinking and using heuristics) are deliberately taught (Ministry of Education, 2013) and teachers make these processes visible by thinking aloud during the problem solving process. Ideas, skills or concepts are introduced using the Concrete-Pictorial-Abstract (CPA) approach or Brunner’s (1957) enactive, iconic, and abstract phases of representation. This approach introduces students to an idea or skill through the use of manipulatives in the enactive phase. Students then use the representations from the enactive stage to form pictorial/visual representations of the concrete situation in the iconic phase. Finally students use words and mathematical notation to represent the problem situation in the abstract phase. The CPA approach facilitates the understanding of concepts, the ability to see connections and develops mathematical thinking and problem solving skills (Ginsberg et al., 2005). Class discussion of various solutions methods and reasons for preferences for particular methods are held as part of the learning process. Mathematical ideas are provided to support students who are experiencing difficulty and to provide enrichment for students who require more challenging tasks (Ministry of Education, 2013).

In Korea, only the top 5% of high school graduates are accepted onto education programmes, and mathematics teachers are required to take an employment test, in addition to a demonstration of their teaching as part of the interview
process (Koyama and Lew, 2017). The test consists of general mathematics education and mathematical content knowledge. Approximately 4.1% – 4.4% of applicants are successful in the completion of this process. An approach similar to the CPA approach used in Singapore is used in Korea, this is known as the Reality-Model-Agreement-Method approach. Students usually begin at the reality stage of this approach but the teacher selects the stage which is most appropriate for developing the students’ mathematics levels (Bae et al., 2008). Intuition and manipulative activities are used to introduce mathematical concepts and ideas in the reality stage. In the model stage, students draw diagrams or use mathematical notation to represent the corresponding reality stage of the problem situation (Bae et al., 2008). In the agreement stage, students use mathematical symbols and terms that agree with the model stage. The students discover simpler, more efficient methods for solving problems in the final method stage, this is achieved by letting the students engage in the productive struggle with many problems (Bae et al., 2008). Teachers encourage this discovery process and mathematical thinking is developed through the comparison of multiple approaches to finding a solution to a problem (Bae et al., 2008).

In Japan, teachers have a high level of mathematical content knowledge and they engage in reflective practice to continuously improve their teaching (OECD, 2010). Teachers work collaboratively to design lessons in a process known as “lesson study” (Leung et al., 2015). Lesson study involves collaborative study of the content, methods of instruction and the ways in which students think and learn by the group of teachers (Leung et al., 2015). After the group of teachers have finished designing a particular lesson plan, one of the teachers from this group teaches the lesson to his/her students while the other teachers from the group observe the lesson (OECD, 2010). The observing teachers take notes on teacher and student participation in the lesson for later discussion in a meeting held after the lesson (Leung et al., 2015). The discussion focuses on areas such as students’ responses to the mathematical tasks, and the appropriateness of the questioning utilised by the teacher. After the evaluation of the lesson in the discussion, any recommended revisions are made and the lesson is taught again in another class with observers (Leung et al., 2015). This process is the problem solving cycle of lesson study (Leung et al., 2015).

The main aim of mathematics teachers in Japan is to achieve student engagement (OECD, 2010). Teachers give careful consideration to their lesson planning, and problem solving features heavily in their lessons (OECD, 2010). A practical problem usually starts the lesson and lessons which are organised around a single problem are commonly found in many mathematics classes in Japan (OECD, 2010). Students are not grouped by ability and the class size is large by comparison with Western classrooms (OECD, 2010). Teachers use questioning to stimulate mathematical thinking and the lesson is aimed at developing deep understanding (OECD, 2010). Teachers monitor students’ work and select students to present their work on the board, some of the work selected is correct and some is not. The class is asked to give their opinions of the so-
olution methods used by the students at the board (OECD, 2010). All opinions of the work must be based on mathematical reasoning (OECD, 2010). Class discussions on the approaches taken by different students, reasons for incorrect solutions, and the efficiency of the various solution paths are held (OECD, 2010). The larger class sizes facilitate numerous problem solving strategies from which all can learn.

High expectations are set for all mathematics students in Japan and higher-achieving students can help lower-achieving students in their class or school which facilitates the learning process of both types of student (OECD, 2010). Teachers demand effort and believe achievement comes as a result of effort. A reform in education in Japan in the early 21st century saw a reduction on the emphasis of rote learning in the mathematics curriculum and an increase in problem solving and experimentation (OECD, 2010). The mathematics curriculum has been carefully designed with respect to the logical sequencing of topics and in depth exploration of these topics, and is set at a high level of cognitive demand. The teaching of the curriculum in the classroom is insisted upon (OECD, 2010).

Hungary is a country known for its long tradition of excellence in mathematics education (Matsuura, 2015). Problem solving, communication and mathematical creativity are explicitly emphasized in the curriculum. Teachers select problems for each lesson that achieve the mathematical goal for the lesson. Teachers give careful consideration to the sequencing of the problems to provide a logical coherence to the mathematics learned (Matsuura, 2015). The problems selected stimulate students’ mathematical thinking and students are left to engage in productive struggle towards development of understanding. The aim of mathematics teachers is to facilitate students’ engagement in mathematics and to provide opportunities for mathematical discovery. Class discussions on approaches to problems are held after students have engaged in solving them either individually or in small groups (Matsuura, 2015). Students share their solutions on the board and teachers act as a facilitator of students’ learning, using questioning to help aid understanding and advancement in achieving a solution to a problem. Teachers also aim to create a learning environment that is conducive to sharing mathematical ideas and experiences. After the students have engaged in the problem solving process, the teacher summarises the learning achievements and generalises the findings (Matsuura, 2015).

Common aspects in the teaching of mathematics in these countries include:

- a focus on problem solving;
- the development of mathematical thinking;
- explicit teaching of problem solving;
- high expectations;
• use of manipulatives;
• promoting discovery;
• logical sequencing of mathematical tasks;
• presenting, sharing and discussing multiple solutions.

2.3.2 Summary on Performance of Students in Mathematical Problem Solving

The results from PISA 2012 (OECD, 2014) revealed that on average, twenty percent of 15 year old students across 34 OECD countries are only capable of solving very basic problems (and not all the 20% are capable of solving even these simple problems) on the condition that they refer to situations that the students are familiar with. The students in the highest achieving countries in PISA 2012 are (on average) able to contend with problems of moderate complexity in a systematic, strategise a few steps ahead, and check on the progress of their plan (OECD, 2014). However the students in the lowest achieving countries in PISA 2012 are (on average) able to contend with problems of only a very basic level, where the formulation of a plan is not necessary and the problem situations are familiar (OECD, 2014). The results (OECD, 2014) showed that there are a significant number of 15 year old students who do not have the basic necessary problem solving skills in order to engage with unfamiliar problems, or to think even one step in advance. These basic skills are considered a requirement for successful participation in today’s world (OECD, 2014).

Although general improvements in the performance of Irish students in the TIMSS and PISA assessments have been noted, there is persistent difficulty in assessment items that require the application of knowledge and problem solving skills in both mathematics and science (STEM Education Review Group, 2016). How can Irish students’ current problem solving performance in mathematics be improved? In order to obtain a reliable answer to this question, the interconnected web of cognitive activities that is problem solving, along with the factors which have a bearing on students’ ability to engage in problem solving in mathematics are considered.

2.4 Factors Affecting Students’ Ability to Problem Solve

Epistemological beliefs, which are defined by Schommer as ‘students’ beliefs about the nature and knowledge of learning’ (Muis, 2004 p.320) can have an effect on how a student attempts to solve mathematical problems and also on their level of persistence when faced with challenging tasks. These beliefs are also likely to affect reasoning, learning and decision-making. Kilpatrick (1983) notes that students may not always solve problems in the classroom or in examinations, despite being well capable of doing so. He highlights factors such as a lack of interest, time pressure and the fear of failure as playing a potential
inhibitory role in the demonstration of capability in mathematical problem solving situations, by even the most adept students. A selection of the factors which can affect students’ ability to engage in problem solving are discussed in this section. These factors may also be considered as the competencies that need to be developed for successful engagement in problem solving. These include:

- Knowledge;
  - teacher knowledge;
  - student knowledge;
    * conceptual understanding;
    * relational and instrumental understanding;
    * deductive and inductive reasoning;
    * strategic knowledge - problem solving models and heuristics;
    * expert problem solvers versus novice problem solvers.

- metacognition, mathematical thinking and transfer;
- nature of the tasks;
- classroom environment;
- mindset, beliefs and attitudes;
- interest and perseverance;
- intuition;
- time.

2.4.1 Knowledge

Problem solving requires the application of knowledge to a problem situation in order to resolve it. It is the acquisition of this knowledge (how it is accumulated and organised in the mind) that is fundamentally the backbone of mathematical problem solving. One cannot apply knowledge that one does not possess (Carson, 2007). In order to solve a problem in mathematics, one must apply mathematical knowledge, in order to apply this knowledge one must understand it. Construction of new knowledge in the form of making new connections between existing knowledge is possible through problem solving in mathematics (Hardin, 2002). Both teacher knowledge and student knowledge are discussed in this section.

Teacher Knowledge

A teacher requires an integrated combination of different types of knowledge and skills in order to be effective. A teacher’s knowledge base plays a vital role in what mathematics is taught in the mathematics classroom and how it is
taught, therefore affecting the mathematics developed by one’s students. Teachers formulate plans and make decisions within the constraints of their knowledge base which subsequently affects what and how well students learn (Shavelson, 1983; Fennema, 1992; Ball et al., 2008). Both subject matter knowledge (knowledge of mathematics) and pedagogical content knowledge (knowledge of how to introduce, sequence and teach mathematics to best facilitate students’ understanding of mathematics) are required for effective teaching (Ball et al., 2008). Shulman (1986) proposed three categories of content knowledge that effective teachers should possess: subject matter content knowledge, pedagogical content knowledge and curricular knowledge. Of these three categories, Shulman suggested that subject matter content knowledge is the most important. Acquiring a strong knowledge of mathematics provides the foundation on which the other types of knowledge can be developed. These categories of knowledge should be present as interconnected strata.

Rowland et al., (2005) proposed the knowledge quartet. This model of teacher knowledge includes foundation knowledge, transformation knowledge, connection knowledge and contingency knowledge. Of these, Rowland states foundation knowledge as the most fundamental. Foundation knowledge is the mathematical content knowledge, understanding and beliefs of teachers. Similar to Shulman, Rowland et al. found that the other types of knowledge are built from the basis of mathematical content knowledge, however Rowland et al. also noted the role that a teacher’s beliefs play in forming this foundation knowledge.

Transformation knowledge is how the teacher transfers the knowledge and understanding he has to his/her students. This includes choice of presentation of ideas, representations, examples and explanations in order to best transfer content knowledge to his students.

Connection knowledge involves the ability of the teacher to sequence the teaching of topics in a way which facilitates the coherent, logical addition and reconstruction of mathematical knowledge by his/her students. This aspect of knowledge also involves the ability to present subject matter in a connected way, i.e to form connections within and between topics in mathematics, as well as connections between mathematics and other subject areas. This ability to form connections in mathematics and transfer them to students is heavily reliant on a deep connected knowledge of mathematical content, free from misconceptions. Beliefs, which form part of the foundation knowledge according to Rowland’s model, also play an integral role in what mathematical problem solving is taught by the teacher and how it is taught. As discussed in section 2.2, a single unified definition of problem solving does not exist. A teacher’s perception of what it means to be engaged in problem solving affects his/her approach to the teaching of mathematical problem solving.

Ball (1990) also proposed the absolute necessity for mathematics teachers to possess a connected knowledge base. The interconnected relationship within a
teacher’s knowledge base is also evident in Rowland’s model.

The fourth entity in the knowledge quartet, contingent knowledge, is the flexibility required by a teacher to use his/her knowledge to respond to the unexpected. This involves responding and utilising useful comments/questions/incorrect solutions by students as a meaningful learning experience and realising when and how to change the course of the mathematics lesson in order to achieve better overall understanding among his/her students.

Ball et al., (2008) further refined Shulman’s categories of teacher knowledge by dividing the subject matter knowledge proposed by Shulman into common and specialised content knowledge. Common content knowledge refers to the mathematical knowledge that is not solely specific to teaching, i.e knowing the actual mathematical concepts and procedures that other professionals who use mathematics in their work would also be familiar with. Specialised content knowledge is knowledge and skill that is exclusive to teaching; this includes the ability to clearly explain and justify in a logical way, the reasons why certain procedures work, for example why when dividing fractions, one inverts and multiplies, why the maximum point is found by setting the derivative of a function equal to zero. Others who use mathematics in their work can utilise these procedures in their calculations without having to explain why to others. Ball et al. (2008) also divided pedagogical knowledge further into knowledge of content and students, and knowledge of content and teaching. Knowledge of content and students involves anticipating students’ likely responses and difficulties in advance of them occurring and planning their lessons with this knowledge in mind. The knowledge of common misconceptions is also included here, for example, the misconception that multiplying numbers always results in a bigger number. Knowledge of content and teaching involves knowing the advantages and disadvantages of different examples, representations, demonstrations and sequencing of topics in instruction of mathematics.

O’Meara (2011) added knowledge of applications to the categories of teacher knowledge, stating that a teacher with this knowledge can make mathematics more relevant to students’ lives as they will be able to recognise problem situations in life where mathematics can be applied in order to resolve the problem. Lehman (1977) notes that understanding in mathematics is expressed in terms of one’s ability to apply one’s mathematical knowledge to problem situations, understand the meanings behind the procedures and be able to explain their work and understand logical relationships.

As described in the models of knowledge for effective teaching, subject matter knowledge is commonly proposed as being the most critical aspect of a teacher’s knowledge base, with the ability to form connections within and between mathematical concepts (this is essentially conceptual understanding (explanation in section 2.4.1) and to other subjects also playing a vital role in the foundation of an effective teacher’s knowledge and skills base.
‘Teacher quality is believed to be one of the most important factors affecting students’ learning’ (Ní Riordáin and Hannigan, 2011, p.291). Studies on the relationship between teacher content knowledge (based on results of subject matter competence tests) and student achievement have found a positive correlation between the two variables (Rowan et al., 1997; Wayne et al., 2003). In a study by Hill et al., (2005), of the relationship between teacher knowledge for teaching (providing mathematical explanations, representations and working with non-routine solution methods) and student achievement, teacher knowledge for teaching was found to be positively correlated with students’ gains in mathematical achievement during the first and third grade (Hill, 2005).

In Ireland, a national study conducted by Ní Riordáin and Hannigan (2011) found that 48 percent of teachers (teaching mathematics, number of participants = 324) were not qualified mathematics teachers. These teachers predominantly teach non examination years (particularly first and second year students) at ordinary level, foundation level and resource. The percentage of teachers teaching each year level with or without a teaching qualification is shown in Table 2.

<table>
<thead>
<tr>
<th>Teaching qualification in mathematics</th>
<th>1st year</th>
<th>2nd year</th>
<th>3rd year</th>
<th>4th year</th>
<th>5th year</th>
<th>6th year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes (n = 168)</td>
<td>88 (51%)</td>
<td>100 (51%)</td>
<td>134 (51%)</td>
<td>95 (51%)</td>
<td>133 (51%)</td>
<td>131 (51%)</td>
</tr>
<tr>
<td>No (n = 156)</td>
<td>81 (52%)</td>
<td>94 (60%)</td>
<td>79 (51%)</td>
<td>18 (12%)</td>
<td>45 (29%)</td>
<td>38 (24%)</td>
</tr>
</tbody>
</table>

Table 2: Numbers of teachers teaching each year level for both teachers with and without a teaching qualification (Ní Riordáin and Hannigan, 2011, p.298)

As discussed, the literature reveals the strong connection between the quality of the teacher and the achievement of the students. Mathematics is a subject that is learned by the hierarchical agglomerative clustering of concepts and skills. Mathematical concepts and skills need to be understood and developed at the early stages of a student’s education. This is crucial so that further analysis of the original concepts and skills can be conducted and new concepts and skills can be successfully added to reconstruct students’ knowledge base as they progress through their education (Smith, 2004). Since new knowledge causes a reconstruction of prior knowledge, the importance of complex thinking at all levels of mathematical learning is highlighted in order to facilitate the addition of new knowledge to a well formed construction of prior knowledge. The significant number of students at the early stages of their secondary level education who receive mathematics instruction from out of field teachers offers a possible

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6see section 1.8

7The percentages of qualified/unqualified teachers do not add to 100% as a teacher may teach several groups and therefore may be counted more than once.
explanation for the poor performance in mathematics, problem solving in particular, by second-level students in Ireland. Dina and Pierre van Hiele (1984) noted that mathematical knowledge gained at one level of understanding can be the subject of analysis at a higher level. This concept will be looked at more thoroughly in sections 2.4.2 and 2.9.4. If students have not acquired correct knowledge at a one level of understanding, it impedes the progression to analysis of this knowledge at a higher level of understanding.

“students need their best maths teachers at a young age. Teachers who really know what they are doing and really understand the simplicity of what they are doing. Once confidence is in place at a young age, I think the other issues... will right themselves.”

(NCCA, 2006, p.29)

The probability of the students in the higher level mathematics classes in Ireland being taught by a qualified mathematics teacher is greater than the probability of the students in the ordinary level mathematics classes being taught by a qualified mathematics teacher (Ní Riordáin and Hannigan, 2011, p.299).

<table>
<thead>
<tr>
<th>Teaching qualification in mathematics</th>
<th>2nd year</th>
<th>3rd year</th>
<th>5th year</th>
<th>6th year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes ( (n = 168) )</td>
<td>53 (31%)</td>
<td>71 (42%)</td>
<td>75 (45%)</td>
<td>68 (40%)</td>
</tr>
<tr>
<td>No ( (n = 156) )</td>
<td>7 (4.5%)</td>
<td>4 (3%)</td>
<td>0 (0%)</td>
<td>2 (1%)</td>
</tr>
</tbody>
</table>

Table 3: Numbers of teachers with and without a teaching qualification teaching Higher Level Mathematics (Ní Riordáin and Hannigan, 2011. p.299)
This further compounds the issue, the students who experience difficulty with mathematics are the ones who are more likely to be taught by a teacher who does not have a teaching qualification in mathematics. An ongoing initiative (funded by the Department of Education and Skills from September 2012) is providing a two-year, part-time course (free of charge) for ‘out-of-field’ teachers to gain a teaching qualification in mathematics. A study by Akiba et al. (2007) found that the highest achieving countries in TIMMS 2003 assessment had a greater percentage of students being taught by mathematics teachers who had satisfied their country’s requirements for full certification, had a major qualification in mathematics or mathematics education, and had been teaching mathematics for at least three years. It is hoped that the number of qualified mathematics teachers teaching mathematics in Irish secondary schools would approach 100% in the upcoming years.

Teaching mathematics is a complex task and teaching mathematical problem solving and modelling (discussed in section 2.5) requires a broader range of teaching approaches than most teachers currently use (Burkhardt, 2006). These skills include:

1. **Knowing when to help and adequate timing**
   Teachers need to give students sufficient time to analyse problems completely. Help should only be provided by the teacher when students have attempted to complete the problem several times with different approaches. The teacher should not intervene too early and should not formulate the problem for the student or provide the solution technique to the student. Students’ thinking should be provoked and supported by asking higher order productive questions;

2. **Discussion**
   Teachers need to be able to facilitate meaningful and productive discussion, by playing a supportive role more than an instructive role. Teachers should play more of a collaborative role with their students, in such a way that students feel shared responsibility for the accuracy of their own and others’ reasoning and solutions and do not expect all answers to be given by the teacher. This aims to foster increased self-confidence in students’ own mathematical ability;

3. **Active modelling**
   Teachers need to routinely provide non-routine authentic tasks for students to engage with;

4. **Diversity**
   Students need to be presented with a wide variety of tasks, both during classes and for assessment, the use of project work and experimentation are some such tasks;
5. **Additional Questions**

Teachers should provide mathematical tasks that build on each student's understanding and leads them to progress further.

(Burkhardt, 2006)

This presents difficulty for teachers as traditionally mathematical concepts and skills have been taught by:

1. explaining to students;
2. showing students a worked example of an exercise;
3. asking students to complete imitative exercises.

(Lyons et al., 2003)

In problem solving and modelling, students must figure out and formulate their own responses to problems and so a more student-centered approach is warranted in these problem solving situations. Dealing with these non-routine problems, from real world situations, in a more-student centered classroom environment places the majority of mathematics teachers in a situation which is not routine for them and outside of their past experiences of teaching mathematics (Burkhardt, 2013).

The recent implementation of the Project Maths curriculum in Ireland places greater emphasis on students’ understanding of mathematical concepts with an increased focus on the development of students’ problem solving skills, explanation and communication (NCCA, 2012). Recent changes in mathematics syllabi in many countries reflect similar shifts in attention to problem solving in mathematics education (Eurydice network, 2011). These changes in the syllabi and recommended methodologies are considerably different to some of the current pre-service teachers’ experiences of learning mathematics during the completion of their own second level education (Conway et al., 2013).

Teachers work within ‘frames of reference’ (Kennedy, 1999), that is they generally teach mathematics the way they were taught themselves. Their time as a student in school level mathematics classes gave them a particular ‘apprenticeship of observation’ (Lortie, 1975). Kennedy notes that pre-service mathematics teacher education courses are optimally positioned to provide a new ‘frame of reference’ for these students as the course is after their experience of mathematics education as a student and before their experience of mathematics education as a teacher. Several studies have identified deficiencies in pre-service teachers’ understanding of mathematics (Chazan et al., 1999; Llinares and Krainer 2006; Hourigan and O’Donoghue 2013). A study by Varghese (2009) showed that none of the pre-service secondary mathematics teachers \( n = 17 \) showed an awareness of the use of proof to develop understanding, and did not consider the developing of understanding of mathematics as being promoted/confirmed.
by a group effort. Functional fixedness\(^8\) may impede the teachers’ flexibility in their use of mathematics to foster understanding. When pre-service teachers were provided with situations to reflect on their current teaching methods, they often come to the conclusion that these were not optimal (Varghese, 2009). However even with this realisation, the pre-service teachers are uncertain about other available methods as they did not experience these other methods when they were students themselves (Varghese, 2009). Evidence on pre-service secondary teachers’ \((n = 30)\) performance on two non-routine problem solving tasks (3.72 on a scale of \((1 – 7)\)) and experience with solving such problems (12/30 stated they did not have enough practice with these types of problems), led Felmer and Diaz (2016) to recommend that action be taken on pre-service mathematics teacher education courses. Felmer and Diaz (2016) also suggest professional development courses for teachers.

Foster et al. (2014) propose that the construct of mathematical knowledge for teaching (MKT) developed by Ball et al. (2008) be extended to include mathematical process knowledge (MPK) and pedagogical process knowledge (PPK). Foster et al. (2014) believe that attention to the processes have not been explicitly addressed in the existing constructs of mathematical knowledge for teaching and propose that an understanding of MPK and PPK would aid in facilitating mathematics teachers to ‘improve their skills in teaching mathematical problem solving’ (p.1). Foster et al. (2014) propose replacing the term ‘content’ in the MKT construct developed by Ball et al. (2008) with the terms ‘concepts and processes’ (Figure 2). In a case study working with 3-4 mathematics teachers in each of nine secondary schools (teaching 11-18 year old students) for a little over a year, Foster et al. (2014) found that the teachers’ knowledge and ability to employ ‘effective strategies for teaching problem solving processes was particularly underdeveloped’ (p.7). Teachers’ understanding of process skills and their conceptualisation of what constitutes advancement in learning processes is ‘currently significantly underdeveloped’ (Foster et al., 2014, p.7).

\(^8\)A limitation which prevents an individual from forming the consideration of using an object for a different purpose than it is usually (or has originally been) used for (Duncker, 1945).
The first part of this section discussed current issues relating to teacher knowledge and has revealed the impact a teacher’s knowledge has on their students. The interconnected nature of the knowledge required for the teaching and assessing of problem solving in mathematics is discussed further in the conclusion of this chapter (section 2.10). The next part of this section now considers the mathematical knowledge a student requires to engage in problem solving. This begins with discussing conceptual understanding.

**Student Knowledge**

1. **Conceptual:** Conceptual understanding in mathematics is essentially about forming connections between pieces of mathematical information (Hiebert and Lefevre, 1986), in a way that is not dissimilar to hierarchial agglomerative clustering. The links between the individual items of information are as numerous as the discrete pieces of mathematical information that they are tying together. Conceptual understanding of mathematical knowledge includes the individual packets of mathematical information as well as the overall organizational structure of these packets of knowledge in one’s memory, this network then provides the base for visualising and conceptualising new information (Hardin, 2002). Mathematical thinking is fundamental in attaining conceptual understanding of mathematics. For knowledge of mathematical information to progress to further conceptual
understanding, a way of looking at the distinct packets of mathematical information is required so as to reveal their intrinsic structure (logical/numerical/graphical/symbolical) as well as understanding how they fit (form connections) into the overall network of the complete body of mathematical knowledge one possesses, is required. Hardin (2002) describes conceptual knowledge of a domain as the ability to make meaning of domain specific problems, based on previous knowledge of that particular domain. The ability to recognise opportunities to productively use mathematical knowledge alone or in conjunction with other mathematics in order to make an attempt to solve a problem is heavily dependent on a sound conceptual knowledge base of mathematics.

2. ‘Relational’ and ‘Instrumental’ understanding:

‘Relational’ understanding in mathematics is similar to conceptual understanding in that it is about knowing the connections and relationships within the body of mathematical knowledge one possesses (Skemp, 1976). This type of understanding facilitates both knowing what to do in mathematical problem situations and why the chosen approach is appropriate. This type of understanding can aid motivation among students as it can be more satisfactory to understand relationally than to simply accept rules. Successful mathematics learners are those who consider mathematics as a subject to be interpreted, where sense-making can be conducted by deriving and constructing meaning (Resnick, 1987). They are not content with following a set of rules unless those rules are fully understood by them and it makes sense to them to apply those rules in order to resolve that particular mathematical situation.

‘Instrumental’ understanding in mathematics is not particulary understanding at all, it is more about knowing what to do, for example knowing to use a formula/procedure to obtain a correct solution to an exercise without knowing why/how the formula/procedure is appropriate or why the formula/procedure is structured the way it is (Skemp, 1976).

One can obtain the correct answer to an exercise with ‘instrumental’ understanding, however this type of ‘understanding’ will not adapt well to novel problem situations whereas ‘relational’ understanding is more transferable to unfamiliar problems. Teaching/learning through ‘instrumental’ understanding does not facilitate the formation of quality mental schema like ‘relational’ understanding does. Skemp (1976) notes that ‘instrumental’ understanding can allow for quicker results, i.e. students can obtain the correct answer, whereas ‘relational’ understanding is more difficult to obtain. The pressure placed on one’s memory is greater for ‘instrumental’ understanding than it is for ‘relational’ understanding.

Since what constitutes a problem for one student may not be a problem
for another student (section 2.1) determining whether a student employed instrumental/relational understanding can be quite difficult to ascertain from assessments.

3. Deductive and Inductive Reasoning

Inductive reasoning involves the process of generalisation to reach a conclusion based on a pattern observed in numerous instances (Pólya, 1945).

Deductive Reasoning is making a conclusion based on prior factual knowledge, this type of reasoning is employed when proving theorems in geometry. Each step in the process involves making conclusions which build logically upon previous steps of factual mathematics to form a proof.

Since proficiency in mathematics is dependent on adaptive reasoning, both types of reasoning should be developed by students. Knowledge of problem solving models and heuristics are also useful in problem solving in mathematics.

4. Strategic knowledge: Problem-Solving Models and Heuristics

Heuristics are suggestions and questions which aim to assist problem solvers in their attempts to understand a mathematical problem. These questions and suggestions act as prompts in facilitating problem solvers’ discovery of mathematical properties and solution methods in mathematical problem solving (Pólya, 1985). Using heuristics is employing a thought process and repetitive use of this process may aid one’s ability to think more systematically in their approach to mathematical problem solving (Carson, 2007). Heuristics differ from algorithms; following an algorithm designed for a specific problem will result in the correct answer to a problem whereas the use of a heuristic to solve a mathematical problem may or may not lead to the correct solution (Krulik and Rudnick, 1980). As previously mentioned in section 2.2.2, mechanically following an algorithm is not considered to be engaging in mathematical problem solving. The problem solving of Dewey (1933) and Pólya (1945) are discussed here.

John Dewey 1933

The American Philosopher, John Dewey, formulated a theory on experience and reflective thought (1910, 1925, 1938). Dewey’s theory is grounded in the Darwinian biological theory of evolution (1976), in that an organism adapts to its surrounding environment. In this adaptation, the person (organism) learns to perform tasks in routine ways and these ways become automatic for the individual. It is only when these routine methods for performing tasks fail, a problem arises, i.e. there is an interruption in completing the task. This necessitates an investigation into the problem situation and use of reflective thinking to try to understand why the routine methods fail and to try and find a new method of completing the
This theory has similarities to the ideas and observations of Dina and Pierre van Hiele, (employed by Freudenthal in Realistic Mathematics Education (section 2.6.1)) in that some analysis of basic mathematical concepts can occur at a later time in the learning period. In the primary stage, the concepts may be seen as objects with certain properties, in the secondary stage, further analysis (reflective thinking) of these concepts may be necessary. This theory led to a problem solving approach by Dewey as shown in Figure 3.

![Dewey's Model of Reflective Thought and Action 1933](Miettinen, 2010, p.65)

Dewey notes that the end result consists of both solving the problem and also the production of meaningful knowledge (concept) which may be utilised in future problem solving situations. Dewey is of the opinion that concepts and meanings are constructed as a result of interactions between humans and the practical activities with which they engage. He proposes that it is the quality and uniformity of these interactions, as well as the regularity at which these are experienced which facilitates the transfer of concepts from one problem situation to another (Miettinen, 2010).
Pólya 1945

The Hungarian mathematician, George Pólya, devised a model for problem solving in mathematics in 1945. His model consists of four stages, as shown in Figure 4:

![Figure 4: Pólya How to Solve It Model 1945](www.learnlogic.net)

Pólya’s model is consistent with higher order questioning, similar to assessment for learning. Each of the four stages in the cyclical process, requires the self posing and answering of questions. The teacher may initially have to probe the students with the questions at each stage, but these questions should become part of the student’s own internal processes, when engaged in mathematical problem-solving. As mentioned, this process is cyclical in nature, the stages do not have to be completed in any particular order, the stages are not discrete and a student does not necessarily have to go through each stage in order to solve the problem (Reys et al., 2001).

In the first stage; understanding the problem, students are encouraged to represent the problem in pictorial form when appropriate, with suitable notation (Pólya, 1945). Where a picture is not an appropriate choice of representation, a student should try to organise the information presented in the problem by introducing suitable notation and/or diagrams (e.g. words and arrows or alternative) to represent the information. This is to provide an aid with visualising and understanding exactly what is required in solving this problem. The second stage involves examination of the given information, with the aim of forming connections between the
data presented in the problem, as well as establishing relations between the data given and the unknown required (Pólya, 1945). Determining these connections plays a crucial role in formulating ideas for possible solution plans, which leads into the third stage of this model; carrying out the solution plan. While carrying out the plan, each stage of the solution should be carefully monitored and checked for errors. The main question a problem solver should ask him/herself here is; can you be certain that each step of your solution, and hence that the complete solution is correct? This stage involves self-monitoring and control. The fourth and final stage of the plan involves reflecting on the solution obtained and verifying it (Pólya, 1945). This final stage also involves questioning whether one could have obtained the solution using a different method, in addition to determining whether the method utilised could be used to solve other similar or more general problems (Pólya, 1945).

Dewey’s view of the status of the conceptual knowledge gained from resolving the problem as potentially being of more value to the student than the initial intended result of solving the problem, is similar to Pólya’s view of the discovery of the correct use of the heuristics given in his model as being more valuable to the problem solver than the knowledge gained of any individual mathematical fact. It is the knowledge gained in the doing of mathematics (how to approach resolving the problem and why this approach was successful) that can be utilised in future problem situations.

Resnick (1987) notes the lack of specificity in Pólya’s heuristics as being an inhibitory factor in the usefulness of these heuristics for helping those who are not already adept problem solvers, while others note that to make these more specific would be increasing the difficulty in choosing the most appropriate strategy for a given problem situation (Schoenfeld 1985) and so the need for metacognition in knowing when, why and how to implement these heuristics is required alongside knowledge of heuristics.

Many studies on teaching particular heuristics revealed limited success in improving performance in mathematical problem solving (Schoenfeld, 1992). However a study conducted by Koichu et al. (2007) on the relationship between heuristic literacy development and mathematics achievement \( (n = 37, 8^{th} \text{ grade students}) \) revealed that the positive effect of heuristics on improving weaker students’ mathematical achievement may be somewhat neutralised by the lack of effect on changing more capable students’ mathematical achievement. This meant that overall the mean effect size of interventions involving teaching heuristics could be found to be small, but this may not be representative of the potential that teaching heuristics has on improving students’ performance in mathematical

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9Discussed in section 2.4.2

10able to approach problems using heuristics and use of heuristic vocabulary
problem solving. Lesh and Zawojewski (2007) recommend that Pólya’s heuristics be treated not just as prompts that assist in choosing and executing correct procedures but also as providing a way of looking at things more mathematically and developing the ability for interpreting mathematical situations which advance their thinking in mathematical problem solving.

5. Expert versus Novice

When solving mathematical problems, students depend on their knowledge of previously encountered types of problem from textbooks. They do not utilise their knowledge of problem solving strategies or the inherent properties specific to each problem (Schoenfeld, 1985). In studies comparing expert and novice problem solvers in mathematics, it has been determined that experts make use of their rich extensive knowledge in trying to see the deep structure of the problem. In contrast, novice problem solvers base their reasoning on superficial/trivial content of the problem (Glaser, 1992; Simon, 1990). When faced with novel problems, experts make use of a number of general heuristics. When novice problem solvers/students encounter problems that are outside of the typical design of problems found in textbooks, many are at a loss as to how to proceed (Perkins and Salomon, 1989; SEC, 2005).

Along with a vast, in-depth conceptual understanding of mathematics, expert problem solvers are skillful in their ability to identify and recall knowledge that is pertinent to obtaining a solution to a specific problem. Their knowledge is linked, organized into coherent and sensible patterns and is situated, meaning they can detect the contexts/situations to which their knowledge may be applied (Glaser, 1992). The student also needs to be able to think mathematically, be aware of how one thinks and learns and be able to transfer the knowledge they have gained to mathematical situations.

2.4.2 Metacognition, Mathematical Thinking and Transfer

Metacognition

Cognition is defined in the online Oxford dictionary (freedictionary.com) as, ‘the mental action or process of acquiring knowledge and understanding through thought, experience and the senses’. Meta is a word which, when used as a prefix, describes the word after the prefix about itself (online Oxford dictionary). Therefore metacognition is the cognition of cognition, or thinking about one’s own self thought-process of acquiring knowledge and understanding through different processes. Metacognition is generally defined as self-knowledge of one’s
thinking and learning processes, along with the monitoring and subsequent regulation and implementation of these processes (Schoenfeld, 1983; Lester, 1982; Silver, 1982; Flavell, 1976). Metacognition also includes the ability to reflect on one’s own level of performance and understanding (National Research Council, 2001). Flavell (1979) discussed three components of metacognition: metacognitive knowledge, metacognitive skills and metacognitive experiences. These components are summarised in Figure 5.
Metacognition

- Metacognitive Knowledge
  Knowledge of person, task and strategy variables, that affect cognitive process and overall outcome performance

- Metacognitive Experiences
  Persons awareness and feelings elicited in a problem solving situation

- Metacognitive Skills
  Skills in controlling cognition. Regulating the choice of action/strategy, monitoring and assessing progress. Reflecting on, and revising overall plan of action/choice of strategy. Play a role in cognitive activities (metacognitive experiences can activate both cognitive and metacognitive strategies), such as:
  - Oral communication of information
  - Reading comprehension
  - Memory

Figure 5: Metacognition, (Flavelle, 1979, pp.906-909)
Metacognition can be thought of as:

1. self-knowledge of how one learns, understands and remembers new information and what methods best facilitate the acquisition, understanding and retention of this information for oneself.

2. self-knowledge and understanding of how this new knowledge is linked to existing knowledge.

3. self-knowledge and understanding of how this knowledge can be utilised.

4. self-realisation of appropriateness of application of this knowledge in familiar and novel situations.

5. self-monitoring of application of this knowledge.

6. regular, truthful and meaningful self-reflection of one’s own use of knowledge and one’s own understanding of this knowledge.

7. critical thinking on each of the previous six items and during the acquisition, analysis, interpretation, evaluation, validation and explanation of this knowledge and its application when completing tasks such as problem solving in mathematics.

8. self-generation of questions and self-knowledge and awareness of how to answer and/or how to obtain answers to these questions for oneself.

9. self-awareness of the problem to be solved, the use of mental images in representation of the problem as an aid to completion and of strategies which could be implemented to solve the problem.

(Flavell, 1979)

The heuristics present in many of the problem solving models described in section 2.4.1 can act as a stimulus for metacognition and mathematical thinking. The self-questioning present in these models, facilitates reflective thinking among students, the questions may force them to stop and consider their progress and the appropriateness of their chosen methods. It has been shown that teaching mathematics, supported with metacognitive skills has led to significant gains, in both conceptual and procedural knowledge of mathematics, among students (third grade level, 8 – 9 years old) (Mevarech and Fridkin, 2006; Goldberg and Bush, 2003).

Schoenfeld (1992, p.42) identifies five competencies (or components of cognition) which are significant in problem solving in mathematics: the “knowledge base, problem solving strategies, monitoring and control (or metacognition), beliefs and affects, and practices” (each of these competencies are discussed in this section (2.4)). Schoenfeld (1992) stated that metacognition, and, beliefs and practices are vital components of developing mathematical thinking, noting
that metacognition helps students to overcome difficulties experienced during the problem solving process.

Research has confirmed that metacognition is a viable predictor of performance in mathematics at both primary and secondary level (Verschaffel, 1999). A study conducted by Cornoldi et al., (1995, cited in Schneider and Artelt, 2010) revealed that increases in metacognition corresponded to increased achievement in problem solving and logical reasoning in mathematics. Resnick (1987) found that students who think about what they are doing and why they are doing it are more successful than those who just follow the rules they have been taught.

Self-explanation is one method of stimulating metacognition. In studies on self-explanation, it has been shown that, regardless of the type of instruction, the use of self-explanation improved transfer among mathematics students (Atkinson et al., 2003; Wong et al., 2002). Encouraging students to self-explain while engaged in a discovery learning environment can improve procedural transfer on novel/unfamiliar problems (Siegler, 2002).

**Mathematical Thinking**

Devlin (2012) describes mathematical thinking as thinking about things from a mathematical point of view. It involves considering things in their entirety, as well as noticing their numerical, structural and/or logical component parts. It also consists of analysing the components, their relationships to each other and how they relate to the complete object/situation, in addition to the analysis of the overall object/situation mathematically. In Schoenfeld’s (1992) description of mathematical thinking, he notes the importance of valuing the process of mathematisation and abstraction, while possessing the preference of applying these processes over other non-mathematical processes, to gain understanding of problem situations/objects/mathematics. He also stresses the willingness of developing competence in mathematics and its tools as a means to enhance understanding. Mason et al. (2010) share a similar view to Schoenfeld in their definition of mathematical thinking as a ‘dynamic process’ (p.144) which, by providing the means for individuals to increase the complexity of ideas they can work with, allows for the expansion of their understanding.

All thinking happens when neurons in the brain are stimulated to form connections. Mathematical thinking is no different, it uses the same cerebral resources that are available for thinking in general (Tall, 2013). Neurons in the brain are designed to make contact, they are built for connection (O’Connor, 2011). Each person’s brain has a unique make up of these connections. As links between neurons change, the biochemistry of these links change. Tall (2013) describes how the more these neuronal links are used, the more structured the thought process and the more richly the knowledge is connected. Over time, these strengthened neuronal connections lead to new and more instantaneous thoughts, often leading to ‘compression of knowledge’ (p.3) so that a mathematical process can be
replaced by a conceptual connection (e.g. the process of 4+5 is immediately outputted as 9, without the need to count). This compression of knowledge, aligns with Mason et al.’s (2010) definition of mathematical thinking, in that this compression may facilitate the enabling of understanding of increasingly complex mathematical ideas. The long term development of mathematical thinking is this continuous rebuilding of neuronal connections that evolve to build increasingly complex knowledge arrangements (Tall, 2013).

Tall (2013) formulated a framework to explain and predict how humans learn to think mathematically. In this framework three progressions of mathematical thinking, (which are in line with Devlin’s 2012 description of mathematical thinking) which develop in the long-term, are identified as the ability to think about objects in relation to:

1. their structural properties (e.g. structure of shapes in geometry, graphical representation of mathematical functions)
2. the operational properties of actions on them (e.g. arithmetic, algebraic)
3. the formal properties of mathematical objects given by formal definition and proof (e.g. set theory)

Both Tall (2013) and Devlin (2012) describe mathematical thinking as the ability to see mathematical (and other) objects from multiple perspectives in such a way that these representations are seen as equivalent in the student’s mind.

Gray and Tall (1994) (cited in Tall, 2013) named symbols which act as both a concept and a process, a procept, for example the procept 4+5 is both a process (addition) and a concept (the sum). Gray and Tall (1994) named the thinking involved in the recognition and use of alternative symbolism to represent the same procept and the adaptability of the thinker to derive new connections and alternatives in mathematics as proceptual thinking. Gray et al. (1999) state that proceptual thinking is essential in the long-term development of a proficient mathematical thinker. Recognising the same procept in its many forms, the thinking involved in this recognition, and the ability to express the same procept in various ways facilitates “flexible use of symbolism in forming new relationships” (Tall, 2013 p.11). The flexibility of the thinking which led to the formation of these relationships gives rise to a variety of alternative representations/processes/ways of thinking for the mathematical thinker to use in problem situations (Tall, 2013). This flexibility associated with proceptual thinking also facilitates ‘compression of knowledge’ in manipulating algebraic expressions (for example) so that one can carry out a mathematical process in a more sophisticated way rather than being restricted to a step by step procedural approach. Krutetskii (1976 p.350) also refers to this ‘curtailment of mathematical reasoning’ in his description of mathematical ability. This compression of knowledge (curtailment of reasoning) facilitates the recognition of situations where shortening the number of operations required to obtain a solution to a problem is
possible along with the ability to use this shorter process. Gray and Tall (2001) discuss three ways in which mathematical thinking develops:

1. **conceptual embodiment:**
   This includes mental perceptions of objects, which has similarities to the visualisation level in Van Hiele’s level of reasoning; the learner associates a mental image with a particular mathematical object or a set of mathematical objects, be they actual objects or ideas. For example, the word triangle, being ‘embodied’ as a figure with three line segments may evoke an image of an isosceles triangle in the mind of one learner while to another learner this may be visualised as a right angled triangle. This image may then serve as a representation for the entire set of triangles. It includes the perception and meaning that learners attribute to certain mathematical properties within their own minds, and the thinking that occurs about these objects/properties.

2. **operational/proceptual symbolism:**
   This develops from conceptual embodied mathematical thinking, through actions and practice of these actions until they can be executed with minimal conscious effort, effectively developing ‘procedural fluency’, which is one of the objectives of Leaving Certificate mathematics (2015 Syllabus, p.6). This development of mathematical thinking includes learners who simply know the procedures to follow and also those learners, who can appreciate the flexibility of the symbols within the procedures as being ‘operators’ which act on the procedures and also the procedures as being an operational process which is adaptable through choice and combinations of calculations.

3. **axiomatic formalism:**
   Mathematical thinking increases in sophistication in the transition from proceptual symbolism to axiomatic formal mathematics. The axiomatic formal mathematics involves making logical deduction from axioms to formal proof, in situations where the mathematics and thinking in the proof are not dependent on particular embodiments or specific calculations (Tall, 2013). The mathematical thinker at the axiomatic formal mode of thinking has a network of mathematics in their mind that has been built up through conceptual embodiment and proceptual symbolism. Exactly like the van Hiele’s levels of thinking, the structure has been formed from concrete to abstract such that it allows for the development of a network which permits reversibility from the abstract to the concrete by the mathematical thinker themselves. Tall (2013, p.17) states that ‘formal mathematics can lead to proving theorems which specify the structure of an axiomatic system, giving it new forms of embodiment and symbolism’.

The development of high quality mathematical thinking requires a well connected knowledge base. Tall’s framework shows three different ways in which
this knowledge is acquired and developed (see Figure 6). Knowledge of different subject areas within the subject of mathematics progresses in different ways. Tall (2013, p.5) describes how knowledge of geometry initially progresses through the study of objects and their properties, facilitating a growth of ‘mental imagery’. The learning of arithmetic primarily progresses from studying actions on the objects (counting, grouping etc.). The mathematical thinking developed in acquiring knowledge in both of the previous two ways advances with the study of “formal definitions of axiomatic systems and the deduction of properties by mathematical proof” (Tall, 2013, p.5). The growth of mathematical thinking at each stage includes earlier stages in Tall’s framework. An individual working in the axiomatic formal stage may thus revert back to thinking in either/both of the other two stages (Tall, 2013). The mathematical thinking developed in the proceptual symbolic stage includes mathematical thinking developed from working in both the proceptual and embodiment stage together. In problem solving or thinking about a concept in mathematics, the simultaneous engagement with proceptual and conceptual embodiment thinking should occur where possible. Tall (2013a, p.2) states that the advancement to formal thinking and reasoning occurs from two “parallel” forms of development (conceptual embodiment and operational symbolism). Understanding mathematics at secondary school level develops by increasing sophistication of mathematical thinking through “perception, operation and reason” (Tall, 2013a, p.2). This development of mathematical thinking (Figure 6) subsequently leads to engagement with theoretical mathematics; definition and proof in Euclidean geometry and symbolic proof based on the ‘rules of arithmetic’ in number and algebra (Tall, 2013a).
The degree of evolution of the proceptual thinking and compression of knowledge among individual learners can result in a range of outcomes (ability to arrive at a solution to a problem and choice of solution) among a class of learners (Tall, 2008). The range of outcomes are presented in Figure 7. This models the development of the learners in their ability to progress from employing a strictly procedural approach to solving a problem, to a more flexible approach in arriving at a solution. As a student’s symbolic compression increases, their understanding and use of the procedure advances from a step by step approach for solving a particular problem/type of problem to understanding the overall process. This change in focus may result in the overall process being compressed as a procept (Tall, 2008). The ability to see the overall process as a procept may then result in an increase in the adaptability of the learner to use procedures in increasingly flexible ways in solving problems. This aids the learner in the application of appropriate mathematical knowledge to various problem situations and in the provision of efficient solutions (Tall, 2008).
A curriculum focused on symbolism without the associated embodiments may restrict the learner to procedural approaches rather than approaching a problem with conceptual understanding (Tall, 2008). This way of learning mathematics also impedes the progression of the thinking of the learner to consider the procedure as a process and denies them the opportunity of conceiving the process as an ‘object’ (Tall, 2008). The symbolic compression involved in advancing from thinking in a procedural way, to envisaging the procedure as a process, to conceiving of the process as an ‘object’ has a corresponding advancement in embodiment (actions on the objects have an effect on the objects). Moving the focus of attention from the steps of a procedure to the effect of that procedure allows for compression of knowledge from procedure to process (Tall, 2008). Reflection on the solution from the procedure is therefore vital in condensing the procedure into an ‘object’, with further reflection organising related ‘actions’, ‘processes’ and ‘objects’ into ‘schemas’ (reflective abstraction and APOS theory (Asiala, 1996)).

Thinking mathematically enhances the ability to successfully engage in mathematical problem solving, while engaging in mathematical problem solving also provides the opportunity to develop mathematical thinking (Isoda and Katagiri, 2012).

Factors affecting one’s ability to think mathematically, when engaged in problem solving/modelling activities include the following:
1. competence in the employment of mathematical inquiry.
2. confidence in managing the emotional and psychological situations involved in mathematical problem solving, and making productive use of these aspects in engaging with the problem.
3. understanding of the mathematical content required to solve problem/model task and also of the information presented in the task which may necessitate the application of mathematics to a subject area one is unfamiliar with.

(Mason et al., 2010)

Resnick (1989) notes that the acquisition of mathematical habits such as interpretation and application of a rational thought process, play an equal if not more important role to the acquisition of knowledge, skills and strategies in the development of mathematical thinking (p.58). Thinking is driven by questions, the greater the quality of the questions in relation to provoking further questions, the greater the quality of the thinking needs to be in order to address these questions productively. Similarly the quality of mathematical thinking is driven by the quality of the mathematical tasks one is engaging with. The greater the quality of the mathematical tasks, the greater the potential is for provoking high quality mathematical thinking.

Transfer

Kaminski et al. (2009) found that although asking questions in context is a viable instrument for testing how well students have learned mathematics, teaching the concept using abstract symbols is more likely to result in transfer of mathematical knowledge in different problems and contexts. Mathematics in context may facilitate initial learning more successfully than abstract symbolism, by being more engaging to students, however learning mathematical concepts does not necessarily result in transfer of these concepts to novel problem situations. Kaminski et al. (2009) suggest that if learning concepts by employing contextual mathematics, then the ‘deep structural information’ of the mathematics problem should be made explicitly visible to students. The extra contextual information, which is not mathematically relevant may inhibit the learner from recognising relational similarities between analogous mathematical problems.

This opinion of representing mathematical knowledge in a generic way as a means of supporting transfer is also shared by Halpern (1998). She states that the method of learning should be designed to enable students to access and utilise knowledge when appropriate situations arise, irrespective of the context or subject area. The nature of the tasks play a significant role in the development of students problem solving skills (NCCA, 2015).
2.4.3 Nature of the tasks

The tasks that students engage with in the classroom lead to their understanding of what it means to ‘do’ mathematics. Different mathematical problems facilitate different thinking and learning. Good mathematical tasks are those that broaden students’ mathematical knowledge in a connected manner and which provoke questions (Hiebert and Wearne, 1993). Swan and Burkhardt (2012) classify a good mathematical task as one which assesses the mathematical proficiency that you want students to develop. Both mathematical content knowledge and problem solving processes should be assessed in the one task (Swan and Burkhardt, 2012). In the solving of the mathematical task, it should be necessary for students to represent, analyse, interpret and communicate (Swan and Burkhardt, 2012). A classification of problems by O’Donoghue (1993) and the mathematical theory of communication system developed by Weaver (1949, p.3) offer a way of thinking about the ways in which the nature of problems can be made more difficult by presenting a problem to students for which the mathematics has not yet been learned or for which the context (domain) in which the mathematics is embedded is unfamiliar to the student (Figure 11). Different combinations of restricting/unrestricting the domain combined with known/unknown mathematics allows the nature of the tasks to be changed with varying degrees of complexity. In the communication system represented by Weaver, the noise source can come from the classroom environment or factors relating to the individual student, whereas the noise source in the classification of problems developed by O’Donoghue refers more to the complexity of the problems themselves (noise coming from unfamiliar context/domain or unfamiliar mathematics).
The classroom environment and the student’s own beliefs, confidence and mindset also have a bearing on a student’s ability to engage in problem solving.

2.4.4 Classroom Environment

An enriched learning environment (challenging environment) facilitates the formation of more neuronal connections than an environment which is deficient in stimulation (O’Connor, 2010). A mathematics environment which consists of challenging problem solving activities is therefore considered such an enriched environment. Considering that experts have a knowledge base which is rich in connections, teaching mathematics in such a challenging environment may facilitate the development of a densely connected knowledge base in students’ brains. O’Connor (2011) describes a deprived learning environment as one which is monotonous for students, where students work individually and there are low expectations.
2.4.5 Mindset, Beliefs and Attitudes

Mindset
According to Dweck (2007), mindsets are beliefs, which people have about themselves; beliefs about their qualities, such as their intelligence and talents. There are two types of mind-sets, which Dweck has observed among students; fixed mindset and growth mindset. Students who have a fixed mindset, irrespective of their ability are predominantly concerned with how their ability is viewed by others i.e. with how they will be judged. These students believe that it is ability, not effort that determines success. Students with this mindset have a fear of failure, as they view it as a direct measure of their ability. This fear of failure, among students with a fixed mindset, carries also a fear of making a real effort, as failure after putting in a genuine effort makes these students judge their own ability as being even more inadequate. This mindset can therefore cause clever students to abandon efforts when faced with challenging tasks. Challenging tasks, like problem solving in mathematics, can be viewed with fear (Hong et al., 1999) among students with a fixed mindset, as these students value how their ability appears to others, more than they value opportunities to learn.

A study conducted by Mueller and Dweck (1998), on the effect of praising students for their intelligence versus the effect of praising students for their effort revealed results which could have implications for the effective teaching of mathematical problem solving. Fifth grade (age 10−12 years) students (n=128) were given a set of problems to complete from Raven’s progressive matrices (RPM) test, a non-verbal test which is designed to measure ‘fluid’ (analytic) reasoning, independently of ‘crystallised’ (practised routine) knowledge (Prabhakaran et al 1997, p.43). On completion of the first set of tasks students were praised for intelligence, or praised for effort, or were part of the control group, who were not praised for either. Students praised for intelligence were more likely to form a fixed mindset, whereas students praised for effort were more inclined to form a growth mindset. Students were then offered more challenging problems from the RPM test. The students who were praised for intelligence after the first set of problems, experienced loss in confidence in their ability and enjoyment of the task, when they struggled to complete the problems in the second set. The students who were praised for effort displayed greater perseverance, a higher level of enjoyment and better performance, when confronted with difficulty, than the students praised for intelligence. The third set of problems presented to students were at a similar level to the first set. Students praised for intelligence, performed worse, as a group than they had initially. Students praised for effort displayed excellent performance and continued to improve.

A longitudinal study and intervention, based on teaching students (12-13 years old) about the growth mindset (Blackwell et al., 2007) resulted in a three-fold increase in perseverance and engagement in mathematics among students who received growth mindset training, in comparison to corresponding groups who received other instruction. Also a similar study conducted by Good et al. (2003)
revealed that, students who received instruction on growth mindset (compared to control groups of similar size and ability, who received other instruction) showed significant increases in their mathematics achievement test results.

Beliefs and Attitudes

McLeod and Adams (1989) observed students engaged in mathematical problem solving, and found that the students emotions’ played a role in their cognitive process of solving the problem. A student’s belief system can limit their perception of reality (of both the problem and their ability to solve it) to a sub-set of the solution space, which does not contain/allow for the answer (Campbell, 2003). Students’ self-confidence and belief in their ability to engage successfully in the mathematical problem solving process, can significantly impede their progression in solving mathematics problems, in some cases, completely obstructing their ability to initiate a productive response (Shaughnessy, 1983).

2.4.6 Interest and Perseverance

Perseverance has been found to be mildly positively correlated ($r = .250$) with performance in problem solving (OECD, 2014). In the PISA problem solving assessment 2012, it was determined that a one point increase in perseverance corresponded with a 29.7 point increase in problem solving performance among fifteen year old Irish students. The strength of this correlation increases for students who perform at the higher percentiles (1 point increase in perseverance with a 38.5 point increase in performance at the 90th percentile, compared with a 20 point increase in performance at the 10th percentile (OECD, 2014)). As previously mentioned, epistemological beliefs and mindset can affect how students persist with challenging tasks (Muis, 2004). Interest is another factor which can affect how students persevere with challenging tasks.

Interest

The level of interest a student has for the task they are completing/subject content they are learning about, has been shown to have a strong impact on their learning (Alexander, 1997; Ainley et al., 2002; Durik and Harackiewicz, 2003). A four phase model of interest development proposed by Hidi and Renninger (2006) describes varying levels of interest from situational interest to well developed individual interest. Situational interest is the affective response which can be initiated by environmental stimuli. This level of interest may be transient (Krapp et al., 1992). Individual interest denotes a student’s disposition to engage with subject content over prolonged time, usually with personal value attached to their endeavor (Schiefele, 1999). Most cases of individual interest were first experienced as a situational interest which developed further. Lipstein and Renninger (2006) found that varying degrees of effort, self-efficacy, setting challenges for oneself and monitoring and controlling one’s behavior, distinguishes
each phase of the interest development model. Fluctuations in these variables occur as interest develops or subsides.

Hidi and Renninger's four phase model will be briefly discussed here, with implications for educational instruction in mathematical problem solving.

**Phase 1: Triggered Situational Interest**

Interest is triggered by presenting surprising information or information which is of personal relevance. Learning environments which include puzzles, computers and group work have been found to initiate situational interest. This level of interest development is usually externally supported.

**Phase 2: Maintained Situational Interest**

This state of interest involves focused attention and perseverance which is maintained over a significant time period. Teaching and learning environments which provide meaningful tasks, which require personal involvement can help to maintain this interest. This level of interest, similar to phase one, is also usually supported externally.

**Phase 3: Emerging Individual Interest**

This phase of interest is characterised by positive feelings, good knowledge base and value associated with engaging in tasks in/related to the particular subject of emerging individual interest. Students begin to formulate questions based on their curiosity in the particular subject. Students begin to work independently and explore connections within the subject of emerging interest and related content from other disciplines. As a result of these questions and independent work, students may “redefine and exceed task demands in their work with an emerging individual interest” (p.115). Learning environments which provide opportunities to arouse curiosity and presents challenging tasks to students, with the aim of facilitating generation of questions by students, can contribute to students’ increased understanding of content of the emerging individual interest. Such an environment can also facilitate the development of an emerging interest. External support in the form of encouragement, may be needed to allow students to persevere when faced with difficult tasks. This phase of interest development is, however, usually self-generated.

**Phase 4: Well-developed Individual Interest**

This phase of interest development is classified by “positive feelings, increased stored knowledge and more stored value for particular content, than for other activities, including emerging individual interests” (p.115). Students with well-developed individual interest will continue to persevere on a problem, even when it causes continued frustration. These students also have the capacity to generate increased types and deeper levels of strategies to solve problems. Both content and context are taken into consideration when formulating a solution to
a problem. Although this phase of interest is usually self-generated, it may also benefit from external support. Providing opportunities to complete challenging interactive tasks that leads to construction of knowledge can support this phase of interest development.

(Hidi and Renninger, 2006, pp.114-115)

Well-developed individual interest facilitates sustained effort and perseverance, even when students encounter intense frustration and repeated failure to arrive at the correct solution. As interest development increases in mathematical problem solving, so too does the resourcefulness of the student in obtaining information and formulating strategies to help them to solve the problem (Hidi and Renninger, 2006). This, along with mindset offers an explanation as to why some students give up at the first sign of difficulty, while others will maintain intense concentration and increased effort over an extended time period, even when presented with what seems to be an impossible task.

Since cases of individual interest are preceded by situational interest (Hidi and Renninger, 2006), the learning environment in which mathematical problem solving is being conducted is of significant importance. To enable students to experience and maintain situational interest, the mathematical problems chosen should be authentic and relevant to the students. Also the way in which a topic is introduced should, where possible, include surprising or unexpected information. This could include the history behind the mathematics or providing information on where the mathematics is actually used in real life situations. The methods used to complete the tasks should include personal involvement; modeling eliciting activities\(^\text{11}\) involving group work, is one such method. The nature of the problem is again of importance in the individual interest phase. The problems should be appropriately challenging, which initially causes students to generate questions in their search for a solution path. Both the search for the answers to these generated questions, as well as the formulation of the solution to the given problem by the students themselves may lead to construction of knowledge which is a feature of the well-developed individual interest phase. Construction of knowledge is facilitated by inquiry based learning (Education Development Centre, Inc, 2012; Chang, 2011) by working with problems that arise from personal experience and observation.

Students’ perception of the difficulty level of the mathematics problem may influence their cognitive activity and their persistence (Montague and Applegate, 1993). Schoenfeld (1989) found that students expected all mathematical problems to be solved in a matter of minutes. Students generally believed that failure to solve a mathematical problem within approximately twelve minutes meant that the problem was impossible. In a study conducted by Higgins (1997) students who were taught problem solving heuristics for an academic year and given

\(^{11}\text{Described in section 2.6.1}\)
a weekly challenge to solve displayed greater perseverance in solving problems compared to a control group of students who received traditional mathematics instruction. The use of challenging tasks undertaken for extended periods of time is also advised by the National Council of Teachers (NCTM 1989) as a method of enabling students to understand that problems which cannot be solved quickly can be solved with time and sustained effort.

2.4.7 Intuition

Poincaré (1952) states that the fundamental goal of mathematics education is to develop capacities of the mind. Poincaré mentions intuition as one of these capacities. He describes intuition as the mechanism through which the mathematical and the real world remain in contact. Hestenes (2015 p.6) defines scientific (mathematical) thinking as a “realignment of intuition with experience”. Lester and Kehle (2003, cited in Lester, 2013) include intuition among the coordinating factors in their definition of problem solving in mathematics. Ben-Zeev and Star (2002) state that intuition can and should be a central component in the learning of mathematics. There are primarily two views of mathematical intuition, namely: classical intuition and inferential intuition (Ben-Zeev and Star, 2002).

The classical intuitionist view is that of mathematical intuition as being distinct from formal reasoning. Students may be able to identify the solution to a problem immediately without having to analyse the problem and because the answer is so obviously correct to them, they see no need to justify their solution (Westcott 1968). This intuitive knowledge is knowledge, which is independent of knowledge gained/developed formally through education/experience or informally through experience. Students who employ this form of intuition may therefore not justify their solution as they see no reason to do so.

The inferential intuitionist perspective is that of mathematical intuition as being a reasoning activity, which is based on and guided by mathematical knowledge, skills and experiences (Ben-Zeev and Star, 2002). This view of mathematical intuition therefore, suggests that mathematical intuition can be acquired. However, similar to previous misconceptions that students may have due to incorrect informal mathematical intuition, intuition gained in a formal setting may also be incorrect if mathematics and its concepts are taught in an ambiguous way. This again highlights teacher knowledge as being paramount to developing students’ understanding of mathematical concepts and their connections, since students may and do use this intuition in their attempt to understand and solve problems in mathematics. Fischbein(1973) suggests that the classical intuitionist perspective may be of more use in examining primary intuitions (informal knowledge which young children possess), whereas the inferential intuitionist view may be better suited in examining secondary intuitions (accumulated throughout formal schooling).
Intuition is defined by Hersh (2013) as

“a belief (possibly mistaken), arising from internal inspection of a mental image or representation - a ‘model’. It may be assisted by subconscious plausible reasoning, based on the availability of that mental image”.

(p.22)

Hersh (2013) states that in empirical science and life in general, these mental models represent actual objects, or objects that have potential to be constructed physically. In mathematics, these mental models may also include models that have no equivalent object (actual or potential to become actual) in the physical sense. Tall (1980) describes intuition in mathematics as the mental activity involving the macro combination of micro processes from the existing cognitive structure in the brain of an individual, which is activated as a direct response to a stimulus by a unfamiliar problem situation. In both of these definitions, the inferential intuitionist perspective is evident. Poincaré (1952) states that pure logic cannot provide a holistic view of the science of mathematics, intuition is also required. To cultivate one aspect without the other would provide an inaccurate view of mathematics, he notes that logic is the choice of method for proof, but it is by intuition that discovery in mathematics is made and it is intuition which teaches one to be able to see the end result before it is available through proof. This resonates with Pólya’s (1945) statement that it is only when one is sure that a theorem must be true will they begin the task of proving it.

Intuition can be useful in providing a conjecture and in helping to initiate a possible response to a problem. Intuition, along with metacognition is useful in knowing when, where and how to apply certain mathematics and also to trigger the identification of incorrect approaches to problems or incorrect parts of a solution path as they occur, or either prospectively or retrospectively. Intuition is also important for the mathematics teacher in again, making decisions in real time as when/where/how and what questions to ask students during problem solving in order to enhance their learning experience and advance their thinking. In developing students’ intuition, the process of solving the problem should be stressed as more important than obtaining the correct solution, the discussion of approaches, argumentation and justification on reaching a consensus on an answer is essential in facilitating the students in building upon and trusting their intuition (Jung et al., 2007). Tall (1980) states that students’ intuitions should be respected by teachers and other students. Jung et al. (2007) state that when students’ intuitions are respected and appreciated and when they are encouraged to explain their solutions methods, as well as listening carefully to the solution paths of others, they naturally absorb more advanced methods of solving problems. The use of manipulatives can help in the development of
secondary intuitions (Mason, 1996 in Ben-Zeev and Star, 2002) by helping students to specialise and generalise mathematical concepts and situations. The processes of specialising and generalising are of more importance than the use of the manipulatives in facilitating the development of secondary intuitions (Ben-Zeev and Star, 2002). Engaging in discovery learning programs such as the Moore Method of Instruction in mathematics also allows for the development of intuition. In the discovery method, the processes of experimentation and reflection should be evident (Ben-Zeev and Star, 2002).

The discovery method can take more time than the ‘traditional’ approaches to teaching. The instruction time available in mathematics is another factor affecting students’ ability to engage in problem solving.

### 2.4.8 Time

The instruction time allocated to mathematics in lower secondary education (years 1-3) in Ireland is less than the OECD average instruction time given to mathematics at lower secondary education (OECD, 2014).

<table>
<thead>
<tr>
<th>School Level</th>
<th>Primary</th>
<th>Lower Secondary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Instructional Time in Mathematics Ireland</td>
<td>17% (933.3 hours)</td>
<td>12% (336.72 hours)</td>
</tr>
<tr>
<td>OECD Average</td>
<td>15% (682.95 hours)</td>
<td>12% (350.64 hours)</td>
</tr>
<tr>
<td>Total Instruction Time Ireland</td>
<td>5490</td>
<td>2806</td>
</tr>
<tr>
<td>OECD Average Total Instruction Time</td>
<td>4553</td>
<td>2922</td>
</tr>
</tbody>
</table>

Table 4: Instructional Time (in hours) in Mathematics as a percentage of Total Instruction Time

A study by Prendergast and O’Meara (2016) showed that there is a large variation in the instruction time that students in secondary school in Ireland receive in mathematics. The school students are enrolled in, the year they are currently in and the level at which they study mathematics were among the factors which determined the amount of instruction time students receive in mathematics (Prendergast and O’Meara, 2016). The variation\(^\text{12}\) that was found to exist (184 secondary schools) is shown in Table 5. This study suggests that although the instruction time in mathematics at secondary level is approximately

\(^{12}\text{There were 26 different variations given among the 184 schools for first and second year groups. The most common was } 5 \times 40 \text{ minute classes in first year and second year (51\% of schools for first year and 53.8\% of schools for second year respectively). The results for the other years are presented in this way also.}\)
in line with the OECD average, some students may receive substantially less instruction time in mathematics than others (Prendergast and O’Meara, 2016). The times given in Table 5 are in minutes and 2nd yr (o) refers to second year ordinary level whereas 2nd yr (h) refers to second year higher level.

This problem with variation in instruction time in mathematics is not unique to Ireland, research has shown that students receive different instruction time in other countries also (Corey et al. 2012). A study of 15-year-old students from over 50 countries showed that increased instructional time has a positive significant effect on test scores in reading, mathematics and science (Lavy, 2015). Given that some students in Ireland receive a total of 659 hours of maths instruction time over a five-year period while other students receive 487 hours (26% less) instruction time in mathematics (Donnelly, 2016), and that the recommended approaches to learning mathematics in the Project Maths syllabus take more time (Jeffes, 2013), there are some students who are at a distinct disadvantage in terms of learning mathematics with understanding and when sitting examinations in mathematics. To learn mathematics with understanding, class periods of longer duration than the 40-minute classes shown in Table 5 are required (IMTA, 2012).

Mathematical modelling tasks which may also take considerable time to engage in may aid in developing the problem solving processes of students.
2.5 Mathematical Modelling and Rationale for Inclusion in Mathematics Education

The negative attitudes towards mathematics of numerous students around the world has been shaped by the experience of learning mathematics in a procedural way, with little rationale given to them for the potential applicability of mathematics in their present and future lives (Vorholter et al., 2014). The use of mathematical modelling in mathematical instruction and learning is a current approach which provides students with an appreciation of mathematics as having many and varied potential uses in the analysis of phenomenon in the world, both at a micro and macro level (Greer et al., 2007, cited in English and Sriramen, 2010).

Theoretically, and in numerous cases, practically, the role of the mathematics teacher has changed from that of main instructor: teaching rules, formulae and correcting related exercises, to that of a facilitator of mathematical activities that promote understanding of mathematics, mathematical thinking and reasoning abilities. Educators today are aiming to provide students with a mathematical experience which will equip students with the advanced thinking skills needed to acquire mathematical knowledge and solve complex problem situations both in and beyond school (Kulikowich and DeFranco, 2003).

Mathematical modelling has been incorporated into second level curricula for many years in some countries (Blum, 1993). In Ireland however, apart from a mathematical modelling workshop organised by the Mathematics Applications Consortium for Science and Industry (MACSI) for senior cycle secondary school students at Clongowes Wood College (secondary school), there is little research available on the actual incorporation of mathematical modelling in the Irish second level curriculum (Charpin and O’Hara, 2014).

Mathematical modelling involves mathematisation, this fosters mathematical thinking among students, enabling the acquisition and development of knowledge of mathematical concepts, skills and strategies. Mathematical modelling also enables the enrichment of students’ appreciation of mathematics as a ‘toolkit’ for understanding and solving problems in the real world (Burkhardt, 2006).

What are the distinctions between problem solving and modelling? Can one be considered a subset of the other, do the two intersect or are they mutually exclusive? The perspective of Lesh and Zawojeski (2007) on models and modelling is that the learning of mathematics occurs through mathematical modelling, such that problem solving (in the traditional story-type sense) is a subset of mathematical modelling (applied problem solving) as can be seen in Figure 5. In this perspective, students develop ‘conceptual systems’ at the beginning or intermediate stage of their learning experience in order to make sense of situations where a mathematical way of thinking needs to be formed or revised (Lesh and Zawojeski, 2007, p.783). Students interpret a problem using their own
perspective that makes sense to them and then they test their interpretation of the problem situation through mathematical modelling. It is expected that students will gain increased understanding of both the problem situation and of the mathematical modelling process that they used to formulate the particular problem (Lesh and Zawojewski, 2007). The more traditional type of problems become a subset of the modelling tasks through which students learn mathematics (Lesh and Zawojewski, 2007). In this perspective mathematical modelling equates to the co-development of mathematical knowledge and problem solving skills through engagement with mathematical modelling tasks (Lesh and Zawojewski, 2007). Students learn mathematics through engagement in problem solving and students develop problem solving through creating mathematics (mathematical models) (Lesh and Zawojewski, 2007).

There are various perspectives on mathematics modelling in mathematics education with respect to the aims that an educator intends on achieving by including modelling in their practice. Kaiser and Sriraman (2006) suggest the classification of the different perspectives of mathematical modelling as shown in Table 6.

![Figure 9: Models-and-Modelling Perspective on Problem Solving](Lesh and Zawojewski, 2007, adapted from Lesh and Doerr, 2003)
<table>
<thead>
<tr>
<th>Name of perspective</th>
<th>Central aims</th>
<th>Relations to earlier perspectives</th>
</tr>
</thead>
<tbody>
<tr>
<td>Realistic or applied modelling</td>
<td>solving real world problems, understanding of the real world, promotion of modelling competencies</td>
<td>Pragmatic perspective of Pollak</td>
</tr>
<tr>
<td>Contextual modelling</td>
<td>subject-related and psychological goals, i.e solving word problems</td>
<td>Information processing approaches leading to systems approaches</td>
</tr>
<tr>
<td>Educational modelling: differentiated in a) didactical modelling and b) conceptual modelling</td>
<td>Pedagogical and subject-related goals: a) Structuring of learning processes and its promotion b) Concept introduction and development</td>
<td>Integrative perspectives (Blum, Niss) and further developments of the scientific-humanistic approach</td>
</tr>
<tr>
<td>Socio-critical modelling</td>
<td>Pedagogical goals such as critical understanding of the surrounding world</td>
<td>Emancipatory perspective</td>
</tr>
<tr>
<td>Epistemological modelling</td>
<td>Theory-oriented goal, i.e. promotion of theory development</td>
<td>Scientific-humanistic perspective of “early” Freudenthal</td>
</tr>
<tr>
<td>Cognitive modelling *</td>
<td>Research aims: a) Analysis of cognitive processes taking place during modelling processes and understanding of these cognitive processes Psychological goals: b) Promotion of mathematical thinking processes by using models as mental images or even physical pictures or by emphasising modelling as mental process such as abstraction or generalisation</td>
<td>No relation to earlier perspective but background in cognitive psychology</td>
</tr>
</tbody>
</table>

Table 6: Classification of perspectives on modelling (Kaiser and Sririman, 2006, p.304) * considered a ‘kind of meta-perspective’).
The perspective on modelling taken in this research aligns with the educational modelling perspective and the cognitive modelling perspective in structuring the learning process and developing conceptual understanding by stimulating mathematical thinking processes. Blum and Borromeo Ferri (2009) refer to the following processes as problem solving activities in mathematical modeling:

- Understanding and simplify the problem: understanding text, tabular, diagram and graphical information, using previous mathematical knowledge appropriately to understand the problem;
- Simplifying the problem structuring it;
- Developing a model: Looking for the data that is required, recognising irrelevant data making assumptions and determining mathematical relations of the data in the problem situation, using representations/organisation in the formulation of the solution;
- Using appropriate mathematising (expressing the real model as a mathematical model) to find results;
- Explaining and generalising results;
- Verifying the solution by referring back to the original problem situation, comparing solutions from different perspectives/methods. Check validity of the developed model.

These processes are shown in the perspective of mathematical modelling by Lesh and Doerr (2003) (Figure 10).
The past and continuing development of technology affects the nature of skills required in certain jobs. As stated by the National Research Council (NRC, 2001), these changes are more likely to elevate the skills required, by increasing the demands on cognitive, communication and interactive skills.

In mathematical problem solving tasks the emphasis is usually on the search for, and the use of, the appropriate procedure to solve the problem (Zawojewski, 2010). As the solution path is not immediately obvious to the problem solver/group, he/she may use problem solving strategies/heuristics as an aid to finding the correct procedure to go from the given problem state to the goal state. The problem solving task presented is typically well formulated, that is, there is usually little need to interpret the problem situation or solution outcome. Mathematical modelling tasks place the focus on the problem. The emphasis is on the interpretation of the information presented in the task and the formulation of the mathematics, which model the problem situation (Haines et al., 2007). Mathematical modelling tasks represent a real world task. These tasks are often more complex than some of the problem solving tasks, which students find in a typical textbook. As previously mentioned, Schraw et al. (1995) found that the thinking required to solve well-defined problems differs from the thinking employed when engaged with ill-defined problems, therefore solving the problem solving tasks typically found in most mathematics textbooks does not necessarily lead to improved proficiency in solving the complex problems encountered beyond the classroom. Hamilton (2007) concurred with this find-
ing, noting that there was little evidence to suggest the contrary. The modeling tasks have not been constructed so that the information required to solve the problem is clearly available, as is the case in some problem solving tasks. These modelling tasks thus more closely resemble the mathematical problem situations found beyond school. Mathematical modelling eliciting activities (section 2.6.1) provide opportunity for students to make sense of the problem situation and use mathematics which are meaningful to them to model this situation mathematically (Doerr and Lesh, 2003). As modelling tasks usually involve working in groups, questions, arguments, solutions and revisions are all part of the completion of the task (English and Doerr, 2004). These activities can also aid teachers in gaining an understanding of students’ thinking. This is a cyclic process composed of interpreting the problem situation and information provided, choosing relevant quantities and creating meaningful representations. This process is shown in Figure 11.

Figure 11: Mathematical Modelling Cycle by Blum and Leiβ as adapted and presented by Borromeo Ferri, 2006a, p.92)

The perception of mathematics of 88% of the students (n = 35) who participated in the modelling workshop at a second level private male boarding school13 changed positively (Charpin and O’Hara, 2014). These positive changes included discovery of real world applications of mathematics and increased awareness of the ability to express creativity in mathematics. The awareness of mathe-

13Female students, students studying ordinary level mathematics or Junior Cycle students did not participate in this study.
matics as an interesting subject, despite its complexity at times, was also noted by students. The participants in this study were among the top physics and mathematics students in fourth and fifth year\textsuperscript{14}. Pre and post-tests of problem solving and/or modelling ability of the students who participated in this workshop were not conducted.

The author’s perspective on mathematical modelling was stated in this section. The benefits of including mathematical modelling in mathematics education was discussed. The author uses model eliciting activities as part of the intervention phase of this research. The next section considers instructional practices which may help to facilitate problem solving in mathematics.

2.6 Analysis of Instructional Practices of Mathematics Education

This section critically examines five significant initiatives in mathematics education, focusing in particular on the way these methods of instruction successfully facilitate effective conceptual understanding and problem solving in mathematics, with the aim of identifying possible implications of including aspects of these models to refine the instruction and application of problem solving in the Irish post-primary mathematics curriculum.

2.6.1 Realistic Mathematics Education (RME)

Realistic Mathematics Education (RME) originated in the Freudenthal Institute in the Netherlands in the 1970s. It is based on Freudenthal’s idea of mathematics as a “human activity” (Van den Heuvel-Panhuizen, 2003, p.11), not as a list of rules. Freudenthal believed that each individual discovers mathematical properties in their own surroundings and makes sense of these properties/structures in their own unique way, thereby creating their own personal concept of mathematics. The fundamental elements of RME are that mathematics should be “connected to reality, stay close to children’s experiences and be relevant to society” (Van den Heuvel-Panhuizen, 2003, p.9). In RME, children learn mathematics by engaging in problem-solving in contexts that are meaningful to them. The term ‘realistic’ refers to problems set in contexts which the students can imagine/visualise in their minds. The contexts, therefore, are not solely restricted to real world scenarios.

The particular components of RME that are closely aligned to the objectives of mathematical problem solving are the following:

- Initially, students devise their own methods for solving a problem, based on their intuition. Through teacher chosen examples and interventions,

\textsuperscript{14}Students in fourth and fifth year are 16-17 years old approximately.
they learn to generalise and thus develop a deeper understanding of the mathematical concepts involved in the problem (Dickinson and Hough, 2012)

- Both horizontal and vertical mathematisation (Freudenthal, 1991) are employed, that is to move from mathematics in life to representing mathematics in symbolic form and to move within the symbolic form of mathematics respectively. Vertical mathematisation facilitates the formation of connections between concepts and strategies and the use of these connections. Freudenthal was of the opinion that both types of mathematisation are not mutually exclusive activities. He also believed that both types could occur on all levels of mathematical activity, from the most basic to the advanced level.

- The problems chosen in RME should include model eliciting activities, that is that the problem should necessitate the need for model building as part of the formulation of a solution.

Model Eliciting Activities (MEAs) are constructed based on six principles of instructional design:

  - Reality Principle:
    The situation should be meaningful to the students and should build upon their pre-existing knowledge.
  - Model Construction Principle:
    The problem situation should necessitate the need to create mathematical models.
  - Self-Evaluation Principle:
    The problem situation should require the students to continuously assess their own elicited models.
  - Construct Documentation Principle:
    The problem situation should require the students to reveal their thinking while solving the problem.
  - Construct Generalisation Principle:
    The students should figure out if their elicited model can be generalised to similar situations.
  - Simplicity Principle:
    The problem situation itself should be simple, the thinking required to solve it can be complex.

(Doerr and Lesh, 2003)

2.6.2 Mathematics in Context (MiC)

Mathematics in Context (MiC) is a middle school mathematics curriculum for grades 5 through 8 (Romberg and Meyer, 2001). It was developed between 1991
and 1998 as a result of a collaborative effort between research teams at the University of Wisconsin-Madison and the Freudenthal Institute at the University of Utrecht, The Netherlands, and a team of middle school teachers. The programme was developed to teach students to solve non-routine problems. It is based on perceiving mathematics as being a rational subject. The programme is essentially a combination of three elements (Educational Development Centre, 2001): the research accumulated to date on a problem-solving approach to the teaching of mathematics, the NCTM Curriculum and Standards, and the RME (Freudenthal) approach.

Engaging in solving real problems is central in MiC, the interpretation and understanding involved in solving the problems provides the foundation for the mathematical development of the students. Problems are presented to the students before they have acquired the knowledge to solve them. This generates a sense of purpose to the mathematics that the student needs to learn. This instructional approach is similar to that which is proposed in cognitive apprenticeship. Also since the problems employed in MiC are contextualised realistically, it gives further authenticity to the need for learning about and making sense of the necessary mathematics. Poincaré (1952) notes that the rote learning of mathematics without any associated meaning attached to the mathematics by students is a result of students being taught mathematics for which their mind perceives no need.

One of the most significant elements in MiC is that students are given the freedom to formulate and make use of their own strategies to solve the problems which they are engaging with (Educational Development Centre, 2001). This is similar to aspects of Cognitively Guided Instruction (discussed in subsection 2.6.3), particularly utilising students’ natural intuitive knowledge of mathematics in their learning experience of mathematics.

### 2.6.3 Cognitively Guided Instruction (CGI)

Cognitively Guided Instruction (CGI) was developed by Carpenter et al., (1989). It is an approach to teaching mathematics based on the combination of results from research on: how students’ mathematical thinking develops and instruction on facilitating this development, how the knowledge and beliefs of teachers influence their approach to teaching mathematics, and how their approach to teaching is shaped by their understanding of students’ mathematical thinking. CGI is guided by the following:

1. helping teachers to understand the natural intuitive knowledge that children already possess before beginning formal education and how to utilise this knowledge as the basis for developing formal mathematics instruction.

2. basing instruction on the relationship between problem solving and computational skills; children are asked to solve problems (verbally and in
writing) using their own strategies. They are also asked to explain and justify their own ways of solving the problems. Both the teacher and the students share responsibility for deciding whether a solution is correct or not. This facilitates a problem solving emphasis in the mathematics classroom.

(Carpenter et al., 1996; 1999)

Fennema et al. (1996) conducted a study on the effectiveness of CGI at primary school level. This study involved 21 teachers (of first, second and third grade) and their students over a four year period (longitudinal study). Results from this study revealed the following changes in the beliefs of 17 of the 21 teachers (who participated in this study) in relation to what it means to teach mathematics:

- teachers more firmly believed that students had the ability to solve problems without being predisposed to procedures for solving them.
- teachers more convinced that mathematics can be learnt by engaging with many purposefully chosen/devised problems (i.e to learn mathematics via problem solving)
- their role as a mathematics teacher as providing a learning environment in which students’ mathematical learning can be developed through engagement in problem solving rather than the teacher presenting procedures for the students (effectively telling the students how and what to think).

The students who participated in this study showed significant increased ability to solve problems and understand mathematical concepts (greater than usual with traditional instruction), while their computational ability improved at an equal rate to that expected from engaging with traditional instruction. This study gives support to the view of improving mathematics teaching and learning by increasing the awareness among teachers of the mathematical thinking of their students (Cobb et al., 1990).

2.6.4 Cognitive Apprenticeship (CA)

In the past, apprenticeship was the medium employed for transmitting the knowledge and skills required for expert practice in many varied fields from carpentry to medicine. Cognitive apprenticeship is a model of instruction that reverts back to apprenticeship while incorporating elements of schooling (Collins et al., 1989).

In traditional apprenticeships, the master makes the desired process explicitly visible to the apprentice (modeling). In cognitive apprenticeship the teacher’s thinking should be made visible to the students and, similarly, the students' cognitive processes must be made visible to the teacher. By explicitly displaying the cognitive and metacognitive processes that constitute expertise to students,
they can then perceive, utilise and refine these processes with help from the teacher and fellow students. In mathematics the teacher can model the process of problem solving by having the students bring difficult problems for him/her to solve (Schoenfeld, 1983). The use of heuristics and metacognition can be made visible to the students during this modeling process and an important element of problem solving can be seen: how to proceed when one’s chosen strategies fail. Generally, in secondary schools, only the correct method/solution path is shown by the teacher and also in textbooks the correct method is the only solution presented to students. Observing how experts deal with problems that are challenging for them is reassuring for students and helpful in developing belief in their own mathematical ability (Collins et al., 1989). This also allows students to see that solving problems takes time and if they are attempting a problem, they may have increased perseverance after witnessing that even experts can experience difficulties and have failed attempts before arriving at the correct solution.

2.6.5 Authentic Pedagogy (AP)

Authentic Pedagogy (AP) is based on the work of Fred Newmann and associates at the Center on Organisation and Restructuring Schools. It is a model which emphasises higher order thinking, conceptual understanding and problem-solving in students’ learning. It has its basis in constructivism. Authentic work requires the “original application of knowledge and skills, rather than just routine use of facts and procedures” (Newmann et al., 2007, p.3). The product of Authentic work should have meaning/value, more than just meeting the criteria for success in school. AP consists of three main features (Newmann et al., 2007):

- **Construction of Knowledge:**
  This involves the application of knowledge and basic skills to solve complex problems which may be unfamiliar to the students. Solving these non-routine problems, necessitates the organisation, interpretation, evaluation and synthesis of prior knowledge to arrive at new knowledge with the aim of employing this to solve the problem. Newmann et al. (2007) believe that construction of knowledge is best learned by engaging in activities which demand this type of cognitive work, rather than the direct teaching of these thinking skills.

- **Disciplined Inquiry:**
  This feature of AP aims to maintain the quality of both the work in progress and the end result produced, at a high standard. Students’ work should be validated by inquiry into the relevant existing knowledge on that particular topic they are working on. The construction of knowledge should be guided by disciplined inquiry, that is students should make use of prior knowledge, aim for in-depth understanding and develop and express their ideas and findings through elaborated communication.
1. Prior Knowledge Base
   Students should have a solid base of knowledge, along with accomplished basic skills. These will assist them in participating in disciplined inquiry.

2. In-depth understanding
   Students’ prior knowledge base should consist of both conceptual and procedural understanding and this should be connected.

3. Elaborated Communication
   Students should use correct mathematical terminology and symbols in the presentation/communication of their work

• Value Beyond School:
   The result in an examination or obtaining approval by the teacher for a correct answer should not be the only value derived by completing work in mathematics. Mathematical work should include tasks that link mathematics learnt in school to situations outside of school to which the mathematics can be applied. This aims in providing a meaningful learning experience for students.

Research has shown that when teachers teach through AP and set work which meets the AP criteria, students (8th grade, n = 2, 100) score significantly higher on assessments of complex intellectual performance as well as on examinations of basic knowledge and skills, compared to students in classes where teaching does not meet AP criteria (Newman et al., 1996)

2.7 Selected Instructional Practices

Allowing students to use their own strategies to solve a problem is common practice in RME, MiC and CGI. Fennema et al. (1996) found that when teachers taught mathematics using CGI at primary school level, the teachers came to strongly believe that students did not have to be taught the procedures for solving problems in advance of solving the problems. This resonates with the findings of Boaler (1997) and Jung et al., (2007) that young children have innate ability for solving problems. The teachers viewed their role as providing an environment where students learn mathematics through engagement with problem solving rather than providing an environment where the teachers provide the solutions for the students. The gains to students of using their own strategies/intuitions in a problem solving environment are increased ability to solve problems and greater understanding of mathematical concepts.

Making mathematical thinking visible is common in both CGI and CA and this works both ways in that students’ thinking is made visible to the teacher and the teacher’s thinking is made visible to students.

Engaging in activities which necessitate the use of the mathematical thinking/ aspects of mathematical proficiency that a teacher is trying to develop in
his/her students is the optimal way for students to construct their mathematical knowledge, this is a vital aspect of AP. The author intends to use tasks to “do the teaching”, that is to develop conceptual understanding and the other four strands of mathematical proficiency while also developing mathematical thinking through careful design and selection of the mathematical tasks. The author intends to use the Model Eliciting Activities from RME, the realistic contextualisation of problems (in MiC), and presentation of some problems for which the pre-service teachers do not have the complete knowledge required to solve them. The shared responsibility by the students and teacher for the decision on whether a solution is correct or not is an aspect of CGI and AP that the author intends to include in the intervention component of this research. In addition the author will use the common aspects of the instructional practices that have been summarised here.

The author will provide information to the pre-service teachers on mindset, mathematical thinking and metacognition, with the information given on mathematical thinking developed through the use of tasks which necessitate the use of the specific mathematical thinking. The author intends to facilitate the development of the pre-service teachers' ability to ask themselves metacognitive questions. These aspects were selected for inclusion in instruction in an attempt to build on students’ mathematical competencies in an effort to form their overall competency in mathematics. The next section discusses both of these terms.

2.7.1 Competence and Competencies in Mathematics

When a person possesses competency in a particular field, this person is able to “master the essential aspects of that field effectively, incisively, and with an overview and certainty of judgement” (Niss and Højgaard, 2011, p.49). Mathematical competency includes possessing a knowledge and understanding of mathematics, in addition to being able to do and use mathematics, while also being able to have an opinion about mathematics and mathematical activity in a multitude of contexts and situations where mathematics is or could be involved (Niss and Højgaard, 2011). The ability to relate one’s knowledge and understanding of mathematics to the context/situation in which the mathematics/mathematical idea is embedded is as vital a component of mathematical competence as is the knowledge and understanding of mathematics itself (Cobb, 1986 in Taplin, 1998).

There are several mathematical competencies and the integration of these competencies (which themselves cannot usually be acquired separately from the other competencies) forms an individual’s competence in the field of mathematics (Niss and Højgaard, 2011). Niss and Højgaard (2011, pp.50-51) have identified eight competencies and have classified these eight competencies into two groups, which consist of:
• “the ability to pose and answer questions in mathematics and with mathematics:

  – mathematical thinking competence (ability to pose questions and be aware of the possible answers);
  – problem tackling competence (ability to answer given questions in mathematics by using mathematics);
  – modelling competence (ability to formulate answers to posed questions in mathematics by using mathematics);
  – reasoning competence (ability to understand, assess, and produce arguments to solve mathematical questions (including understanding when reasoning constitutes a proof or not)).

• the ability to contend with mathematical language and tools:

  – representing competence (ability to deal with different representations of mathematical situations (including understanding the relations between different representations of the same mathematical situation) and being able to choose and switch between representations depending on the given situation);
  – symbol and formalism competence (ability to contend with symbolic and formulaic representations in mathematics and translate between natural language (e.g., English) and mathematics (including insight into the nature of the “rules” of formal mathematical systems));
  – communicating competence (ability to communicate in, with and about mathematics (including interpreting written, oral, or visual communication and being able to express oneself mathematically using these various media also);
  – aids and tools competence (knowledge of available technical aids for mathematical activity and an ability to use and interpret such aids).”

A visual representation of the integration of the eight mathematical competencies is shown in Figure 12.
Figure 12: A visual representation of the eight mathematical competencies (Niss and Højgaard, 2011, p.51).

The visual representation (Figure 12) highlights the interconnected nature of the eight competencies in forming an individual’s overall competency in mathematics. These competencies have been discussed as part of the factors that affect students’ ability to engage in problem solving in mathematics (section 2.4 (and also in section 2.5)). The implementation of instructional practices, in addition to teaching for developing mathematical competency has been found to be influenced by what is assessed in examinations.

2.8 Assessment

Historically, in mathematics, the emphasis in assessment has been more on testing students’ knowledge, rather than their understanding. This method of assessment has often failed at identifying talented students, who may obtain low grades, while also failing to display weaknesses in students’ understanding among those who obtain high grades (Lesh and Sriraman, 2005a). The ‘goal of education is to improve minds’ (Devlin, 2003 p.38), enabling them to increase their understanding of the information they are learning about, which facilitates the connection of knowledge, experiences and skills (KES) to previously gained KES, as well as to KES that are being acquired simultaneously. The importance is then on how these minds process, form connections between, and utilise their KES in mathematical problem situations.
What is assessed in the state examinations has an influence on the teaching and learning that occurs in the mathematics classroom (Jürges et al., 2012). The successful implementation of Project Maths, the problem solving emphasis in particular, is partly dependent on the inclusion of the assessment of problem solving in the state examinations. If non-routine problems are included in the examination then it is likely that teachers and students will work with non-routine problems in the classroom (O’Donoghue, 1997). To date, Project Maths has included non-routine problems in the contexts and applications section (questions 7, 8 and 9) of the examination, however in 2014 the inclusion of non-routine problems was limited to part of question 9 in the first examination paper and again only to question 9 in the second (of two) papers. The non-routine problems assessed in the initial phases of examination of Project Maths (2010-2012) were of a more difficult standard than those which were assessed in 2014 (Bielenberg, 2013; Donnelly, 2014).

A study conducted by Iversen and Larson (2006) involved the development of a model eliciting activity (MEA) as a way of measuring students’ performance on a complex real-world problem. Data was collected from ≈ 200 (178 pre-test, 201 MEA and 199 post-test) first year university students (≈ 18 – 19 years old) in a calculus course, over a 7 week period. The students’ performance on the traditional style pre-test, the MEA and on the traditional style post-test was analysed. The results of the analysis (and also observations by several teachers and researchers of students’ performance) gave support to Lesh’s claim that many students who are often categorised as low ability by performance in traditional assessments, can describe and formulate mathematical models of complex problem situations (modelling tasks) and also that students who are noted as being very capable by their performance in traditional assessments can be quite inept at applying their mathematical knowledge to complex real-life problems in a productive way (Iverson and Larson, 2006).

Limiting the assessment of mathematical problem solving to one form of assessment therefore reduces the probability of certain students achieving high results despite them being highly capable in mathematics and able to display this ability more effectively through other methods of assessment. This means that inferences drawn from students’ performance in certain examinations cannot in general be used as a true representation of their overall ability in mathematical problem solving and/or modelling situations (Watt, 2005). There is a breakdown in validity in assessing students when using only one form of assessment resulting in a favorable outcome for some students, while placing others at a distinct disadvantage (Stephens, 1988).

Since what is assessed affects what is taught and learnt in mathematics classrooms, assessment needs to be closely aligned with the mathematical syllabus, teaching methods and goals (O’Donoghue, 1997) of Project Maths. Since the emphasis in Project Maths is on the development of students’ conceptual understanding and problem solving skills, the Project Maths examination should
then reflect this emphasis by assessing these valued competencies.

One limitation of the study conducted by Iversen and Larson (2006) was that the MEA was solved in groups of 3, whereas the pre and post-tests were completed individually. Analysis of the results in the search for correlation between individual pre-test scores and individual MEA scores and also between group MEA scores and the sum of the group members’ individual pre-test scores were conducted. The sum of the individual performances on a traditional examination need not necessarily equate to the whole performance of the group in the same examination. This finding alone raises the issue of the problems experienced in assessing group problem activities in mathematics education.

Success in problem solving is heavily dependent on one’s knowledge and understanding of mathematics. The next section examines models and theories of developing this understanding.

2.9 Models and Theories of Mathematical Understanding

Essentially problem solving is thinking: about the information given in the problem, about one’s knowledge base relative to the information present in the problem, synthesising information and forming associations (Hardin, 2002; Pólya, 1945). Developing understanding in mathematics involves the connecting of concepts, ideas and representations to a well structured network (Hardin, 2002; van Hiele, 1984(c); O’Connor, 2011). This process requires the recognition of relationships between the discrete packets of knowledge to the network of mathematical knowledge one possess, as well as the knowledge of the cognitive network as a whole system of relationships (Devlin, 2012; O’Connor, 2011). This section discusses models and theories for developing understanding in mathematics. The information gained in the study of these models and theories informed the development of the author’s framework for teaching and assessing problem solving in mathematics.

2.9.1 Dubinsky - Action-Process-Object-Schema (APOS) Theory

APOS theory is concerned with modelling the cognitive activity which individuals may engage in when they are learning particular mathematical concepts (Arnon et al., 2014). These models of cognitive activity are subsequently utilised to inform the design of instructional activities and/or to assess students in their performance in mathematical problem solving situations (Arnon et al., 2014). The framework used in APOS theory is cyclical, consisting of three components:

1. theoretical analysis - outlining what it actually means to understand a particular mathematical concept. This is achieved through a “genetic decomposition” of the mathematical concept, which is a description of how the
understanding of this particular concept may be specifically constructed
in the mind of the learner;

2. design and implementation of instruction - the theoretical analysis of the
mathematical concept informs the design and implementation of the in-
struction program;

3. observations and assessments - the implementation of the instructional
program facilitates the collection of data which can then be employed to
reflect on the initial theoretical analysis of the concept. This reflection
then informs any subsequent revisions to the theoretical analysis and the
instruction (which may be deemed appropriate with respect to the inform-
ation gained in the collected data).

(Asiala et al., 1997)

APOS theory is a constructivist theory. The perception of an individual’s math-
ematical knowledge and understanding in APOS theory is their inclination to
respond to perceived mathematical problem situations by engaging in reflec-
tion on problems and their solutions, and also by constructing and/or rebuild-
ing mathematical ‘Actions’, ‘Processes’ and ‘Objects’ and organising these into
‘Schemas’ for use in problem situations (Asiala et al., 1997). This theory of
“what it means to learn and know something in mathematics” (Asiala et al.,
1997, p.5) takes into consideration that a learner may possess the knowledge
and the capability to solve a particular mathematical problem but may not
necessarily access this knowledge at a particular time or in a given problem
situation. APOS theory was developed by Dubinsky, as a result of trying to
gain an understanding of the term “reflective abstraction” which was utilised
by Piaget to describe how logical thinking develops in children (Dubinsky and
McDonald, 2002). Reflective abstraction as utilised by Piaget consists of two
parts:

• Reflection; conscious thinking (through engagement with mental or phys-
ical actions) about the properties of particular mathematical content and
procedures/operations on that content;

• Reconstruction and reorganisation: the content and operations which have
been reflected on, by the individual, results in the individual being able
to consider the mathematical content and operations in a more advanced
way of thinking. The operations can now be perceived as also being con-
tent to which further operations can be applied. Thus, these need to be
reorganised to fit coherently into the knowledge structure in the mind of
the individual in order to facilitate this more advanced level of thinking.

(Aron et al., 2014)

Dubinsky (1991) notes five types of workings in the mind (‘interiorisation’, ‘co-
ordination’, ‘reversal’, ‘encapsulation’ and ‘generalisation’) which are involved
in reflective abstraction, which give rise to the construction of the mental structures: ‘Actions’, ‘Processes’, ‘Objects’ and ‘Schemas’. Dubinsky (1991) states that the ‘mental mechanisms’ of reflective abstraction are important in the development of mathematical thinking. These five types of ‘mental mechanisms’ as described by Dubinsky (1991) are outlined as follows:

*Interiorisation* involves the construction of mental processes through the use of symbols, pictures and language in order to make sense of mathematical concepts. There is a mental construction formulated in the mind of an individual which relates to the action carried out. An individual has conscious awareness of the action, which allows them to think/reflect on it and combine it with other actions. A representation of the mathematical concept is formulated in the mind of the learner.

*Coordination* is the coordination of two or more processes to attain a new process.

*Encapsulation* is the ability to see a ‘dynamic structure’ as being a ‘static structure’ (Arnon et al., 2014) to which actions can be applied (similar to Sfards’ (1991) structural conception as opposed to an operational conception only). It is the individual’s awareness of a process as a whole and their ability to apply actions to this static structure in their mind, explicitly or both. Piaget (cited in Dubinsky) refers to encapsulation as the ‘thematisation’ of actions or objects into objects of thought.

*Generalisation* is the ability of an individual to apply a schema they already possess, to a broader collection of phenomena. This can occur as a result of increased awareness by the individual of the more extensive applicability of the schema or as a result of a process being ‘encapsulated’ as an object.

*Reversal* is the ability of an individual to think of a process in reverse, that is to construct a new process which is the result of reversing the original process. This can occur once a process exists internally in the mind of an individual.

Dubinsky (1991) holds that the mental activities of reflective abstraction, as just described, are important in mathematical thinking. The extent of an individual’s understanding of a mathematical concept depends on their ability to make connections between the mental structures of which it is composed (Arnon et al., 2014). These connections form the foundation of a schema whose consistency in the logical relating of these connections is fundamental to an individual’s capacity to make sense of mathematical problem situations related to the concept (Arnon et al., 2014). Each of these mental structures are described as follows:

*Actions* are transformations of objects which are considered by an individual from an external point of view, that is they can complete a step by step transformation only when guided by external cues (Arnon et al., 2014; Asiala et
al., 1997). An individual with this type of understanding cannot skip a step in a provided procedure, or conceive of the steps without having to carry them out (Arnon et al., 2014). For example; when asked to solve a mathematical problem, where the use of a certain mathematical proof or idea/concept would be useful, the individual would not think to use this particular proof or idea/concept to complete the transformation of the object unless they are provided with the external cue to do so (Asiala et al., 1997). An individual who is limited to an Action conception of a mathematical concept is not able to recognise a particular concept in a mathematical problem situation without receiving external cues.

Processes are the mental structures which result from ‘interiorisation’ of an Action; that is that the individual can complete the same transformation of the object as in the Action structure but can do so without being dependent on external signs or prompts (Asiala et al., 1997). The repetition of the Action with reflection may facilitate this ‘interiorisation’ of an Action to a Process (Asiala et al., 1997). This type of conception has at its disposal an “internal construction” related to the steps of the transformation, which facilitates the reflection on, and the description and reversal of the transformation without having to actually complete the steps (Asiala et al., 1997. p.7). In essence, an individual can imagine completing the transformation without actually explicitly doing it. An individual who has a Process conception of a mathematical concept has the ability to recognise a particular concept in a previously met mathematical problem situation. The particular mathematical concept can now be associated with this mathematical problem situation and the individual has the ability to use this concept without requiring external instruction to do so.

Objects are the mental structures which result from an individual’s awareness of a Process as a whole entity, along with the realisation that transformations (Actions or Processes) can be applied to this whole entity (Arnon et al., 2014). An individual who has an Object conception of a mathematical concept has the ability to construct these transformations in their mind or explicitly. Dubinsky (2005a) refers to this as ‘encapsulation’ of a Process to an Object. This is considered to be a very difficult mental processes. Sfard (1991) also mentions the difficulty of this process which she refers to as reification. It is an ability to see something familiar in a entirely new way.

Schemas are the mental structures of a mathematical concept which are formulated as a result of the coherent connection of a collection of mental structures (Actions, Processes and Objects) which have been constructed by an individual in relation to a particular mathematical concept (Dubinsky and McDonald, 2002). The more coherent the Schema is in terms of its connections, the greater the ability the individual has in determining the appropriateness of its use in resolving a particular mathematical problem solving situation (Arnon et al., 2014). A Schema which is highly coherent in its connection of the collection of structures can be “transformed” into a fixed structure (Object) and/ or utilised.
as a “dynamic structure that assimilates other related Objects or Schemas” (Arnon et al., 2014, p.25).

In summary, APOS theory is based on the principle that an individual can learn any mathematical concept if the structures required to comprehend the concept have been constructed in their mind (Dubinsky 1991, in Arnon et al., 2014). It is the organisation and coherence of this structure which facilitates the understanding of the mathematical concepts and also determines the degree of success of an individual’s engagement in mathematical problem solving situations.

2.9.2 Tall Framework of Mathematical Thinking

Discussed in section 2.4.2

2.9.3 Sfard - Operational and Structural Conceptions

Sfard (1991) discusses the capacity to conceive abstract mathematical concepts in one’s mind as being a vital part of mathematical ability. Sfard (1991, p.3) defines a concept in mathematics as being a “theoretical construct” or an official mathematical idea or notion, whereas a conception consists of the entire “cluster of internal representations and associations evoked by the concept”, in the mind of the individual. Three phases of abstraction are described by Sfard, with the highest of these being reification (discussed in this section). Sfard states that reification improves problem solving and learning abilities as it facilitates a more structural approach to solving a problem which increases the confidence of the learner in the appropriateness of their chosen solution approach.

Sfard (1991) considers mathematical conceptions from a dualistic point of conception; structural and operational. Structural conceptions refer to conceptions which perceive mathematical ideas in terms of some abstract object. Operational conceptions include thinking of mathematical concepts in terms of a computational process or algorithm or action (Sfard, 1991). Operational conceptions occur both at the physical and mental level whereas structural conceptions refer more to mental conceptions of a non physical mathematical entity. These definitions by Sfard (1991) are similar to the definition of concept put forward by Vinner (1991) and the description of concept image by Tall and Vinner (1981). While there is a difference in the level of abstraction between structural and operational conceptions of a mathematical concept, the two are not mutually exclusive (Sfard, 1991) since a mathematical concept has both structural and operational aspects. In general, Sfard(1991) states that operational conceptions occur before structural conceptions but this is not always the case (in geometry, the reverse is possible). Sfard (1991) argues that the ability of perceiving a mathematical concept as both a process and an object is key in developing a deep understanding of mathematics. Sfard (1991) states that structural and operational conceptions of mathematical concepts complement each
other and both are essential for the development of conceptual understanding of mathematics. Sfard (1991) notes that structural conceptions are supported with mental images, this visualisation facilitates a holistic consideration of the concept, with the conception of these ideas as almost physical objects, allowing for observations from multiple perspectives.

Sfard (1991) discusses three phases of abstraction in progressing from operational to structural conceptions of a mathematical concept: interiorisation, condensation and reification. That is a process is performed/action made on a mathematical object which one is familiar with, repetitions of these actions/processes then leads to recognition of the process as an independent entity in itself, which advances further to the ability to think of this process as an “integrated whole object-like” structure (Sfard, 1991. p.18).

These phases are described by Sfard (1991) as follows:

- **interiorization:** learner becomes accustomed with the processes which leads to proficiency in the use of these processes → ability to perform the process mentally and to consider, analyse and compare the process without actually carrying out the process. That is the ability to hold a mental representation of the process in one’s mind and think about it;

- **condensation:** condensing/shortening of the process so that the sequence of steps which up to this point would have had to be conducted to complete the process can now be carried out in fewer computations (thought of from the initial input to product of process without the linking steps between), allowing the learner to combine the process with other computational operations. The learner still conceives of the condensed version of the process as a process and not an object at this stage. The more the learner is able to use a condensed version of a process without having to consider its particular values, the more advanced they are in the condensation phase;

- **Reification:** the learner conceives the condensed process as an object. The familiar condensed process is now seen as an object (rather than being connected to a process) which can be further operated on. Reification involves seeing something familiar from an entirely new point of view. It is the “sudden ability to see something familiar in a totally new light” (Sfard, 1991 p.19). This new perception of the condensed process as an object allows the learner to conceive the object separately from the process from which it originated. This understanding facilitates the study of the object’s properties and its various representations in relation to it being an element of a certain category, rather than it being the result of some process. This conception in turn enhances the range of possible problem situations to which this object may be applied by the individual.

As stated, both operational and structural conceptions are necessary in order to understand and ‘do’ mathematics. However, while it is possible to work well
in mathematics with an operational understanding, this type of understanding places greater demands on the human working memory than conceiving of the mathematics in a structural way (Sfard, 1991). The processing of operational information is not easily assimilated and is more susceptible to being stored in a disjoint (or very weakly connected) manner in the mind than conceiving the information in a structural way. Attempting to solve a complex mathematical problem from a purely operational approach places excessive strain on an individual's working memory (Sfard, 1991). This is due to the organisation and storage of information in such a schema being inadequate in allowing for the efficient retrieval and use of information, as well as being ineffective in its capacity to add new information to stored information in a coherent and connected way (Sfard, 1991). Sfard (1991) states it is usually more productive to think of the structural conceptions of the mathematical information one possesses when solving a complex problem rather than the operational conceptions. This is also seen in the ways an expert problem solver focuses on the structural components of the problem rather than the novice problem solvers who place emphasis on superficial information in the problem (Glaser, 1992; Simon, 1990). Reification results in the conception of processes as objects. Reification compresses the operational information to reorganise the cognitive schema of objects and processes into a hierarchial tree like structure (Sfard, 1991). These objects can be utilised by an individual to conceive of processes (without having to complete them) or conceive of an object without thinking of the process which produced it, at various phases of engagement in problem solving in mathematics. These representations can help a learner in conceiving new information as well as optimising the precision and rate of retrieval and storage of mathematical information. Sfard (1991) states that problem solving in mathematics consists of a complex interactive process between operational and structural representations of the same mathematical concept, in an attempt at optimising the proficient use of one's available mathematical knowledge. She notes that at certain points in one's learning, inability to conceive of an item structurally may impede further growth in understanding due to an operational conception reaching its saturation point in relation to the coherent addition of new knowledge.

Sfard notes the co-dependent relationship between reification of a particular process to an object and the interiorisation of the processes which can now be applied to the 'new' object as being a reason why many students, despite having completed their mathematics education at secondary level, view the mathematics they learned at school as a set of arbitrary rules to be followed in order to arrive at the correct answer. This co-dependent relationship between knowing and being skilled in the operations well enough for them to be interiorised and reificated as an object and understanding the object so that the operations on them are meaningful to the learner, is an on going issue for mathematics educators. Trying to teach mathematics to facilitate deep structural understanding, while also developing skills in proficiency with operations, is comparable to the chicken and egg scenario; which came first? What should be developed first, since the development of a skill is closely entwined to understanding the con-
cept underlying the skill (Carpenter et al., 1980 cited in Sfard, 1991). Sfard (1991) notes that this complex co-dependency may result in the structural conception lagging behind the operational conception at times during the learning of mathematical concepts. This means that a learner may have to contend with the knowledge that their understanding is superficial, without being aware of the fact that this situation is temporary for most people. The whole understanding of concepts may not come together until some time after they have been studied. This delay in reification may be detrimental to the mathematical development of some students as they may attribute this delay to their own inability to understand mathematics and/or form the belief that mathematics is a subject they will never understand. This delay in reification is similar to the way in which individuals working at different levels in the van - Hiele theory have an inability to understand each other, except that this mismatch between operational and structural understanding goes on in the mind of the individual, potentially leading to feelings of uneasiness with respect to their self satisfaction with their degree of understanding of the mathematics (Sfard, 1991). A summary of operational and structural conceptions is presented in Table 7.

<table>
<thead>
<tr>
<th></th>
<th>Operational conception</th>
<th>Structural conception</th>
</tr>
</thead>
<tbody>
<tr>
<td>General characteristic</td>
<td>a mathematical entity is conceived as a product of a certain process or is identified with the process itself</td>
<td>a mathematical entity is conceived as a static structure - as if it was a real object</td>
</tr>
<tr>
<td>Internal representations</td>
<td>is supported by verbal representation</td>
<td>is supported by visual imagery</td>
</tr>
<tr>
<td>Its place in concept development</td>
<td>develops at the first stages of concept formation</td>
<td>evolves from the operational conception</td>
</tr>
<tr>
<td>Its role in cognitive processes</td>
<td>is necessary, but not sufficient for effective problem solving and learning</td>
<td>facilitates all the cognitive processes (learning, problem solving)</td>
</tr>
</tbody>
</table>

Table 7: Summary of operational and structural conceptions (Sfard, 1991 p.33)

As described by Sfard (Table 7), verbal and visual representations and imagery support the development of operational and structural conceptions of mathematical concepts. Since both of these conceptions are necessary for effective problem solving, both of these types of stimuli should therefore be used in in-
structural methods to aid in the development of mathematical thinking and problem solving ability among students.

Sfard (1994) notes that the perception of algebraic symbols present/applied in a problem situation is dependent on:

- the requirements of the problem to which the algebraic symbols are applied;
- the perception ability of the individual solving the problem;
- what one is able to notice/ give attention to.

Sfard (1994) notes the agreement reached by several researchers of this flexibility of knowledge in switching between operation and structural interpretations as being a sign of algebraic competence (Mason 1989; Gray and Tall 1991; Moschkovich et al., 1992). Sfard (1994) states that this flexibility of perspective of the symbols is a function of two variables: the versatility of students' thinking in relation to their currently available interpretations of a particular mathematical entity depending on the context in which it is presented and the adaptability of students' thinking to employ different perspectives, depending on the requirements of the problem situation. Sfard (1994) notes that the operational conception determines the actions which are undertaken to solve the problem, while the structural conception results in compression of the information with an expanded view of the problem situation. Sfard (1994) notes the consistent display by students of an inability to solve problems which are not solvable by the algorithms they have previously seen. This is often the result of an inability to perceive the underlying process as an object (reification does not occur). The student is thus unable to imagine the “intangible entities” and uses a particular picture and/or symbol to represent all types of a particular “intangible entity” (Sfard, 1994 p.117). This type of conception, called by Sfard “pseudostructural”, is when a student conceives of a sign as the actual object it is signifying. The student’s thinking is inflexible and their understanding is instrumental rather than relational in nature. The inability to form a structural conception through reification results in the student’s new knowledge being completely separate from its operational foundation and so an inflexible mechanical approach to solving problems is utilised. Without being challenged (to justify reasons or explain problem and/or solution), a student may adopt this perspective permanently and cease to try and understand what they are actually doing and the reason why this approach would resolve the problem. Sfard notes that to combat pseudostructural conceptions, students need to be motivated to engage in seeking and formulating meaning at every phase in the learning process, regardless of how difficult this may be.

2.9.4 The van Hiele Model of the development of abstraction in understanding mathematics (levels of thinking in geometry)

van Hiele (2002) cited in Collignates (2014) notes that three levels of thinking occur in most subjects: visual, descriptive and theoretical.
Pierre Marie van Hiele and Dina van Hiele Geldof (1984c) formulated a model of the development of mathematical thought. This was in response to noticing their students’ difficulties in gaining an understanding of particular areas within mathematics persist. These difficulties consistently presented themselves repeatedly among many classes of students over several years despite providing numerous refined explanations. It seemed to the van Hieles as though they were speaking a different language to that of their students (van Hiele, 1984c). This led to them considering the different levels of thinking in mathematics.

The van Hiele model was inspired by Piaget’s idea of levels of understanding in mathematics, however van Hiele’s model is based on the idea that advancing from one level of thinking to the next is dependent more on experience gained and teaching received (akin to apprenticeship) by students than on their age (van Hiele, 1984c). The model of the development of mathematical thought developed by van Hiele uses geometry as the subject area to base the model on, however the model is not just restricted to the development of geometric thought. van Hiele studied mathematical insight in general, while placing a particular emphasis on the development of insight in geometry (van Hiele 1984c). The van Hiele model is the basis of Tall’s (2013) framework of mathematical thinking. The model has also been adapted and extended to model the development of mathematical thinking (learning) in other subject areas within mathematics, such as trigonometry (Walsh, 2015) and analysis (Isoda, 1996) and also to subjects outside mathematics such as computer science (Wey Chen, 2005).

The van Hiele model of thinking levels emphasises the importance of the development of mathematical insight. Berghuys (1952, cited in van Hiele (1984a; 1984c) discusses the concept of mathematical insight and the processes which develops the mind to a sufficient level for the instigation of mathematical insight to occur. Berghuys (1952 in van Hiele 1984c, p.45) states that “sensory perception images” can have multiple subtle differences in meaning associated with them, depending on the individual and the situation in which this perception was acquired. Berghuys notes that the human mind needs to link these perceptions together into simpler arrangements so as to make sense of them. The multiple perceived meanings associated with these various mathematical objects/ideas/concepts “can be made simpler by focusing only on some specific phenomena” of the mathematical object/idea/concept at particular times while ignoring other parts of the phenomena that are not significant to a particular problem situation at those particular times (Berghuys 1952, cited in van Hiele 1984a p.45). Berghuys (1952, cited in van Hiele 1984a p.46) refers to this focused attention as “conscious attention given to the scheme”, that is attention is focused on how mathematical items are linked rather than what is being linked. Berghuys states that this conscious attention is how mathematics is brought

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15“image” does not refer only to visual image, all sensory perceptions are included.
to life and also that the mathematical understanding of individuals progresses from their “sensory perception images to form mathematical systems” (1952, cited in van Hiele 1984a p.45).

Mathematical insight as defined by Berghuys is composed of two parts of knowledge; empirical and intellectual, that is knowledge of the mathematical data gained through and verified by observation or experience and also knowledge of the mind which has the ability to concentrate on the mathematical objects/ideas/concepts for sufficient time in order to fully understand the mathematics (van Hiele 1984a). This definition of mathematical insight involves experiential, theoretical and metacognitive knowledge. The evidence that a student has gained mathematical insight is in the appropriate application of mathematical knowledge and skills to an unfamiliar situation (van Hiele 1984a). Van Hiele states that inference of this indicator of mathematical insight can only be made if the application of the appropriate mathematics to resolve the unfamiliar situation was the result of “deliberate intent” (van Hiele 1984c p.242), i.e. no chance was involved in the application of the appropriate mathematics.

In van Hiele’s work (1984c) the idea of constructing networks of associations is fundamental in gaining an understanding of the subject of mathematics. The stronger the network of association of mathematical components (in relation to a structure), the more flexible the student’s ability is to generalise to extended structures, specialise to study components of the structure, recognise and study similar components from another structure and compare to the original structure and also to compare similar structures (van Hiele, 1984a). In this way, the formulation of associations of mathematical concepts and ideas facilitates more sophisticated thinking and increased flexibility in the application of knowledge in solving problems.

The development of an organised, interconnected sensible network of relationships is dependent on the process of learning experienced. van Hiele states that this development of networks of relationships is the aim of mathematical education. This network should be formed from concrete to abstract such that it allows for the development of a network which permits reversibility from the abstract to the concrete by the students themselves. This type of an established network is necessary for the productive application of mathematical knowledge (van Hiele, 1984c).

The van Hiele model consists of five levels, each of which outlines the degree of progression of thought in relation to geometry (van Hiele, 1984a; Usiskin, 1982). Advancement through the thinking levels in the model occurs as one’s thought process in relation to the geometric figures/properties/ideas/concepts and hence understanding of geometry as a system expands, facilitating increased sophistication in one’s degree of abstraction (in the area of geometry). The levels progress from sensory perception to abstract thinking. The hierarchical nature of the levels in the van Hiele model of the progression of mathematical thought
reflect increasing levels of ability to specialise and generalise, with higher degrees of sophistication and accuracy in the flexibility and reversibility of the thought process. Krutetskii’s (1976) research on mathematical abilities concurs with the van Hiele levels of thought. Krutetskii (1976) found that more capable students in mathematics have:

- greater ability in generalising mathematical information;
- higher degrees of flexibility in the provision of solutions to problems, being able to switch to different methods when conventional methods fail and being able to reconstruct previously known thought patterns;
- greater ability to recognise reversible associations if presented with problems that are essentially the opposite of previously met problems.

The van Hiele model “of mathematical reasoning has become a proved descriptor of the progress of students’ reasoning in geometry and is a valid framework for the design of teaching sequences in school geometry” (Jaime and Gutierrez, 1995, p. 592). Reasoning is defined as the process of thinking about something in a logical way in order to form a conclusion or judgement, while thinking is defined as the action of using one’s mind to produce thought (Merriam-Webster Dictionary). Therefore mathematical reasoning encompasses mathematical thinking and the van Hiele model has potential for use in modelling one’s growth of mathematical thinking and reasoning in respect to problem solving.

The development of thought levels of pupils at each of the van Hiele levels of thinking (with respect to geometry) is as follows:

**Level 0:**
Sensory experiences facilitate the recognition of the forms by their shapes and so names can now be assigned to the various types of forms according to their shape (van Hiele, 1986). The sensory experiences have facilitated the initiation of geometric thinking. This type of thinking is described by Dina van Hiele (1986, p.230), as a thinking which is leaning towards the structure and properties of the forms, in addition to being a thinking which is focused towards the awareness of “the structure of thinking itself in order to arrive at the structure of thinking” (comparative, symbolic and logical). However, at this level, only certain aspects of comparative thinking are achieved by students (recognition and classification). At this stage, classification of the shapes depends solely on appearance. Students classify by comparative thinking based on recognition of likeness and difference. A shape that is oriented differently to another of the same type may not result in the two being placed into the same classification. This is due to recognition being based solely on appearance, not the properties which define that actual shape as students have yet to fully notice, think about and understand the properties (Burger and Shaughnessy, 1986).

**Level 1:**
To progress further into the structure of geometric thinking from the initiation
in level 0, language needs to be used to represent the thinking structure that students need to develop. The methodologies utilised at this level to advance students’ learning involves the introduction of the geometric symbol “ to follow from” through the use of a “genealogical tree” (van Hiele, 1986, p.237). This is a branching chart showing the lines of advancement from the foundation of geometry and the relationships between certain components in the structure. This is essentially an explicit representation of the implicit thought structure which one aims to make explicit. Transition from level 0 to level 1 involves the reconstruction of existing concepts and recognition of certain relationships between these concepts (De Villiers, 2010). Students at this level of thought are capable of simple deductions in order to classify the structure of the forms based on their properties. By classifying the shapes according to their properties, students are now applying geometric thinking. As the simple deductive process becomes an habitual way of thinking, the thought process becomes reversible and associative (van Hiele, 1986). van Hiele characterises the geometric thinking at this level as thinking which:

- uncovers the structure of geometry;
- leads to the development of insight of the forms;
- further develops the initiation of the geometric thinking experienced at level 0.

At this level, as a result of the deductive method being applied in order to classify the shapes, mathematical thinking now becomes an acquired process in the students’ minds (van Hiele, 1986). At this level, there is an “explicit lack of understanding of the concept of proof” (Burger and Shaughnessy, 1986 p.44).

Level 2:
At level 2, the structure of mathematics is realised by students (van Hiele, 1986). Insight into geometry further develops as the geometric thinking of students is now more established as a way of thinking. There is also initiation into the use of logic as a way of thinking (van Hiele, 1986). As a result of the learning experienced at levels 0 and 1, students have developed associations of figures and concepts in their mind which have a logical order. This conceptual structure built in their mind allows students at level 2 to explicitly use ‘if, then’ conditional statements in situations including solving problems. Students at this level can also “form correct informal deductive arguments with implicit use of logic such as the chain (‘if and if, then’) statements (Burger and Shaughnessy, 1986 p.44). The logical relationships between the properties are part of the network of relations at level 2 (at level 1, the network of relations between the properties is associative (De Villiers, 2010)). There is an inability to differentiate correctly between the meanings (roles) of the terms used in formal proofs such as ‘axiom’ and ‘theorem’ (Burger and Shaughnessy, 1986). Students at level 2 can follow a formal proof as it is presented to them but cannot develop a proof on their own, unless it is already familiar to them or starts from a familiar point (Pandiscio
Level 3:
At level 3, the structure of logic can be arrived at through ‘exact thinking’. Insight into mathematics is developed as a result of the development of logical thinking (van Hiele, 1986). Students understand theorems and have the ability to begin proofs with unfamiliar starting points (Pandiscio and Knight, 2010). Students understand the meaning of “necessary and sufficient conditions” (Mason, 2002 p.4). Students can rephrase mathematical problems and clarify vague questions precisely. Mathematical proof is the only acceptable form of establishing irrefutable truth of a mathematical proposition by students at this level (Burger and Shaughnessy, 1986). Students form their own conjectures and attempt to prove them by deduction, Euclid’s postulates are accepted by students at this level (Burger and Shaughnessy, 1986 p.45). Students at this level also understand the roles of the terms axiom, proof, definitions and theorems (Burger and Shaughnessy, 1986 p.45).

Level 4:
Insight into logic can be achieved at level 4, with a person operating on this level having the ability to arrive at the structure of exact thinking, that is that the mind is at the background of the mathematical analysis (van Hiele, 1986). The thinking processes of the individual mathematical thinker become objects of analysis for this same individual, now thinking as a logician at level 4. There is awareness and utility of the psychology of thought, as well as the mathematical thinking process driven by the psychological thinking of the mathematical thinker (in oneself), by individuals at this level of thinking (van Hiele 1986). At this level, individuals have complete understanding and use of formal deduction to compare different mathematical systems and form mathematical systems (Mason, 2002). Mason (2002) also notes that understanding and use of proof by contrapositive can be performed and non-Euclidean systems can be understood by individuals at this level.

Level four is the ultimate thinking level, this thinking level is generally not reached at secondary level. It is more usually formed at tertiary level. The names assigned to each level of thought is shown in Figure 13 (sourced at http://dungeongeometry.tumblr.com/).
A significant aspect of the van Hiele theory is the provision of “conceptual structuring” of mathematical information in the minds of learners (De Villiers, 2010, p.11). De Villiers (2010, p.11) notes that the “informal activities at levels 0 and 1 should provide appropriate conceptual substructures for the formal activities at the higher levels”. There are five sequential phases which a learner must pass through (in an apprenticeship type setting) in order to advance from one level to the next level. These five levels are (van Hiele, 1986):

- **inquiry/information**: during the information/inquiry phase pupils become familiar with the topic to be studied. Conversations between the teacher and the pupils along with activities engaged in by the pupils allows the teacher to gain knowledge about the students’ current knowledge and capability of and in this topic. These conversations and activities also facilitate the students’ discovery of the aspect of mathematics they are about to develop further and gives them an idea of how this progression of development will be structured;

- **directed orientation**: during this phase, students explore the topic through engagement with presented material. The intention of the presented material is to facilitate the gradual coherent discovery by the students, of structures within the topic. The teacher should be able to explicitly see the understanding of the students from their elicited response to the material presented. The presented material should also allow the students to discover more and more of the structures of the topic;

- **explication**: students relate the knowledge and experience they have gained to the mathematical terminology used to describe this knowledge and ex-
experience. Students express their opinions in correct mathematical language, the teacher oversees this and insists upon it. Systems of relations are formed during this phase;

- **free orientation**: The topic of investigation is mostly known to the students now but they must be able to construct this knowledge for themselves at a convenient rate. Problems which can be completed in different ways should be presented to the students during this phase. These allow the opportunity for the students to find their own solutions and choose the appropriate combination of knowledge and experience from those they currently possess;

- **integration**: during this phase the student needs to consolidate their learning, that is to condense the knowledge, experience, methods and representations which they have gained so that they fit into a coherent structure. During this stage, learning is summarised. No new information should be presented by the teacher. A teacher may help at this stage in trying to help in the organisation of the summarised information but each student may have their own individual perception of how the information makes sense to them.

At the end of the integration phase, the students now possess a network of relations which is related to the entire topic they have just studied. This network replaces the previous network which occurred at the prior mathematical thinking level, and the new current network will be replaced by network which is formed at the end of the integration phase of the subsequent mathematical thinking level. Knowledge is reconstructed to reflect the higher level of sophistication of the mathematical thinking of the learner as they advance through the thinking levels (van Hiele, 1986).

Through the use of the learning phases outlined by Dina and Pierre van Hiele and an awareness of the current thinking levels of students, a carefully designed instructional program can be provided to the students to facilitate progression of the construction of a logical neuronal network of relationships between mathematical concepts.

The description of the thinking levels at each level in the van Hiele model provided by Dina van Hiele (1986 p.238) leads to the following synopsis. There is progression in thinking at each of the levels from:

- **Level 0**: aspect of geometry → initiation of geometrical thinking;

- **Level 1**: essence of geometry → evolution to structure of geometric thinking → mathematical thinking because of use of deduction;

- **Level 2**: insight into geometry evolves → evolution to structure of mathematical thinking → initiation of logical thinking → aspect of logic;
• Level 3: exact thinking → insight into mathematics evolves → structure of logic;

• Level 4: insight into the subject of logic evolves.

Replacing the term ‘geometry’ with a different subject area within mathematics, using APOS theory to discover the genetic make up of this subject area in mathematics and the teaching phases of the van Hiele model adapted for the new subject area, should in theory produce similar progression in the mathematical thinking levels of students in relation to the new subject area. This theoretical notion has been verified practically and theoretically by Walsh (2015) in the subject area of trigonometry.

2.9.5 Comparisons between models and choice and rationale for choice of model/theory to base framework on

Understanding mathematics develops by increasing sophistication of mathematical thinking (Tall, 2013a p.2). The degree to which individuals can:

• conceive of mathematical concepts in their minds;

• engage in the mathematical thinking processes of generalising, reversing (ability to think in both the forward and reverse direction in relation to the mental processes involved in mathematical thinking and reasoning) and reflection (justification, explanation, integration, summarising) of mathematical objects, relations and operations;

• be flexible (adaptable) in the cognitive processes in mathematical problem solving situations

has an effect on how well they perform in problem solving situations. The mathematical thinking processes of generalising, reversing and reflection, as well as flexibility in mathematical thinking relative to the knowledge one has are common factors in the theories and models of mathematical understanding discussed in section 2.9 (see Table 8).
Knowledge forms the foundation of competency (Voskoglou, 2011) with mathematical thinking being essential in the formation of mathematical proficiency. It is the coherency of the organisation of this mathematical knowledge in one’s mind, along with the ease of accessibility to it that is vital in competent mathematical problem solving (Voskoglou, 2011). Mathematical thinking is required in order to understand and utilise this mathematical knowledge. Tall’s framework of mathematical thinking is based on several theoretical concepts (Tall, 2008) including research findings on the levels of the development of mathematical thinking from the van Hieles, and models of cognitive activity in APOS theory, by Dubinsky, forming its foundation. Tall’s theory incorporates most of the common mathematical thinking processes present in the models and theories of mathematical understanding discussed in section 2.9 (Table 8). The ‘conceptual structuring’ provided by the van Hiele theory (De Villiers, 2010 p.11), incorporated in Tall’s framework aids in the choice and sequencing of appropriate mathematical problems. The genetic decomposition involved in APOS theory is also useful in the sequencing of the mathematical problems presented to students. Tall’s Framework for Mathematical Thinking is central in the Framework for Teaching and Assessing Problem Solving in Mathematics formulated by the author (chapter 4). The mathematical thinking required for the development of mathematical proficiency is the deciding factor in the choice of mathematical tasks, as well as playing a role in linking the mathematical knowledge and intrinsic motivation of the students, in addition to driving instruction in mathematical problem solving.

<table>
<thead>
<tr>
<th></th>
<th>APOS</th>
<th>Sfard</th>
<th>van Hiele</th>
<th>Krutetskii</th>
<th>Tall</th>
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Table 8: Comparison of Theories and Models of Mathematical Understanding
Evident from the discussion on models and theories of mathematical understanding, is the necessity of the co-development of mathematical concepts and processes in the progression of problem solving proficiency in mathematics. Research conducted by Foster et al. (2014) revealed that most classroom practices are focused around concept development, while also identifying that the confines of teachers’ knowledge and understanding of the processes involved in problem solving are potentially adversely affecting students’ learning. These findings by Foster et al. (2014) suggest that teachers’ knowledge and understanding of mathematical problem solving (processes of mathematical thinking in particular) as well as their knowledge and understanding of mathematical problems and the factors affecting students’ and teachers’ ability to engage in problem solving need to be improved. These knowledge and affective factors have also been discussed in Chapter 2 with regard to the affect these factors have on students’ ability to engage in mathematical problem solving. These knowledge and intrinsic motivation factors feed into the central mathematical thinking component of the author’s framework with the overall aim being to improve mathematical proficiency in problem solving situations.

2.10 Conclusion

The review of the literature revealed the interconnected nature of the knowledge required for the teaching and assessing of problem solving in mathematics. Chapman (2015) asserts that teachers’ knowledge of what it means to be proficient in problem solving in addition to their knowledge of how to teach in order to develop problem solving proficiency must include more extensive knowledge than their own knowledge of and ability in solving mathematics problems. The review of the literature discussed in this chapter concurs with Chapman (2015). The author found that mathematical problem solving is a complex phenomenon consisting of an interconnected web of components, which interact with each other in various combinations. The literature reviewed facilitated the study of this complex nature by researching problem solving in mathematics in the completeness of its complexity (Waldrop, 1992 in Grootenboer, 2010). The author identified the following as essential components affecting the teaching and assessing of problem solving in mathematics:

- Knowledge of what a problem is and what is meant by problem solving;
- Knowledge of the current performance of both students and teachers in problem solving;
- Knowledge of the knowledge that both teachers and students require in order to teach and learn problem solving;
  - Conceptual understanding;
  - Relational and instrumental understanding;
  - Deductive and inductive reasoning (and mathematical thinking);
- Strategic knowledge - problem solving models and heuristics;
- Knowledge of the factors affecting students’ ability to engage in problem solving:
  - Mathematical thinking, Metacognition, and transfer;
  - Nature of the problems;
  - Classroom environment;
  - Mindset, beliefs and attitudes;
  - Interest and perseverance;
  - Intuition, transfer and time.
- Knowledge of how experts solve problems;
- Knowledge of instructional practices;
  - how methods of instruction facilitate the development of conceptual understanding and problem solving in mathematics.
- Knowledge of the existing models and theories of developing mathematical understanding.

As discussed in section 2.4.7 Poincaré (1952) states that the fundamental goal of mathematics education is to develop capacities of the mind. A fundamental component of problem solving in mathematics is learning to ask the right questions.

“One of the hardest parts of problem solving is to ask the right questions and the only way to learn to do so is to practice”.

(Halmos, 1980, p.524)

Halmos notes that students of mathematics should be engaged in asking questions and solving problems. Engaging in problem solving is the only means of acquiring the ability to ask the right questions (Halmos, 1980). Like many endeavors in life, there is no substitute for engaging in the practice of the art you are trying to master (Halmos, 1980). The teacher should model the use of questioning by making their own internal thinking visible to their students. The use of metacognitive journals may also be useful in developing students’ ability to ask the right questions (McCrisdle and Christensen, 1995).

The approach to teaching problem solving that Halmos et al. (1975, p.467) advocates is the “Moore Method” or “Modified Moore Method”. This is a method of teaching which is a “method of creating the problem-solving attitude in a student, that is a mixture of what Socrates taught us and the fiercely competitive spirit of the Olympic games” (Halmos et al., 1975, p.467). The problems that should be used in teaching are problems that are written in simple language,
stimulate interest, are not trivial to solve, and the process of determining the solution requires the use of mathematics and mathematical thinking that encompass “all the important ideas” of mathematics (Halmos et al., 1975, p.467).

Attempts at improving problem solving in mathematics education require the focus of attention to be on the education of the pre-service mathematics teacher (Zimmermann, 2016), with alignment existing between the pre-service teachers’ education and students’ education (Zimmermann, 2016). Part of the pre-service mathematics teachers’ education should include the solving and posing of problems that are similar to, and some more challenging than those they will be teaching to their students (Chapman, 2015; Zimmermann, 2016).

“The best way to learn is to do - to ask, and to do. The best way to teach is to make students ask, and do. Don’t preach facts stimulate acts. The best way to teach teachers is to make them ask and do what they, in turn, will make their students ask and do”.

(Halmos, 1980, p.524)

The author’s framework for teaching and assessing problem solving in mathematics (F-TAPS) is presented in chapter 4. The author’s F-TAPS is used in conjunction with the Project Maths syllabi and the type of problems advocated by Halmos (1975) to formulate a problem solving pre and post assessment (chapter 4). The use of the advocated Modified Moore Method is further integrated with the author’s F-TAPS in the development and implementation of an intervention with pre-service secondary school mathematics teachers (chapter 6). The next chapter discusses the methodology for this research study.
3 Methodology

3.1 Introduction

This chapter outlines the way in which the research in this study was undertaken. Descriptions of the research paradigms which form the foundation of this research, along with descriptions of the theoretical framework and the methods of data collection and analysis are included in this section.

Research is “the systematic, controlled, empirical and critical” investigative process employed to collect and analyse information (Kerlinger cited in Cohen et al. 2011 p.4). This process aims to increase, validate and/or revise existing knowledge and understanding in order to obtain valid and reliable answers to questions about particular phenomena of interest and their associated relationships (Kerlinger, 1970, cited in Cohen et al., 2011). Formal research generally consists of a combination of experience and reasoning (both inductive and deductive), in the exploratory analysis of information to discover the truth about “hypothetical propositions about the presumed relations among natural phenomena”(Kerlinger, 1970 cited in Cohen et al., 2011 p.4).

High quality research is:

- logical - research is conducted based on valid procedures and principles, utilizing both inductive and deductive reasoning;
- methodological - systematic methods and procedures are conducted free from bias;
- critical - careful and precise judgement is evident throughout the research process; item analytical - proven analytical procedures are employed in the gathering of data;
- replicable - the results obtained by the research should be verifiable by replicating the study;
- cyclical - the process begins with addressing a problem through research and ends with suggestions for further research.

(Kothari, 2004 pp. 20-21)

3.2 Assumptions and Research Paradigms

A paradigm is the underlying intellectual architecture (including assumptions and beliefs) upon which research and development in a field of study is based (Kuhn, 1970). Paradigms facilitate general consensus in the standards of practice expected by individuals conducting research in a particular field of study (Kuhn, 1970). Paradigms facilitate the categorising of beliefs and perspectives (Hughes, 1997). The beliefs and assumptions underlying the intellectual architecture are both ontological and epistemological in nature. The research
paradigm employed in a study has implications for the methodology implemented in the study. The philosophical assumptions underlying this research, along with the corresponding appropriate research methods employed in the development of knowledge in this study are discussed in this section.

Ontological assumptions are assumptions about the nature of the phenomenon being studied (Cohen et al., 2011), i.e. what is true about the subject matter being studied. Schoenfeld (2016) states that individuals develop their understanding of what mathematics is from the experience they encounter with it. This experience predominantly occurs in the classroom (Schoenfeld, 2016). Ernest (2012 p.12) states that the main objects of study in mathematics research are “human beings, their activities and their relationships”. Heidegger (1962) cited in Ernest (2012 p.12), states that one’s understanding of oneself (one’s identity) is something which cannot be fully “articulated”. In this research, problem solving in mathematics is being investigated. An individual’s understanding of what problem solving in mathematics is, is to a large extent determined by their experience with mathematical problem solving. Problem solving is, as defined by Lester and Kehle (2003, cited in Lester, 2013):

‘the coordination of previous experiences, existing knowledge, familiar representations, patterns of inference and intuition in an effort to generate new representations and related patterns of inference that resolve some unknown, which was the basis for the problem solving cognitive activity’.

(Lester 2013, p.249)

The inclusion of ‘previous experience’ in this definition, means that an individual’s understanding of what problem solving is, is dependent on their past experience. Thus to some extent the individual’s understanding of the ‘reality’ of problem solving is influenced by their teacher (Shavelson, 1983; Fennema, 1992; Ball et al., 2008), this is the ontological assumption for this research.

Epistemological assumptions are assumptions about how knowledge of the phenomenon being studied is acquired, communicated and disseminated (Cohen et al., 2011; Stanford dictionary of philosophy, 2009), i.e. how one can figure out the truth about the subject matter being studied. The factors that affect one’s ability to engage in problem solving, along with the teaching, learning and assessment of problem solving in mathematics is examined in this research. There are two main epistemological paradigms employed in educational research: the paradigm of positivism and the interpretivist paradigm (Cohen et al. 2011). Another recent paradigm in the social sciences is the paradigm of complexity theory. These paradigms will now be briefly discussed.

- Positivism - An individual with a positivist view perceives knowledge as an objective entity and utilises observation and measurement in their endeavors to research a particular topic (Weber, 2004). Positivists believe in
employing the scientific method to gain knowledge of reality. They view
reality as being mutually exclusive from the individual who observes it
(Cohen et al. 2011). Positivists use quantitative methods in their search
for the truth of the phenomenon they are studying.

- Interpretivism - An interpretivist perceives reality as being linked to
the individual who observes it. This viewpoint has both objective and subjec-
tive characteristics; subjective in that an individual’s perspective shapes
their perception of the phenomenon being studied and objective in that
an interpretivist perspective facilities an “intersubjective reality” (Weber,
2004 p.v). Interpretivists view knowledge as being socially constructed by
the individual and the activities that the individual engages in, in gain-
ing knowledge, as being inextricably linked to the individual’s experiences
and perspectives of reality (Cohen et al., 2011). Interpretivists mainly use
qualitative methods in their search for the truth of the phenomenon they
are studying (Cohen et al., 2011).

- Complexity Theory - An individual who adopts this theory perceives re-
ality as not being only compatible with “simple cause and effect models”
(Cohen et al., 2007). This theory looks at using holistic approaches to
gain understanding of phenomena rather than using atomistic approaches
(Cohen et al., 2007). Phenomena are viewed as consisting of an inter-
connected web of components which interact with each other in varying
combinations within the phenomenon and also to the phenomenon itself.
These combinations of relationships constitute the complex system and
how the system behaves/responds to an environment needs to be studied
in the completeness of its complexity as to study it alternatively would
diminish the meaning of the findings (Waldrop, 1992 in Grootenboer, 2010).
For example, the phenomenon of the brain consists of a complex combi-
nation of billions of neurons, but the phenomenon of the mind “emerges”
(Mason, 2014 p.1) as much more than just the sum total of these combi-
nations of neurons. Complexity theory is useful in providing insight into
what type of intervention is most likely to be sustained (Manson, 2014).
This theory may use several methods to study complex phenomena (Davis
and Sumara, 2006).

The main differences between the positivist and interpretivist perspectives are
outlined in Table 9. The interpretivist’s perspective considers that humans re-
respond to a stimulus in different ways depending on their interpretation and
experience. This view takes the thinking and construction of meaning by indi-
viduals into account in the construction of knowledge. Mathematics education is
generally viewed as the construction of knowledge by individuals, where both the
individual and the social environment play a role (Steinbring, 2000). The epistemological stance of the researcher is from an interpretivist’s perspective. The
researcher also makes use of complexity theory in analysing the phenomenon of
problem solving to provide an intervention which would likely lead to sustained
change among the participants, and in turn may contribute to the transfer of
Metatheoretical Assumptions About Positivism Interpretivism

Ontology Person (researcher) and reality are separate. Person (researcher) and reality are inseparable (life-world).

Epistemology Objective reality exists beyond the human mind. Knowledge of the world is intentionally constituted through a person’s lived experience.

Research Object Research object has inherent qualities that exist independently of the researcher. Research object is interpreted in light of meaning structure of person’s (researcher’s) lived experience.

Method Quantitative: Statistics, content analysis Mainly qualitative: Hermeneutics, phenomenology, etc.

Theory of Truth Correspondence theory of truth: one-to-one mapping between research statements and reality Truth as intentional fulfillment: interpretations of research object match lived experience of object.

Validity Certainty: data truly measures reality Defensible knowledge claims

Reliability Replicability: research results can be reproduced Interpretive awareness: researchers recognise and address implications of their subjectivity

Table 9: Differences between Positivism and Interpretivism (Sandberg cited in Weber, 2004, p.iv)

this change into school practice. Complexity theory suggests the need for “case study methodology, qualitative research methods based on the interactionist and interpretative accounts” (Cohen et al., 2007 p.30).

The advantages and disadvantages of the interpretivist perspective in conducting research are outlined as follows:

- Advantages:
  - It facilitates an understanding and explanation of the phenomenon being studied, along with the research utilised to study it.
  - It allows for consideration of the complexity of the phenomena being studied, and the context in which the research setting takes place.
  - It allows for the consideration of human change over time and allows for responses to these changes by the researcher.
  - It is a good perspective to use in gaining an understanding of social interactions.
• Disadvantages:

– The data may be complex and the data analysis may be quite difficult to complete, both in terms of time and ability of the researcher.

– The researcher needs to be aware that the analysis of the data collected may not lead to the emergence of identifiable patterns.

– The general consensus of individuals (not involved in research) is that this viewpoint of conducting research is less reliable and plausible than the positivist (or other) viewpoints.

Raddon (2010)

The data analysis employed in a research study aligns with the research paradigm of the study. Qualitative and quantitative approaches in the analysis of data are possible, as well as the combination of these two approaches, namely the mixed methods approach.

3.2.1 Qualitative/Quantitative and the Mixed Methods Approach

There are two ways of gathering and analysing data; qualitatively or quantitatively. The approach employed depends on the philosophical underpinnings of the research (Ary et al., 2010). Quantitative research is rooted in positivism, whereas qualitative research is in alignment aligned with the interpretivist paradigm (section 3.2). Both types of data analysis are discussed in this section.

3.2.2 Qualitative Approach

The qualitative approach is focused on gaining a deep understanding of a phenomenon (Ben-Eliyahu, 2014). This approach involves the comprehensive description of a phenomenon. The qualitative approach is concerned with attempting to answer the ‘how’s’ and ‘why’s’ types of questions about phenomena (Craven, 2008). It consists of the organisation and explanation of the data in terms of the participants’ interpretation/understanding of an situation (Cohen et al., 2011). In general, there is a small number of participants (Ben-Eliyahu, 2014) in this type of research. This allows for the collection of detailed rich data (Cohen et al., 2011), which facilitates a deep understanding of the situation. The data is gathered using open-ended questions, focus groups, interviews, observations, documents and reports (Cohen et al., 2011). The data gathered from the participants then has to be interpreted. This interpretation of data in the qualitative approach is aligned with the interpretivist paradigm. Due to the small sample size of participants, the findings from qualitative data cannot be inferred to the population. This type of research can however contribute to a deeper understanding of the research problem, which can be employed to inform theoretical, practical and specific situations (Ben-Eliyahu, 2014). The findings from the qualitative data analysis can inform larger studies which can then
be undertaken to make inferences about the population (Ben-Eliyahu, 2014). Qualitative analysis of the data can allow the researcher to:

- describe, portray and summarize;
- interpret;
- identify patterns and generate themes;
- understand individuals and features specific to them;
- understand groups and general properties/norms of those groups;
- raise issues, prove or demonstrate;
- explain or seek causality (but cannot infer causality to population);
- explore and test;
- discover commonalities, differences and similarities;
- to examine the application and operation of the same issues in different contexts.

(Cohen et al., 2011 p.538)

The qualitative data arising from the research instruments employed can be quite challenging to analyse (Cohen et al., 2011). The analysis involves the selecting and ordering of rich data by the researcher and this may result in personal bias, which the researcher needs to be aware of (Cohen et al., 2011). In addition, the data provided by the participants needs to be interpreted by the researcher, however this can involve the researcher interpreting participants’ interpretations of a situation which involves a “double hermeneutic process” (Giddens, 1976 cited in Cohen et al., 2011 p.540). This may result in a subjective perspective of a situation being presented (Cohen et al., 2011). This subjective nature has implications on validity and reliability factors of the data analysis. To aid in the provision of a more objective view of the situation (increasing validity and reliability), a range of data are examined and external perspectives of the research situation are included in qualitative data analysis, i.e. inter-rater reliability and reviews by expert panels. Also detailed description of the research process is given to aid with replication of the study by others. The advantages and limitations of the qualitative approach are outlined as follows:

- Advantages:
  - Facilitates the identification of new phenomena;
  - Can result in a more comprehensive deep understanding of the phenomenon and its mechanisms;
  - Provides verbal/written information that may (in some situations) allow for conversion of this data to numerical form;
– May reveal information through open-ended questions that might not have been uncovered otherwise.

- Limitations:
  – Does not allow for inference of findings from the sample to the population;
  – Applying statistical methods are challenging;
  – Difficult to assess relations between characteristics;
  – Can be difficult to replicate.

(Ben-Eliyahu, 2014 p.1)

3.2.3 Quantitative Approach

The quantitative approach is focused on describing a phenomenon with respect to a large number of participants. It is often focused on finding evidence to either support or contradict ideas or claims about phenomena (Ben-Eliyahu, 2014). The quantitative approach is concerned with attempting to answer the “what’s” and “how many” types of questions about phenomena (Griffiths, 2008). Quantitative data analysis is a systematic approach to investigating phenomena. It involves the collection of numerical data or the transformation of collected/observed data into numerical data (Griffiths, 2008). This approach applies statistical analysis to the data to identify patterns in the data. The data is gathered using questionnaires, structured interviews, observations/measurements of specified tasks/actions in a controlled environment, records and documents. The statistical analysis in the quantitative approach is in alignment with the positivist paradigm. Due to the large sample size of participants in a sample that is representative of the population, inferences from the findings from quantitative data can be made from the sample to the population. Quantitative analysis of the data can allow the researcher to:

- quantify the problem in order to describe and explain phenomena;
- measure incidences of opinions, attitudes or behaviors of a sample;
- establish causality;
- generalise results from a representative sample;
- hypothesise.

(Sukamolson, 2007 p.9)

The analysis involves the use of statistical tests which results in an objective perspective of a situation being presented (Cohen et al., 2011). This objective nature of quantitative data analysis lends itself to representative, valid and reliable results. It is possible for other researchers to replicate the study in similar situations. The advantages and limitations of the quantitative approach are outlined as follows:
• Advantages:
  – Facilitates the comparison between groups;
  – Allows inferences from the findings of the sample to the population to be made;
  – Facilitates the determination of relationships between variables.

• Limitations:
  – Difficult in recognising new phenomena;
  – Interpretation is difficult without a control group.

(Ben-Eliyahu, 2014 p.1)

One approach is no more important than the other (Cohen et al., 2011). Each has benefits and limitations as outlined (Ben-Eliyahu, 2014). The approach chosen is dependent on the fitness of the approach in achieving the purpose of what the researcher intends to show and in adhering to the epistemological and ontological assumptions related to the particular research (Cohen et al., 2011). The main differences between qualitative and quantitative approaches are summarised in Table 10.
<table>
<thead>
<tr>
<th>Criteria</th>
<th>Qualitative</th>
<th>Quantitative</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Aim</strong></td>
<td>To understand and interpret social interactions.</td>
<td>To test hypothesis, look at cause and effect.</td>
</tr>
<tr>
<td><strong>Sample Size</strong></td>
<td>Smaller and not random selection.</td>
<td>Larger and random selection.</td>
</tr>
<tr>
<td><strong>Variables</strong></td>
<td>Study of the whole.</td>
<td>Specific variables studied.</td>
</tr>
<tr>
<td><strong>Type of Data Collected</strong></td>
<td>Words, images or objects.</td>
<td>Numbers and statistics.</td>
</tr>
<tr>
<td><strong>Form of Data Collected</strong></td>
<td>Qualitative data - open-ended responses, interviews, participant observations, field notes &amp; reflections.</td>
<td>Quantitative data - based on precise measurements, using structures and validated research instruments.</td>
</tr>
<tr>
<td><strong>Type of Data Analysis</strong></td>
<td>Identify patterns, characteristics &amp; themes.</td>
<td>Identify statistical relationships.</td>
</tr>
<tr>
<td><strong>Objectivity and Subjectivity</strong></td>
<td>Subjectivity is expected.</td>
<td>Objectivity is critical.</td>
</tr>
<tr>
<td><strong>Role of the Researcher</strong></td>
<td>Researcher &amp; their biases may be known to participants &amp; participant characteristics may be known to the researcher.</td>
<td>Researcher &amp; their biases are not known to participants &amp; participant characteristics are not made known to the researcher.</td>
</tr>
<tr>
<td><strong>Results</strong></td>
<td>Particular or specialised findings that is less generalisable</td>
<td>Generalisable results that can be applied to other populations</td>
</tr>
<tr>
<td><strong>Scientific Method</strong></td>
<td>Exploratory - the researcher generates theory from the data collected.</td>
<td>Confirmatory - the researcher tests the hypothesis and theory with the data.</td>
</tr>
<tr>
<td><strong>View of Human Behaviour</strong></td>
<td>Dynamic, situational, social, &amp; personal.</td>
<td>Regular &amp; predictable.</td>
</tr>
<tr>
<td><strong>Most Common Research Objectives</strong></td>
<td>Explore, discover &amp; construct.</td>
<td>Describe, explain &amp; predict.</td>
</tr>
<tr>
<td><strong>Focus</strong></td>
<td>Wide-angle lens; examines the breadth &amp; depth of phenomena.</td>
<td>Narrow-angle lens; tests a specific hypothesis.</td>
</tr>
<tr>
<td><strong>Nature of Observation</strong></td>
<td>Study behavior in a natural environment.</td>
<td>Study behavior under controlled conditions; isolate causal effects.</td>
</tr>
<tr>
<td><strong>Nature of Reality</strong></td>
<td>Multiple realities; subjective.</td>
<td>Single reality; objective.</td>
</tr>
<tr>
<td><strong>Final Report</strong></td>
<td>Narrative with contextual description &amp; direct quotations from research participants.</td>
<td>Statistical report with correlations, comparisons of means, &amp; statistical significance of findings.</td>
</tr>
</tbody>
</table>

Table 10: Differences between Qualitative and Quantitative Research (Xavier University Library, 2012)
As evident in Table 10, both approaches are informative. They can be used in combination to provide a more comprehensive understanding of the phenomenon, while also hearing the viewpoints of the participants (Guetterman et al., 2015). The collection, analysis and integration of both quantitative and qualitative data is called the Mixed Methods Approach (Guetterman et al., 2015). This approach utilises the strengths of both the qualitative and quantitative approaches in gaining an understanding of the phenomenon being studied (Ben-Eliyahu, 2014).

3.2.4 The Mixed Methods Approach

The mixed methods approach has been used by researchers to investigate many topics, such as gaining an understanding of the implementation and viability of interventions (Guetterman et al., 2015). The triangulation of approaches in collecting and analysing data is an attempt to provide a more comprehensive explanation/description of the complexity of phenomena by studying it from more than one perspective (Cohen et al., 2011). As each individual approach to collecting and analysing data has different (sometimes opposing) limitations, combining one approach with a different approach may help to neutralise (or at least minimise) the deficiencies in the single methods (Cresswell, 2003). For example; a limitation of the qualitative approach is that it is difficult to assess relations between characteristics, whereas this is a strength of the quantitative approach. The strengths and weaknesses associated with the mixed methods approach are outlined in Table 11:
<table>
<thead>
<tr>
<th>Strengths</th>
<th>Weaknesses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Words, pictures &amp; narrative can add meaning to context and numbers.</td>
<td>Practically more complex on researchers’ skills set. A researcher may only be familiar with one type of research and only able to explain that one.</td>
</tr>
<tr>
<td>Numbers can be used to add meaning to Words, pictures &amp; narrative.</td>
<td>Relatively new &amp; therefore good models are difficult to find, e.g. difficulty in determining how to analyse inconsistent results.</td>
</tr>
<tr>
<td>Answer a broader &amp; more complete range of questions</td>
<td>Complexity in relation to data collection and analysis.</td>
</tr>
<tr>
<td>Use the strengths of an additional method to overcome weaknesses in another</td>
<td>Time and resource intensive.</td>
</tr>
<tr>
<td>Stronger evidence for a conclusion through triangulation</td>
<td></td>
</tr>
<tr>
<td>Add insight &amp; meaning that might otherwise be missed with a single method approach</td>
<td></td>
</tr>
<tr>
<td>Increase generalisability of the results</td>
<td></td>
</tr>
</tbody>
</table>

Table 11: Strengths and Weaknesses of a Mixed Methods Approach (Hughes, 2012 pp.4-5)

There are different ways in which the quantitative and qualitative approaches can be combined in the mixed methods approach (Onwuegbuzie and Johnson, 2006). The following conditions hold in basic concurrent mixed methods designs:

- both qualitative and quantitative data are collected separately at approximately the same point in time;
- during the data analysis stage, neither the qualitative nor the quantitative data analysis builds on each other;
- at the data interpretation stage, the results from each type of analysis are not consolidated until both sets of data have been collected and analysed separately.

(Onwuegbuzie and Johnson, 2006 p.53)
Cohen et al., (2011) state that triangulation is significant in demonstrating concurrent validity, for qualitative research in particular. Concurrent validity is established if there is consistency between the data obtained from one research instrument and the data obtained from another research instrument, which is measuring the same construct (Haynes et al., 1995; Vogt et al., 2004). There is greater validity in making a conclusion based on the concurrent results than by using the results from one instrument alone. This study adopts the embedded design of the concurrent mixed methods approach. This design of this mixed methods approach is illustrated in Figure 14:

![Concurrent Mixed Methods Designs](image)

Figure 14: Concurrent Mixed Methods Designs

(Bulsara, 2015 p.11)

The embedded design aims to answer different questions that require different types of data (Bian, 2014). The collection and analysis of one set of data may take place before, during or after the collection and analysis of the first data set (Bulsara, 2015; Bian, 2014). This type of design is helpful in gaining an understanding of the effectiveness of an intervention on participants (Bian, 2014). In this study, two assessments were designed to collect both qualitative and quantitative data from the participants. One assessment was given to the participants prior to a designed intervention, and the other assessment (deemed similar in structure and level by an expert panel) was given after the intervention. Questionnaires on mindset and beliefs were given pre and post intervention. These questionnaires consisted of closed questions and were designed to collect quantitative data (use of a Likert scale) from the participants. Focus groups were conducted to collect qualitative data and journal entries also consisted of qualitative data.
3.3 Research Problem

The main aim of this research study was to improve the teaching, learning, and assessing of problem solving. The literature on problem solving in Ireland show that there are a considerable number of students leaving second level education who are struggling to engage with problem solving in unfamiliar contexts (SEC, 2015). These difficulties experienced by students have an effect on their engagement with mathematics at third-level as the development of the conceptual and transferrable skills required to solve unfamiliar problems have not been nurtured in some classrooms (Ní Fhloinn, 2007; Hourigan and O’Donoghue, 2007). The literature reviewed revealed that despite the considerable attention problem solving in mathematics has received in educational research, the research conducted in problem solving has not easily translated into improving classroom practice (Lester and Kehle, 2003). The literature shows that mathematics teachers are having difficulties teaching for and developing understanding and have deficiencies in their own understanding. The aim of this research was thus narrowed to focus on the creation of a framework for the teaching and learning of problem solving which nurtures conceptual and transferable skills and which would be transferable to the classroom. The author’s framework was used in conjunction with the 10 Steps to Complex Learning Model developed by van Merriërnboer and Kirschner (2007) to create an intervention with pre-service second level mathematics teachers to aid in transferring research into practice. The combination of the framework with van Merriërnboer and Kirschner’s Model takes all aspects of teaching and learning problem solving into consideration. This combination also facilitates the reusability of the intervention by others by clearly revealing the design/choice of learning tasks, decisions on how to sequence those tasks, and how the analysis of cognitive structures and mental models were used in the inclusion criteria of the tasks (chapter 6). While similar frameworks exist in the review of the literature conducted, the author found none that encompassed all of the necessary components that the author’s developed intervention does.

3.3.1 Aims and Objectives of the Research

The aim of this research is to develop a framework for teaching and assessing problem solving in mathematics. This framework needs to consider the knowledge and affective factors which have an impact on students’ ability to engage in problem solving. Theories of developing mathematical thinking need to be researched. Research into current methods of teaching and assessing problem solving in mathematics needs to be conducted, with consideration given to the knowledge and skills required by teachers to teach and assess problem solving in mathematics.

The objectives of this research study based on the aims are to:

- conduct a review of the current literature in order to gain an in-depth understanding of mathematical thinking and problem solving, the knowledge
and affective factors that are associated with the learning and teaching of problem solving, theories of developing mathematical thinking, and current methods of teaching and assessing problem solving;

- develop a theoretical framework for teaching and assessing problem solving in mathematics which will provide the basis for the development of the assessment and teaching intervention components of this research;

- develop the assessment component\(^ {16}\) based on findings from the literature and adjusted to incorporate significant findings from the pilot test (aligned with the Project Maths syllabus);

- design an intervention to develop and evaluate pre-service teachers’ problem solving abilities in mathematics, based on the framework developed and findings from the assessment component;

- implement the intervention with a sample of pre-service mathematics teachers to evaluate the effectiveness of the framework in the teaching and assessing of problem solving;

- determine the effectiveness of the intervention in achieving its aims by analysing the gathered quantitative and qualitative data.

### 3.4 Research Questions

The research questions associated with the aims, objectives and phases (section 3.7.3) of this study are:

1. How should mathematics be taught and assessed in order to facilitate the development of learners’ problem solving proficiency in mathematics?

2. How can the research on the teaching and assessing of problem solving be brought into practice?
   
   (a) Who should the researcher implement the intervention with to best facilitate the transfer of research into practice?

   (b) What activities and teaching approaches should be used in the intervention?

3. What evidence is there to show the effectiveness of the framework employed in the intervention?

4. Is the framework and the developed intervention transferrable to a mathematics class in a secondary level school?

The theoretical and practical research conducted aimed to answer these research questions. In order for the answers to these questions to emerge from the research, a theoretical framework which provides the foundation for this research was formulated.

\(^ {16}\)A problem solving assessment discussed in chapter 4, section 4.4
3.5 Theoretical Framework

A theoretical framework lies at the core of a research study, it is the organising structure of the research (Ennis, 1999). The theoretical framework is the structure that identifies and describes the relationships among the various elements; existing theories and concepts of the particular research study (Ennis, 1999). The theoretical framework emerges from the research focus, guides the design and research decisions of the study, and gives coherent structure to the complete work. The appropriateness of the theoretical framework to the research topic being studied can be a determinant of the success of the overall study (Businessdictionary.com).

3.5.1 Theoretical Framework for this study

The literature review conducted informed the development of the framework of problem solving instruction and assessment. The main issues which were identified in relation to problem solving were the formulation of a rich knowledge base, the development of mathematical thinking, the influence which affective factors have on students’ ability to engage in problem solving, the consideration of mathematics as a design science, the benefits of extending problem solving to include modelling activities and the influence that assessment has on the mathematics that is actually engaged with in the classroom. These aspects form the theoretical basis for the proposed framework for teaching and assessing problem solving in mathematics.

The developed framework for teaching and assessing problem solving in mathematics addresses the complexity of problem solving in mathematics by integrating theories in relation to the discussed significant knowledge and affective factors which affect the growth of problem solving ability in individuals. This framework takes the discussed complexities into consideration. Mathematical thinking (Tall 2001; Mason et al., 2010) is central in this framework. Tall’s framework (2008) for the development of mathematical thinking is based on “perception, operation and reason”. The perception formed of a problem situation by the problem solver is in the form of a “mental model” (Johnson-Laird, 2004). Making sense of this problem situation in one’s mind involves mentally operating on this internal model (Johnson-Laird, 2004). The more sophisticated the mathematical thinking of the individual, the more they can work with increasingly complex ideas as their ability to form different internal representations of the same problem situation and their flexibility to manipulate these mental models in their mind develops (Tall, 2001; van Hiele, 1984c; Dubinsky and McDonald, 2001). This may be due to the development of the ability to shorten the problem solving processes involved in determining a solution (Krutetskii, 1976; Sfard, 1991; Tall 2001), in addition to the ability to conceive of these processes as “objects” (Sfard, 1991; Dubinsky and McDonald, 2001).

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17 discussed in chapter 2
as well as processes. An “object” perception is the mental model which results from the individual’s awareness of a mathematical process as a whole entity, as well as the realisation that further actions and processes can be applied to this whole entity (Arnon et al., 2014). Sfard (1991) refers to this ability as being able to perceive mathematical ideas both structurally and operationally, and states that both are essential in the development of conceptual understanding of mathematics. Reflecting on the outcomes of the operations along with providing explanations and justifications for the chosen solution method aid in the development of a structural conception, which facilitates all the cognitive processes required for effective problem solving and learning in mathematics (Sfard, 1991).

The mathematical thinking processes of specialisation and generalisation (Mason et al. 2008) along with Tall’s framework (2008) incorporates most of the common mathematical thinking processes (flexibility, generalising and determining structural relationships) present in several of the models and theories of mathematical understanding18, which have been developed by various researchers (Dubinsky and McDonald, 2001; Sfard, 1991; van Hiele 1984c; Krutetskii, 1976). This study also uses this created framework to inform the design and implementation of an intervention focused on improving pre-service teachers’ performance in problem solving. The design and choice of the problems utilised in the assessments and during the intervention are based on the requirement and development of mathematical thinking and proficiency in the acquisition of knowledge, while also catering for the emotional engagement of the learners. The proposed structure of the framework in chapter four (Figures 16 and 17) combines knowledge and affective factors in the design and choice of mathematical problems which are aimed at provoking mathematical thinking with the goal of developing mathematical proficiency in problem solving situations. The components which form the framework will be further discussed in chapter four.

The aim of the research methodology is to show how the data was collected and how it was analysed. The reasons for the chosen methods for data collection and analysis are also clarified. The next section discusses the methodology pertaining to this research.

### 3.6 Research Methodological Approach

This section describes the process of the research undertaken for this study, specifically how the research information was collected and how the data was analysed. Explanations for the choice of data collection and analysis are also provided. To begin this section, the methodological approach of Educational Design Research is described as this is the methodological approach for this study.

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18discussed in chapter 2
3.6.1 Educational Design Research

As previously mentioned in the introduction, although problem solving has received substantial attention in research, this has not translated easily into improving school practice (Lester and Kehle, 2003). This is an issue not only confined to problem solving, there is often a disconnect between educational research and the issues that occur in general practice (Design-Based Research Collective, 2003). The need for a research design that is relevant to, and can be used to address problems in educational practice has been recognised (Design-Based Research Collective, 2003) by those involved in all aspects of education (research, practice or both).

The primary aim of Educational Design Research (van den Akker et al., 2013) is to:

- attend to complex problems which exist in educational practice by creating research-based solutions;

or to:

- formulate or validate theories about learning processes and learning environments.

Barab and Squire (2004 p.1) define Educational Design Research as:

"a series of approaches with the intent of producing new theories, artefacts, and practices that account for and potentially impact learning and teaching in naturalistic settings".

Educational Design Research takes the theoretical knowledge of teaching and learning, together with an understanding of the relationships existing between theory, practice and designed products into consideration when formulating interventions.\(^1\) Research on these designed interventions can also provide knowledge to the theories of teaching and learning (van den Akker et al. 2013). Educational Design Research, like other forms of scientific research, uses existing theory in the pursuit of contributing to theoretical understanding, however a distinguishing feature of Educational Design Research is that the existing theory and understanding gained from the research of this theory is also used to structure the designs of solutions to real problems in educational practice (McKenna and Reeves 2012). The methodological approach of Educational Design Research contributes to educational practice.

Lesh and Sriraman (2005) describe the following as features of design science, which relate to mathematics education:

\(^{19}\)Intervention refers to all entities that can be designed or developed; learning processes, learning environments, teaching learning materials, products, & systems\(^5\) (van den Akker et al. 2013, p.11).
• items being investigated have an aspect of human creativity associated with them (conceptual understanding, mathematical thinking);

• the subjects being studied are complex systems (studying conceptual models created by students or researcher, which may undergo successive cycles of re-development, the conceptual system which led to the creation of the model is also assessed by assessing the model);

• the subjects for which one is trying to gain understanding of are constantly changing as well as the conceptual systems which are trying to understand them;

• an object which is designed for some purpose generally undergoes several revisions over the course of development.

Amit (2010) in her discussion on the treatment of mathematics education as a design science, states that realistic solutions to real complex problems requires the integration of ‘concepts and procedures drawn from more than one grand theory of education’ (p.147).

Educational Design Research fits the purpose of this study as the author wanted to use the review of existing theory to formulate a framework, which is based on the integration of concepts and theories from the areas of mathematical understanding and affective domains to inform the design of an intervention which could provide a potential solution to the problem of problem solving in educational practice. The investigation of the students’ solutions to problems have mathematical thinking associated with them and the research environment of the classroom is dynamic.

The research process of Educational Design Research varies with respect to the “frameworks and models that describe and at times guide the process” (McKenna and Reeves, 2013 p.12). However common distinguishing characteristics (McKenna and Reeves, 2013 p.12) of the phases involved in Educational Design Research include:

• analysis and orientation phase
  - analysis through research (literature review, needs analysis) of practical problems and their causes (development of theoretical framework);

• design and development phase
  - development of solutions which are informed by existing theory and design principles;

• evaluation and reflection phase
  - testing of solutions in practice with refinement and reflection to produce design principles and improve the implementation of the solutions.

  (McKenna and Reeves, 2006, cited in van den Akker et al. 2013, p.18)
These phases of the research process of Educational Design Research were employed during the process of completing this research study. The needs assessment was established from a review of the current literature and also from the findings of the pre-test. The needs assessment provided the theory and evidence which informed the development of the Framework for teaching and assessing problem solving in mathematics. This framework then informed the design, development and implementation of the intervention. The proof of concept approach (section 3.6.2) was used to test the performance of the intervention and assessments in a small scale naturalistic setting.

3.6.2 Proof of concept Approach

A proof of concept approach which has its origins in the field of engineering, is the practical implementation of a specific program, process, method, principle, model, or idea to demonstrate its feasibility, i.e. to test that the program, in practice works as it is theoretically designed to do (K12 Blueprint, 2014). The proof of concept approach was used in the evaluation and reflection stage of the Educational Design Research process by administering the pre and post-tests, along with implementing the intervention based on the Framework for Teaching and Assessing Problem Solving on a small scale with pre-service mathematics teachers. While implementing fieldwork on a large scale is optimal in educational research, this is not always achievable on a practical level (due to participant, cost, and other constraints). The proof of concept approach provides evidence for the feasibility of the Framework for Teaching and Assessing Problem Solving, pre and post-tests, and intervention in performing as they should on a small scale, which gives credibility to the concept of these research components also working on a larger scale.

3.7 Research Design

This section describes what type of a study this is and who took part in this study. To begin this section, the participants in the study will be described.

3.7.1 Participants

Students who were enrolled on the BSc. second-level mathematics teacher training programme (year 1 - year 4) in the University of Limerick Ireland took part in this study. Participation was voluntary.

\[20\] In at least one of the phases of the research
Seven fourth-year pre-service teachers completed the pilot of the pre-test. Thirty-two pre-service mathematics teachers completed the pre-test (21 first-year students, 8 second-year students and 3 third-year students). Sixteen pre-service teachers attended the intervention classes in the 2016/17 academic year (11 first-year students and 5 second-year students (who were in third-year at the time of the intervention)). Ten first-year pre-service teachers (who were in second year at the time of the intervention) were completing teaching practice at the time of the intervention, and were unavailable to participate in the intervention. Thirteen pre-service teachers completed the post intervention assessment (8 first-year and 5 third-year students). Three first-year students completed both the pre and post-tests (in second-year at time of post-test) without participating in the intervention.

### 3.7.2 Design

The design of this study is categorised as a quasi-experimental within subjects “repeated” (problem solving pre and post-test not identical) measures design. The study is quasi-experimental as the participants were not randomly selected. A within subject factor is where each of the participants is measured at all levels of the factor. It distinguishes measurements made on the same subject. In this study, test is the within subject factor consisting of two levels, pre and post corresponding to the pre and post-tests (problem solving test and mindset test both given before the intervention and after intervention) respectively. This study is prospective as it collects data from participants while following their progress over time (Thiese, 2014). This study measures the problem solving ability, and mindset of participants before and after an intervention is implemented. Pre and post-test study designs possess the strength of temporality (linear progres-
sion of time), which allows for making the suggestion that the outcome of the post-tests is impacted by the intervention (Thiese, 2014). However, this type of a design cannot control other factors which may be changing during the intervention implementation period (potentially confounding the findings) (Thiese, 2014). Trying to hold certain variables constant in a learning environment is almost impossible, given its dynamic nature (Cohen et al. 2007). Therefore changes in problem solving ability and mindset during the course of this study cannot be completely explained by the intervention. Multiple studies involving different samples of the population need to be considered in forming assessments of causation. This is further discussed in chapter 8.

Crotty (1998) outlines the following four essential aspects to consider in research design:

- the epistemology which informs the research;
- the philosophy underlying the methodology;
- the methodology;
- the techniques, instruments and procedures employed to collect data.

These four aspects are all discussed in this chapter.

3.7.3 Phases of the Research

The phases of the research are shown in Figure 15. There were seven phases of the research in the completion of this first cycle of the research.
Research Phases in the development of the F-TAPS and Intervention

Began with

Informed

Phase 1
Consisted of

Phase 2
Consisted of

Phase 3
Consisted of

Phase 4
Consisted of

Phase 5
Consisted of

Phase 6
Consisted of

Phase 7
Consisted of


Review of teacher knowledge and the effect this has on student achievement.

Analysis of instructional practices in mathematics education.

Analysis of models and theories of the development of mathematical thought.

Formulating the F-TAPS and the pre-intervention assessment of problem solving.

Pilot test assessment. Adjustments to assessment based on findings from pilot test. Distribution of assessment (pre-test) to participants.

Analysis of data from assessment (pre-test).

Development of the intervention and post-test.

Implementation of the intervention.

Post-test after intervention, analysis of data from post-test and evaluation of the intervention.

Figure 15: Research Phases

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Phase 1

This phase of the research identified problem solving in mathematics as an area of concern in mathematics education both in Ireland and internationally. The researcher conducted a review of the literature to investigate possible reasons for this problem. The literature review revealed the difficulties experienced by students, teachers and pre-service teachers in problem solving in mathematics. The review also revealed that the research conducted in problem solving does not easily translate into improving school practice (Lester and Kehle, 2003). The findings from the literature review conducted in phase 1 informed the development of the framework for teaching and assessing problem solving in mathematics (Figures 17 and 18). The main issues which were identified in relation to problem solving were the following:

- formulation of a rich knowledge base (metacognition, heuristics, Diene, Bruner);
- development of mathematical thinking (van Hiele, Tall, Mason, Katagiri);
- development of mathematical proficiency (five strands);
- influence which affective factors (mindset, interest and perseverance, respecting students’ own intuition and strategies) have on students’ ability to engage in problem solving;
- consideration of mathematics as a design science;
- influence that assessment has on the mathematics that is actually engaged with in the classroom.

It was noted that little research has been completed on “teachers as problem solver”, particularly at second level (Felmer and Perdomo-Diaz, 2016, p.289). Also Zimmermann (2016) notes that attempts at improving problem solving in mathematics education require the focus of attention to be on the education of pre-service mathematics teachers, with alignment existing between pre-service teachers’ education and students’ education (Zimmermann, 2016). Part of pre-service teachers’ education should include the solving and posing of problems that are similar to, and some more challenging than those they will be teaching to their students (Chapman, 2015; Zimmermann, 2016). Kilpatrick (1983) concluded that an instructional program for developing problem solving in mathematics will not be successful if it does not take the effects of students’ attitudes and beliefs about themselves as problem solvers into consideration. The findings from the research conducted in this phase were used in the methodological decisions for the remaining phases of this research study. Phase 1 completed the analysis stage of the Educational Design Research methodology.
Phase 2

This phase is part of the design stage of the Educational Design Research methodology. During this phase of the research, the researcher formulated the Framework for Teaching and Assessing Problem Solving in Mathematics based on the research findings in Phase 1. Phase 1 of the research revealed that considerable research had been done in relation to problem solving. The researcher noted the findings of Lester (2013), that the tendency to conceptualise the complex phenomenon of problem solving in a more simplistic manner than its nature would suggest, limited the effectiveness of instruction on improving students' ability to problem solve. The framework for teaching and assessing problem solving reflects this by integrating several theories in relation to the factors identified in Phase 1. The researcher noted the development of mathematical thinking as central to problem solving in mathematics, with mathematical thinking being fundamental in attaining conceptual understanding of mathematics. The ability to recognise opportunities to productively use mathematical knowledge alone or in conjunction with other mathematics, in order to make an attempt to solve a problem is heavily dependent on a sound conceptual knowledge base of mathematics. Mathematical thinking is thus the central component of the researcher’s F-TAPS in mathematics. Key knowledge and affective factors are taken into consideration during the development of mathematical thinking in the researcher’s F-TAPS in mathematics. The proposed structure of the framework in Figure 17 (and more detailed version Figure 18) combines knowledge and affective factors in the design and choice of mathematical problems, which are aimed at provoking mathematical thinking with the goal of developing mathematical proficiency in problem solving situations. The goal of the F-TAPS in mathematics is to result in the development of mathematical proficiency, which is necessary for optimal problem solving in mathematics. The goals of the framework also are to immerse the pre-service mathematics teachers in a problem solving environment and to provide them with information and ideas which they may use in their teaching. This was based on the research findings in Phase 1. While other disparate frameworks exist, the author’s framework is unique in the combination and associated magnitude of significance of the models and theories utilised in its formation. The pre-test component was also developed during this phase of the research. The researcher’s framework is discussed further in Chapter 4.

Phase 3

The pre-test was reviewed by an expert panel of Mathematicians. The expert panel consisted of five expert mathematicians (one who is also a doctor of mathematics education and three of whom are applied mathematicians and professional problem solvers), along with two qualified secondary mathematics teachers.

The panel of experts were asked to check the problems for accuracy in relation
to each of the following:

- wording of the problem;
- mathematical accuracy of the problem;
- fit for purpose of assessing problem solving ability.

The panel was also asked to rate the problems into the following levels of difficulty:

- easy;
- intermediate;
- difficult.

The initial assessment instrument was finalised following the reviews by the panel of experts. The pre-test was piloted with a group of seven fourth year (final year undergraduate) pre-service mathematics teachers. Adjustments to the pre-test based on findings from pilot test included:

- Explicitly ask if students know of alternative solutions to each problem after each problem, instead of only writing this in the instructions on the first page and stating this at the start.

- Change the ratio $3 : 2 : 1$ in question five, students compared this to the length of time needed to do the work by each shop. Shop 1, 2 and 3 takes 10, 30 and 15 days respectively so essentially shop 2 is twice shop 3 and 3 times shop 1. The ratio of the books is irrelevant in this question, but the link between both sets of ratios may make this less clear. Also changing the ratio of the books should ease in correcting the solutions to this problem (coincidentally the method of comparison mentioned gave the same result as the correct answer - but reasoning not correct).

- Insert extra rows in the table for students to rate each part of the problem, if there is more than 1 part to a problem.

The results of the pilot tests were evaluated by the researcher and three independent mathematics education experts using the analytic scoring framework (Appendix J). There was substantial agreement amongst the raters (Kappa > 0.6). The pre-test was then distributed to participants.

**Phase 4**

During this phase, corrections of the pre-test and an analysis of the findings from the pre-test was conducted. The researcher used the developed analytic scoring framework (revised after collaboration with mathematics education specialists Appendix K) and comparative analysis of solutions to mark the pre-tests. The
author also wrote a detailed account of each participant’s response to each problem.

**Phase 5**
The researcher designed the intervention component of the research during this phase of the research. This phase forms part of the design stage of the Educational Design Research methodology. The author also developed the post-test (Appendix T) and analytic scoring framework (for the post-test, Appendix M) during this phase of the research which was given to the same panel of experts for review and subjected to the same inclusion criteria for the tasks.

**Phase 6**
The intervention component was implemented with two groups of pre-service mathematics teachers (11 first years and 5 third years) in February, March and April 2016. Each of these participants had completed the pre-test, questionnaires on mindset, and had provided the researcher with their top three interests/pastimes.

**Phase 7**
The post-test was then distributed to participants after completion of the intervention. The effectiveness of the intervention in relation to whether the developed and implemented intervention was successful in achieving its aims was analysed. Shapiro’s 1987 model was used to evaluate the effectiveness of the intervention. The four components of the intervention examined by Shapiro’s model showed that the intervention was effective in achieving its aims, was delivered in adherence to its design, was deemed relevant, acceptable and useful by the participants. A focus group was held with each of the two groups of pre-service teachers who participated in all aspects of the research. The suggestions of including this intervention as part of their studies indicates the acceptability of the intervention by the participants. During this phase the post-tests were also marked and analysed.

**Phase 8**
The researcher continued with the analysis of the findings and the write up of the full thesis.

3.7.4 **Chronology of the Research**

**Phase 1: Aug 2013 - Sept 2014**
Review of up to date literature relevant to topic.

**Oct-November 2014**
Write up of Confirmation report.
Phase 2: Nov 2014 - Feb 2015

Phase 3: March 2015 - May 2015
Pilot test assessment. Adjustments to assessment based on findings from pilot test. Distribution of assessment to participants. Commencement of analysis of data.

Phase 4: April - Aug 2015
Analysis of data from assessment (pre-test).

Phase 5: Sept 2015 - Jan 2016
Design of intervention based on findings from assessment.

Phase 5 and 6 : Feb 2016 - April 2016
Conduction of intervention component and analysis of findings during intervention.

Phase 7: April 2016 - Sept 2016
Post-test after intervention, analysis of data from post-test and evaluation of the intervention.

Phase 8: Oct 2016- June 2017
Findings and write up of full thesis.

3.7.5 Sampling
A convenience sample, which was also a purposive sample (typical case sampling), was employed in this research study. A convenience sample is a non-probability sampling technique where the sample of participants are selected from a population based on the ease of access to that sample of participants (laerd.com). Typical case sampling is a purposive sampling technique where the sample is “typical” of the type of people, contexts the researcher is interested in studying (laerd.com). The purposive sample was chosen from pre-service teachers who were enrolled on the BSc. Physical Education and Mathematics Teacher Education course (year 1 - year 4) at the University of Limerick. The non-probability technique does not allow for inferences from the sample to the population, however purposive sampling allows for comparisons between similar samples to be made. A convenience sample is useful to “illustrate the application of some new method or new technique” (Ferber, 1977 p.58), which was done in this research study. The researcher choose to work with pre-service secondary level mathematics teachers, as they have demonstrated deficiencies in their understanding of mathematics (Chazan et al., 1999; Llanares and Krainer 2006; Tsamir and Ovodenko, 2005; Alvery et al., 2016; Hourigan and O’Donoghue 2013) and the teacher is the “single most powerful influence on student achievement” (Hattie, 2003, p.4). Furthermore many Irish second level mathematics teachers receive no instruction in problem solving in mathematics as part of
their teacher education program at third level (O’Meara, 2015). It has been recommended that secondary mathematics education teacher education programs should consist of at least three modules that focus on secondary level mathematical content from an advanced perspective (Conference Board of the Mathematical Sciences (CBMS), 2012).

The advantages of a convenience sample is that it is easy to carry out with cost and time involved in obtaining the sample small in comparison to probability techniques (laerd.com). The disadvantages are that due to the sample not being selected at random, generalisations from the sample to the population is not possible. In educational settings, similar samples would have participants of similar abilities in mathematical problem solving and the intervention would be carried out in similar contexts. It cannot be said that the results obtained in this study are generalisable to all groups of pre-service mathematics teachers but similar results may be obtained in similar samples of pre-service teachers in similar contexts.

3.8 Research Methods

This section describes how the information was collected.

3.8.1 Research Instruments

Research instruments are measurement devices which the researcher uses to collect data (Cohen et al., 2007), which relates to the research questions (Crotty, 1998). The research instruments employed in this study consisted of the following:

1. a questionnaire on mindset
2. a problem solving assessment (pre and post-tests21.)
3. a research journal of the intervention
4. focus group

This section describes the use of questionnaires, assessments, journal entries and focus groups as a means of providing data.

3.8.2 Assessment

A problem solving assessment22 was used in this study to provide an indicator of the problem solving ability of pre-service mathematics teachers.

21The post-test was not identical to the pre-test as this would not be conducive to assessing problem solving ability. However the problems on the post-test was deemed to be comparable to the problems on the pre-test, both in terms of difficulty and on the kind of thinking required to solve them, by a panel of 4 expert mathematicians
22The construction of this assessment is discussed in chapter 4 section 4.4
In the construction of the assessment, Cohen et al., (2011, p.481) state that the following items need to be determined:

- the purpose of the assessment;
- the type of assessment;
- the objectives of the assessment;
- the content of the assessment;
- analysis of items included in the assessment;
- presentation of the assessment;
- time to complete assessment;
- validity and reliability of the assessment;
- standard for scoring the assessment items.

As previously mentioned, the purpose of the assessment instrument was to obtain an individual problem solving profile for each pre-service teacher. This provides information about the initial problem solving performance of each pre-service teacher, which gives an indication of their problem solving ability before the intervention. The information provided by the participants’ responses to the problems in the assessment was used in the development of an intervention aimed at improving problem solving ability.

The assessment instrument is a written assessment consisting of nine problems. The possible mathematical thinking, along with the potential mathematical proficiency required in order to formulate a solution to these problems was at the foundation of the selection criteria for inclusion. The problem solving skills and content from the Irish mathematics syllabi for both junior and senior certificate levels also informed the design of the assessment. Each problem is individually discussed under specific headings (relating to skills and abilities, mathematical proficiency, mathematical thinking and content form syllabus) after the presentation of the model for teaching and assessing problem solving in chapter 4.

The presentation of the written assessment was subject to review by an expert panel of mathematicians (one who is also a doctor of mathematics education23). Suggested changes to initial design(s) and presentation of the assessment were made following the review by the expert panel. The written assessment instrument was not subject to a time constraint as this provided information about students’ perseverance. However, the maximum time spent by any individual student from the pilot-test was 2 hours 17 minutes, while the maximum amount of time spent by the mathematics teachers was 2 hours 31 minutes. A time of

23Full details on the expert mathematicians given in section 4.4.2.
3 – 4 hours was allocated for the assessment but students were informed that there was no time limit. All students completed the pre-assessment (pre-test) within 2.5 hours, with a median time of 87 minutes (interquartile-range=19 minutes ([79 – 98])).

The researcher is aware that recorded interviews have been utilised in studies on problem solving but decided to use a written assessment (while asking students to write down any information relating to how they felt about the problem, their performance, if they knew/ tried an approach but could not do it at this particular moment in time so choose an alternative approach instead). This decision was made in consideration to optimising the potential sample of voluntary participants from the population of pre-service teachers. Recording students during an assessment may have deterred many of the pre-service teachers from participating in this study.

The scoring framework for the assessment items was devised from the analytic scoring system formulated by Charles et al. (1987), (Appendix G) using the solutions provided by members of the expert panel, the mathematics teachers, then refining with the solutions provided by the seven pre-service teachers who completed the pilot-test. This analytic scoring framework was further refined based on the responses provided by the larger sample of all the participants (Appendix H; Appendix J).

In a pilot test of the pre assessment, the inter-rater reliability was determined. This indicated substantial agreement between the raters (the researcher and three mathematics education specialists). The pilot-test and inter-rater reliability are discussed in more detail in chapter 4. The inter-rater reliability was also calculated for the post-test (this test was not identical to the pre-test but was judged (by the expert panel) to be similar in the thinking required by the participants and in the difficulty of the problems)\(^{24}\).

### 3.8.3 Focus Groups

A focus group is fundamentally a group interview that makes use of interactions within the group, based on discussions of topics which are supplied by the researcher (Morgan, 1997). The primary characteristic of a focus group is its use of the group interaction to gain data (on perceptions, attitudes, experiences) and insights from the participants that would be less available without the interaction of a group dynamic (Morgan, 1997). A primary advantage of focus groups is that they “yield a large amount of information over a relatively short period of time” (University of Oregon, 2015 p.51). Focus groups are also useful for assessing a wide range of perspectives on a specific topic (University of Oregon, 2015). The focus group was semi-structured, i.e participants answer preset open-ended questions, the guide of preset questions helps to keep the

\(^{24}\)although it was noted that problems 3 and 9 were more difficult on the post-test than these corresponding problems on the pre-test.
focus of the interview on the desired topics to be discussed (DiCicco-Bloom and Crabtree, 2006) while also allowing for a conversational manner of discussion to arise. Open-ended questions are a good way of getting in-depth honest answers from participants, these type of questions do not limit the range of responses a participant could give and give the participants an opportunity to explain their thoughts, feelings and opinions on a topic (University of Oregon, 2015).

The researcher used information by Krueger (2002) on principles of designing and conducting focus group interviews to aid in the formulation of the open-ended questions for the focus group. A set of questions for the focus group (Appendix I) was prepared by the researcher in order to gain qualitative feedback from the participants in relation to their thoughts on participating in a problem solving environment that was in coherence to the Framework for Teaching and Assessing Problem Solving. These questions were administered to two focus groups (one for each of the first and third year groups). Two groups were used as each focus group should consist of similar individuals (Barnett, 2008). Four first year students who had participated in all components of the research (pre-test, intervention and post-test) participated in the focus group for first years. Four third year students who had participated in all components of the research (pre-test, intervention and post-test) also participated in the focus group for third years. Participation in the focus group was voluntary.

3.8.4 Research Journals

The researcher wrote journal entries during the intervention. These journals documented events during the intervention classes. The main purpose of these journals were to monitor the progression of the classes and to alert the researcher of potential ways to improve the classes if necessary. As these journals were written by the researcher, there may be potential bias in the contents of them and therefore the use of data from these journals is limited to a minimum within the research. The researcher attempted to remain objective in the writing of these journals. Where use is made of data from these journals, it is usually to elaborate on particular points so as to provide an insight into the intervention classes.
3.8.5 Instruments and Mixed-Methods Approach

The instruments used in the research yielded both qualitative and quantitative data:

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Qualitative</th>
<th>Quantitative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-assessment</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Intervention</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>Post-assessment</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Focus Groups</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>Mindset Questionnaire</td>
<td></td>
<td>✓</td>
</tr>
</tbody>
</table>

Table 13: Research Instruments & the Mixed Methods Approach in this Research Study.

3.9 Data Collection

This section describes how the data was collected. On completion of the design of the research instruments, in addition to feedback from panel of experts and pilot-tests, the researcher continued with Phase 3 of the research (section 3.7.3). Pre-assessment data was collected from first, second, third and fourth year pre-service second-level mathematics teachers in April 2015. Pre-test data was also collected from first-year pre-service teachers in February 2016. At the time of collecting the pre-assessment data, data on mindset was also collected before the intervention.

Journal entries were written during the intervention stage in Phase 6. Post-test data was collected from first and third year participants during Phase 7 in April 2016 upon completion of the intervention. The post-test was not identical to the pre-test as this would not be conducive to assessing problem solving ability. However the problems on the post-test were deemed to be comparable to the problems on the pre-test, both in terms of difficulty and on the kind of thinking required to solve them, by a panel of 4 expert mathematicians. Each of these experts had also reviewed the pre-tests and were familiar with the purpose of the assessments. Again, at the time of collecting the post-assessment data, data on mindset was also collected after the intervention. The questionnaire on mindset was identical pre and post intervention. The focus groups also took place in Phase 7 after the completion of the intervention, there were four participants in each of the two groups (one group for first years and one for third years).

3.10 Data Analysis

This section will describe how the data was analysed.
3.10.1 Assessment Data

The problem solving assessment was analysed by the researcher\textsuperscript{25}. The responses given by all pre-service teachers were examined on a problem by problem basis to form a comparative judgement on the types of responses provided by the pre-service teachers to each individual problem. This informed the researcher of the different methods employed by the pre-service teachers to solve the same problem. These methods were then coded for convenience to provide information on the efficiency of the solution method employed. The solution method employed by the expert mathematicians were considered in the coding of the efficiency of the solution methods (for example solving problems 1 and 2 (in the pre-test) by trial and error is a less efficient method than forming an algebraic equation to solve it). The actual responses by students to each problem were also compared on a problem by problem basis and assessed according to the focused analytic scoring point framework (Appendix H) adapted from Charles et al. (1987). This resulted in the following analytic scoring point framework per question:

Understanding the Problem:

- **Type 0**: No attempt;
- **Type 1**: Written work reveals complete misinterpretation of the problem situation;
- **Type 2**: Written work reveals basic understanding of the information presented in the problem situation but with no/unsuitable introduction of notation/representation/organisation of work to show mathematical understanding of relations among the data;
- **Type 3**: Written work reveals understanding of the information presented in the problem situation with suitable notation/representation/organisation of work introduced for part of the problem but there are substantial omissions or errors in the mathematical understanding of the relations among the data;
- **Type 4**: Written work reveals substantial understanding of the information presented in the problem situation with suitable notation/representation/organisation of work introduced for the problem but there are minor errors in the mathematical understanding of the relations among the data;
- **Type 5**: Written work reveals complete understanding of the problem situation.

Planning/Solving the Problem:

\textsuperscript{25}After a collaborative review with three expert mathematicians, who corrected the pilot tests and a sample of the post-tests.
• **Type 0**: No attempt/notation or work presented does not lead to any structured plan of action;
• **Type 1**: Inappropriate/illogical/incorrect plan;
• **Type 2**: Correct plan with high error in solving;
• **Type 3**: Correct plan with medium error in solving;
• **Type 4**: Correct plan with low error in solving;
• **Type 5**: Plan leading to a correct solution with no arithmetic errors.

**Solution and Checking:**

• **Type 0**: No solution presented due to no attempt or no/incorrect solution presented as a result of an inappropriate/illogical/incorrect plan;
• **Type 1**: No/incorrect solution based on appropriate plan but insufficient progression/high or medium error made;
• **Type 2**: Incorrect answer based on an appropriate plan implemented correctly but with minor computational errors in solving;
• **Type 3**: Correct answer but not justified (not checked to see make sense in context of problem) or not communicated clearly;
• **Type 4**: Complete well explained (well communicated) justified solution.

The responses provided by the pre-service teachers in the pilot test were analysed in relation to the above analytic scoring point framework to provide a more specified analytic scoring point framework for each individual problem (Appendix H). This was done to aid in the provision of a more detailed marking scheme in order to facilitate increased reliability among multiple raters. Efficiency of the solution method employed by students was also rated (Appendix H).

The assessment data also provided data on the following:

1. whether the pre-service teacher had seen the exact same or a similar problem previously;
2. whether the pre-service teacher knew how to solve the problem immediately after reading it or not;
3. whether the fact that the pre-service teacher had seen exact/similar problem before was helpful in solving the current problem or not;
4. the rating the pre-service teacher gave the problem on a scale of very easy - very difficult;
5. if the pre-service teacher could suggest an alternative solution method.

The objective of gaining this data was as follows:

1. Information on whether the pre-service teacher had seen the same or similar problem provides information about the ability of the student to retain mathematical information and also indicates the depth of knowledge gained by a previous encounter with a similar problem (Kruteskii, 1976);

2. Determining whether the pre-service teacher knew how to solve the problem immediately after reading it or not reveals whether this problem is an exercise or a problem for the pre-service teacher (Jonnasen, 2000);

3. Ascertaining whether the fact that the pre-service teacher had seen exact/similar problem before and whether it was helpful in solving the current problem or not again indicates the depth of retention of knowledge gained by a previous encounter with a similar problem (Kruteskii, 1976);

4. Knowing the rating the pre-service teacher gave the problem (from a scale of very easy - very difficult) allows a comparison to be made between the pre-service teacher in relation to perseverance. It also provides a measure of the difficulty rating of the problems for the particular group of pre-service teachers;

5. A pre-service teacher who can (does) suggest an alternative solution method indicates a higher flexibility in problem solving than a pre-service teacher who cannot (does not) suggest alternative solution method (Rittle-Johnson and Star, 2007).

The actual responses provided by the pre-service teachers in the pre and post-tests facilitated the elaboration of the above scoring framework, responses were utilised to provide a more precise description of the types of answers for each individual problem (Appendix J). An expert mathematician and a mathematics education specialist were consulted to aid in the assigning of ordering of the specific types of answers in the scoring framework, along with reviewing the elaboration of the types of answers.

In order to provide a more holistic profile of the problem solving ability of the participants (pre-service teachers who completed pre and post-tests and intervention), the solutions provided by the participants\(^{26}\) were also examined by considering each of the following problem solving phases described by Pólya:

- understand the problem;
- devise a plan;
- carry out the plan and look back.

\(^{26}\)who participated in the pre-test, intervention and post-test only
in relation to each of the following aspects:

- heuristics (Pólya);
- mathematical thinking (Mason, Tall, Katagiri);
- problem solving abilities (Krutetskii);
- problem solving skills (Leaving Certificate Project Maths syllabus);
- mathematical proficiency (Kilpatrick, Leaving Certificate Project Maths syllabus);
- van Hiele levels of mathematical thought.

This is in order to provide a more precise description of the performance of the participants in relation to the phase(s) of the problem solving process, along with the strand(s) of mathematical proficiency (rubric (adapted by Docktor and Heller, 2009) given in Appendix S).

Where a pre-service teacher provided more than one method of solution, the method which resulted in the most correct solution was analysed using the analytic scoring point framework. The description of the phases (as they pertain to this research) are as follows:

1. Understanding the Problem;
   - This involves evaluating the understanding of the wording of the problem as well as understanding of the relations among the data in the problem situation (Singer and Voica, 2012). This may involve some or all of the following: representation, explanation, introduction of variables and formulation of mathematical expressions/equations etc. Conceptual understanding, strategic competence, adaptive reasoning are among the elements of mathematical proficiency examined here.

2. Planning/Solving the Problem;
   - This involves evaluating the student’s ability to correctly put the parts from 1. together to form a solution method. The accuracy to which this is carried out is also examined here, along with noting if there were more than one attempt made if the student encountered difficulties or noting any display of productive disposition. Procedural Fluency is the main element of mathematical proficiency examined here.

Again the accuracy of carrying out the solution plan is looked at here, along with noting if the student made any attempt to justify their solution. Adaptive reasoning in the justification of the solution method is the main element of mathematical proficiency examined here.

The responses provided by the pre-service teachers were evaluated using the analytic scoring framework (Appendix J). The type of response given by the pre-service teachers in the pilot-test are shown in Table 14. This shows a specific vector of response types for each pre-service teacher for each of the phases of the problems from 1 to 9. The response types provided by the pre-service teachers were entered into SPSS (Statistical Package for Social Sciences) 24.0 for Windows. The three phases, reading and understanding (R&U), planning and solving (P&S), and solution and checking (S&C), of the problem solving process were analysed separately. The percentage of pre-service teachers at each response type was determined (Tables 15, 16, 17, and 19).

The following scoring scale shows how marks were assigned to the types of responses for the reading and understanding, the planning and solving, and the solution and checking phases (in the scoring framework)

- a type 0 or a type 1 response was awarded 0 marks;
- a type 2 response was awarded 1 mark;
- a type 3 response was awarded 2 marks;
- a type 4 response was awarded 3 marks;
- a type 5 response was awarded 4 marks;

The types of responses entered into SPSS were recoded to give the marks associated with each type. The sum of each participant’s score for each of the phases of the problem solving cycle was calculated in SPSS. For example (looking at the types of responses in Table 14 and using the scoring scale) the sum of the marks for the R&U phase for participant Pi3 is:

\[1 + 4 + 4 + 4 + 4 + 1 + 4 + 2 + 4 + 4 + 4 + 4 + 2 = 42\]

The sum of each participant’s score for each of the phases of the problem solving cycle is presented in Table 21. The analysis presented after Table 14 first concentrates on the percentage of pre-service teachers at each response type over each of the phases of the problem solving cycle.
Table 14: Types of Responses provided by participants in the Pilot-test

<table>
<thead>
<tr>
<th>Student</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pi1</td>
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<td>5</td>
<td>0</td>
<td>1</td>
<td>4,4,3,4</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>1,1</td>
</tr>
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<td>Gender = 0</td>
<td>P&amp;S</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L.C = HC1</td>
<td>S&amp;C</td>
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<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
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<td></td>
</tr>
<tr>
<td></td>
<td>E(iii)</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>R&amp;U</td>
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<td>5</td>
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<td>5</td>
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<td>5</td>
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<td>5</td>
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<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
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<td>1</td>
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<td></td>
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<tr>
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<td>E(iii)</td>
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<td></td>
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</tr>
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<td>Pi3</td>
<td>R&amp;U</td>
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<td>5,5,2,5</td>
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<td>5</td>
<td>5,3</td>
<td>5</td>
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<td>P&amp;S</td>
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<td>7</td>
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<td>5</td>
<td>5,5</td>
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<td>L.C = HB2</td>
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<td>3</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>4,3</td>
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<td>0</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td></td>
<td>E(iii)</td>
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<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Pi4</td>
<td>R&amp;U</td>
<td>5</td>
<td>4</td>
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<td>5,5,4,5</td>
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<td>3</td>
<td>0</td>
<td>5,0</td>
</tr>
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<td>P&amp;S</td>
<td>2</td>
<td>2</td>
<td>5*</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>5,0</td>
<td>3</td>
</tr>
<tr>
<td>L.C = HB2</td>
<td>S&amp;C</td>
<td>3</td>
<td>1</td>
<td>4*</td>
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<td>3</td>
</tr>
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<td></td>
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<tr>
<td></td>
<td>E(iii)</td>
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<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pi5</td>
<td>R&amp;U</td>
<td>3</td>
<td>5</td>
<td>5</td>
<td>5,5,4,4</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>3,0</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S</td>
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<td>2</td>
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</tr>
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<td>L.C = HB2</td>
<td>S&amp;C</td>
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<td></td>
</tr>
<tr>
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<td>E(iii)</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pi6</td>
<td>R&amp;U</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>5,5,5,5,5</td>
<td>5</td>
<td>5</td>
<td>5,5</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S</td>
<td>5</td>
<td>4</td>
<td>5*</td>
<td>6(h)</td>
<td>5</td>
<td>5</td>
<td>5,5</td>
<td>3</td>
</tr>
<tr>
<td>L.C = HA1</td>
<td>S&amp;C</td>
<td>3</td>
<td>2</td>
<td>4*</td>
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<td>4</td>
<td>4,4</td>
<td>1</td>
<td>3</td>
</tr>
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<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td></td>
<td>E(iii)</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pi7</td>
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<td>4</td>
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<td>5</td>
<td>5,1</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S</td>
<td>5</td>
<td>0</td>
<td>2</td>
<td>6(a)</td>
<td>0</td>
<td>3</td>
<td>5,1</td>
<td>3</td>
</tr>
<tr>
<td>L.C = HB1</td>
<td>S&amp;C</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4,0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Time = 110</td>
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<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Table 14: Types of Responses provided by participants in the Pilot-test

27Pi1 = Pilot Participant 1; Gender: 0 = Male, 1 = Female; L.C = Leaving Certificate grade achieved: HC1 = Higher Level C1 grade. Time = Time (in minutes) spent on the assessment, by the participant. R&U = Read and Understand, P&S = Plan and Solve, S&C = Solution and Checking. G(i-ii) = Generalising from part (i) to part (ii) for Q3. E(iii) = Explanation for part (iii) in Q6. Q4* = For Q4: R&U = Understand and Represent the problem situation. P&S = Understand the relationship between the symbolic representation of the problem situation and the problem situation, and Solve. See (Appendix J) for description of type of solution. Types and scores: A Type 0 or 1 response = 0 marks, a type 2 response = 1 mark, a type 3 response = 2 marks, a type 4 response = 3 marks and a type 5 = 4 marks.
The percentage of the fourth year pre-service teachers at each type of response for the reading and understanding phase of the problem solving cycle is shown in Table 15. The percentage of 1st years (2015) at each response type for reading, understanding and representing the problem situation in problem 4 (a) is shown in Table 16.

The analysis of problem 4 is separate from the other problems due to the nature of the problem. Part (a) requires the sketching of a graph to model a particular situation and provide an explanation of how the graph sketched models the situation. The independent and dependent variables are provided in the question. Part (b) provides the formulae for each of the situation in part (a) and asks the respondents to match the situation to the correct formula and then solve some problems based on these. The fact that students were given the formulae makes this problem less open in nature than the other problems and for this reason (in addition to the marking of part (b) being different to the marking of the other problems) it is looked at separately in the analysis.

<table>
<thead>
<tr>
<th>Pilot</th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
<th>Type 5</th>
</tr>
</thead>
<tbody>
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<td>Problem 1</td>
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<td>0</td>
<td>14.3</td>
<td>14.3</td>
<td>0</td>
<td>71.4</td>
</tr>
<tr>
<td>Problem 2</td>
<td>28.6</td>
<td>0</td>
<td>0</td>
<td>14.3</td>
<td>14.3</td>
<td>42.9</td>
</tr>
<tr>
<td>Problem 3</td>
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<td>14.3</td>
<td>0</td>
<td>0</td>
<td>42.9</td>
<td>42.9</td>
</tr>
<tr>
<td>Problem 4</td>
<td>28.6</td>
<td>14.3</td>
<td>0</td>
<td>28.6</td>
<td>14.3</td>
<td>14.3</td>
</tr>
<tr>
<td>Problem 5</td>
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<td>0</td>
<td>28.6</td>
<td>14.3</td>
<td>57.1</td>
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<td>Problem 6</td>
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<td>Problem 7</td>
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<td>28.6</td>
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<td>28.6</td>
</tr>
<tr>
<td>Problem 7(ii)</td>
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<td>28.6</td>
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<td>14.3</td>
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<td>Problem 8</td>
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<td>14.3</td>
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<tr>
<td>Problem 9</td>
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<td>14.3</td>
<td>14.3</td>
<td>0</td>
<td>71.4</td>
</tr>
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</table>

Table 15: Percentage of the Fourth Year Pre-service Teachers at each Response Type for the Reading and Understanding Phase of the Pilot-Test.

<table>
<thead>
<tr>
<th>Pilot</th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
<th>Type 5</th>
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</thead>
<tbody>
<tr>
<td>Problem 4 a(i)</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>14.3</td>
<td>85.7</td>
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<td>Problem 4 a(ii)</td>
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<td>0</td>
<td>0</td>
<td>28.6</td>
<td>71.4</td>
</tr>
</tbody>
</table>

Table 16: Percentage of the Fourth Year Pre-service Teachers at each Response Type for Reading, Understanding and Representing the Problem Situation in problem 4 of the Pilot-Test.
This group experienced most difficulty with the reading and understanding of problem 5. Two of the pre-service teachers made no attempt at this problem. There was incorrect understanding of the relations among the data shown by another two and one showed complete misunderstanding by giving a result which was longer than the quickest shop working solo. The extra information (which was not necessary to determine the solution) presented in the problem caused difficulty for the pre-service teachers. Problem 7 (ii) also gave difficulty to this group. The pre-service teachers showed difficulties in adhering to the restrictions set in the problem despite being capable of adhering to them for part (i) of the problem. The participants demonstrated good ability in the graphical representations of the problem situations in problem 4. The problem situation (4a(iii)) which required the participants to sketch a graph to model the cooling of a kettle, caused the most difficulty (of the four situations in problem 4) for this group, with 28.6% demonstrating only a very basic understanding or else substantial error in the graphical representation. Participant Pi1 gave correct reasoning in his explanation stating that the cooling is possibly faster at the beginning than at the end, however the graph drawn does not reflect this reasoning. He drew a concave down graph to model the situation, which had a steeper slope towards the end than at the start and also failed to notice that the curve would level off towards room temperature after some time. Participant Pi3 gave some correct reasoning in his explanation, but failed to realise that the kettle will cool quicker initially than it does as it approaches the temperature of the room so he drew a decreasing linear graph instead of a curve. He did make note of the temperature remaining constant at end, once it has reached room temperature (only realised this after seeing the formula - stated that he made a change to the graph after seeing the formula).

The percentage of the fourth year pre-service teachers at each type of response for the planning and solving phase of the problem solving cycle is shown in Table 17. The percentage of the fourth year pre-service teachers at each type of response for understanding the relationship between the symbolic representation of the problem situation and solving problem 4(b) is shown in Table 18.
Table 17: Percentage of the Fourth Year Pre-service Teachers at each Response Type for the Planning and Solving Phase of the Pilot-Test.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
<th>Type 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>0</td>
<td>14.3</td>
<td>14.3</td>
<td>14.3</td>
<td>0</td>
<td>57.1</td>
</tr>
<tr>
<td>Problem 2</td>
<td>28.6</td>
<td>0</td>
<td>28.6</td>
<td>0</td>
<td>28.6</td>
<td>14.3</td>
</tr>
<tr>
<td>Problem 3</td>
<td>0</td>
<td>14.3</td>
<td>14.3</td>
<td>0</td>
<td>14.3</td>
<td>57.1</td>
</tr>
<tr>
<td>Problem 5</td>
<td>28.6</td>
<td>42.9</td>
<td>14.3</td>
<td>0</td>
<td>0</td>
<td>14.3</td>
</tr>
<tr>
<td>Problem 6</td>
<td>0</td>
<td>14.3</td>
<td>14.3</td>
<td>28.6</td>
<td>0</td>
<td>42.9</td>
</tr>
<tr>
<td>Problem 7</td>
<td>0</td>
<td>28.6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>71.4</td>
</tr>
<tr>
<td>Problem 7(ii)</td>
<td>28.6</td>
<td>28.6</td>
<td>0</td>
<td>14.3</td>
<td>0</td>
<td>28.6</td>
</tr>
<tr>
<td>Problem 8</td>
<td>0</td>
<td>14.3</td>
<td>14.3</td>
<td>57.1</td>
<td>0</td>
<td>14.3</td>
</tr>
<tr>
<td>Problem 9</td>
<td>0</td>
<td>42.9</td>
<td>28.6</td>
<td>0</td>
<td>0</td>
<td>28.6</td>
</tr>
</tbody>
</table>

Table 18: Percentage of the Fourth Year Pre-service Teachers at each Response Type for Understanding the Relationship between the Symbolic Representation of the Problem Situation and the Problem Situation and Solving.

This group demonstrated high difficulty in formulating a correct plan for problems 5 and 9, almost half of them formed an inappropriate plan for each of these problems. The proportional reasoning, along with the extra information caused high difficulty in problem 5. The failure to consider of all of the combinations of squares caused difficulty and also there was evidence of difficulty among some of the pre-service teachers with monitoring of the solution process in problem 9. One of the pre-service teachers demonstrated substantial misunderstanding of the relationship between the symbolic representations of the problem situation and the problem situation for problem 4(b). He correctly matched the formula for the ferris wheel but failed to match any other formula to the correct situation. He matched the linear formulae for the candle to kettle situation, despite previously stating in part (a) that the kettle situation was modelled by a curve not a line. Matched the formula for the kettle cooling to the car value situation, he gave reason for this as being that it starts off decreasing rapidly. He did not think of looking at the magnitude of the numbers and relate to room temperature, initial value of car etc. He did state that he guessed most of the answers and was not sure of his guesses, he did not progress to solving the situations. The other six showed good understanding of the relationship between the symbolic representations of the problem situation and the problem situation and solved most of the parts appropriately.
The percentage of the fourth year pre-service teachers who completed the pilot-test, at each type of response for the solution and checking phase of the problem solving cycle is shown in Table 19.

<table>
<thead>
<tr>
<th>Pilot</th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>14.3</td>
<td>0</td>
<td>0</td>
<td>42.9</td>
<td>28.6</td>
</tr>
<tr>
<td>Problem 2</td>
<td>28.6</td>
<td>28.6</td>
<td>28.6</td>
<td>14.3</td>
<td>0</td>
</tr>
<tr>
<td>Problem 3</td>
<td>14.3</td>
<td>14.3</td>
<td>14.3</td>
<td>0</td>
<td>57.1</td>
</tr>
<tr>
<td>Problem 5</td>
<td>71.4</td>
<td>14.3</td>
<td>0</td>
<td>14.3</td>
<td>0</td>
</tr>
<tr>
<td>Problem 6</td>
<td>14.3</td>
<td>42.9</td>
<td>0</td>
<td>0</td>
<td>42.9</td>
</tr>
<tr>
<td>Problem 7(i)</td>
<td>28.6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>71.4</td>
</tr>
<tr>
<td>Problem 7(ii)</td>
<td>57.1</td>
<td>0</td>
<td>14.3</td>
<td>14.3</td>
<td>14.3</td>
</tr>
<tr>
<td>Problem 8</td>
<td>14.3</td>
<td>71.4</td>
<td>0</td>
<td>0</td>
<td>14.3</td>
</tr>
<tr>
<td>Problem 9</td>
<td>42.9</td>
<td>28.6</td>
<td>0</td>
<td>28.6</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 19: Percentage of the Fourth Year Pre-service Teachers at each Response Type for the Solution and Checking Phase of the Pilot-Test.

The pre-service teachers showed most difficulty in obtaining the correct solution to problems 2, 5 and 8, with only one response being correct for each of these problems. Failure to relate the times taken by the two trains when they had traveled the same distance, and incorrect use of the extra information caused difficulties in problem 2. Incorrect use of extra information (difficulty understanding the structure of the problem) and difficulty with proportional reasoning was evident in the responses to problem 5. Difficulty in the understanding of the concept of proof was shown by one member of this group, who believed showing that the statement was true for one case was sufficient. The majority showed a lack of flexibility in algebraic manipulation in their responses to problem 8.

The percentage of the fourth year pre-service teachers who completed the pilot-test with the highest type of response over each phase of the problem solving cycle is shown in Table 20 and Figure 16.
Table 20: Percentage of the Fourth Year Pre-service Teachers at the Highest type of Response over each Phase of the Problem Solving Cycle of the Pilot-Test (* includes those who gave a correct plan and solution based on an incorrect interpretation).

Although a high percentage of the pre-service teachers showed complete understanding of problems 8 and 9 (85.7% and 71.4% respectively), the percentage forming a correct plan free from error for each of these problems was substantially lower (14.3% and 28.6% respectively). There was also a considerable drop in the percentage of the pre-service teachers who formed a correct plan (14.3%) from those who demonstrated complete understanding (42.9%) of problem 2.
Figure 16: Percentage of the Fourth Year Pre-service Teachers at the Highest type of Response over each Phase of the Problem Solving Cycle in the Pilot-Test
Charles et al. (1987) summed the points $x_u$, $x_p$, $x_a$ assigned for each of the problem solving phases $U$ (understand the problem), $P$ (planning a solution) and $A$ (getting an answer) respectively (Appendix H), to give an overall score of $x_u + x_p + x_a$. Similarly in this research study, the points assigned from the scoring scale for each phase (as described in the scoring framework) were summed up to give a overall score for each of the phases of the problem solving process for each participant (i.e the points assigned for the R&U phase for each of the nine problems were summed to give an overall score for R&U, similarly for the overall score for the P&S, and the S&C phases (Table 21). However due to the potential overlap in the planning and solving and solution and checking phases, it was decided to calculate the total score across the phases for each participant’s performance in the pre and post-tests as described in Table 25.

The scoring scale for assigning marks to the types of answers provided by the participants of the pilot-test was used to calculate the total score for each participant:

- a type 0 or a type 1 response was awarded 0 marks;
- a type 2 response was awarded 1 mark;
- a type 3 response was awarded 2 marks;
- a type 4 response was awarded 3 marks;
- a type 5 response was awarded 4 marks;

The maximum score that can be achieved per phase is:

- Total R&U = $(13 \times 4) = 52$
- Total P&S = $(9 \times 4) = 36 + 6$ (for prob 4)
- Total S&C = $(9 \times 3) = 27$

The total score of each participant for each of the phases is shown in Table 21. Differences in the total scores for each phase were analysed using the Kruskal-Wallis H Test in SPSS (Statistical Package for Social Sciences) 24.0 for Windows to determine if there was a statistical significant difference in the total score obtained between:

- gender (not for the Pilot-test as only one female in group)
- Leaving Certificate grade obtained
- Year of BSc. course the participants are currently in (not for the Pilot-test as all in same year)
The Kruskal-Wallis Test is a nonparametric (rank-based) test used to determine if there are statistically significant differences between two or more groups of the categorical independent variable (e.g., Year of BSc. course) on an ordinal dependent variable (e.g., total R&U score) (Laerd Statistics, 2013). In order to use this test, four assumptions must be met:

1. the dependent variable must be measured on ordinal or continuous level (e.g., Likert scale data is ordinal);
2. the independent variable consists of two or more categorical independent groups (each participant belongs to one and only one group, e.g., A, B or C grade in Leaving Certificate);
3. independence between observations (e.g., independence between the total scores of the participants);
4. homogeneity of variance of the distribution of the ordinal dependent variable for each group of the independent variable.

(Laerd Statistics, 2013)

To determine if homogeneity of variance was met, a one way analysis of variance (ANOVA) test on the absolute value of the ordinal dependent variable by the categorical independent variable was conducted. To assess whether the condition of homogeneity of variance was met between the groups (A grade, B grade, C grade), a one way analysis of variance (ANOVA) test on the absolute value of total reading and understanding (the ranked TRU score – the mean rank TRU score) by grade was carried out. The ANOVA test ($F = 1.857$, $p$-value $= 0.269$) showed that the condition of homogeneity of variance between the different grade groups was met. The Kruskal-Wallis test showed no statistically

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28 Mann Whitney U Test usually used if only two groups

---

<table>
<thead>
<tr>
<th>Participant</th>
<th>Pre-intervention Total Score</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total R&amp;U Score (TRU)</td>
<td>Total P&amp;S Score (TPS)</td>
<td>Total P&amp;S Score (TPS prob 4)</td>
<td>Total S&amp;C Score (TSC)</td>
</tr>
<tr>
<td>Pi1</td>
<td>22</td>
<td>5</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Pi2</td>
<td>46</td>
<td>23</td>
<td>6</td>
<td>14</td>
</tr>
<tr>
<td>Pi3</td>
<td>42</td>
<td>22</td>
<td>6</td>
<td>13</td>
</tr>
<tr>
<td>Pi4</td>
<td>40</td>
<td>18</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>Pi5</td>
<td>34</td>
<td>11</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Pi6</td>
<td>51</td>
<td>33</td>
<td>5</td>
<td>19</td>
</tr>
<tr>
<td>Pi7</td>
<td>39</td>
<td>14</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 21: Individual results of each phase of the pilot problem solving assessment for the pre-service teachers
significant difference in the median reading and understanding score between the different grade groups ($\chi^2(2) = 4.821$, p-value = 0.090). The median reading and understanding score of the different grade groups is shown in Table 22.

<table>
<thead>
<tr>
<th>TRU</th>
<th>N</th>
<th>Median</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>48.5</td>
<td>-</td>
<td>46</td>
<td>51</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>39.5</td>
<td>6.25</td>
<td>34</td>
<td>42</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 22: Average Performance in TRU by Grade Groups for Pilot-test

To assess whether the condition of homogeneity of variance was met between the groups (A grade, B grade, C grade), a one way analysis of variance (ANOVA) test on the absolute value of total planning and solving (the ranked TPS score − the mean rank TPS score) by grade was carried out. The ANOVA test ($F = 1.857$, p-value = 0.269) showed that the condition of homogeneity of variance between the different grade groups was met. The Kruskal-Wallis test showed no statistically significant difference in the median planning and solving score between the different grade groups ($\chi^2(2) = 4.821$, p-value = 0.090). The median planning and solving score of the different grade groups is shown in Table 23.

<table>
<thead>
<tr>
<th>TPS</th>
<th>N</th>
<th>Median</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>28</td>
<td>-</td>
<td>23</td>
<td>33</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>16</td>
<td>9.25</td>
<td>11</td>
<td>22</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 23: Average Performance in TPS by Grade Groups for Pilot-test

To assess whether the condition of homogeneity of variance was met between the groups (A grade, B grade, C grade), a one way analysis of variance (ANOVA) test on the absolute value of total solution and checking (the ranked TSC score − the mean rank TSC score) by grade was carried out. The ANOVA test ($F = 1.857$, p-value = 0.269) showed that the condition of homogeneity of variance between the different grade groups was met. The Kruskal-Wallis test showed no statistically significant difference in the median solution and checking score between the different grade groups ($\chi^2(2) = 4.821$, p-value = 0.090). The median solution and checking score of the different grade groups is shown in Table 24.
In this research it was noted that the possible dependence of the score assigned to the S&C phase on the score assigned to the P&S phase (particularly in the area of accuracy of completion) may result in a potential ‘double-counting’ in the calculation of the total score (across the phases) in the pre and post-test for each participant. The overall score assigned for each of the phases for each participant as presented in Table 21, highlights the specific areas of strengths and weaknesses for each participant and is therefore worth including in the analysis of results (Charles et al., 1987). The analysis of the results for the pre-tests focuses on the percentage of pre-service teachers at each response type over each phase, and the overall scores per individual phase (so the potential dependence and double counting does not arise). However when calculating the total score across the phases (shown as an example next here) for the pilot-test and in the analysis of the difference between the participants’ performance in the pre and post-tests (Chapter 7, section 7.7), the total score does not include the overall score for the S&C phase\textsuperscript{29}. i.e. the total score is:

$$TS = R\&U + P\&S + P\&S \text{ (for Problem 4)}$$

The maximum score that can be achieved is:

$$TS = (13 \times 4) + (9 \times 4) + 6$$

$$TS = 52 + 36 + 6 = 94$$

The total score and percentage for each of the participants is shown in Table 25.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
TSC & N & Median & IQR & Min & Max \\
\hline
A & 2 & 16.5 & - & 14 & 19 \\
B & 4 & 7.5 & 8 & 4 & 13 \\
C & 1 & - & - & - & - \\
\hline
\end{tabular}
\caption{Average Performance in TSC by Grade Groups for Pilot-test}
\end{table}

\textsuperscript{29}It should be noted that the author originally checked the sum across each of the three phases and the results (including the total score for solution and checking (TSC)) were the same in terms of statistical significance as the results (not including the total score for solution and checking (TSC)).
Table 25: Individual results of the pilot problem solving assessment for the pre-service teachers

To assess whether the condition of homogeneity of variance was met between the groups (A grade, B grade, C grade), a one way analysis of variance (ANOVA) test on the absolute value of TS (the ranked TS score – the mean rank TS score) by grade was carried out. The ANOVA test \( F = 1.857, \ p\text{-value} = 0.269 \) showed that the condition of homogeneity of variance between the different grade groups was met. The Kruskal-Wallis test showed no statistically significant difference in the median total score between the different grade groups \( \chi^2(2) = 4.821, \ p\text{-value} = 0.090 \). The median total score of the different grade groups is shown in Table 26.

<table>
<thead>
<tr>
<th>Participant</th>
<th>Pre-intervention</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total Score (TS)</td>
<td>Percentage</td>
</tr>
<tr>
<td>Pi1</td>
<td>22 + 5 + 1 = 28</td>
<td>30%</td>
</tr>
<tr>
<td>Pi2</td>
<td>46 + 23 + 6 = 75</td>
<td>80%</td>
</tr>
<tr>
<td>Pi3</td>
<td>42 + 22 + 6 = 70</td>
<td>74%</td>
</tr>
<tr>
<td>Pi4</td>
<td>40 + 18 + 4 = 62</td>
<td>66%</td>
</tr>
<tr>
<td>Pi5</td>
<td>34 + 11 + 4 = 49</td>
<td>52%</td>
</tr>
<tr>
<td>Pi6</td>
<td>51 + 33 + 5 = 89</td>
<td>95%</td>
</tr>
<tr>
<td>Pi7</td>
<td>39 + 14 + 5 = 58</td>
<td>62%</td>
</tr>
</tbody>
</table>

Table 26: Average Performance in TS by Grade Groups for Pilot-test

In addition to the marking described in the example of the pilot-tests, it was necessary to compare the pre and post-test results for each participant. This comparison was done using SPSS 24.0.

### 3.10.2 Data Analysis of the Pre and Post-Tests

The statistical analysis was carried out using SPSS. In this study, the variables consisted of the categorical nominal variables; gender and year, and the categorical ordinal variables; Leaving Certificate grade, type of response for reading and understanding, planning and solving and solution and checking. Total scores for
each participant were calculated by using the scoring scale to recode the types of responses into marks in SPSS and then summing the marks achieved by the participant to each problem, for each of the phases total score for reading and understanding (TRU), and total score for planning and solving (TPS).\footnote{As stated previously, it should be noted that the author originally checked the sum across each of the three phases and the results (including the total score for solution and checking (TSC)) were the same in terms of statistical significance as the results (not including the total score for solution and checking (TSC)).}

The Wilcoxon signed rank test is a nonparametric test used to compare two sets of measures on the same participants (i.e. pre and post-test results (Pallant, 2005)). The Wilcoxon signed rank test can be used when the measured data is not normally distributed (Pallant, 2005). This test has three assumptions which must be met in order for the results from the test to be valid:

1. the dependent variable must be measured on ordinal or continuous level (e.g. Likert scale data is ordinal);
2. the independent variable consists of two categorical groups, where the same participants are present in both groups (pre-test group and post-test group) and where each participant has been measured twice on the same dependent variable (each participant has a total score (for each phase and for overall total) on the pre-test and on the post-test);
3. the distribution of the differences between the two groups (differences between the pre and post-test scores) must be symmetrical.

(Laerd Statistics, 2013)

The Wilcoxon signed rank test was employed to compare the pre and post-test results of the participants of this study as the responses to the pre and post-tests were assigned a particular type of score similar to “Likert type” data (Clason and Dormody, 1994). These individual types of responses are ordinal variables. The dependent variable was the sum of these Likert type variables. The second assumption of the Wilcoxon signed rank test was also met and the third assumption was checked by determining the differences between the pre and post-test total scores (for each of the phases of the problem solving cycle) and then testing the distribution of these differences for normality in SPSS.

There has been controversy in agreement between experts on the question of whether ordinal data which has been converted to numbers can be treated as interval data or not (Sullivan and Artino, 2013). In this study this translates into the question: could the total score assigned (for each of the phases of the problem solving cycle) to the participants be considered interval data or not? What would the mean of a type 1 response (inappropriate plan) and a type 2 response (correct plan with high error) for the P&S phase be? No controversy existed with using frequencies, medians and non-parametric tests with Likert data, as the experts agreed that the ordinal data can be ranked but these experts say that the distances between the rankings are not a meaningful measure.
and that the distances between any two successive ranks are not necessarily equal (Sullivan and Artino, 2013).

Other experts state that if the data is normally distributed and the sample size consists of at least 5-10 participants/observations per group then parametric tests may be used with Likert scale ordinal data (Sullivan and Artino, 2013). Norman (2010) states that the parametric tests are more powerful than the non-parametric tests and can be used with ordinal data, even in cases where the data may not be normally distributed. Boone (2012) states that there is a difference between Likert-type data and Likert-scale data. Clason and Dormody (1994 cited in Boone, 2012) state that Likert-type data are individual questions to which there is a Likert-type of response whereas Likert scale data is the “sum or average (mean) of four or more Likert-type items” (p.3). Clason and Dormody (1994) state that medians and frequencies should be employed with Likert-type items, and that means and standard deviations should be used to describe Likert-scale data.

Sullivan and Artino (2013) state that when grouping Likert-type items to form Likert-scales, Cronbach’s alpha or the Kappa test or factor analysis should be used and suggest that researchers state clearly how they are analyzing the Likert data.

There is no definitive consensus in the literature on the analysis of Likert-type and Likert-scale data. The author used medians and frequencies to describe the individual Likert-type items as there is consensus in the literature reviewed on this aspect of analysing Likert data. The author also used medians and non-parametric tests to describe the Likert-scale data but also cross checked the results obtained from these tests with the parametric equivalent tests. The results were the same for both (although in the case of using parametric tests for the total planning and solving by year, and total solution and checking by year, the removal of P6, and the removal of P6 and P11 respectively was necessary (P6 and P11 were extreme outliers) so as not to violate assumptions of normality with the parametric tests).

3.10.3 Focus Group Data

Transcription of the focus groups was completed by the researcher before conducting an analysis of the data. QSR NVivo 11 was employed to analyse the qualitative data from the transcriptions. Thematic Analysis was utilised in the analysis of the transcriptions. Thematic analysis is a method for “identifying, analysing, and reporting patterns (themes) within data” (Braun and Clarke, 2006 p. 5). Thematic analysis was used to examine the qualitative data presented by the participants themselves in the focus groups conducted. The data was classified into themes which emerged from the qualitative data provided by the participants, this allowed the author to interpret the various experiences of

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the participates in a “rich and detailed way” (Braun and Clarke, 2006 p. 5).

3.10.4 Journal Entry Data

The researcher maintained research journals during the intervention (Phase 6) for the purpose of documenting what occurred in the classes. Due to potential researcher bias data from these journal entries are referred to on rare occasions to elaborate on information, or in order to provide a clear overall picture of the research process (the intervention in particular) to the reader. An in-depth analysis of these journal entries was not conducted.

3.11 Research Issues

During the process of completing this research study, ethical issues arose in consideration to gaining permission to work with individuals, and gaining their consent in participating in the study. Issues of validity and reliability arose in relation to the research instruments employed in this study, in addition to the methodology used to collect the data. Maintaining researcher distance and limitations of the research are also discussed in this section. The consideration of these issues in advance of conducting the research is a vital aspect in the production of valid, reliable work, which is a requirement for all research studies (Cohen et al. 2011).

3.11.1 Ethics

An application for ethical approval (Appendix E) was sent to the Ethics Committee on the 12th of January 2015. This application adhered to the ethical practices outlined by the University of Limerick’s Research Ethics Committee (ULREC). The following ethical guidelines were specified on this application:

- participants for this research would be recruited solely from the mathematics students on the mathematics teacher training programmes in the University of Limerick;
- participation is completely voluntary;
- A consent form (Appendix F) will be provided to all participants;
- all data will be confidential and any reports the author will make will not make any reference to anything about the individuals who supplied that information;
- all data will be stored on the author’s supervisor’s password protected computer.

Ethical approval was granted for this research on the 22nd of January 2015 by ULREC.
3.11.2 Researcher Distance

The researcher attempted to obtain researcher distance by maximising objectivity throughout the research. The mindset questionnaire employed in the study was sourced from the literature. The pre and post-assessments contained problems which adhered to conditions specified in the literature. The evaluation method of these assessments was adapted from a scoring scale which was designed by Charles et al., 1987. This aimed to ensure an objective evaluation of the responses provided in the assessments. Where subjectivity was a possible issue, in the case of assigning an order to particular types of answers, the researcher sought aid from an expert mathematician and a mathematics education specialist. The intervention was developed using a model of instruction design from the literature and the delivery was in alignment with a method from the literature. The research journals were written with the aid of recordings of the groups in the intervention so as to help the researcher maintain objectivity and accuracy in her writing. These are only referred to on a small number of occasions to elaborate on a point. The thematic analysis of the transcribed data from the focus group was an analysis of the data provided directly from the participants and thus objective in nature.

3.11.3 Validity

Validity is the degree of accuracy to which a measurement instrument actually measures what it is supposed to measure (Cohen et al., 2011), or the degree of honesty and depth of the data obtained in representing or explaining a phenomenon. Validity is a requirement in research, while it is not possible to guarantee 100% validity in a research study (due to subjectivity of respondents or standard error etc.), there are steps that can be taken to ensure maximum validity in the research conducted. There are several forms of validity in research. The types of validity employed depends on the research paradigm of the study (Cohen et al., 2011). Content, descriptive, theoretical and internal validity were employed in this research.

Content validity is the degree of accuracy to which a measurement instrument measures what it is supposed to measure, in addition to being a fair, instrument which is comprehensive in its inclusion of elements which are representative of the particular aspect being measured (Cohen et al., 2011). The design of the assessments and the design and delivery of the intervention were in alignment with the Framework for Teaching and Assessing Problem Solving in Mathematics, which was developed and informed by the theoretical and empirical findings of the literature review conducted. The review of the assessments by the expert panel of mathematicians also aided in the provision of content validity. The mindset questionnaires were previously validated in the research. The design of the focus group was informed by the review of theory on designing and conducting focus group interviews.
Descriptive validity is the degree of accuracy in the truthful description of the actual objective facts of the study (Cohen et al., 2011), i.e., what actually happened in the research study. The researcher maintained a high level of objectivity in the relating of information from this study, both in conducting the research and in analysing the data. Furthermore, the description provided throughout the research document is written in as scientifically objective a style of writing as possible.

Theoretical validity is the degree to which the theory explains the phenomenon, i.e., the degree to which the research instrument aligns with the theoretical context in which it is set (Strauss and Smith, 2009). In designing an instrument to measure problem solving ability, the first necessary thing to do was to define what exactly is meant by problem solving (Strauss and Smith, 2009). Otherwise there is no way of knowing exactly what you are measuring. Theoretical validity was ensured in this research by the extensive review of the literature in informing the development of the Framework for Teaching and Assessing Problem Solving in Mathematics, and by the presence of this framework in informing every aspect of the research process; from the selection of the assessment items to the marking of these assessments, to the implementation of the intervention.

Internal validity refers to the degree to which observed changes can be explained by the implementation of the intervention, i.e., the accuracy of inferences made in respect to causal relationships (Trochim, 2008). That is, what evidence exists to support the notion that what you did, caused what you observed? Trochim (2008) states that there are three criteria which have to be met in order to demonstrate evidence of a causal relationship: “temporal precedence, covariation of the cause and effect, and no plausible alternative explanations” (p.2).

Temporal precedence is showing that the cause happened in advance of the effect. As stated previously pre and post-test study designs possess the strength of temporality, which allows for making the suggestion that the outcome of the post-tests is impacted by the intervention (Thiese, 2014).

Covariation of the cause and effect is showing that there is at least some relationship between the cause and effect; that is if you participate in the intervention, then problem solving ability increases /(attitudes, mindsets, perseverance -some factor affecting problem solving ability improves) whereas if you don’t participate in the intervention, then problem solving ability does not increase /(attitudes, mindsets, perseverance - no factor affecting problem solving ability improves). This in itself is not sufficient to show evidence of causality as there could be a confounding factor(s) involved in the relationship (Trochim, 2008).

To demonstrate evidence of a causal relationship, the ruling out of no plausible alternative explanations is necessary (Trochim, 2008), that is to rule out the threats to internal validity. In this study two single groups were formed, Trochim (2008) lists the following as threats to internal validity (these items are usually
more specific to quantitative data) in a single group case:

- History - some historical event that occurred at the same time as the intervention was being implemented may have caused the outcome, not the intervention itself;

- Maturation - the natural maturation of the participants may have caused the same outcome to occur with or without them participating in the intervention;

- Testing - completing the pre-test may have prepared the pre-service teachers for the kind of intervention that was implemented, they may respond better to the intervention than they would have had they not completed the pre-test;

- Instrumentation - the change from pre-test to post-test may be as a result of the change in the test instrument, and not to the intervention;

- Mortality threat - the participants who drop out of the study may have been the lower achievers on the pre-test and not having them at the post-test phase may over-estimate the post-test results;

- Regression threat - if the sample is non-random and is made up of participants who achieve low pre-test scores, then the mean score for this sample will appear to increase (relative to the population) irrespective of participating in the intervention.

The evaluation of the pre and post-tests by the same panel of expert mathematicians, aimed at maximising the comparability of the pre and post-tests in an attempt to minimise the threat of instrumentation to internal validity. The regression threat of low attainers was not present in this study as the participants were mixed in respect to their pre-test scores. The researcher was aware of the potential mortality threat and compared the pre-test results of the dropout group to the non dropout group (Trochim, 2008). There were no major differences. Plausibility and credibility are important in achieving internal validity in qualitative data. Lincoln and Guba (1985, cited in Cohen (2011)) suggest that triangulation of methods can aid in achieving credibility in the research conducted, this was done in this research study.

3.11.4 Reliability

Reliability refers to the consistency of a measurement. Research which is reliable should produce similar results if it were conducted with a similar group of participants, in a similar context over a similar period of time as in the original study (Cohen et al., 2011). Research instruments are thus considered reliable if reproduction of the results of the study, under a similar methodology, can be achieved (Joppe, 2000).
The ability to repeat a study in the same way as completed by the author depends on the degree to which the original work has been defined, documented and presented (Schoenfeld, 2008). This study is transparent in how it was conducted. The assessments, scoring frameworks and scoring scales are clearly presented. The book of problems utilised in the intervention classes is included for other researchers to use. This book also details clearly the way in which the classes were delivered, with a note to both teachers and participants included. Any presentations used in the intervention are also included. Solutions to all the problems are also available for use by other instructors.

Pictures were taken of all of the work completed by the groups of pre-service mathematics teachers in the intervention classes. This was primarily to facilitate the emailing of class work to the participants so they could continue working on the problems outside of the classes. These also provide information to other researchers on the workings of the pre-service teachers.

For an assessment to be valid, it must also be reliable (William, 2001). Interrater reliability for the pre-test shows substantial agreement between raters ($\kappa > 0.6$) with a 95% confidence interval showing agreement between the raters as ranging from mid moderate to almost perfect agreement (section 4.4.5). Interrater reliability for the post-test shows mostly substantial agreement between raters ($\kappa > 0.6$) for seven out of nine calculated statistics with a 95% confidence interval showing agreement between the raters as ranging from moderate to almost perfect agreement (Table 40).

### 3.11.5 Triangulation

Triangulation is a method employed in research to assess the validity of the results obtained (Patton, 2002). It is the examination of a research phenomenon using multiple perspectives (Cohen et al., 2011; Denzin and Lincoln, 2000). The study of the research phenomenon through these different perspectives facilitates an increase in the confidence of the accuracy of the research results obtained (Patton, 2002). The combination of the research results obtained also provides a more complete picture of the phenomenon being studied (Patton, 2002). Thurmond (2001) states that the purpose of employing triangulation in a research study, is to decrease or neutralise the weaknesses of any single strategy and hence increase the potential for the findings to be interpreted. There are various types of triangulation employed in research studies, these include triangulation of methods, observers, theories, data sources, space/cultural and time (Cohen et al., 2011). Theoretical triangulation, data triangulation and methodological triangulation were employed in this research study.

Theoretical triangulation - is the use of two or more theories in an attempt to gain a comprehensive understanding of data and information (Thurmond, 2001). The researcher conducted an extensive review of literature while trying
to determine answers to the research questions. This resulted in the integration of several theories during the formulation of the Framework for Teaching and Assessing Problem Solving in Mathematics. These theories have been discussed in Chapter 2 and are further elaborated in the discussion of the development of the Framework for Teaching and Assessing Problem Solving in Mathematics in Chapter 4.

Data triangulation - is the use of data from different sources of information (Guion et al., 2002). The credibility (validity) of the research is enhanced by complementing one source of data with another (Guion et al., 2002). This type of triangulation was employed in this study through the collection of data from pre and post assessments, focus groups, and journal entries (which utilised photographs and recordings of participants work completed during the intervention stage). These findings are presented and discussed in Chapters 5 and 7.

Methodological triangulation - is the use of several quantitative, qualitative or a mixture of both qualitative and quantitative methods in the study of a phenomenon (Guion et al., 2002; Thurmond, 2001). In this study, triangulation ‘between methods’ (Cohen et al., 2011) was used, this type of triangulation is the use of multiple methods in the same study (Thurmond, 2001; Cohen et al., 2011). The ‘between methods’ approach facilitates a check on validity by comparing independent measures of the same construct (Campbell and Fiske, 1959 cited in Cohen et al., 2011). The mixed methods approach employed in this research satisfies the criteria for methodological triangulation.

3.12 Limitations of the Research

There were a number of limitations encountered during this research:

- the nature of the assessment may have possibly caused a reduction in the sample size of students. The fact that the time required to complete the assessment exceeded the time of a lecture may have put some students off of participating in the research.

- the sample size of pre-service teachers who completed all three components of this research (pre-test, intervention and post-test) is quite low (n = 13). Thirty-two pre-service teachers completed the pre-test, sixteen pre-service teachers completed the intervention, and thirteen completed the post-test. The small sample size subsequently limited the choice of data analysis.

- the time to complete the intervention was limited as the pre-service teachers had their own course work to complete as part of their degree. However problem solving interventions of six weeks duration have produced statistically significant results (Woodward et al., 2012).

- the number of pre-service teachers who formed a control group was limited to three and so, although the data is presented, it is limited in the contribution it makes to the research.
the participants did not fully engage with the homework aspect of the intervention. Those that did showed higher gains in performance in problem solving (P1 in the third year group). The intervention was voluntary and the workload (in terms of their university course) of the pre-service teachers was high. Comments made in relation to this in the focus group was that if this had been a compulsory intervention, they would have set aside time for homework (section 7.10.1, Appendix I)

3.13 Conclusion

This chapter discussed the ontological and epistemological stance of the researcher and the implications of this stance on the decisions made in relation to the methodology implemented in this study. The process of the research undertaken for this study was presented, along with a description of the rationale for the choice of data collection and analysis. The phases of the research were described. Validity, reliability, sampling and ethical issues that were encountered during the research process were discussed. Possible limitations of the research study were identified and discussed. Having identified the appropriate methodology to understand and address the research problem, with necessary consideration given to validity and reliability, the researcher now focuses on the development of the Framework for Teaching and Assessing Problem Solving in Mathematics. This framework is at the core of this research study and provides the coherent structure for the complete work.
4 Framework for Teaching and Assessing Problem Solving (F-TAPS)

4.1 Introduction

The difficulties experienced by students in problem solving in mathematics were discussed in Chapter 2. The literature review indicates that students have substantial difficulty in applying their mathematical knowledge to solve problems in unfamiliar contexts (Chief Examiner’s Report, 2005; 2015). Students are also showing a decline in the time they are willing to persevere in solving a problem (Chief Examiner’s Report, 2005; 2015). The literature review also revealed the difficulties experienced in attempts to implement the research conducted in problem solving to date, into school practice (Lester and Kehle, 2003). The findings from the literature review conducted in Chapter 2, informed the development of the Framework for Teaching and Assessing Problem Solving in mathematics (F-TAPS) (Figure 17). The main issues which were identified in relation to problem solving were the:

- formulation of a rich knowledge base;
- development of mathematical thinking;
- influence which affective factors have on students ability to engage in problem solving;
- consideration of mathematics as a design science;
- benefits of extending problem solving to include modelling activities;
- influence that assessment has on the mathematics that is actually engaged with in the classroom.

These aspects form the theoretical basis of the framework for teaching and assessing problem solving in mathematics. Each of these will be briefly discussed before presenting the framework.

4.2 Formulating the F-TAPS Framework

The purpose of this study was to create a framework for teaching and assessing problem solving in mathematics, which addresses the complexity of problem solving in mathematics, by integrating theories in relation to the discussed significant knowledge, and affective factors, which affect the growth of problem solving proficiency in individuals. The author created a framework entitled Framework for Teaching and Assessing Problem Solving (F-TAPS) in mathematics, which considers the discussed complexities (Chapter 2). The proposed structure of the framework in Figure 17 and Figure 18 integrates knowledge and affective factors in the design and choice of mathematical problems, which are
aimed at provoking mathematical thinking with the goal of developing mathematical proficiency in problem solving situations. Mathematical thinking (Tall 2001; Mason et al. 2010) is central in this framework. While similar frameworks exist in the review of the literature conducted, the author found none that encompassed all of the necessary components that the author’s developed intervention does. The author’s F-TAPS comprises 6 integrated components:

- knowledge;
- intrinsic motivation;
- mathematical thinking;
- mathematical proficiency;
- assessment;
- teaching strategies.

4.2.1 Theoretical Basis for Framework

1. Knowledge: The following theoretical aspects underpin the knowledge component of the proposed model:

   Bruner (1957) theory is built upon social constructivism which highlights the importance of the construction of meaningful knowledge by humans through the transmission of language and shared understanding. Bruner’s theory supports the learning of new concepts through three phases of representation (irrespective of the age of the learner):

   - enactive - learning new concepts through the use of manipulative based activities;
   - iconic - using diagrams, pictures, graphs and other visual representations/organisations to represent the concrete situation. The iconic representation should resemble the concrete situation it represents;
   - symbolic - uses words and symbols to represent the problem situation.

   Although these stages are presented by Bruner sequentially, some students may be simultaneously presented with all three stages (Wood et al., 1976). The time to progress through the different stages of representation vary per individual student. Representation is an important resource in problem solving as it structures and modulates action and perception (Vergnaud, 2013). Bruner’s theory of learning is about facilitating and developing students’ mathematical thinking and problem solving skills (Bruner, 1961). Mason (2011) notes that the manipulating process facilitates the development of understanding underlying structural relationships.

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Diene’s (1960) multiple embodiment principle is about presenting students with many problems of various contexts but with similar structural/conceptual components so as to facilitate the individual perception and formation of the underlying concept by each student. What one mathematical concept means to an individual student is dependent on how they have connected it to their overall body of knowledge (Hiebert & Lefevre, 1986). If a student can recognise the same mathematical concept in several different representations, then they are building up their repertoire of approaches to solving problems. Well connected representations facilitate thinking which allows for knowledge to be adapted to various problem situations (Minsky, 1986). Diene’s (1960) multiple embodiment principle is incorporated into the author’s F-TAPS, with the aim of improving the flexibility of students’ thinking in order to facilitate the use of an alternative method to solve a problem when they are unable to proceed with a specific method. Diene’s principle supports the development of understanding of the underlying structure of a problem while allowing for the recognition of the same mathematical concept in different representations.

The problems included in the intervention phase of this research consisted of problems that had similar structural components but varied in context (e.g. problems 12 - 17 in the Course Notes). Problems which used manipulative based activities were also included in the intervention phase of the research (e.g. problem 11 - the participants used matches to form patterns in the sequence in an attempt to generalise; problem 15 - participants used computer simulation to aid in the generalising of the pattern; problem 24 - participants used Geogebra for varying the conditions of the problem to observe the resulting change in the appearance of the graph for a particular case and to aid in determining if this would also be true for the general case). The majority of the problems required the formulation of a symbolic representation with many requiring the formulation of iconic representations (e.g. problems 7 and 14 require the drawing of several representations of the situation, problem 18 prompted the use of an iconic representation and problems 16 and 22 required graphical representations). Problems requiring different representations were also included (e.g. problem 33 - participants were asked to match verbal, visual and symbolic representations of situations and where a particular representation was missing, the participants were asked to form the missing representations to match the other representations).

The thought processes in the van Hiele (1984c) thinking levels are “visual, descriptive, relational, deductive and rigor”. Levels are hierarchical and sequential and are utilised in the sequencing
of problems presented to students during the intervention. For the purpose of contributing to this model, these thought processes (levels) are also employed in more of a simultaneous manner during the solving of mathematical problems (similar to Anderson’s and Krathwohl’s (2001) taxonomy in that higher order thinking does not necessarily occur after lower order thinking). Problems should be described by the students and represented visually (i.e. diagram/picture drawn) or organised in some way to externally represent the internal mental model associated with the initial perception of the problem situation. Students should be aware of the relationship within and between problems and solutions, make logical deductions and advances in problem solving and be rigorous in the completion of their solution. Teachers should also model this practice of making thinking explicit along these processes (similar to cognitive apprenticeship). For learners with weak mathematical proficiency, the external representation of the internal “mental model” and manipulations on this external representation should help them to “interiorise” this model for themselves and these activities should then build up their repertoire of interiorised “mental models” of more and more problem situations. This involves working at the van Hiele level 0, which uses sensory experiences to facilitate the recognition and classification of mathematics for problem solving situations. High expectations should be promoted.

The Modified Moore Method (section 6.4.2 (2)) was the teaching strategy employed in the intervention. This facilitated the practice of making thinking explicit between teacher-participant and participant-participant. The presentation of participants’ work as part of this strategy also facilitated the promotion of high expectations and the development of rigor in the completion, explanation and defence of participants’ solutions.

Metacognition and heuristics are also important in the acquisition of mathematical knowledge, logical reasoning and achievement in problem solving (Mevarech and Fridkin, 2006; Schneider and Artelt, 2010). Metacognition is self-knowledge of how one learns, understands and remembers information along with self-knowledge of which methods best facilitate the acquisition, understanding and retention of this information. It involves knowing and understanding how new information links to existing knowledge as well as realising the appropriateness of application of knowledge in familiar and unfamiliar situations, along with self-monitoring of the application of this knowledge (Flavell, 1979).

Heuristics are suggestions and questions which aim to assist prob-
lem solvers in their attempts to understand a mathematical problem (Pólya, 1945). The heuristics present in many of the problem solving models that have been developed can act as a stimulus for metacognition and mathematical thinking. The self-questioning present in these models facilitates reflective thinking among students. These questions may force them to stop and consider their progress and the appropriateness of their chosen methods. “One of the hardest parts of problem solving is to ask the right questions and the only way to learn to do so is to practice” (Halmos, 1980, p.524).

During the intervention stage of this research, participants were presented with and informed of the mathematical thinking model developed by Mason et al. (2010) and the problem solving models developed by Pólya (1945) and Schoenfeld (1980). The heuristics given by Schoenfeld (1980) based on Pólya’s work were also presented to the participants. Guidelines on using the “perception, action, and reason with reflection” aspect of Tall’s framework were provided to the participants to use when solving problems (Tall, 2008, p.1). The participants were also asked to complete a metacognitive journal entry for a selected number of problems with the aim of facilitating and developing the practice of asking the right questions.

2. Intrinsic motivation: The theoretical aspects/models which follow form the foundation of the intrinsic motivation component of the proposed model:

Mindsets are beliefs which people have about themselves: beliefs about their qualities, such as their intelligence and talents (Dweck, 2008). There are two types of mindsets which Dweck has observed: fixed mindset and growth mindset. Students who have a fixed mindset, irrespective of their ability believe that it is ability, not effort that determines success. They also believe that their intelligence is fixed and cannot be changed. Students who have a growth mindset irrespective of their ability, believe that it is effort and perseverance, not ability that determines success. They believe that their intelligence can be developed through dedication and hard work. Challenging tasks like problem solving in mathematics can be viewed with fear (Hong et al., 1999) among students with a fixed mindset. A longitudinal study based on teaching students (12-13 years old) about the growth mindset (Blackwell et al., 2007) resulted in a threefold increase in perseverance and engagement in mathematics among students who received growth mindset training, in comparison to corresponding groups who received other instruction. There is a strong positive correlation between perseverance and problem-solving performance (OECD, 2014), with the strength of this correlation increasing for those students who perform at the higher percentiles.
(1 point increase in perseverance with a 38.5 point increase in performance at the 90th percentile, compared with a 20 point increase in performance at the 10th percentile (OECD, 2014)).

The participants were given a presentation on mindset (Appendix U) informing them of both mindsets, the education implications of both mindsets and how to develop the growth mindset at the beginning of the first class in the intervention.

**Intuition** Poincaré (1952) states that the fundamental goal of mathematics education is to develop capacities of the mind. Poincaré includes intuition as one of these capacities. Jung et al. (2007) state that when students’ intuitions are respected and appreciated and when they are encouraged to explain their solution methods, as well as listening carefully to the solution paths of others, they naturally absorb more advanced methods of solving problems. The use of manipulatives are utilised in the intervention as they can help in the development of secondary intuitions (Mason, 1996 in Ben-Zeev and Star, 2002)) by helping students to specialise and generalise mathematical concepts and situations. The processes of specialising and generalising are of more importance than the use of the manipulatives in facilitating the development of secondary intuitions (Ben-Zeev and Star, 2002).

Problems which required the processes of specialisation and generalisation during the completion of them were included in the intervention. As part of The Modified Moore Method the participants were required to present, explain and defend their solutions to problems. Participants were also required to listen to the solutions of their peers, to ask questions (where necessary/appropriate) about the presented solutions and to reach consensus on whether a solution was correct or not.

**Interest** The four stages of interest development (Hidi and Renninger, 2006 discussed in section 2.4.6) are included in the author’s F-TAPS in mathematics. Situational interest was considered by including puzzles on the assessments and during the intervention, and by including groupwork and tasks requiring the use of computers during the intervention. The interests of the participants were taken into consideration for the contexts of the tasks during the intervention to aid with maintaining situational interest. The participants were asked to state their interests at the time of the pre-assessment and the author developed the problems for the intervention using these interests as contexts for some of the problems. The Modified Moore Method was employed as the mode of delivery of the intervention to
aid with emerging individual interest, this method presents challenging tasks with the aim of facilitating the generation of questions by the participants which leads to the construction of knowledge.

This completes the intrinsic motivation component of the author’s F-TAPS in mathematics. The next component is at the centre of the author’s F-TAPS in mathematics and integrates the knowledge and intrinsic motivation components of the F-TAPS while driving the proficiency component.

3. **Mathematical thinking**: is the central component of the author’s F-TAPS in mathematics. Tall’s framework (2008) for the development of mathematical thinking is based on “perception, operation and reason”. The perception formed of a problem situation by the problem solver is in the form of a “mental model” (Johnson-Laird, 2004). Making sense of this problem situation in one’s mind involves mentally operating on this internal model (Johnson-Laird, 2004). The more sophisticated the mathematical thinking of the individual, the more they can work with increasingly complex ideas as their ability to form different mental representations of the same problem situation and their ability to manipulate these mental models in their mind develops (van Hiele, 1984c; Tall, 2001; Dubinsky and McDonald, 2001). This may be due to the development of the ability to shorten the problem solving processes involved in determining a solution (Krutetskii, 1976; Sfard, 1991; Tall 2001), in addition to the ability to conceive of these processes as “objects” (Sfard, 1991; Dubinsky and McDonald, 2001) as well as processes. An “object” perception is the mental model that results from the individual’s awareness of a mathematical process as a whole entity, as well as the realisation that further actions and processes can be applied to this whole entity (Arnon et al., 2014). Sfard (1991) refers to this ability as being able to perceive mathematical ideas both structurally and operationally and states that both are essential in the development of conceptual understanding of mathematics. Reflecting on the outcomes of the operations along with providing explanations and justifications for the chosen solution method aid in the development of a structural conception, which facilitates all the cognitive processes required for effective problem solving and learning in mathematics (Sfard, 1991).

The mathematical thinking processes of specialisation and generalisation (Mason et al. 2008) along with Tall’s framework (2008) incorporates most of the common mathematical thinking processes (flexibility, generalising and determining structural relationships) present in several of the models and theories of mathematical understanding, which have been developed by various researchers (Krutetskii, 1976; van Hiele 1984c; Sfard, 1991; Dubinsky and McDonald, 2001). The design and choice of the problems utilised in the assessments and during the intervention are based on the requirement and development of mathematical thinking and proficiency.
in the acquisition of knowledge, while also catering for the emotional engagement of the learners.

The next component is the mathematical proficiency and assessment component of the author’s F-TAPS in mathematics. To assess for competency in problem solving in mathematics, the participants\footnote{only those participants who completed pre-test, intervention and post-test} demonstration of mathematical proficiency, along with the degree of sophistication displayed in mathematical thinking, problem solving ability, problem solving skills, and competency shown during the completion of the phases of the problem solving process (Pólya) were examined.

4. **Mathematical proficiency**: The objective of the Leaving Certificate Project Maths syllabus is to develop mathematical proficiency in students. There are five components of mathematical proficiency (National Research Council, 2001) listed in the Leaving Certificate Project Maths syllabus:

- **Conceptual understanding** involves understanding mathematics in an interconnected way, which facilitates the learning of new mathematical ideas by relating them to previously gained mathematical knowledge. It also facilitates the productive use of mathematics in problem situations (Kilpatrick et al., 2001).
- **Procedural fluency** involves the appropriate use of procedures and accuracy in executing these procedures in a flexible and efficient way.
- **Strategic competence** is the ability to devise, represent and solve problems in contexts which one is accustomed to and also in unfamiliar contexts.
- **Adaptive reasoning** is the ability to engage in logical thought, explanation, reflective processes, communication and justifying one’s reasoning (Kilpatrick et al., 2001).
- **Productive disposition** is the consideration of mathematics as a useful body of knowledge which makes sense, along with the belief in the importance of persistence in one’s work in mathematics and also to believe in one’s own ability (Kilpatrick et al., 2001).

Competency is measured by an individual’s ability to use and adapt their current cerebral resources to contend with new problem situations (Vergnaud, 2009) hence the problems in the assessment should be unfamiliar to the students in order to measure their overall competency in problem solving.

Schoenfeld (2007) identifies four elements of proficiency in mathematics: knowledge base, strategies, metacognition and beliefs and dispositions. His reference to Pólya in the strategies component of proficiency and also his reference to
metacognition are not specifically\textsuperscript{32} mentioned in the view of mathematical proficiency given in the Leaving Certificate 2015 syllabus but are incorporated in the author’s Framework for Teaching and Assessing Problem Solving in Mathematics. Figure 17 shows the key components of the author’s F-TAPS in mathematics. Figure 18 shows a more detailed version of the author’s F-TAPS in mathematics. The teaching strategies that were used in conjunction with the author’s F-TAPS in mathematics in the provision of the intervention phases of this research are discussed in chapter 6. The author’s F-TAPS is used in conjunction with the Project Maths syllabi and the type of problems advocated by Halmos (1975) to formulate a problem solving pre and post assessment (section 4.4.4). The next section looks at how the author’s F-TAPS in mathematics relates to the teaching and assessment of mathematics (problem solving in particular) in Ireland.

\textsuperscript{32}Not explicitly mentioned in the case of Pólya’s Strategies and Heuristics
Knowledge

- Diene’s (1960) multiple embodiment principle
- Bruner’s (1957) modes of representation
- Van Hiele’s (1957) levels of thinking
- Metacognition (Flavell, 1976)
- Heuristics (Polya, 1945)

Intrinsic Motivation

- Mindset & Perseverance (Dweck, 2007)
- Hidi & Renninger (2006) model of interest development
- Intuition & own strategies (Fischbein, 1987)
- Beliefs & attitudes (McLeod and Adams, 1989)

Mathematical Thinking

Figure 17: Framework for Teaching and Assessing Problem Solving in Mathematics
Many problems of various contexts but with similar structural/conceptual components to facilitate individual perception of underlying concept by each student.

Manipulative based activities, followed by visual images, and then proceeding to represent images using words and symbols to develop students’ mathematical thinking and problem solving skills.

Presentation on mindset to aid in students’ perseverance levels in the problem solving tasks. Tasks related to interest to also aid in perseverance levels of students engaged in problem solving.

Intuition and Own Strategies
Process stressed over solution. Manipulatives to help with specialising and generalising, Augmentation and justification in reaching consensus on solution. Experimentation & Reflection

Beliefs and Attitudes
Mindset.
Respecting intuitions.
Valuing mistakes.

Conceptual Understanding
Relating new mathematics ideas to previously gained mathematical knowledge.
Productive use of mathematics.

Procedural Fluency
Appropriate, flexible and accurate use of procedures.

Strategic Competence
Formulate, represent and solve problems in contexts which one is accustomed to and also in unfamiliar contexts.

Adaptive Reasoning
Logical thought in explanation and in justification of reasoning. Ability to reflect and communicate one’s reasoning.

Productive Disposition
More than one attempt if unsuccessful first attempt.

Heuristics and Metacognition
Presentations & using the problem solving template with heuristics included when solving problems. Reasoning made explicit during problem solving and reflection on understanding & performance.

Intuition and Mindset
Presentation on mindset to aid in students’ perseverance levels in the problem solving tasks. Tasks related to interest to also aid in perseverance levels of students engaged in problem solving.

Heuristics and Metacognition
Presentations & using the problem solving template with heuristics included when solving problems. Reasoning made explicit during problem solving and reflection on understanding & performance.

Beliefs and Attitudes
Mindset.
Respecting intuitions.
Valuing mistakes.

Design of Problems

Figure 18: Framework for Teaching and Assessing Problem Solving in Mathematics (detailed)
4.3 Relating the Framework to the Irish syllabus

The learning outcomes in the syllabus state that the learning that students experience during their study of mathematics should be a contributing factor in the development of their problem solving skills (NCCA, 2015). Applications of mathematical knowledge and skills is referred to as a means of contributing to this growth in problem solving skills. The Leaving Certificate syllabus (for Foundation level in particular) advocates the consideration of the interests and different learning styles (visual, spatial and numerical) of students in the teaching of mathematics (NCCA, 2015). The Leaving Certificate syllabus also proposes that the study of mathematics should be suited to the interests of the learners at all levels. The knowledge and intrinsic motivation components of the author’s F-TAPS directly relate to these aspects of the mathematics syllabus in Ireland and also make explicit the means of how to achieve these aspects (chapter 6).

The Junior Certificate syllabus proposes that students work together in discussing ideas relating to problems and their solutions to facilitate the development of the processes of explanation and justification of their thinking (NCCA, 2015). The Leaving Certificate syllabus states that the teaching and learning strategy employed in the delivery of the mathematics course gives high value to the development of the communication and collaborative abilities of the learners (NCCA, 2015). The Leaving Certificate syllabus states that students should be able to explain the mathematical appropriateness of the procedures and strategies used in the provision of their solutions, in addition to being able to justify the properties of mathematical concepts (NCCA, 2015). It is envisaged that students’ confidence in their ability to communicate their ideas in mathematics will increase by working in an environment that is open to novel ideas and approaches and allows for the consideration of the views of others. Both syllabi refer to the quality of the mathematical tasks as having a significant role in the development of problem solving skills. The syllabi state that “problem solving tasks activate creative mathematical thinking processes as opposed to imitative thinking processes” (NCCA, 2015, p10).

The author’s F-TAPS in mathematics places mathematical thinking as the central most significant component. The stimulation of mathematical thinking was the main focus in the design of new problems, and one of the deciding factors (along with mathematical proficiency) for inclusion of tasks in both the intervention and the assessments. The participants were given a presentation on various types of mathematical thinking in the first class of the intervention. Tall’s Framework of Mathematical Thinking, van Hiele’s Model of the development of Mathematical Thought and aspects of Mason’s theories on mathematical thinking are utilised in the sequencing of tasks for the intervention and are promoted in the completion of the problems by the participants. The use of the Modified Moore Method provided the problem solving environment as described in the mathematics syllabus. The author explicitly explains this method of teaching
and learning at the beginning of the book of course notes for the intervention
and also in Chapter 6.

The proficiency component of the author’s F-TAPS are the five intertwining
strands of proficiency proposed by Kilpatrick et al. (2001). These are also
the strands of proficiency defined in the Irish mathematics syllabi. There is
thus a direct relationship between the proficiency component of the author’s
F-TAPS and the Irish mathematics syllabi. The author assesses the partici-
pants’ solutions in relation to the demonstration of mathematical proficiency,
mathematical thinking, problem solving ability (as given in Leaving Certificate
syllabus), problem solving skills (as given in Leaving Certificate syllabus) and
competency shown during the completion of the phases of the problem solving
process.

4.3.1 Teaching Strategies
The teaching strategies that were used in conjunction with the author’s F-TAPS
in mathematics are discussed in detail in chapter 6. The next section looks at
the formulation of the assessment component for this study.

4.4 Formulating the Assessment
An initial assessment was designed in March 2015 after a set of problems were
reviewed by an expert panel (section 4.4.2). A pilot test of this assessment
was then carried out with fourth (final year undergraduate) year pre-service
teachers. The assessment instrument was finalised in April 2015. The initial
design of the assessment, review by expert panel, and individual items included
in the assessment are discussed.

4.4.1 Initial design of the assessment
The selection of problems for the assessment was based on the following criteria
by Goos and Galbraith (1996):

1. The focus is on examining the ability of the student to use what they
know in an efficient and logical way to solve/attempt to solve a math-
ematical problem. The emphasis is on assessing mathematical thinking
and understanding over procedural knowledge;

2. The problems must therefore be challenging enough to elicit problem solv-
ing abilities while also being within the capability of the pre-service teacher
to solve with existing knowledge and experience;

3. The possible problem solving abilities elicited in the solving of the chosen
problem must be determined in advance of giving the assessment to par-
ticipants so that a suitable assessment instrument can be formulated to
evaluate these elicited abilities when they are made visible;
4. The processes involved in the solving of the problems should allow for the development (display of in assessment, development of in intervention) of mathematical proficiency in students, i.e. the development of:

- conceptual understanding;
- procedural fluency;
- strategic competence;
- adaptive reasoning;
- productive disposition.

5. The problems should necessitate the employment of mathematical thinking in the application of knowledge, experiences and skills;

6. The problems should contain knowledge from different areas which are of interest\(^{33}\) to the students and may also contain knowledge which overlaps with other school subjects;

7. Requiring students to present full solutions to each problem facilitates the provision of clear evidence of the level of development of mathematical problem solving ability by each participant for the researcher to assess (Wee and Laoi 2009).\(^{34}\)

The following principles for assessing mathematics (Swan and Burkhardt, 2012) were also adhered to:

- The tasks on the assessment should allow opportunities for students to show performance of their problem solving solving skills in the content areas of number and algebra as outlined in the Project Maths curriculum;
- The problems should be considered as ones worth solving from an interest or usefulness point of view;
- The nature of the tasks should reflect the purpose of the assessment, that is the problems should be fit for purpose. Each task should assess students’ mathematical knowledge, conceptual understanding, problem solving strategies and fluency with the mathematical procedures needed to solve the problem. These components should not be assessed in isolation;
- Reasoning should be rewarded rather than solely rewarding a correct solution, therefore the tasks should necessitate the use and display of varying networks of the phases of problem solving, i.e. interpretation, formulation, representation, manipulation, evaluation, communication and verification. The scoring system employed should also correspond to the purpose of the assessment.

\(^{33}\)At the time of the pre-assessment, the participants were asked to state their interests so that these could be taken into consideration for the problems in the intervention

\(^{34}\)This note is in addition to the criteria outlined by Goos and Galbraith (1996)
• Inclusion of authentic tasks, some of which may contain insufficient and or surplus information;

• Inclusion of open-ended tasks which allow students to formulate their own solution to a problem and allow for several solution paths.\textsuperscript{35} This allows for the identification and evaluation of problem solving abilities;

• It should be made clear to the students, what types of responses are valued in the assessment.

The following attributes which have an impact on the difficulty of problems in context (Heller et al., 1992) were also taken into consideration in the selection of problems for the assessment:

• Context of the Problem: Problems in which the context is familiar to the students (through experience) are easier then problems in which the context is not familiar.

• Problem Cues: Problems that provide cues as to how the problem may be solved; (for example: form two equations and then solve the system of equations provides a cue to use simultaneous equations to solve the problem) are easier than problems in which such cues are not provided;

• Given information: Problems with extra irrelevant or insufficient information are more difficult than problems with no unnecessary/superfluous information.

• Explicitness of Question: Problems that specifically ask for the unknown variable (e.g. at what distance will the second train be in line with the first train?) are easier than problems in which the unknown must be determined (e.g. explain possible reasons why the formula you determined confirms/does not confirm the number of reported cases);

• Number of Approaches: (more physics related) - problems that can be solved with one set of related principles are easier than problems that require the application of more than one set of related principles;

• Memory Load: Problems that require the solution of 5 or less equations are easier than problems that require the solution of more than 5 equations.

After a selection of such problems were made, these were reviewed by an expert panel before conducting the pilot-test.

\textsuperscript{35}The problems in the pre and post-tests allowed for several solution paths to the same result.
4.4.2 Review by Expert Panel

Twenty one problems\textsuperscript{36} which adhered to the criteria outlined in section 4.4.1 were reviewed by the expert panel. The expert panel consisted of five expert mathematicians employed at the University of Limerick (one who is also a doctor of mathematics education and three of whom are applied mathematicians and professional problem solvers (two of these three are the director and co-director of the Mathematics Application Consortium for Science and Industry (MACSI)). One of the expert mathematicians and the doctor of mathematics education each taught a class group from which participants for the assessment were obtained. Thus these two members of the expert panel had valuable knowledge of the performance of these students and provided feedback which related the wording and level of the problems to the potential participants. The panel of experts were provided with criteria outlined in section 4.1.1 (criteria by Goos and Galbraith, principles for assessing mathematics, and attributes which affect difficulty). The author also provided the expert panel with the information given in Figure 19 as these aspects also featured in the selection of the problems since they include components of mathematical thinking, problem solving abilities, heuristics and mathematical proficiency which the problems were to elicit.

\textsuperscript{36}These problems were sourced from Krutetskii, 1976, pp.110-121; Swan and Burkhardt, 2012; Johnson, 2011; standard mathematical textbooks, D’angelo et al., 2000, p.xvii and some were created by the author.
### Task Difficulty

<table>
<thead>
<tr>
<th>Concept-What</th>
<th>Mathematics-How (calculations)</th>
<th>Context- familiar/not</th>
<th>Irrelevant surplus info/not Abstraction</th>
<th>Number of steps</th>
</tr>
</thead>
</table>

### Mathematical Thinking (Mason)

- **Specialising** (particular)
- **Generalising**
- **Induction**, deduction, analogical, abstract, developmental.
- **Looking for structural relationships.**
- **Metacognition** (c.i.a)
  - Conjecturing
  - Imagining
  - Attention to what and when, order of completion
- **Classification and characteristics** (even whole no divisible by 2, ends in 0,2,4,6,8).
- **Action**-what are you doing, what is reverse action and what does reverse do? Why and when use it?

### Abilities (Krutetskii)

- **Obtain Information**
  - Formulate a perception of the mathematical material in order to grasp the structure of a problem.
- **Process Information**
  1. Think logically in relation to quantitative and spatial relationships and number and letter relationships. Also being able to think in mathematical symbols.
  2. Ability to quickly and extensively generalise mathematical objects, relations and operations.
  3. Ability to shorten the number of operations required.
  4. **Being flexible (adaptable)** in cognitive processes in Problem Solving.
  5. **Endeavour to use rational thinking**, clarity and simplicity in the provision of effective and efficient solutions.
  6. **Being able to switch from and reconstruct the direction of mathematical thinking.**
- **Retain Information**
  - Possess a general memory for mathematical relationships, main features involved in approaches to solutions, arrangements and structure of arguments and proofs and, models and heuristics of problem solving.

### Heuristics/Strategies (Polya)

1. **Read, Understand, Perceive, Interpret**
   - Identify the key information (words, expressions, conditions & figures) presented in this question and in your own words explain exactly what you are being asked to do/find. Is there more than one way of looking at this problem?
2. **Make a Plan**
   - What combination of mathematical information, methods, formulae, theorems or procedures will help you complete the task you are being asked to do in this question? Explain your reasoning where possible.
3. **Carry out the Plan**
   - By implementing the plan you outlined in 2, can you come up with some possible solutions to the problem? Does each part of your solution plan make sense when implemented? All workings should be clearly shown.
4. **Verify your Solution**
   - Do your answers make sense in the context of the question? Is there any check that you can use to verify your solutions, your argument? If so, carry out this check.

### van Hiele

1. **Visual**
   - Recognise the mathematics in the problem situation.
2. **Descriptive**
   - Be able to picture problem accurately in mind, draw diagram/some way of representing problem situation on paper. Be able to describe the problem situation in one’s own words and know what exactly one is being asked to do in the task.
3. **Relational**
   - Relate givens to goals, problem situation to mathematical information required to resolve situation. Relate previously acquired mathematical knowledge and experience with new insight from reading problem situation.
4. **Deductive**
   - Logical deductions and advances made in solving the problem using mathematical thinking.
5. **Rigor**
   - Rigorous in completion of problem, aware of thought process (metacognition), monitoring progress at each stage of process and verifying solution(s).

### Proficiency (L.C Syllabus)

- **Conceptual Understanding**
- **Productive use of mathematics in problem situations**
- **Interconnected understanding of mathematics**
- **Relating** new mathematical ideas to previously gained mathematical knowledge.

- **Adaptive Reasoning**
  - Flexibility
  - Logical thought
  - Explanation
  - Reflection
  - Justify
  - Communicate

- **Procedural Fluency**
  - Appropriate use of procedure
  - Accuracy in execution of procedure in flexible and efficient way
  - Recognise appropriateness for and of use

- **Strategic Competence**
  - Representation
  - Approaches to Solving

- **Productive Disposition**
  - Persistence

---

Figure 19: Problem Selection
The panel of experts were asked to check the problems for accuracy in relation to each of the following:

- wording of the problem;
- mathematical accuracy of the problem;
- fit for the purpose of assessing problem solving ability.

The panel was also asked to rate the problems according to difficulty:

- easy;
- intermediate;
- difficult,

and also to highlight any problem that they particularly liked for its potential to assess problem solving ability. Full solutions to all problems were provided by two of the expert mathematicians. The feedback received was used to select the problems for the initial assessment instrument and to make the changes to the phrasing of the questions as advised by the expert panel. Nine of these problems were selected for inclusion in the initial assessment. These problems were chosen based on the following criteria:

- consistency of the rating of difficulty level by the panel of experts;
- highlighted for potential to assess problem solving ability by the panel of experts;
- to include problems at each of the difficulty levels;
- to allow for the elicitation of various combinations of problem solving abilities, skills, heuristics, and mathematical thinking;
- to allow for varying combinations of the five strands of proficiency to be elicited.

The initial assessment instrument (Appendix C) was finalised following the reviews by the panel of experts. The following changes were made to the initial assessment (Appendix C):
<table>
<thead>
<tr>
<th>Expert</th>
<th>Changes Made</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Expert Mathematician and Doctor of Mathematics Education</td>
<td>Reword problem 2: To make it more clear what is meant by the term “overtake”.</td>
</tr>
<tr>
<td>1 Expert Mathematician and Doctor of Mathematics Education</td>
<td>Reword problem 3: To make it more clear what is meant by the maximum number of little cubes - use the word distinct and also explain the point from which the box is viewed more clearly.</td>
</tr>
<tr>
<td>1 Expert Mathematician and Doctor of Mathematics Education</td>
<td>Reword problem 7: Change the phrase “is slightly lighter than” to the phrase “weighs slightly less than”.</td>
</tr>
<tr>
<td>1 Expert Applied Mathematician and Professional Problem Solver</td>
<td>Inform students which problems contain surplus information.</td>
</tr>
</tbody>
</table>

Table 27: Reviews By Expert Panel

After these changes were made, the author then needed to pilot-test this revised assessment instrument (Appendix B). A small scale pilot-test was conducted before proceeding to a larger scale pilot-test.

### 4.4.3 Pilot-Tests

Pilot testing is a scaled down version of the full study which is to be performed (van Teijlingen and Hundley, 2001). Pilot-testing is also the pre-testing of specific research instruments before employing these instruments in the study (van Teijlingen and Hundley, 2001). Thus, pilot-testing can serve many functions: improving internal validity of research instruments as well as identifying prospective potential problem areas in the study (De Vaus, 1993; Fink and Kosekoff, 1985). In this research, the pre-assessment was piloted among a representative sample before being implemented in the study. This was in order to identify any potential problems with implementing it in practice.

A small scale pilot-test was conducted among a sample consisting of two qualified secondary mathematics teachers (who are also completing PhD studies in mathematics education). The teachers completed the assessments and provided feedback to the author. The assessments were then piloted with a small sample of seven fourth (final) year pre-service mathematics teachers. The solutions to the problems were assessed using a scoring framework as discussed in section 3.10.1.
The solutions provided by the sample of pre-service teachers who completed the pilot-tests facilitated the generation of a more specified scoring framework which more closely reflected the solutions given by this sample. After the pilot-tests had been assessed and modifications made to the pilot-tests and to the scoring framework, the researcher had a finalised assessment and scoring framework to use for the pre-assessments part of the research. The finalised scoring framework is presented in Appendix H. The finalised assessment is described in section 4.4, each problem in the assessment is discussed in relation to the aspects of mathematical proficiency, problem solving skills and abilities, and mathematical thinking it assesses. This discussion shows how the problems in the assessment link to the framework for teaching and assessing problem solving in mathematics.

Findings from the pilot test, which informed changes to be made to the revised assessment (Appendix C):

- Explicitly ask if students know of alternative solutions to each problem after each problem, instead of only writing this in the instructions on the first page and stating this at the start.
- Change the ratio 3 : 2 : 1 in question five, students compared this to the length of time needed to do the work by each shop. Shop 1, 2 and 3 takes 10,30 and 15 days respectively so essentially shop 2 is twice shop 3 and 3 times shop 1. The ratio of the books is irrelevant in this question, but the link between both sets of ratios may make this less clear. Also changing the ratio of the books should ease in the correcting the solutions to this problem (coincidentally the method of comparison mentioned gave the same result as the correct answer - but reasoning not correct).
- Insert extra rows in table for students to rate each part of the problem, if there is more than 1 part to a problem.

The changes made to the revised assessment (Appendix C) resulted in the final version of the assessment (Appendix B).
4.4.4 Assessment Items

The formulation of the framework for teaching and assessing problem solving, in particular the mathematical thinking required to solve the problems along with the potential components of mathematical proficiency displayed in the solution process, were employed in formulating the problems for the assessment. The problem solving skills and content from the Irish Mathematics syllabi for both Junior and Senior Certificate levels were also considered in the design of the assessment problems. Each of the problems included in the assessment will now be described under the headings of:

- Problem Solving Skills;
- Problem Solving Ability;
- Mathematical Proficiency;
- Mathematical Thinking;
- Content from Irish Mathematics Syllabus.
Problem 1: Level: Easy:
There are some hamsters and some cages. If one hamster is put into each cage, two hamsters will be left without a cage. If two hamsters are put into each cage, two cages remain empty. How many hamsters and how many cages are there? (Krutetskii, 1976, p.121)

Problem 1 assesses the following:

<table>
<thead>
<tr>
<th>Problem Solving Skills</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Apply knowledge and skills to solve problems in familiar and unfamiliar contexts;</td>
</tr>
<tr>
<td>• Analyse information presented verbally and translate it into mathematical form;</td>
</tr>
<tr>
<td>• Use appropriate mathematical techniques to process information;</td>
</tr>
<tr>
<td>• Justify conclusions.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem Solving Ability</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Obtain mathematical information: Formulate perception to grasp the structure of the problem;</td>
</tr>
<tr>
<td>• Processing mathematical information: Ability to think logically in relation to number and letter relationships and think in mathematics symbols;</td>
</tr>
<tr>
<td>• Flexibility: Suggest different solution(s). If started one method, did lack of flexibility in that method cause them to abandon that method? Were they then able to use an alternative method?;</td>
</tr>
<tr>
<td>• Processing mathematical information: Rational thinking in providing an efficient solution, one linear equation or simultaneous;</td>
</tr>
<tr>
<td>• Retaining information: Was knowledge of a similar problem helpful to the student?</td>
</tr>
</tbody>
</table>
Proficiency

- Conceptual understanding: productive use of mathematics in a problem situation;
- Conceptual understanding: relating new mathematical ideas to previously gained mathematical knowledge;
- Adaptive reasoning: logical thought in explanation and justification;
- Adaptive reasoning: Flexibility (as described in problem solving ability above);
- Procedural fluency: accuracy in completion;
- Productive disposition: if failed on first attempt, was a second attempt made;
- Strategic Competence: formulation, representation and approach to solving the problem.

Mathematical Thinking

- Determining structural relationships.

Content From Syllabus: Algebra

- Find formulae to express relationship arising from context;
- Solve simultaneous linear equations with two unknowns and interpret the result
- (Junior Certificate Higher Level).

Problem 2: Level: Easy:
The back of a train passed the Eiffel Tower in Paris on its route to Berlin at a constant speed of 48km/hr. Two hours later, the back of a second train passed the same point on the Eiffel Tower in Paris on its way to Berlin at a constant speed of 56km/hr on a parallel track. At what distance from the Eiffel Tower will the back of the second train be in line with (at same level as) the back of the first train, if the distance between the Eiffel Tower and Berlin is 1200km and there are twice as many carriages on the first train as there are on the second? (Krutetskii, 1976, p.110)

Problem 2 assesses the following:
**Problem Solving Skills**

- Apply knowledge and skills to solve problems in familiar and unfamiliar contexts;
- Analyse information presented verbally and translate it into mathematical form;
- Use appropriate mathematical techniques to process information;
- Justify conclusions: Check that the distance from the Eiffel Tower to the point where the trains meet $\leq 1200\text{km}$ (the distance between the Eiffel Tower and Berlin).

**Problem Solving Ability**

- Obtain mathematical information: Formulate perception to grasp the structure of the problem, this problem contains surplus information which is not necessary to solve the problem;
- Processing mathematical information: Ability to think logically in relation to quantitative and number and letter relationships;
- Flexibility: Suggest different solution(s). If started one method, did lack of flexibility in that method cause them to abandon that method? Were they then able to use an alternative method?
- Processing mathematical information: Rational thinking in providing an efficient solution, equating distance or finding difference in speed;
- Retaining information: Was knowledge of a similar problem helpful to the student?
<table>
<thead>
<tr>
<th>Proficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Conceptual understanding: productive use of mathematics in a problem situation;</td>
</tr>
<tr>
<td>• Adaptive reasoning: logical thought in explanation and justification;</td>
</tr>
<tr>
<td>• Adaptive reasoning: flexibility (as described in problem solving ability above);</td>
</tr>
<tr>
<td>• Procedural fluency: accuracy in completion;</td>
</tr>
<tr>
<td>• Productive disposition: if failed on first attempt, was a second attempt made?;</td>
</tr>
<tr>
<td>• Strategic Competence: formulation, representation and approach to solving the problem.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mathematical Thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Determining structural relationships</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Content From Syllabus: Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Find formulae to express relationship arising from context;</td>
</tr>
<tr>
<td>• Form and solve linear equations;</td>
</tr>
<tr>
<td>• Express one variable in terms of another;</td>
</tr>
<tr>
<td>• Number: distance, speed and time.</td>
</tr>
<tr>
<td>• (Junior Certificate Higher Level).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Metacognition</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Attention to what and in what order?</td>
</tr>
</tbody>
</table>
Problem 3: Level: Intermediate :
216 identical little cubes have been stacked into a large box which is also a cube. The little cubes fit exactly into the large box, with no space remaining. The large box is see-through, with an open top, such that the little cubes can still be seen from outside the box.

(i) You view the large box from a point that allows you to see the maximum number of distinct little cubes. What is the maximum number of distinct little cubes that you can see from this point?

Now suppose that \( n^3 \) identical little cubes have been arranged into one larger cube. You view the larger cube from a point that allows you to see the maximum number of distinct little cubes.

(ii) Determine a formula for the maximum number of distinct little cubes that can be seen from this viewing point.

Problem 3 assesses the following:

<table>
<thead>
<tr>
<th>Problem Solving Skills</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Apply knowledge and skills to solve problems in familiar and unfamiliar contexts;</td>
</tr>
<tr>
<td>• Use appropriate mathematical techniques to process information;</td>
</tr>
<tr>
<td>• Justify conclusions: Student should check that the 216 case is correct as per their generalised solution in part(ii), substitute ( n = 6 ) into solution for (ii).</td>
</tr>
</tbody>
</table>
## Problem Solving Ability

- Obtain mathematical information: Formulate perception to grasp the structure of the problem;
- Processing mathematical information: Ability to think logically in relation to spatial, quantitative and number and letter relationships;
- Flexibility: Suggest different solution(s). If started one method, did lack of flexibility in that method cause them to abandon that method? Were they then able to use an alternative method? Also this problem focuses on number of cubes not number of faces of cubes (the latter would be more familiar to students - flexible in adaptation to unfamiliar?);
- Processing mathematical information: Rational thinking in providing an efficient solution, total minus common cubes or count cubes on each side;
- Retaining information: Was knowledge of a similar problem helpful to the student?

## Proficiency

- Conceptual understanding: productive use of mathematics in a problem situation (correct use of $\sqrt[3]{216}$);
- Adaptive reasoning: logical thought in explanation and justification (notice counting some cubes twice if certain method used);
- Adaptive reasoning: flexibility (as described in problem solving ability above);
- Procedural fluency: accuracy in completion;
- Productive disposition: if failed on first attempt, was a second attempt made?;
- Strategic Competence: formulation, representation and approach to solving the problem.
<table>
<thead>
<tr>
<th>Mathematical Thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Specialising and generalising.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Content From Syllabus: Number (Area and Volume) and Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Find formulae to express relationship arising from context;</td>
</tr>
<tr>
<td>• Perform calculations to solve problems involving the volume/surface area of rectangular solids ;</td>
</tr>
<tr>
<td>• (Junior Certificate Higher Level - Leaving Certificate Higher Level (due to way in which question posed and combination of questions asked).</td>
</tr>
</tbody>
</table>
Problem 4: Level: Intermediate:

(a) Sketch a graph to model each of the following situations and provide an explanation of how your graph models each situation

(b) Then answer the following questions: (Swan and Burkhardt, 2012)

(i) The formulae given are models for the situations (Candle, Ferris Wheel, Kettle Cooling and Car Value). Match each situation to the correct formula.

<table>
<thead>
<tr>
<th>Formulae</th>
<th>Situation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 12 - 0.5x$</td>
<td></td>
</tr>
<tr>
<td>$y = 20 + 69.5e^{\frac{x}{7}}$</td>
<td></td>
</tr>
<tr>
<td>$y = 3000 \times (0.8)^x$</td>
<td></td>
</tr>
<tr>
<td>$y = 30 + 30\sin(18x)$</td>
<td></td>
</tr>
</tbody>
</table>

(ii) How long will the candle last before it is completely burnt away?

(iii) How long does it take for the ferris wheel to complete one full turn?

(iv) What is the temperature of the kitchen where the kettle is cooling?

(v) How much will my car be worth after 3 years?

Problem 4 assesses the following:

<table>
<thead>
<tr>
<th>Problem Solving Skills</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Apply knowledge and skills to solve problems in familiar and unfamiliar contexts;</td>
</tr>
<tr>
<td>• Analyse information presented verbally and represent graphically;</td>
</tr>
<tr>
<td>• Use appropriate mathematical techniques to process information - select appropriate mathematical models to process information;</td>
</tr>
<tr>
<td>• Communicate mathematics in verbal and written form - Students should provide explanations of how the graphs they drew models each situation;</td>
</tr>
<tr>
<td>• Justify conclusions: in the provision of the explanation, and after completing (b) should check solutions to part (a) again to see if they make sense.</td>
</tr>
</tbody>
</table>
### Problem Solving Ability

- Obtain mathematical information: Formulate perception to grasp the structure of the problem (b);
- Processing mathematical information: Ability to think logically in relation to graphical, number and letter relationships;
- Processing mathematical information: Rational thinking in providing an efficient solution;
- Processing mathematical information: Being flexible in cognitive processes in graphing the everyday functions and in answering (b);
- Retaining information: Was knowledge of a similar problem helpful to the student?

### Proficiency

- Conceptual understanding: productive use of mathematics in a problem situation;
- Adaptive reasoning: logical thought in explanation and justification;
- Adaptive reasoning: flexibility in reasoning abstractly and quantitatively;
- Procedural fluency: accuracy in completion;
- Productive disposition: if failed on first attempt, was a second attempt made?;
- Strategic competence: formulation, representation and approach to solving the problem.
**Mathematical Thinking**

- Reason abstractly and quantitatively;
- Looking for structural relationships.

**Content From Syllabus: Number (Area and Volume) and Algebra**

- Articulate and find formulae to express relationships between variables arising from everyday contexts;
- Represent situation graphically and connect graphical and symbolic representations of algebraic concepts;
- Analyse functions using different representations;
- Relations without formulae - use graphs to represent formula quantitatively;
- Examining algebraic relationships;
- Construct and compare linear, exponential and trigonometric models;
Problem 5: Level: Intermediate:

A library needs to bind some English, French and German books whose number is in the ratio of 5 : 2 : 1. Three shops who bind books were contacted. The first said it could have the books bound in 10 days, the second said it would have the books bound in 30 days and the third could have the books bound in 15 days. In order to have the books bound as quickly as possible, it was decided to give the job to all three shops at once. In how many days will the shops do the job, working simultaneously? In your solution, have you made any assumptions about this problem situation? If you have, please state these assumptions. (Adapted from Krutetskii, 1976, p.121)

Problem 5 assesses the following:

<table>
<thead>
<tr>
<th>Problem Solving Skills</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Apply knowledge and skills to solve problems in familiar and unfamiliar contexts;</td>
</tr>
<tr>
<td>• Analyse information presented verbally and translate it into mathematical form;</td>
</tr>
<tr>
<td>• Use appropriate mathematical techniques to process information;</td>
</tr>
<tr>
<td>• Communicate mathematics verbally and in written form: explain assumptions;</td>
</tr>
<tr>
<td>• Justify conclusions: Student should check that the solution they obtained is ( \leq 10 ) days (the 3 shops working together should be quicker than the quickest shop working alone).</td>
</tr>
</tbody>
</table>
**Problem Solving Ability**

- Obtain mathematical information: Formulate perception to grasp the structure of the problem, this problem contains surplus information which is not necessary to solve the problem;

- Processing mathematical information: Ability to think logically in relation to quantitative, number and letter relationships: Finding fraction or percentage of the job done in 1 day by each of the three shops;

- Flexibility: Suggest different solution(s). If started one method, did lack of flexibility in that method cause them to abandon that method? Were they then able to use an alternative method?;

- Processing mathematical information: Rational thinking in providing an efficient solution;

- Retaining information: Was knowledge of a similar problem helpful to the student?

**Proficiency**

- Conceptual understanding: productive use of mathematics in a problem situation;

- Adaptive reasoning: logical thought in explanation and justification;

- Adaptive reasoning: flexibility (as described in problem solving ability above);

- Procedural fluency: accuracy in completion;

- Productive disposition: if failed on first attempt, was a second attempt made?;

- Strategic Competence: formulation, representation and approach to solving the problem.
Mathematical Thinking

- Specialising and generalising.

Content From Syllabus: Number and Algebra

- Find formulae to express relationship arising from context;
- Examining algebraic relationships: proportional relationships;
- (Junior Certificate Higher Level - Leaving Certificate Higher Level (due to way in which question posed, extra information not necessary in order to solve question and identification and explanation of assumption made).
Problem 6: Level: Intermediate:
A 250mg dose of the antibiotic ampicillin is given to a patient to treat his bronchitis. Ampicillin leaves the body at a rate of 40% per hour.

(i) Determine a formula for the amount, $A$ (in mg), of ampicillin in the body $t$ hours after the dose is given.

(ii) How much ampicillin is left in the body after 8 hours?

(iii) How many times a day should the patient be given a 250mg dose? Explain your answer fully, describing the factors which influenced your decision.

Problem 6 assesses the following:

<table>
<thead>
<tr>
<th>Problem Solving Skills</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Explore patterns and formulate conjectures - find pattern arising after filling in for 1 hour, 2 hours ..;</td>
</tr>
<tr>
<td>- Analyse information presented verbally and translate it into mathematical form - from the information in the problem and exploring the pattern, determine a general formula;</td>
</tr>
<tr>
<td>- Apply knowledge and skills to solve problems in familiar and unfamiliar contexts - knowledge of the formula for compound interest is useful here;</td>
</tr>
<tr>
<td>- Use appropriate mathematical models and techniques to process information - compound interest model;</td>
</tr>
<tr>
<td>- Communicate mathematics verbally and in written form: communicate reasoning for decision of how many times dose should be given;</td>
</tr>
<tr>
<td>- interpret and justify conclusions - justify the decision of dosage by explaining clearly the factors involved in the decision including interpreting the results obtained for (i) and (ii).</td>
</tr>
</tbody>
</table>
Problem Solving Ability

- Obtain mathematical information: Formulate perception to grasp the structure of the problem;
- Processing mathematical information: Ability to think logically in relation to quantitative, number and letter relationships and to think in mathematical symbols;
- Flexibility: Suggest different solution(s). If started one method, did lack of flexibility in that method cause them to abandon that method? Were they then able to use an alternative method? In this question, flexibility in the use of the ‘compound interest’ formula;
- Processing: Generalising: Ability to generalise to a formula for a time $t$ from the pattern explored/information presented;
- Processing mathematical information: Rational thinking in providing a efficient solution: using table of specific cases to determine formula for general case or noticing similarity to compound interest formula;
- Retaining information: Was knowledge of a similar problem helpful to the student?: In this problem, the use of the compound interest formula is helpful in formulating a formula for the amount of ampicillin in the body.

Proficiency

- Conceptual understanding: productive use of mathematics in a problem situation;
- Adaptive reasoning: logical thought in explanation and justification;
- Adaptive reasoning: flexibility (as described in problem solving ability above) and in explanation of solution;
- Procedural fluency: accuracy in completion;
- Productive disposition: if failed on first attempt, was a second attempt made;
- Strategic Competence: formulation, representation and approach to solving the problem.
<table>
<thead>
<tr>
<th>Mathematical Thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Specialising and generalising.</td>
</tr>
<tr>
<td>• Looking for structural relationships: in formulating specific and general formula;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Content From Syllabus: Number and Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Find formulae to express relationship arising from context;</td>
</tr>
<tr>
<td>• Examining algebraic relationships</td>
</tr>
<tr>
<td>• Compound interest formula/exponential</td>
</tr>
<tr>
<td>• (Junior Certificate Higher Level - Leaving Certificate Higher Level (due to way in which question posed - Determining a general formula, explanation of solution to (iii) giving description of the factors which influenced decision).</td>
</tr>
</tbody>
</table>
Problem 7: Level: Difficult:
You have 8 €1 coins, all of which look identical, however one is a fake and weighs slightly less than the others. Explain how you could use a balance scale to figure out which coin is the fake one in exactly:

(i) 3 weighings.
(ii) 2 weighings.

Problem 7 assesses the following:

**Problem Solving Skills**
- Explore patterns and formulate conjectures - try with fewer coins and build up to 8 and/or try more weighings and then reduce to 3 in trying to formulate solution;
- Communicate mathematics verbally and in written form: communicate explanation clearly with diagram.
**Problem Solving Ability**

- Obtain mathematical information: Formulate perception to grasp the structure of the problem;
- Processing mathematical information: Ability to think logically in relation to quantitative relationships: relationship between number of coins and restrictions on the number of weighings;
- Generalising: Ability to generalise from part (i) to part (ii);
- Retaining information: Was knowledge of a similar problem helpful to the student?

**Proficiency**

- Adaptive reasoning: logical thought in explanation and communication;
- Productive disposition: if failed on first attempt, was a second attempt made?;
- Strategic Competence: formulation, representation and approach to solving the problem.
Mathematical Thinking

- Specialising and generalising.

Content From Syllabus: Number and Algebra

- General independent of syllabus.

Problem 8: Level: Difficult:
Prove that the product of 4 consecutive numbers, added to 1, results in a square number.

Problem 8 assesses the following:

Problem Solving Skills

- Explore patterns and formulate conjectures - show true for filling in for different sets of four numbers;

- Analyse information presented verbally and translate it into mathematical form - from the information in the problem and exploring the pattern, form a general equation to prove;

- Apply knowledge and skills to solve problems in familiar and unfamiliar contexts - knowledge of square numbers, quadratic expressions, squaring out a bracket;

- Communicate mathematics verbally and in written form: communicate reasoning for proof clearly.

- Interpret and justify conclusions - justify the proof.
### Problem Solving Ability

- Processing mathematical information: Ability to think logically in relation to quantitative, number and letter relationships and to think in mathematical symbols;

- Flexibility: Suggest different solution(s). If started one method, did lack of flexibility in that method cause them to abandon that method? Were they then able to use an alternative method? In this question, flexibility in the use of the division, use of quadratic expression squared and equating like coefficients;

- Processing mathematical information: Rational thinking in providing an efficient solution;

- Retaining information: Was knowledge of a similar problem helpful to the student?: In this problem, the use of previously met expressions are useful.

### Proficiency

- Conceptual understanding: productive use of mathematics in a problem situation;

- Adaptive reasoning: logical thought in explanation and justification;

- Adaptive reasoning: flexibility (as described in problem solving ability above);

- Procedural fluency: accuracy in completion;

- Productive disposition: if failed on first attempt, was a second attempt made?;

- Strategic Competence: formulation, representation and approach to proof.
### Mathematical Thinking
- Specialising and generalising.
- Looking for structural relationships: in formulating specific and general expressions and equation;

### Content From Syllabus: Number and Algebra
- Find formulae to express relationship arising from context;
- Examining algebraic relationships;
- Proof.
- (Leaving Certificate Higher Level).
Problem 9: Level: Difficult:
If you counted all the possible squares of all sizes on a standard chessboard (from size $1 \times 1$ to size $8 \times 8$), you would obtain an answer of 204 squares. How many squares of all sizes would there be on a $n \times n$ chessboard? (D’angelo, 2000, p.xvii)

Problem 9 assesses the following:

<table>
<thead>
<tr>
<th>Problem Solving Skills</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Explore patterns and formulate conjectures - generate arithmetic progression from repeating pattern;</td>
</tr>
<tr>
<td>• Analyse information presented verbally and translate it into mathematical form - from the information in the problem and exploring the pattern, determine a general formula for the number of squares in the $n \times n$ case;</td>
</tr>
<tr>
<td>• Apply knowledge and skills to solve problems in familiar and unfamiliar contexts - knowledge of solving simultaneous equations and the common differences for a cubic equation is useful here;</td>
</tr>
<tr>
<td>• Use appropriate mathematical models and techniques to process information - compound interest model;</td>
</tr>
<tr>
<td>• Interpret and justify conclusions - justify the solution for the $n \times n$ case by substituting in $n = 8$ to check if this does give the answer of 204.</td>
</tr>
</tbody>
</table>
**Problem Solving Ability**

- Obtain mathematical information: Formulate perception to grasp the structure of the problem;
- Processing mathematical information: Ability to think logically in relation to quantitative, number and letter relationships;
- Flexibility: Suggest different solution(s). If started one method, did lack of flexibility in that method cause them to abandon that method? Were they then able to use an alternative method?;
- Generalising: Ability to generalise to a formula for the $n \times n$ case from the pattern explored/information presented in the $8 \times 8$ case;
- Processing mathematical information: Rational thinking in providing an efficient solution: using table of specific cases to determine formula for general case;
- Retaining information: Was knowledge of a similar problem helpful to the student? In this problem, the use of summing for the $8 \times 8$ case is helpful in determining the number of squares in the $n \times n$ case.

**Proficiency**

- Conceptual understanding: productive use of mathematics in a problem situation;
- Adaptive reasoning: logical thought in explanation and justification;
- Adaptive reasoning: Flexibility (as described in problem solving ability above) and in explanation of solution;
- Procedural fluency: accuracy in completion;
- Productive disposition: if failed on first attempt, was a second attempt made?;
- Strategic Competence: formulation, representation and approach to solving the problem.
### Mathematical Thinking
- Specialising and generalising.
- Looking for structural relationships: in formulating specific and general formula;

### Content From Syllabus: Number and Algebra
- Generate arithmetic progression from repeating pattern;
- Examining algebraic relationships;
- simultaneous equations;
- sequences and series;
- (Leaving Certificate Higher Level).

The pilot-test was employed to check the design of the assessment in practice, revisions were made to the pilot-test which resulted in the production of the final pre-test (Appendix B). In addition the pilot test was also employed to check the devised analytic scoring framework (Appendix J) for reliability among different raters. The next section evaluates the inter-rater reliability of the devised analytic scoring framework (Appendix J) between different raters.
4.4.5 Pilot-test: Inter-rater Reliability

The pilot tests completed by the seven fourth year pre-service mathematics teachers were evaluated using the analytic scoring framework. Three inter-rater reliability tests were conducted; two between doctors of mathematics education and the researcher, and the third test between a mathematics education specialist and the researcher. The researcher provided the mathematics education specialists with the analytic scoring framework along with the solutions to the assessment and a blank table to fill in the scores. The researcher informed the mathematics education specialists of the slight difference in the marking of problems four (four parts in (a) so four R&U response types instead of one) and seven (two parts so two R&U response types instead of one) and asked for the education specialists to assign a type of answer for generalising in problem three, and for the explanation in problem six (these are described in the scoring framework). The mathematics education specialists did not receive any other instruction or training in the use of the analytic scoring framework. This highlights the replicable nature of the assessment and the scoring framework.

The inter-rater reliability test of the pilot tests and the scoring framework provided a measure of the reliability of the scoring framework in the types of answers being assigned, irrespective of the individual correcting the assessments.

The kappa statistic ($\kappa$) is a measure of agreement between two raters who are measuring a variable on a categorical scale. It is the percentage agreement, which has been adjusted to account for the possibility of agreement occurring by chance (Bland, 2008). Table 29 shows the number of times the researcher and the first mathematics education specialist agreed on the types of answers assigned for the reading and understanding phase. The intersection of a column and row of the same type show the number of times both raters agreed that a response was of that particular type. For example, in Table 29, the number 48 is at the intersection point of column type 5 and row type 5, this means that both the researcher and the mathematics education specialist deemed a response to be of type 5 on 48 occasions. The number of times the raters disagreed on the type of response assigned can also be seen in this table, at the intersection points of a column of one type with a row of a different type. For example, the number 3 is at the intersection of column type 4 and row type 5, this shows that on 3 occasions, the researcher assigned type 5 to a response, while the mathematics educational specialist deemed the same response to be of type 4. The kappa statistic is approximately Normally distributed provided that the number of comparisons are not less than 30 (McHugh, 2012) and provided that both $np_0$ and $n(1 - p_0)$ are greater than five (Bland 2008), where $n$ is the number of comparisons and $p_0$ is the proportion of comparisons where there is agreement between the raters. The formula for the calculation of the kappa statistic (Cohen 1960) is:
\[ \kappa = \frac{p_0 - p_e}{1 - p_e} \]

where \( p_e \) is the expected agreement between the raters. The interpretation of the kappa statistic was made in reference to the slightly adapted scale given by Landis and Koch (1977) (Table 28).

<table>
<thead>
<tr>
<th>Kappa</th>
<th>Strength of Agreement</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 0.20</td>
<td>Poor</td>
</tr>
<tr>
<td>0.21 – 0.40</td>
<td>Fair</td>
</tr>
<tr>
<td>0.41 – 0.60</td>
<td>Moderate</td>
</tr>
<tr>
<td>0.61 – 0.80</td>
<td>Substantial</td>
</tr>
<tr>
<td>0.81 – 1</td>
<td>Almost Perfect</td>
</tr>
</tbody>
</table>

Table 28: Interpretation of the Kappa Statistic (Landis and Koch 1977, p.165)

The mathematics education specialists independently assessed the seven assessments from the pilot-test. A total of 63 solutions to problems were evaluated. This required a total of 238 judgements of types of answers to be made by each of the mathematics education specialists. After the specialists had marked the seven assessments, a comparison between the types of answers assigned by the researcher and each of the mathematics education specialists was made. There was 79\% agreement in the types of answers assigned by the researcher and the first mathematics education specialist. This collaborative review revealed that:

- In the reading and understanding section
  - \( \approx 81\% (13/16) \) of the discrepancies between the researcher and the mathematics education specialist were due to misreading the scoring framework or misreading the students’ answers (due to them being not presented clearly).
  - \( \approx 13\% (2/16) \) of the discrepancies between the researcher and the mathematics education specialist resulted in the advice of the specialist being to adjust the scoring framework.
  - \( \approx 6\% (1/16) \) of the discrepancies between the researcher and the mathematics education specialist were as the result of human error in the choice of the appropriate type of answer.
Table 29: Agreement in the Types of Answers assigned by the Researcher and Mathematics Education Specialist 1 for the Reading and Understanding Phase

The observed agreement $p_0$ is:

$$p_0 = \frac{48 + 5 + 6 + 4 + 6 + 6}{91} = \frac{75}{91} = 0.82$$

$n p_0 = (91)(0.82) = 74.62$ and $n(1 - p_0) = (91)(0.18) = 16.38$.

The expected agreement $p_e$ is calculated as follows:

$$p_e = \left( \frac{48}{91} \right) \left( \frac{59}{91} \right) + \left( \frac{10}{91} \right) \left( \frac{8}{91} \right) + \left( \frac{8}{91} \right) \left( \frac{8}{91} \right) + \left( \frac{9}{91} \right) \left( \frac{5}{91} \right) + \left( \frac{10}{91} \right) \left( \frac{6}{91} \right)$$

$$+ \left( \frac{6}{91} \right) \left( \frac{6}{91} \right) \approx 0.38$$

$$\kappa = \frac{p_0 - p_e}{1 - p_e}$$

$$\kappa = \frac{0.82 - 0.38}{1 - 0.38} \approx 0.71$$

This value corresponds to substantial agreement. The standard error was calculated using the formula given in McHugh (2012 p.281):

$$SE(\kappa) = \sqrt{\frac{p_0(1 - p_0)}{n(1 - p_e)^2}}$$

$$SE(\kappa) = \sqrt{\frac{0.82(1 - 0.82)}{91(1 - 0.38)^2}} \approx 0.06496$$
The 95% confidence interval for \( \kappa \) was then calculated as follows:

\[
[\kappa \pm 1.96(SE(\kappa))] = [0.71 \pm 1.96(0.06496)] = [0.5826, 0.8373]
\]

This interval corresponds to an interval of agreement ranging from the higher end of moderate to almost perfect agreement.

It is noted in Table 29 that the mathematics education specialist 1 seems less likely to assign a type 5 response than the researcher is. The mathematics education specialist 1 was ‘harsher’ than the researcher in assigning types of responses in 14 of the 16 discrepancies noted between the researcher and the mathematics education specialist 1. It was also noted that in 6 of the 16 discrepancies, the difference in the response types assigned was greater than one.
Cohen (1968) recommends that if using the weighted kappa, a panel of experts should assign weights to the categories. However Viera and Garrett (2005) state that assigning the weights is a subjective issue on which even experts may disagree in some cases. Vierra and Garrettt (2005, p.362) suggest using weighted kappa if a researcher is more interested in the agreement across categories where there is “meaningful difference”. For example in medical research where the categories may be ‘normal’, ‘benign’ and ‘cancer’, a researcher may not be concerned about a difference in the rating of ‘normal’ by one radiologist and a rating of ‘benign’ by another when examining the same radiograph. However the researcher would be concerned about a disagreement of ‘normal’ versus ‘cancer’ (Vierra and Garrettt, 2005, p.362). Vierra and Garrett (2005) suggest that weighted kappa is useful if there are many disagreements of more than one category apart between the raters. The author is interested in each difference between the categories and as the number of disagreements of more than one category apart are small in number in this research, she chose to report the kappa statistic, while also being aware of its limitations.

The collaborative review between the researcher and the mathematics specialist resulted in the following reconciliation of the different ratings given in the reading and understanding section (where the researcher gave a type 5 response and the specialist did not):

- On the three occasions where the researcher gave a response of type 5 and the specialist gave a response of 4, the researcher moved to a type 5 on two of these occasions (one reason was misjudgement of future work in relation to demonstration of understanding; the second was due to the graph drawn by a student being misinterpreted due to the work not being presented clearly). The specialist moved to a type 5 on one occasion (due to misinterpreting student’s work).
On the one occasion where the researcher gave a response of type 5 and the specialist gave a response of 3, the researcher moved to a type 3 and took the specialist’s advice of removing type 5 from the analytic scoring framework.

On the three occasions where the researcher gave a response of type 5 and the specialist gave a response of 2, the specialist moved to a type 5 on two of these occasions (one reason was misinterpretation of student’s work due to unclear presentation; the second was due to the specialist misreading the framework). Both the researcher and the specialist moved to a type 4 on one occasion and the researcher took the advice of the specialist to adjust the framework to state that the candle does not burn down 1cm per hour.

On the three occasions where the researcher gave a response of type 5 and the specialist gave a response of 1, the specialist moved to a type 5 (the reason for two of these moves was due to unclear work presented by students; the reason for the third move was due to an alternative solution given by a student was not recognised by the specialist.

The rating of the planning and solving section by the researcher and the mathematics specialist 1 is presented next.

In the planning and solving section

- ≈ 77%(10/13) of the discrepancies between the researcher and the mathematics education specialist were due to misreading the scoring framework or misreading the students’ answers (due to them being not presented clearly).
- ≈ 15%(2/13) of the discrepancies between the researcher and the mathematics education specialist were due to focusing on the overall problem rather than the planning and solving section (not using the scoring framework correctly).
- ≈ 8%(1/13) of the discrepancies between the researcher and the mathematics education specialist was as the result of a previous error made in assigning a mark in the reading and understanding section.

\[ \kappa = \frac{0.79 - 0.22}{1 - 0.22} \approx 0.73 \]

The 95% confidence interval for \( \kappa \) is:

\[ [\kappa \pm 1.96(SE(\kappa))] = [0.73 \pm 1.96(0.06579)] = [0.6011, 0.8589] \]

This interval corresponds to an interval of agreement ranging from the higher end of moderate to almost perfect agreement.
<table>
<thead>
<tr>
<th>Mathematics Education Specialist</th>
<th></th>
<th>Type of Answer</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Researcher</td>
<td></td>
<td>5</td>
<td>21</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>5</td>
<td>0</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>12</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Total</td>
<td>21</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>17</td>
<td>7</td>
<td>63</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 31: Agreement in the Types of Answers assigned by the Researcher and Mathematics Education Specialist 1 for the Planning and Solving Phase

- In the solution and checking section
  - \( \approx 44\%(7/16) \) of the discrepancies between the researcher and the mathematics education specialist were due to misreading the scoring framework or misreading the students’ answers (due to them being not presented clearly).
  - \( \approx 44\%(7/16) \) of the discrepancies between the researcher and the mathematics education specialist were due to human error in assigning an incorrect type of answer or as a result of an earlier mistake in the assigning of a type of answer (not using the scoring framework correctly).
  - \( \approx 12\%(2/16) \) of the discrepancies between the researcher and the mathematics education specialist resulted in the advice of the specialist being to adjust the scoring framework.
<table>
<thead>
<tr>
<th>Mathematics Education Specialist 1</th>
<th>Type of Answer</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Researcher</td>
<td>4</td>
<td>11</td>
<td>5</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
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<td>0</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>11</td>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>14</td>
<td>8</td>
<td>5</td>
<td>15</td>
<td>21</td>
<td>63</td>
<td></td>
</tr>
</tbody>
</table>

Table 32: Agreement in the Types of Answers assigned by the Researcher and Mathematics Education Specialist 1 for the Solution and Checking Phase

\[
\kappa = \frac{0.75 - 0.24}{1 - 0.24} \approx 0.67
\]

The 95% confidence interval for \(\kappa\) is:

\[
[\kappa \pm 1.96(SE(\kappa))] = [0.67 \pm 1.96(0.07178)] = [0.5293, 0.8107]
\]

This interval corresponds to an interval of agreement ranging from moderate to almost perfect agreement.
The inter-rater reliability between the researcher and each of the mathematics specialists (for each of the phases) was calculated (as shown for mathematics specialist 1). The results are shown in Table 33. The values of $\kappa$ and the corresponding confidence intervals show an interval of moderate to almost perfect agreement between the raters.

<table>
<thead>
<tr>
<th>R&amp;U</th>
<th>$\kappa$</th>
<th>$SE(\kappa)$</th>
<th>95% C.I</th>
</tr>
</thead>
<tbody>
<tr>
<td>MS1</td>
<td>0.71</td>
<td>0.06496</td>
<td>[0.5826, 0.8373]</td>
</tr>
<tr>
<td>MS2</td>
<td>0.79</td>
<td>0.06083</td>
<td>[0.6708, 0.9092]</td>
</tr>
<tr>
<td>MS3</td>
<td>0.68</td>
<td>0.06970</td>
<td>[0.5434, 0.8166]</td>
</tr>
<tr>
<td>P&amp;S</td>
<td>MS1</td>
<td>0.73</td>
<td>0.06579</td>
</tr>
<tr>
<td></td>
<td>MS2</td>
<td>0.76</td>
<td>0.06337</td>
</tr>
<tr>
<td></td>
<td>MS3</td>
<td>0.69</td>
<td>0.06898</td>
</tr>
<tr>
<td>S&amp;C</td>
<td>MS1</td>
<td>0.67</td>
<td>0.07178</td>
</tr>
<tr>
<td></td>
<td>MS2</td>
<td>0.73</td>
<td>0.06664</td>
</tr>
<tr>
<td></td>
<td>MS3</td>
<td>0.62</td>
<td>0.07878</td>
</tr>
</tbody>
</table>

Table 33: Kappa Statistic for the Pilot-test of the Pre-test (MS1 = Mathematics Specialist 1 compared to Researcher, C.I = Confidence Interval, Results based on a sample of seven fourth year pre-service teachers who completed the pilot-test).

The two-way tables of the rating given by the mathematics specialist 2, and 3 with respect to the researcher are now presented for each of the phases:

There was 82% agreement in the types of answers assigned by the researcher and the second mathematics education specialist. This collaborative review revealed that:

- In the reading and understanding section
  - $\approx 82\%(9/11)$ of the discrepancies between the researcher and the mathematics education specialist were due to misreading the scoring framework or misreading the students’ answers (due to them being not presented clearly).
  - $\approx 18\%(2/11)$ of the discrepancies between the researcher and the mathematics education specialist resulted in the advice of the specialist being to adjust the scoring framework.

38It is noted that the inter-rater reliability between the researcher and the third mathematics specialist is lower than the inter-rater reliability between the researcher and the specialists 1 and 2. The third mathematics specialist stated that he had difficulty solving problem 9 in the assessment himself which caused difficulty in his marking of the assessments.
Table 34: Agreement in the Types of Answers assigned by the Researcher and Mathematics Education Specialist 2 for the Reading and Understanding Phase

<table>
<thead>
<tr>
<th>Mathematics Education Specialist 2</th>
<th>Type of Answer</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Researcher</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>55</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>59</td>
</tr>
<tr>
<td>4</td>
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<td>6</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
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<td>1</td>
<td>6</td>
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<td>8</td>
</tr>
<tr>
<td>2</td>
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<td>1</td>
<td>1</td>
<td>2</td>
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<td>0</td>
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</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Total</td>
<td>58</td>
<td>9</td>
<td>11</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>91</td>
</tr>
</tbody>
</table>

- In the planning and solving section
  - \( \approx 75\% (9/12) \) of the discrepancies between the researcher and the mathematics education specialist were due to misreading the scoring framework or misreading the students’ answers (due to them being not presented clearly).
  - \( \approx 17\% (2/12) \) of the discrepancies between the researcher and the mathematics education specialist were as the result of human error in the choice of the appropriate type of answer.
  - \( \approx 8\% (1/12) \) of the discrepancies between the researcher and the mathematics education specialist was as the result of a misjudgement made in relation to the depth of students’ work.
Table 35: Agreement in the Types of Answers assigned by the Researcher and Mathematics Education Specialist 2 for the Planning and Solving Phase

- In the solution and checking section
  - \( \approx 54\% (7/13) \) of the discrepancies between the researcher and the mathematics education specialist were due to misreading the scoring framework or misreading the students’ answers (due to them being not presented clearly).
  - \( \approx 38\% (5/13) \) of the discrepancies between the researcher and the mathematics education specialist were due to human error in assigning an incorrect type of answer or as a result of an earlier mistake in the assigning of a type of answer (not using the scoring framework correctly).
  - \( \approx 8\% (1/13) \) of the discrepancies between the researcher and the mathematics education specialist resulted in the advice of the specialist being to adjust the scoring framework.
Table 36: Agreement in the Types of Answers assigned by the Researcher and Mathematics Education Specialist 2 for the Solution and Checking Phase

There was 77% agreement in the types of answers assigned by the researcher and the third mathematics education specialist. This collaborative review revealed that:

- In the reading and understanding section
  - \( \approx 76\% (13/17) \) of the discrepancies between the researcher and the mathematics education specialist were due to misreading the scoring framework or misreading the students’ answers (due to them being not presented clearly).
  - \( \approx 24\% (4/17) \) of the discrepancies between the researcher and the mathematics education specialist were as the result of human error in the choice of the appropriate type of answer.
Table 37: Agreement in the Types of Answers assigned by the Researcher and Mathematics Education Specialist 3 for the Reading and Understanding Phase

- In the planning and solving section
  - $\approx 80\% (12/15)$ of the discrepancies between the researcher and the mathematics education specialist were due to misreading the scoring framework or misreading the students’ answers (due to them being not presented clearly).
  - $\approx 13\% (2/15)$ of the discrepancies between the researcher and the mathematics education specialist were as the result of human error in the choice of the appropriate type of answer.
  - $\approx 7\% (1/15)$ of the discrepancies between the researcher and the mathematics education specialist was as the result of a misjudgement made in relation to the depth of student’s work.
In the solution and checking section

- $\approx 65% (11/17)$ of the discrepancies between the researcher and the mathematics education specialist were due to misreading the scoring framework or misreading the students’ answers (due to them being not presented clearly).

- $\approx 29% (5/17)$ of the discrepancies between the researcher and the mathematics education specialist were due to human error in assigning an incorrect type of answer or as a result of an earlier mistake in the assigning of a type of answer (not using the scoring framework correctly).

- $\approx 6% (1/17)$ of the discrepancies between the researcher and the mathematics education specialist resulted in the advice of the specialist being to adjust the scoring framework.

Table 38: Agreement in the Types of Answers assigned by the Researcher and Mathematics Education Specialist 3 for the Planning and Solving Phase
Table 39: Agreement in the Types of Answers assigned by the Researcher and Mathematics Education Specialist 3 for the Solution and Checking Phase

<table>
<thead>
<tr>
<th>Mathematics Education Specialist 3</th>
<th>Type of Answer</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Researcher</td>
<td>4</td>
<td>17</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>5</td>
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<td>1</td>
<td>6</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td></td>
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<td>10</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>16</td>
<td>20</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>22</strong></td>
<td>1</td>
<td>3</td>
<td>15</td>
<td>22</td>
<td>63</td>
<td></td>
</tr>
</tbody>
</table>

Overall the 95% confidence intervals for the \( \kappa \) statistic show agreement between the raters as ranging from mid moderate to almost perfect agreement.

As a result of the comparative analysis, the following minor adjustments were made to the scoring framework (Appendix J):

- For problem 2 - in the types of answers for the reading and understanding phase, remove 5*, as this is already included in type 3.
- For problem 3 - include a note about type 5* for the planning and solving phase in advance in order to avoid misinterpretation of the marking scheme.
- For problem 4 - in the types of answers for the reading and understanding phase, include candle graph shows 1cm burning down each hour in type 4.
- For problem 6 - in the types of answers for the solution and checking phase, remove type 3 as it is not necessary for this problem.

Changes were also made to the types of some of the answers assigned by the researcher after discussing the differences with the mathematics education specialists and employing their expert advice. The adjusted scoring framework (Appendix K) was then used to re-evaluate the assessments from the pilot-test, and to evaluate the pre-assessments.

For the post-tests (Appendix T) the inter-rater reliability between the researcher and each of the mathematics specialists (for each of the phases) was calculated. The results are shown in Table 40. The values of \( \kappa \) and the corresponding confidence intervals show an interval of moderate to almost perfect agreement between the raters.
As a result of the comparative analysis, the following minor adjustments were made to the scoring framework (Appendix L):

- For problem 4 include a note stating that if no written explanation is provided assign appropriate type with a * in advance in order to avoid misinterpretation of the marking scheme;

- For problem 5 leave type 1* as type 5 in the R&U phase and put this in as type 1(a) in the P&S phase.

Changes were also made to the types of some of the answers assigned by the researcher after discussing the differences with the mathematics education specialists and employing their expert advice. The adjusted scoring framework (Appendix M) was then used to re-evaluate the post-tests.

### Table 40: Kappa Statistic for the Post-Tests, (MS1 = Mathematics Specialist 1 compared to Researcher, C.I = Confidence Interval, Results based on a sample of seven of the pre-service teachers who completed the post-test)

<table>
<thead>
<tr>
<th></th>
<th>( \kappa )</th>
<th>( SE(\kappa) )</th>
<th>95% C.I</th>
</tr>
</thead>
<tbody>
<tr>
<td>R&amp;U</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MS1</td>
<td>0.65</td>
<td>0.08705</td>
<td>[0.4794, 0.8206]</td>
</tr>
<tr>
<td>MS2</td>
<td>0.69</td>
<td>0.08509</td>
<td>[0.5232, 0.8568]</td>
</tr>
<tr>
<td>MS3</td>
<td>0.50</td>
<td>0.10908</td>
<td>[0.2862, 0.7138]</td>
</tr>
<tr>
<td>P&amp;S</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MS1</td>
<td>0.65</td>
<td>0.09153</td>
<td>[0.4706, 0.8294]</td>
</tr>
<tr>
<td>MS2</td>
<td>0.67</td>
<td>0.09065</td>
<td>[0.4923, 0.8477]</td>
</tr>
<tr>
<td>MS3</td>
<td>0.61</td>
<td>0.09607</td>
<td>[0.4217, 0.7983]</td>
</tr>
<tr>
<td>S&amp;C</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MS1</td>
<td>0.75</td>
<td>0.06823</td>
<td>[0.6163, 0.8837]</td>
</tr>
<tr>
<td>MS2</td>
<td>0.57</td>
<td>0.08259</td>
<td>[0.4085, 0.7315]</td>
</tr>
<tr>
<td>MS3</td>
<td>0.69</td>
<td>0.07677</td>
<td>[0.5395, 0.8405]</td>
</tr>
</tbody>
</table>
4.5 Conclusion

This chapter discussed the development of the author’s F-TAPS in mathematics. The teaching strategies employed in conjunction with the author’s F-TAPS are described in chapter 6. The development of the pre and post-assessments and analytic evaluation scales were discussed and the inter-rater reliability between different raters (of the assessments) was evaluated. The pre assessment was given to a sample of pre-service mathematics teachers at the University of Limerick. The findings and results of these assessments in addition to the findings from the literature review were used in conjunction with the author’s F-TAPS in mathematics to develop and implement an intervention in problem solving to improve the problem solving ability of this group of pre-service mathematics teachers and to develop their awareness of the many factors involved in the teaching, learning and assessing of problem solving. An analyses of the results of the pre-assessment in problem solving is presented in the next chapter.
5 Pre-Intervention Assessment Results and Findings

5.1 Introduction

The framework for teaching and assessing problem solving discussed in Chapter 4, along with the findings from the pre-intervention assessment were utilised in the formulation and implementation of the intervention. The findings from the pre-intervention assessment are discussed in this chapter. The results and findings\textsuperscript{39} from the pre-intervention assessment informed the intervention phase of this research.

The pre-intervention assessment was administered to three groups; first year \((n = 10)\), second year \((n = 8)\) and third year \((n = 3)\) pre-service mathematics teachers in the academic year 2014/2015, after it had been piloted with fourth year \((n = 7)\) pre-service mathematics teachers. Five of the second year group who completed the pre-intervention assessment chose to participate in the intervention the following year (2015/2016). The pre-intervention assessment was administered to first year \((n = 11)\), pre-service mathematics teachers in the academic year 2015/2016. Eleven of the first year group who completed the pre-intervention assessment in the academic year 2015/2016 chose to participate in the intervention.

The pre-intervention assessments were evaluated by the researcher, using the assessment scoring framework developed (Appendix K). This framework provides detailed information of the specific type of solution provided by the participants for each problem in the assessment. Specific types of solutions are described for each of the following phases of the problem solving process:

- Understanding the problem;
- Planning and solving the problem;
- Solution and checking;

Each participant thus has a specific vector of types of solution provided at each of the above phases for each problem in the assessment (the efficiency of the solution method employed was also described by specific types (e.g. Type 1 : Trial and Error only)). The findings and results from the evaluation of the pre-intervention assessments are discussed in this chapter. The results of the pre-tests, relative to the post-tests for the pre-service teachers who participated in each phase of the research are discussed further (in more detail) in Chapter 7.

\textsuperscript{39}The findings refer to the types of solution given by the participant at each phase of the problem solving process. The results refer to the scores calculated using the scoring scale in section 3.10.1.
The findings presented in Table 41 are the types of answers that the particular first year (2015) pre-service teacher provided to each phase of the problem solving cycle. Table 41 shows the types of answers given by the pre-service teachers for each problem in the pre-assessment. For convenience the scoring framework showing the scoring types (as presented in section 3.10.1) is shown here again:

Understanding the Problem:

- **Type 0**: No attempt;
- **Type 1**: Written work reveals complete misinterpretation of the problem situation;
- **Type 2**: Written work reveals basic understanding of the information presented in the problem situation but with no/unsuitable introduction of notation/representation/organisation of work to show mathematical understanding of relations among the data;
- **Type 3**: Written work reveals understanding of the information presented in the problem situation with suitable notation/representation/organisation of work introduced for part of the problem but there are substantial omissions or errors in the mathematical understanding of the relations among the data;
- **Type 4**: Written work reveals substantial understanding of the information presented in the problem situation with suitable notation/representation/organisation of work introduced for the problem but there are minor errors in the mathematical understanding of the relations among the data;
- **Type 5**: Written work reveals complete understanding of the problem situation.

Planning/Solving the Problem:

- **Type 0**: No attempt/notation or work presented does not lead to any structured plan of action;
- **Type 1**: Inappropriate/illogical/incorrect plan;
- **Type 2**: Correct plan with high error in solving;
- **Type 3**: Correct plan with medium error in solving;
- **Type 4**: Correct plan with low error in solving;
- **Type 5**: Plan leading to a correct solution with no arithmetic errors.
Solution and Checking:

- **Type 0**: No solution presented due to no attempt or no/incorrect solution presented as a result of an inappropriate/illogical/incorrect plan;
- **Type 1**: No /incorrect solution based on appropriate plan but insufficient progression/high or medium error made;
- **Type 2**: Incorrect answer based on an appropriate plan implemented correctly but with minor computational errors in solving;
- **Type 3**: Correct answer but not justified (not checked to see make sense in context of problem) or not communicated clearly;
- **Type 4**: Complete well explained (well communicated) justified solution.

The scoring scale for assigning a score to each type of response is presented in the footnote under Table 41. Tables 42, 43, and 44 show the types of responses given by the second year (2015), third year (2015) and first year (2016) groups of pre-service teachers respectively.
5.2 Responses given by each Year Group of Pre-service Teachers

First year pre-service teachers 2015:

<table>
<thead>
<tr>
<th>Student</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>F1</td>
<td>R&amp;U</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>5.5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>P&amp;S</td>
<td>2</td>
<td>5</td>
<td>5*</td>
<td>4</td>
<td>1(a)</td>
<td>5</td>
<td>5.5</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>S&amp;C</td>
<td>1</td>
<td>4</td>
<td>4*</td>
<td></td>
<td>0</td>
<td>4</td>
<td>4.4</td>
<td>1</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>L.C = HC2</td>
<td>Time = 99</td>
<td>G(i-ii)</td>
<td>E(iii)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F2</td>
<td>R&amp;U</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>5.0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>P&amp;S</td>
<td>5</td>
<td>0</td>
<td>5*</td>
<td>4</td>
<td>1(a)</td>
<td>1</td>
<td>5.0</td>
<td>0</td>
</tr>
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<td></td>
<td>S&amp;C</td>
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<td>0</td>
<td>4*</td>
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<td>1(d)</td>
<td>4.0</td>
<td>0</td>
</tr>
<tr>
<td>Gender = 1</td>
<td>L.C = HB1</td>
<td>Time = 59</td>
<td>G(i-ii)</td>
<td>E(iii)</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F3</td>
<td>R&amp;U</td>
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<td>1(a)</td>
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</tr>
<tr>
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<td>Time = 101</td>
<td>G(i-ii)</td>
<td>E(iii)</td>
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<td></td>
</tr>
<tr>
<td>F4</td>
<td>R&amp;U</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>5.5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>P&amp;S</td>
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<td>4</td>
<td>4</td>
<td>6</td>
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<td>3</td>
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</tr>
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<td>4</td>
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<td></td>
<td>0</td>
<td>1(d)</td>
<td>4.3</td>
<td>0</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>L.C = HB2</td>
<td>Time = 101</td>
<td>G(i-ii)</td>
<td>E(iii)</td>
<td></td>
<td></td>
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<tr>
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<td>E(iii)</td>
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<td>E(iii)</td>
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</table>

F1= Participant 1; Gender: 0 = Male, 1 = Female; L.C = Leaving Certificate grade achieved: HC1 = Higher Level C1 grade. Time = Time (in minutes) spent on the assessment, by the participant. R&U = Read and Understand, P&S = Plan and Solve, S&C = Solution and Checking. G(i-ii) = Generalising from part (i) to part (ii) for Q3. E(iii) = Explanation for part (iii) in Q6. Q4* = For Q4: R&U = Understand and Represent the problem situation, P&S = Plan and Solve. See appendix for description of type of solution. Types and scores: A Type 0 or 1 response = 0 marks, a type 2 response = 1 mark, a type 3 response = 2 marks, a type 4 response = 3 marks and a type 5 response = 4 marks.
The findings presented in Table 42 are the types of answers that the particular second year pre-service teacher provided to each phase of the problem solving cycle. Table 42 shows the types of answers given by the pre-service teachers for each problem in the pre-assessment. The scoring scale for assigning a score to each type of response is presented in the footnote under Table 42.
Second year pre-service teachers 2015:

<table>
<thead>
<tr>
<th>Student</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4*</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
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<td>5</td>
<td>5</td>
<td>5,5</td>
<td>5</td>
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</tr>
<tr>
<td>Gender = 1</td>
<td>P&amp;S 2</td>
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<td>3</td>
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<td>5</td>
<td>5</td>
<td>5,5</td>
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<td>5</td>
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<td>4, 4</td>
<td>4, 4, 1</td>
<td>3</td>
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<td></td>
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<td>5</td>
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<td>1(c)</td>
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<td>1(a)</td>
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<td>4</td>
<td>3</td>
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<td>1(c)</td>
<td>4</td>
<td>0</td>
<td>1</td>
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<tr>
<td>Time = 98</td>
<td>G(i-ii) 2</td>
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</tr>
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<td>5,5,3,5</td>
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<td>3</td>
<td>5,1</td>
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</tr>
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<td>4(a)</td>
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<td>5,1</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td></td>
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<td>0</td>
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<tr>
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<td>5,5</td>
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</table>

41S1= Participant 1; Gender: 0 = Male, 1 = Female; L.C = Leaving Certificate grade achieved: HC1 = Higher Level C1 grade. Time = Time (in minutes) spent on the assessment, by the participant. R&U = Read and Understand, P&S = Plan and Solve, S&C = Solution and Checking. G(i-ii) = Generalising from part (i) to part (ii) for Q3. E(iii) = Explanation for part (iii) in Q6. Q4* = For Q4: R&U = Understand and Represent the problem situation, P&S = Understand the relationship between the symbolic representation of the problem situation and the problem situation, and Solve. See appendix for description of type of solution. **Types and scores:** A Type 0 or 1 response = 0 marks, a type 2 response = 1 mark, a type 3 response = 2 marks, a type 4 response = 3 marks and a type 5 response = 4 marks.
Table 42: Types of Solutions provided by second year participants (2015)

<table>
<thead>
<tr>
<th>Student</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
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<td>5,5</td>
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<tr>
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</table>

Table 43: Types of Solutions provided by third year participants (2015)

<table>
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<tr>
<th>Student</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
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<tbody>
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<td>5</td>
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<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>T2</td>
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<td>5,5,4</td>
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<tr>
<td>Time = 79</td>
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</tr>
<tr>
<td>T3</td>
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<td>6</td>
<td>0</td>
<td>4</td>
<td>3(b)</td>
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<td>Gender = 0</td>
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<tr>
<td>Time = 92</td>
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</tbody>
</table>

The findings presented in Table 43 are the types of answers that the particular third year pre-service teacher provided to each phase of the problem solving cycle. Table 43 shows the types of answers given by the pre-service teachers for each problem in the pre-assessment. The scoring scale for assigning a score to each type of response is presented in the footnote under Table 43.

Third year pre-service teachers 2015:

T1= Participant 1; Gender: 0 = Male, 1 = Female; L.C = Leaving Certificate grade achieved: HC1 = Higher Level C1 grade. Time = Time (in minutes) spent on the assessment, by the participant. R&U = Read and Understand, P&S = Plan and Solve, S&C = Solution and Checking. G(i-ii) = Generalising from part (i) to part (ii) for Q3. E(iii) = Explanation for part (iii) in Q6. Q4* = For Q4: R&U = Understand and Represent the problem situation, P&S = Understand the relationship between the symbolic representation of the problem situation and the problem situation, and Solve. See appendix for description of type of solution. Types and scores: A Type 0 or 1 response = 0 marks, a type 2 response = 1 mark, a type 3 response = 2 marks, a type 4 response = 3 marks and a type 5 response = 4 marks.
The findings presented in Table 44 are the types of answers that the particular first year (2016) pre-service teacher provided to each phase of the problem solving cycle. Table 44 shows the types of answers given by the pre-service teachers for each problem in the pre-assessment. The scoring scale for assigning a score to each type of response is presented in the footnote under Table 44.

**First year pre-service teachers 2016**

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\footnote{T1= Participant 1; Gender: 0 = Male, 1 = Female; L.C = Leaving Certificate grade achieved: HC1 = Higher Level C1 grade. Time = Time (in minutes) spent on the assessment, by the participant. R&U = Read and Understand, P&S = Plan and Solve, S&C = Solution and Checking. G(i-ii) = Generalising from part (i) to part (ii) for Q3. E(iii) = Explanation for part (iii) in Q6. Q4* = For Q4: R&U = Understand and Represent the problem situation, P&S = Understand the relationship between the symbolic representation of the problem situation and the problem situation, and Solve. See appendix for description of type of solution. Types and scores: A Type 0 or 1 response = 0 marks, a type 2 response = 1 mark, a type 3 response = 2 marks, a type 4 response = 3 marks and a type 5 response = 4 marks.}
<table>
<thead>
<tr>
<th>Student</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>4</td>
<td>5,5,5,4</td>
<td>3</td>
<td>4</td>
<td>2.0</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 1</td>
<td>S&amp;C 0</td>
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<td>1</td>
<td>1(a)</td>
<td>1</td>
<td>1.0</td>
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<tr>
<td>L.C = HC1</td>
<td>Time = 87</td>
<td>G(i-ii)</td>
<td>E(iii)</td>
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<td>0</td>
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<td>R&amp;U 5</td>
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<td>5,5,4,1</td>
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<td>5</td>
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<td>0.0</td>
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<tr>
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<td>E(iii)</td>
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</tr>
<tr>
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<td>Time = 87</td>
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<td>3</td>
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<td>5(b)</td>
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<td>Time = 87</td>
<td>G(i-ii)</td>
<td>E(iii)</td>
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</tr>
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<td>5*</td>
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<td>5</td>
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<td>5(b)</td>
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<td>Time = 87</td>
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<tr>
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<td>5</td>
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<td>5*</td>
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<td>3</td>
<td>5</td>
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<td>5(b)</td>
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<td>Time = 87</td>
<td>G(i-ii)</td>
<td>E(iii)</td>
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<td>4</td>
<td>5.0</td>
<td>5</td>
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</tr>
<tr>
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<td>P&amp;S 5</td>
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<td>1</td>
<td>3</td>
<td>1(a)</td>
<td>3</td>
<td>5.0</td>
<td>2</td>
<td>5(a)</td>
</tr>
<tr>
<td>L.C = HC1</td>
<td>Time = 87</td>
<td>G(i-ii)</td>
<td>E(iii)</td>
<td>1</td>
<td></td>
<td></td>
<td>3</td>
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233
<table>
<thead>
<tr>
<th>Student</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
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<td>4</td>
<td>5,4,5,5</td>
<td>3</td>
<td>5</td>
<td>5,0</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S 3</td>
<td>2</td>
<td>5*</td>
<td>6(c)</td>
<td>1(a)</td>
<td>5</td>
<td>5,0</td>
<td>3</td>
<td>5(a)</td>
</tr>
<tr>
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<td>S&amp;C 1</td>
<td>1</td>
<td>3*</td>
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<td>4</td>
<td>4,0</td>
<td>1</td>
<td>4</td>
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<td>Time = 52</td>
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<td></td>
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<tr>
<td></td>
<td>E(iii)</td>
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<td>P16</td>
<td>R&amp;U 5</td>
<td>4</td>
<td>4</td>
<td>5,5,1(a),1(a)</td>
<td>3</td>
<td>5</td>
<td>5,0</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S 5</td>
<td>2</td>
<td>5*</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>5,0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>L.C = HB2</td>
<td>S&amp;C 4</td>
<td>1</td>
<td>3*</td>
<td></td>
<td>0</td>
<td>4</td>
<td>4,0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Time = 87</td>
<td>G(i-ii) 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>E(iii)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Table 44: Types of Solutions provided by first year participants (2016)
5.3 Results for All Pre-service Teachers who Completed the Pre-Test

A fine-grained analysis of the responses provided by each of the individual year-groups of pre-service teachers was undertaken and tables showing the percentage of the pre-service teachers at each response type for each of the phases of the problem solving cycle were formulated for each individual year group. The percentage of each individual year group at the highest type of response over each phase was also evaluated. The fine-grained analysis did not yield any additional outcomes of significance to the findings from the analysis of the whole group of pre-service teachers (discussed next), and are therefore not presented here but are available in the researcher’s notes.

5.3.1 Total Score for the Reading and Understanding Phase

The percentage of all the pre-service teachers who completed the pre-test, at each type of response for the reading and understanding phase of the problem solving cycle is shown in Table 45.

<table>
<thead>
<tr>
<th>Pre-test (All)</th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
<th>Type 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>0</td>
<td>0</td>
<td>6.3</td>
<td>28.1</td>
<td>12.5</td>
<td>53.1</td>
</tr>
<tr>
<td>Problem 2</td>
<td>6.3</td>
<td>0</td>
<td>6.3</td>
<td>18.8</td>
<td>12.5</td>
<td>56.3</td>
</tr>
<tr>
<td>Problem 3</td>
<td>0</td>
<td>6.3</td>
<td>6.3</td>
<td>3.1</td>
<td>43.8</td>
<td>40.6</td>
</tr>
<tr>
<td>Problem 5</td>
<td>3.1</td>
<td>6.3</td>
<td>3.1</td>
<td>62.5</td>
<td>0</td>
<td>25</td>
</tr>
<tr>
<td>Problem 6</td>
<td>0</td>
<td>3.1</td>
<td>0</td>
<td>12.5</td>
<td>31.3</td>
<td>53.1</td>
</tr>
<tr>
<td>Problem 7</td>
<td>6.3</td>
<td>0</td>
<td>6.3</td>
<td>3.1</td>
<td>0</td>
<td>84.4</td>
</tr>
<tr>
<td>Problem 7(ii)</td>
<td>56.3</td>
<td>3.1</td>
<td>0</td>
<td>6.3</td>
<td>0</td>
<td>34.4</td>
</tr>
<tr>
<td>Problem 8</td>
<td>3.1</td>
<td>6.3</td>
<td>6.3</td>
<td>15.6</td>
<td>0</td>
<td>68.8</td>
</tr>
<tr>
<td>Problem 9</td>
<td>15.6</td>
<td>0</td>
<td>9.4</td>
<td>3.1</td>
<td>0</td>
<td>71.9</td>
</tr>
</tbody>
</table>

Table 45: Percentage of All the Pre-service Teachers at each Response Type for the Reading and Understanding Phase.
Overall, the group of pre-service teachers who completed the pre-test demonstrated most difficulty in the reading and understanding of problem 5 with 71.9% of the responses showing complete misinterpretation of the problem situation or no notation to show mathematical understanding of the relations among the data or some substantial omissions/errors in the mathematical understanding of relations among the data. The group showed least difficulty in the reading and understanding of problem 7(i) with 84.4% of responses showing complete understanding. Difficulty in the reading and understanding of problems 1, 2 and 8 was shown by this group, with 34.4%, 25.1% and 28.2% respectively of the responses for each of these problems showing complete misinterpretation of the problem situation (problem 8), or no introduction of suitable notation/representation to show mathematical understanding of relations among the data, or showing some substantial omissions/errors in the mathematical understanding of relations among the data. Approximately 56% of the pre-service teachers in this group made no attempt at problem 7(ii).

Statistical analysis was carried out using SPSS (Statistical Package for Social Sciences) 24.0 for Windows. In this study, the variables consisted of the categorical variables; gender, grade and year, and the numerical variables; reading and understanding score, planning and solving score and solution and checking score.

The total score for the reading and understanding phase of the problem solving cycle (TRU) for each participant across the nine problems was computed. A Kruskal-Wallis test was considered to determine whether there were any statistically significant differences between the medians of the different year groups. However one of the assumptions of Kruskal-Wallis is homogeneity of variance between the year groups. To assess whether the condition of homogeneity of variance was met, a one way analysis of variance (ANOVA) test on the absolute value of TRU (the ranked TRU score – the mean rank TRU score) by year was carried out. The ANOVA test ($F = 0.063$, p-value = 0.979) showed that the condition of homogeneity of variance between the different year groups was met. The Kruskal-Wallis test showed no statistically significant difference in the median reading and understanding score between the different year groups ($\chi^2(3) = 0.958$, p-value = 0.811). The median reading and understanding score of the different year groups is shown in Table 46. It was noted that both of the first year groups scored a higher median reading and understanding score than the second and third-year groups with the first-year group (2016) slightly higher than the first-year (2015) group.
To assess whether the condition of homogeneity of variance was met between the groups (A grade, B grade, C grade), a one way analysis of variance (ANOVA) test on the absolute value of TRU (the ranked TRU score – the mean rank TRU score) by grade was carried out. The ANOVA test \( (F = 0.124, \ p-value = 0.945) \) showed that the condition of homogeneity of variance between the different grade groups was met. The Kruskal-Wallis test showed no statistically significant difference in the median reading and understanding score between the different grade groups \( (\chi^2(2) = 5.286, \ p-value = 0.152) \). The median reading and understanding score of the different grade groups is shown in Table 47.

<table>
<thead>
<tr>
<th>TRU</th>
<th>N</th>
<th>Median</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>First year 2015</td>
<td>10</td>
<td>40.5</td>
<td>15.25</td>
<td>25</td>
<td>47</td>
</tr>
<tr>
<td>Second year 2015</td>
<td>8</td>
<td>39</td>
<td>16.75</td>
<td>27</td>
<td>51</td>
</tr>
<tr>
<td>Third year 2015</td>
<td>3</td>
<td>36</td>
<td>-</td>
<td>34</td>
<td>38</td>
</tr>
<tr>
<td>First year 2016</td>
<td>11</td>
<td>41</td>
<td>4</td>
<td>30</td>
<td>48</td>
</tr>
<tr>
<td>Total</td>
<td>32</td>
<td>39</td>
<td>9.75</td>
<td>25</td>
<td>51</td>
</tr>
</tbody>
</table>

Table 46: Average Performance in TRU by Year Groups and Total Group

To assess whether the condition of homogeneity of variance was met between males and females, a one way analysis of variance (ANOVA) test on the absolute value of TRU (the ranked TRU score – the mean rank TRU score) by gender was carried out. The ANOVA test \( (F = 0.301, \ p-value = 0.588) \), showed that the condition of homogeneity of variance between males and females was met. The Mann Whitney U test showed no statistically significant difference in the median reading and understanding score between males and females \( (U = 76, \ p-value = 0.057) \). The median reading and understanding score of the different grade groups is shown in Table 48. Males obtained higher scores in TRU than females.

<table>
<thead>
<tr>
<th>TRU</th>
<th>N</th>
<th>Median</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3</td>
<td>47</td>
<td>42</td>
<td>51</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>22</td>
<td>39</td>
<td>11.25</td>
<td>25</td>
<td>48</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>38</td>
<td>10</td>
<td>34</td>
<td>47</td>
</tr>
</tbody>
</table>

Table 47: Average Performance in TRU by Grade Groups

To assess whether the condition of homogeneity of variance was met between the groups (A grade, B grade, C grade), a one way analysis of variance (ANOVA) test on the absolute value of TRU (the ranked TRU score – the mean rank TRU score) by grade was carried out. The ANOVA test \( (F = 0.124, \ p-value = 0.945) \) showed that the condition of homogeneity of variance between the different grade groups was met. The Kruskal-Wallis test showed no statistically significant difference in the median reading and understanding score between the different grade groups \( (\chi^2(2) = 5.286, \ p-value = 0.152) \). The median reading and understanding score of the different grade groups is shown in Table 47.

<table>
<thead>
<tr>
<th>TRU</th>
<th>N</th>
<th>Median</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3</td>
<td>47</td>
<td>42</td>
<td>51</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>22</td>
<td>39</td>
<td>11.25</td>
<td>25</td>
<td>48</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>38</td>
<td>10</td>
<td>34</td>
<td>47</td>
</tr>
</tbody>
</table>

Table 47: Average Performance in TRU by Grade Groups

To assess whether the condition of homogeneity of variance was met between males and females, a one way analysis of variance (ANOVA) test on the absolute value of TRU (the ranked TRU score – the mean rank TRU score) by gender was carried out. The ANOVA test \( (F = 0.301, \ p-value = 0.588) \), showed that the condition of homogeneity of variance between males and females was met. The Mann Whitney U test showed no statistically significant difference in the median reading and understanding score between males and females \( (U = 76, \ p-value = 0.057) \). The median reading and understanding score of the different grade groups is shown in Table 48. Males obtained higher scores in TRU than females.

<table>
<thead>
<tr>
<th>TRU</th>
<th>N</th>
<th>Median</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3</td>
<td>47</td>
<td>42</td>
<td>51</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>22</td>
<td>39</td>
<td>11.25</td>
<td>25</td>
<td>48</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>38</td>
<td>10</td>
<td>34</td>
<td>47</td>
</tr>
</tbody>
</table>

Table 47: Average Performance in TRU by Grade Groups

To assess whether the condition of homogeneity of variance was met between the groups (A grade, B grade, C grade), a one way analysis of variance (ANOVA) test on the absolute value of TRU (the ranked TRU score – the mean rank TRU score) by grade was carried out. The ANOVA test \( (F = 0.124, \ p-value = 0.945) \) showed that the condition of homogeneity of variance between the different grade groups was met. The Kruskal-Wallis test showed no statistically significant difference in the median reading and understanding score between the different grade groups \( (\chi^2(2) = 5.286, \ p-value = 0.152) \). The median reading and understanding score of the different grade groups is shown in Table 47. The median reading and understanding score of the different grade groups is shown in Table 47. Males obtained higher scores in TRU than females.
<table>
<thead>
<tr>
<th>TRU</th>
<th>N</th>
<th>Median</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>18</td>
<td>42</td>
<td>8.75</td>
<td>30</td>
<td>48</td>
</tr>
<tr>
<td>Female</td>
<td>14</td>
<td>35.5</td>
<td>11.75</td>
<td>25</td>
<td>51</td>
</tr>
</tbody>
</table>

Table 48: Average Performance in TRU by Gender Groups

5.3.2 Total Score for the Planning and Solving Phase

The percentage of all the pre-service teachers who completed the pre-test, at each type of response for the planning and solving phase of the problem solving cycle is shown in Table 49.

<table>
<thead>
<tr>
<th>Pre-test (All)</th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
<th>Type 5</th>
</tr>
</thead>
<tbody>
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<td>6.3</td>
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<td>31.3</td>
<td>12.5</td>
<td>3.1</td>
<td>46.9</td>
</tr>
<tr>
<td>Problem 2</td>
<td>25</td>
<td>12.5</td>
<td>9.4</td>
<td>3.1</td>
<td>15.6</td>
<td>34.4</td>
</tr>
<tr>
<td>Problem 3</td>
<td>3.1</td>
<td>18.8</td>
<td>3.1</td>
<td>6.3</td>
<td>15.6</td>
<td>53.1</td>
</tr>
<tr>
<td>Problem 5</td>
<td>6.3</td>
<td>68.8</td>
<td>6.3</td>
<td>0</td>
<td>3.1</td>
<td>15.6</td>
</tr>
<tr>
<td>Problem 6</td>
<td>0</td>
<td>34.4</td>
<td>3.1</td>
<td>18.8</td>
<td>3.1</td>
<td>40.6</td>
</tr>
<tr>
<td>Problem 7</td>
<td>6.3</td>
<td>6.3</td>
<td>0</td>
<td>3.1</td>
<td>0</td>
<td>84.4</td>
</tr>
<tr>
<td>Problem 7(ii)</td>
<td>56.3</td>
<td>6.3</td>
<td>3.1</td>
<td>6.3</td>
<td>3.1</td>
<td>25</td>
</tr>
<tr>
<td>Problem 8</td>
<td>25</td>
<td>9.4</td>
<td>31.3</td>
<td>31.3</td>
<td>3.1</td>
<td>3.1</td>
</tr>
<tr>
<td>Problem 9</td>
<td>28.1</td>
<td>6.3</td>
<td>12.5</td>
<td>3.1</td>
<td>6.3</td>
<td>43.8</td>
</tr>
</tbody>
</table>

Table 49: Percentage of All the Pre-service Teachers at each Response Type for the Planning and Solving Phase.

The group of pre-service teachers demonstrated most difficulty in the planning and solving of problem 5, with 68.8% of the responses forming an inappropriate/illogical/incorrect plan. Difficulties in the planning and solving of problem 6 was also shown by the group, with 34.4% of the responses showing the formation of an inappropriate/illogical/incorrect plan to solve the problem. None of this group formed a correct plan free from error for problem 8. Approximately 84% of the pre-service teachers provided responses which showed the formation of a correct plan free from error for problem 7(i). A little over half of the pre-service teachers provided a correct plan, free from error to problem 3. Less than half of the pre-service teachers provided a correct plan, free from error to the remaining problems.

The total score for the planning and solving phase of the problem solving cycle (TPS) for each participant across the eight problems (problem 4 omitted)
was computed. A Kruskal-Wallis test was considered to determine whether there were any statistically significant differences between the medians of the different year groups. To assess whether the condition of homogeneity of variance was met, a one way analysis of variance (ANOVA) test on the absolute value of TPS (the ranked TPS score – the mean rank TPS score) by year was carried out. The ANOVA test \((F = 1.425, \text{p-value} = 0.256)\) showed that the condition of homogeneity of variance between the different year groups was met. The Kruskal-Wallis test showed no statistically significant difference in the median reading and understanding score between the different year groups \((\chi^2(3) = 2.959, \text{p-value} = 0.398)\). The median planning and solving score of the different year groups is shown in Table 50. It was noted that although both of the first year groups scored a higher median reading and understanding score than the second and third-year groups, in the planning and solving phase the second-year and the first-year (2016) groups scored a higher median score than the first-year (2016) and the third-year groups.

<table>
<thead>
<tr>
<th>TPS</th>
<th>N</th>
<th>Median</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>First year 2015</td>
<td>10</td>
<td>16.5</td>
<td>13.5</td>
<td>5</td>
<td>24</td>
</tr>
<tr>
<td>Second year 2015</td>
<td>8</td>
<td>20.5</td>
<td>17.25</td>
<td>7</td>
<td>29</td>
</tr>
<tr>
<td>Third year 2015</td>
<td>3</td>
<td>12</td>
<td>11</td>
<td>11</td>
<td>18</td>
</tr>
<tr>
<td>First year 2016</td>
<td>11</td>
<td>21</td>
<td>7</td>
<td>1</td>
<td>29</td>
</tr>
<tr>
<td>Total</td>
<td>32</td>
<td>20</td>
<td>12.5</td>
<td>1</td>
<td>29</td>
</tr>
</tbody>
</table>

Table 50: Average Performance in TPS by Year Groups and Total Group

To assess whether the condition of homogeneity of variance was met between the groups (A grade, B grade, C grade), a one way analysis of variance (ANOVA) test on the absolute value of TPS (the ranked TPS score – the mean rank TPS score) by grade was carried out. The ANOVA test \((F = 1.415, \text{p-value} = 0.259)\) showed that the condition of homogeneity of variance between the different grade groups was met. The Kruskal-Wallis test showed no statistically significant difference in the median planning and solving score between the different grade groups \((\chi^2(2) = 3.914, \text{p-value} = 0.141)\). The median planning and solving score of the different grade groups is shown in Table 51.
To assess whether the condition of homogeneity of variance was met between males and females, a one-way analysis of variance (ANOVA) test on the absolute value of TPS (the ranked TPS score – the mean rank TPS score) by gender was carried out. The ANOVA test ($F = 3.728$, p-value = 0.063), showed that the condition of homogeneity of variance between males and females was met. The Mann Whitney U test showed no statistically significant difference in the planning and solving mean rank score between males and females ($U = 78.5$, p-value = 0.071). The median planning and solving score of males and females is shown in Table 52. Males obtained higher scores in TPS than females.

<table>
<thead>
<tr>
<th>TPS</th>
<th>N</th>
<th>Median</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3</td>
<td>24</td>
<td>22</td>
<td>22</td>
<td>29</td>
</tr>
<tr>
<td>B</td>
<td>22</td>
<td>19</td>
<td>12.5</td>
<td>5</td>
<td>29</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>17</td>
<td>15.5</td>
<td>1</td>
<td>24</td>
</tr>
</tbody>
</table>

Table 51: Average Performance in TPS by Grade Groups

<table>
<thead>
<tr>
<th>TPS</th>
<th>N</th>
<th>Median</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>18</td>
<td>21</td>
<td>6</td>
<td>9</td>
<td>29</td>
</tr>
<tr>
<td>Female</td>
<td>14</td>
<td>13.5</td>
<td>14.5</td>
<td>1</td>
<td>29</td>
</tr>
</tbody>
</table>

Table 52: Average Performance in TPS by Gender Groups

5.3.3 Total Score for the Solution and Checking Phase

The percentage of all the pre-service teachers who completed the pre-test, at each type of response for the solution and checking phase of the problem solving cycle is shown in Table 53.
The group of pre-service teachers demonstrated most difficulty in obtaining the correct solution to problems 8, 5, 7(ii) and 2. None of the pre-service teachers were able to provide a correct response to problem 8. The percentage of the pre-service teachers who obtained the correct solutions to problems 5, 7(ii) and 2 was 15.6%, 25.1% and 34.4% respectively. 84.4% of the pre-service teachers obtained the correct solution to problems 7(i). A little over half of the pre-service teachers provided a correct solution to problem 3. Less than half of the pre-service teachers provided a correct solution to the remaining problems.

The total score for the solution and checking phase of the problem solving cycle (TSC) for each participant across the eight problems (problem 4 omitted) was computed. A Kruskal-Wallis test was considered to determine whether there were any statistically significant differences between the median TSC score of the different year groups. To assess whether the condition of homogeneity of variance was met, a one way analysis of variance (ANOVA) test on the absolute value of TSC (the ranked TSC score – the mean rank TSC score) by year was carried out. The ANOVA test ($F = 1.425$, p-value = 0.256) showed that the condition of homogeneity of variance between the different year groups was met. The Kruskal-Wallis test showed no statistically significant difference in the median reading and understanding score between the different year groups ($\chi^2(3) = 2.959$, p-value = 0.398). The median solution and checking score of the different year groups is shown in Table 54. It was noted that the second-year and the first-year (2016) groups scored a higher median score than the first-year (2016) and the third-year groups.
To assess whether the condition of homogeneity of variance was met between the groups (A grade, B grade, C grade), a one way analysis of variance (ANOVA) test on the absolute value of TSC (the ranked TSC score − the mean rank TSC score) by grade was carried out. The ANOVA test ($F = 0.130$, p-value = 0.941) showed that the condition of homogeneity of variance between the different grade groups was met. The Kruskal-Wallis test showed no statistically significant difference in the median solution and checking score between the different grade groups ($\chi^2(2) = 2.491$, p-value = 0.288). The median solution and checking score of the different grade groups is shown in Table 55.

<table>
<thead>
<tr>
<th>TSC</th>
<th>N</th>
<th>Median</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>First year 2015</td>
<td>10</td>
<td>9.5</td>
<td>9.5</td>
<td>2</td>
<td>15</td>
</tr>
<tr>
<td>Second year 2015</td>
<td>8</td>
<td>11</td>
<td>9.5</td>
<td>4</td>
<td>17</td>
</tr>
<tr>
<td>Third year 2015</td>
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<td>6</td>
<td></td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>First year 2016</td>
<td>11</td>
<td>11</td>
<td>3</td>
<td>0</td>
<td>19</td>
</tr>
<tr>
<td>Total</td>
<td>32</td>
<td>10.5</td>
<td>6.75</td>
<td>0</td>
<td>19</td>
</tr>
</tbody>
</table>

Table 54: Average Performance in TSC by Year Groups and Total Group

To assess whether the condition of homogeneity of variance was met between males and females, a one way analysis of variance (ANOVA) test on the absolute value of TSC (the ranked TSC score − the mean rank TSC score) by gender was carried out. The ANOVA test ($F = 0.336$, p-value = 0.566), showed that the condition of homogeneity of variance between males and females was met. The Mann Whitney U test showed no statistically significant difference in the solution and checking mean rank score between males and females ($U = 77.5$, p-value = 0.065). The median solution and checking score of males and females is shown in Table 56. Males obtained higher scores in TSC than females.

<table>
<thead>
<tr>
<th>TSC</th>
<th>N</th>
<th>Median</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3</td>
<td>13</td>
<td>11</td>
<td>16</td>
<td>11</td>
</tr>
<tr>
<td>B</td>
<td>22</td>
<td>10</td>
<td>7</td>
<td>2</td>
<td>19</td>
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<tr>
<td>C</td>
<td>6</td>
<td>9</td>
<td>9.75</td>
<td>0</td>
<td>15</td>
</tr>
</tbody>
</table>

Table 55: Average Performance in TSC by Grade Groups

To assess whether the condition of homogeneity of variance was met between males and females, a one way analysis of variance (ANOVA) test on the absolute value of TSC (the ranked TSC score − the mean rank TSC score) by gender was carried out. The ANOVA test ($F = 0.130$, p-value = 0.941) showed that the condition of homogeneity of variance between the different grade groups was met. The Kruskal-Wallis test showed no statistically significant difference in the median solution and checking score between the different grade groups ($\chi^2(2) = 2.491$, p-value = 0.288). The median solution and checking score of the different grade groups is shown in Table 55.
The percentage of all the pre-service teachers who completed the pre-test with the highest type of response over each phase of the problem solving cycle is shown in Table 57.

<table>
<thead>
<tr>
<th>TSC</th>
<th>N</th>
<th>Median</th>
<th>IQR</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>18</td>
<td>11</td>
<td>4.5</td>
<td>4</td>
<td>19</td>
</tr>
<tr>
<td>Female</td>
<td>14</td>
<td>8</td>
<td>8</td>
<td>0</td>
<td>17</td>
</tr>
</tbody>
</table>

Table 56: Average Performance in TSC by Gender Groups

<table>
<thead>
<tr>
<th>Pre for All</th>
<th>Read and Understand (Type 5)</th>
<th>Plan and Solve (Type 5)</th>
<th>Solution and Checking (Type 3)</th>
<th>Solution and Checking (Type 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>53.1</td>
<td>46.9</td>
<td>18.8</td>
<td>28.1</td>
</tr>
<tr>
<td>Problem 2</td>
<td>56.3</td>
<td>34.4</td>
<td>15.6</td>
<td>18.8</td>
</tr>
<tr>
<td>Problem 3</td>
<td>40.6</td>
<td>53.1</td>
<td>28.1</td>
<td>25</td>
</tr>
<tr>
<td>Problem 5</td>
<td>25</td>
<td>15.6</td>
<td>3.1</td>
<td>12.5</td>
</tr>
<tr>
<td>Problem 6</td>
<td>53.1</td>
<td>40.6</td>
<td>3.1</td>
<td>37.5</td>
</tr>
<tr>
<td>Problem 7</td>
<td>84.4</td>
<td>84.4</td>
<td>9.4</td>
<td>75</td>
</tr>
<tr>
<td>Problem 7(ii)</td>
<td>34.4</td>
<td>25</td>
<td>6.3</td>
<td>18.8</td>
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<tr>
<td>Problem 8</td>
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<tr>
<td>Problem 9</td>
<td>71.9</td>
<td>43.8</td>
<td>34.4</td>
<td>9.4</td>
</tr>
</tbody>
</table>

Table 57: Percentage of All the Pre-service Teachers at the Highest type of Response over each Phase of the Problem Solving Cycle
Approximately 69% of the pre-service teachers demonstrated complete understanding of problem 8, however none of these 68.8% were able to obtain a correct solution to this problem. There was considerable difficulty in proving this statement, with difficulties arising in the manipulation of the algebraic representation of the expression into a perfect square. A quarter of the pre-service teachers demonstrated complete understanding of problem 5, with only 15.6% of these able to provide the correct solution. The group demonstrated substantial difficulty with understanding the structure of the problem (the extra information given in this problem was used in an incorrect way by some of the pre-service teachers) in addition to the proportional reasoning required to solve problem 5. Approximately 25% of the pre-service teachers provided a correct response to problem 7(ii), while 84.4% of the group provided the correct solution to problem 7(i). The pre-service teachers had difficulties in applying/extending the schema they already possess from solving part (i) to the broader problem of part (ii). A correct solution to problem 2 was provided by 34.4% of the pre-service teachers. The pre-service teachers again demonstrated difficulty in understanding the structure of the problem (extra information given in this problem was used in an incorrect way by some of the pre-service teachers) in addition to forming algebraic relationships. Thirty-four percent of the pre-service teachers in this group did not attempt or incorrectly generalised from part (i) to part (ii) in problem 3. The percentage of all the pre-service teachers who completed the pre-test with the highest type of response over each phase of the problem solving cycle is shown in Figure 20.
5.3.4 Problem 4

The percentage of all the pre-service teachers who completed the pre-test with the highest type of response for each of the parts in problem 4 is shown in Figure 21. The pre-service teachers experienced difficulty in demonstrating understanding of, and correct graphical representation of exponential relationships \( y = 20 + 69.5e^{\frac{x}{21}} \) and \( y = 3000 \times (0.8)^x \), while demonstrating good understanding and representation of the linear and trigonometric relationships in parts (i) and (ii) respectively.
5.4 Common Difficulties

As discussed in the analysis of the pre-tests, there were some common findings (difficulties/errors/misconceptions) experienced by the pre-service teachers. These were:

1. in general, the pre-service teachers did not check their solutions to the problems (in contrast to the mathematicians who did);

2. the pre-service teachers experienced difficulty in generalising and there was a lack of evidence of specialising to aid their attempts at generalising (problem 3 and problem 9);

3. the pre-service teachers experienced difficulty with the formulation of and manipulation of algebraic expressions and equations (problems 1, 2, 6 and 8);

4. the pre-service teachers experienced difficulty with proportional reasoning (problem 5);

5. the concept of proof caused difficulty;

6. the structure of the problem (problem 5, problem 2 and problem 8) was not well understood by some of the pre-service teachers;
7. the pre-service teachers displayed a lack of evidence to use an alternative approach to some problems indicating a lack of flexibility (problem 1);

8. the pre-service teachers demonstrated some difficulty with representing mathematical problem situations (exponential relationships in particular);

9. the pre-service teachers indicated poor retention of mathematical information/ideas they had previously encountered in similar problems.

A brief description of the common difficulties is presented here. The frequency at which these difficulties occurred is shown in Table 58\textsuperscript{44}.

The pre-service teachers experienced substantial difficulty with proportional reasoning (problem 5). The difficulties shown by the pre-service teachers demonstrated an inability to formulate a perception to grasp the structure of the problem with many of the pre-service teachers using the surplus information in an incorrect way. They also demonstrated an inability to process mathematical information - to think logically in relation to quantitative, number and letter relationships. These difficulties reflect inadequate problem solving ability. These difficulties also reflect incomplete mathematical proficiency in terms of conceptual understanding, adaptive reasoning and strategic competence. The inability to apply knowledge and skills, analyse information presented verbally and translate it into mathematical form, use an appropriate technique to process information also shows deficiencies in problem solving skills.

The pre-service teachers experienced considerable difficulty with the formulation of, and manipulation of, algebraic expressions and equations (problems 1, 2, 6 and 8). The lack of flexibility demonstrated by the pre-service teachers in the manipulation of algebraic expressions and equations in problem 8 prevented all of the pre-service teachers from fully completing the proof in problem 8\textsuperscript{45}. The inability to form the algebraic equations in problems 1, 2 and 6 meant that many of the participants were unable to solve these problems while a number of them resorted to trial and error (for problems 1 and 2) after failing to produce the equations. They possess the mathematics knowledge necessary to solve problem 1 but cannot productively use it, this demonstrates a lack of conceptual understanding. The use of specialising and generalising would have been very helpful in the formulation of the equation:

\[
\frac{x}{2} + 2 = y
\]

in problem 1, however the use of specialising and generalising was not evident in the participants' solutions. The difficulties experienced by this group with

\textsuperscript{44}The numbers show those who demonstrated substantial difficulty in these aspects of problem solving.

\textsuperscript{45}One student P7 came very close to completing the proof by induction. Only one pre-service teacher was successful in completing the proof, this was P22 in the pilot-test.
problem 1 demonstrate inadequacies in conceptual understanding, adaptive reasoning, strategic competence and for those who did not make more than one attempt it also shows deficiencies in productive disposition. The pre-service teachers experienced difficulty in understanding the structural relationship between the variables in problem 1. These difficulties also reveal deficiencies in problem solving ability, problem solving skills and mathematical thinking. Also in problem 6, a number of pre-service teachers experienced difficulty in communicating their reasoning for the decision made in relation to how many times a dose of a drug should be given.

The pre-service teachers experienced difficulty in generalising and there was a lack of evidence of specialising to aid their attempts at generalising (problem 3 and problem 9). The pre-service teachers demonstrated some difficulty with representing mathematical problem situations (exponential relationships in particular in problem 4). Many students were successful at completing problem 7(i) showing logical thought in the development of the solution but only a small number were able to extend their reasoning for a more difficult similar situation in problem 7(ii).

Two of the seven pre-service teachers who completed the pilot-test scored less than 50% in the pilot-test (26% and 44%), and another achieved 52%. These pre-service teachers were in their final semester of their final year of the BSc. programme in mathematics education.

Half of the ten first year (2015) pre-service teachers, three of the eight second year pre-service teachers (2015), two of the three third year pre-service teachers and one of the eleven first year pre-service teachers (2016) all achieved less than 50% in the pre-test. Another two of the first year pre-service teachers (2016) achieved 51% and 53%.

Of these 16 pre-service teachers who received less than 53% in the pilot and pre-test, 13 did not participate in the intervention (the fourth year group who completed the pilot-test had graduated, the third year (2015) group were in fourth year and chose not to participate, the three second year (2015) pre-service teachers who achieved less than 50% chose not to participate and the first year (2015) group were out on teaching placements). Two of the first year (2016) group initially chose to participate in the intervention but subsequently dropped out (P15 and P16 dropped out after the first class due to workload of university course and P14 dropped out having completed 4 of the 6 classes, she did not give a reason for her decision).
Table 58 shows the frequency at which the common difficulties occurred. The columns in Table 58 are coded as follows:

- Prob - Problem.
- A - Formulation and manipulation of algebraic expressions and equations.
- F - Flexibility to try an alternative approach if first attempt failed.
- S & G - Specialising and generalising.
- E - Efficiency - use of trial and error only.
- S - Structure of the problem (incorrect use of surplus information).
- L - Logical relations among data/logical thinking.
- M - Misinterpretation of problem.
- Comm. - Communication of reasoning/explanation.
- Graph.R - Graphical representation.
- C - Connect algebraic and graphical representation.
- PR - Proportional reasoning.
- R/C - Restriction/Condition (failure to adhere to restriction/condition in problem).
- COP - Concept of proof (incorrectly thinking that showing true for one case is sufficient for a proof).

The numbers show those who demonstrated substantial difficulty in these aspects of problem solving.
<table>
<thead>
<tr>
<th>Prob</th>
<th>A</th>
<th>F</th>
<th>S &amp; G</th>
<th>E</th>
<th>S</th>
<th>L</th>
<th>M</th>
<th>Comm.</th>
<th>Graph.R</th>
<th>C</th>
<th>PR</th>
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<th>COP</th>
</tr>
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<tbody>
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</tr>
</tbody>
</table>

Table 58: Frequency of Common Difficulties in the Pre-Tests (47Linear, 48Trigonometric, 49Exponential, 50Exponential, 51Formulation, 52Manipulation)
5.5 Conclusion

Chapman (2015) states that in order to teach for the development of problem solving proficiency, teachers should be proficient in problem solving themselves, in addition to understanding all of what problem solving encompasses. Chapman (2015) notes the significant role that understanding the structure of a problem plays, in the guiding of students’ solutions. In chapter 2, it was noted that teacher quality is believed to be one of the most important factors affecting students’ learning (Ní Riórdáin and Hannigan, 2011). Chapman (2015) states that this quality should include conceptual and procedural knowledge of the mathematical thinking involved in the various stages of existing problem solving models, in addition to understanding instructional practices aimed at improving students problem solving performance. Thompson (1985, p.292; cited in Chapman, 2015) suggested that teachers need to “experience mathematical problem solving from the perspective of the problem solver before they can adequately deal with its teaching”. The findings from the pre-intervention assessment presented in this chapter, along with the framework for teaching and assessing problem solving in mathematics formed the foundation of the intervention which was developed. The development and implementation of this intervention is discussed in chapter six.
6 Intervention Development and Implementation

6.1 Introduction

To intervene means to become involved in a situation in order to improve it or to prevent it from becoming worse. An intervention in education is defined as a systematic process of assessment and planning employed to remediate or prevent an educational or development problem. Bartholomew et al. (2001, p.1) identifies three main activities involved in developing interventions: “needs assessment, program development, and evaluation”.

In this study the needs assessment was established from a review of the current literature and also from the findings of the pre-test. The needs assessment provided the theory and evidence on which to base the development and implementation of the intervention. Kok et al., (2004) observe that a programme is more likely to be beneficial to participants if it is guided by existing theories in that particular field of interest. The problem with problem solving, required the search for theoretical and empirical data in order to understand the problem and its causal relations more completely (Kok et al., 2004). The author integrated the information gained from the analysis of this data into a coherent form to provide the theoretical framework which underpins this study and identifies the proposed intervention (described in this chapter) as a possible solution to the problem with problem solving in mathematics.

The aims of the intervention were formulated in direct response to the review of the current literature and also from the findings of the pre-test. These aims were formulated with the target audience of pre-service teachers in mind. Consideration was given to the mathematical ideas that the pre-service teachers should understand after participating in this intervention, as well as the mathematical skills they should develop (Stanford University, 2015).

6.2 Aims of the Intervention

The author wanted to create an intervention to address the needs of the pre-service teachers with respect to their own problem solving ability and also to facilitate their ability to teach and assess problem solving. The author designed the intervention to simultaneously address pedagogy, mathematical content knowledge and problem solving. This is an area that requires attention in mathematics education. The provision of problems, presented in a sequence aligned with the development of mathematical thought through the Modified Moore method facilitated this development. The aims of this intervention were to:

- immerse the pre-service teachers in a problem solving environment through the use of the Modified Moore method;
increase the pre-service teachers’ awareness of metacognition, mindset and problem solving models;

• develop the mathematical thinking of the participants;

• develop the problem solving ability of participants;

• increase the perseverance levels of the participants in solving mathematical problems;

• develop the participants’ ability to teach and assess mathematical problems solving;

• provide the pre-service teachers with a resource pack which may be of use to them in their own teaching.

6.3 Program Implementation and Sample

The intervention was conducted over the course of six 80 minute classes at the beginning of 2016. The intervention was delivered by the researcher. The sample for the intervention consisted of sixteen pre-service mathematics teachers; eleven first years and five third year pre-service teachers participated in the intervention. All of the participants in the intervention had completed a pre-assessment in problem solving. Participation in the intervention was voluntary. The types of tasks, along with the teaching and learning pedagogy employed in the intervention classes are detailed in section 6.4. The tasks in the intervention classes, along with supplementary presentations are included in the set of course notes (Appendix W). The set of course notes was provided to each participant during the completion of the intervention.

6.4 Development of Intervention

This section describes the development of the intervention. This development is discussed under two main headings.

1. Organisation of the intervention;

   • The Four Component Instructional Design Model (4C-1D);
     – Inclusion criteria of the tasks for the intervention;
     – Structuring the tasks
       * van Hiele, Tall, Mathematical thinking
       * APOS

2. Mode of delivery of the intervention;

   • Modified Moore Method;
   • The Theory of Totally Integrated Education: TIE theory;
   • Framework for Teaching and Assessing Mathematical Problem Solving
6.4.1 (1) Organisation of the Intervention

The Four Component Instructional Design Model (4C-ID) includes the following components (van Merriërnboer, 2013, p.246):

- Component 1: Learning Tasks;
- Component 2: Supportive Information;
- Component 3: Procedural Information;
- Component 4: Part-Task Practice.

The complexity of the mathematical activity of problem solving necessitates the need to focus on the learning involved in engaging in complex tasks. Complex learning aims to integrate knowledge, skills and attitudes and to facilitate the transfer of the knowledge learned, and the skills and attitudes gained to life and/or work situations (van Merriërnboer et al., 2003). van Merriërnboer and Kirschner (2007) modified the original four-component instructional design (4C-ID) model (proposed by van Merriërnboer in 1997) to make it more practical for use by teachers and those involved in educational/training design. This modified model entitled ‘Ten steps to Complex Learning’ can be utilised to devise educational programs, where the goal is the acquisition of complex cognitive skills. As its name suggests, this model of instructional design involves the construction of educational programs by focusing on four components, with the sum of the steps across these four components equalling ten. The modified 4C-ID model was employed in the design of the intervention for this research. The four components involve 10 steps. These are outlined (relative to this study) as follows:

**Component 1: Learning Tasks**

There are three steps in component 1:

- Step 1: Design Learning Tasks;
- Step 2: Develop Assessment Instruments;
- Step 3: Sequence Learning Tasks

**Step 1: Design Learning Tasks:**

The mathematical problem solving tasks for the intervention were chosen to include tasks from both lower and higher levels of cognitive demand53 (Stein et al. 53mostly higher level cognitive demand)
This was to ensure that there were problems which all the participants could solve. The tasks were chosen/design to elicit mathematical thinking in the development of mathematical proficiency in the solving of them, with consideration given to the components of the author’s F-TAPS in mathematics (chapter 4). Particular attention was given to problems which elicited the use of specialising and generalising processes as these processes are fundamental processes in mathematical thinking (Mason et al. 2010, Pólya, 1945). Mason (1996, p.65) considers generalising to be the “heartbeat of mathematics”, stating that if a teacher is not aware of the process of generalisation and does not include opportunities in their mathematics classes for students to express their own generalisations, then mathematical thinking is not occurring. Mason and Johnston-Wilder (2004) consider it a role of the mathematics teacher to use specialisation in mathematics as a way of overcoming difficulties in conceptual understanding with Mason et al. (2010) stating that specialisation can provide a starting point to solving a problem. It was noted in the pre-tests that there were difficulties experienced among the pre-service teachers in the sample with generalising and then there was a lack of evidence shown of utilising specialising as a means to try to overcome these difficulties. Computer environments were included in some of the specialising tasks to facilitate the generalisation of the mathematical relationship.

The following two problems (problems 13 and 15) are examples of specialising and generalising tasks from the intervention:

**Problem 13.**

The UEFA Champions League begins with three knockout qualifying rounds and a play-off round. The 10 surviving teams then enter the group stage where they join 22 other teams who qualified in advance. These 32 teams are then drawn into 8 groups which consist of 4 teams per group. Within each group, each of these 4 teams plays each other twice, once at home and once away.

(i) How many matches will be played per group of four teams?

(ii) What pattern do you notice? Write an equation in words to represent this relationship between the number of matches and the number of teams. Then write this equation mathematically.

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54 Image sourced: https://www.google.ie/search?q=UEFA+S SYMBOL&rls=com.microsoft:en-IE&tbm=isch&imgil=iFNK8xNhgSuA

55 The use of specialisation is advised here by the tutor.
(iii) How many matches will be played during the group stage in total? (assume no draws occur).

(iv) How many matches would be played during the group stage in total, if there were 96 teams who were drawn into 8 groups of 12 teams? (assume no draws occur).

Problem 15. Tower of Hanoi

In this problem we have a set of \( n \) disks all of different sizes and we have three pegs. All of the disks are on the first peg, and they are in order of size with the largest disk on the bottom. The goal is to move all the disks from the first peg to the third peg, moving only one disk at a time. There is only one catch. You can never put a larger one on top of a smaller one. The question is what is the minimum number of moves needed to achieve this? Write your answer in terms of \( n \).

To get a feel for this problem, you can try moving the disks at:


A solution provided by a third year group is shown as an example (Figure 22).
The pre-service teachers utilised the computer environment to carry out simulations of specific cases (Figure 23).
The author chose to include some modelling eliciting activities (MEA’s) also as engaging in these activities contribute to high levels of understanding of mathematics (Doerr and Lesh, 2003). As stated in chapter 2 (section 2.5), the modelling tasks have not been constructed with the information required to solve the problem clearly available, as is the case in some problem solving tasks. These modelling tasks thus more closely resemble the mathematical problem situations found beyond school. Mathematical modelling eliciting activities provide opportunities for students to make sense of the problems situation and use mathematics which are meaningful to them to model this situation mathematically (Doerr and Lesh, 2003). These tasks were designed using the principles proposed by Doerr and Lesh (2003), described in chapter 2 (section 2.6.1).

The following two problems (problem 34 and 35) are examples of modelling eliciting activities from the intervention:

**Problem 34.**

![Image](image.png)

Morphine is an opioid pain medication. It is used to treat moderate to severe pain. The concentration of morphine in the bloodstream decreases at a continuous rate from the time of absorption of the initial dose. The half life of morphine is 3 hours. Fluoxetine is an anti-depressant used to treat major depressive disorders. The concentration of fluoxetine in the bloodstream decreases at a continuous rate from the time of absorption of the initial dose. The half life of fluoxetine is 5 days.

A 20mg dose of fluoxetine is given to a patient who is suffering with severe depression. A 15mg dose of morphine is administered as a pain reliever, to a patient who has just had a minor medical procedure. The drugs were administered intravenously (through a drip) to allow for immediate absorption. When the concentration of a drug in the bloodstream is 5% or less of the initial medical approved dose, it is not traceable by the testing used by the World Anti-Doping Agency (WADA).
On Saturday 30th January at 8pm, both patients were given the last dose of their respective drugs. Both patients are competitive athletes, who are currently training for major competitions. One competition they both wish to compete in is on Monday 1st February. The 1st drug test for this competition will occur on Sunday 31st January at exactly 8:30pm. Both are fit to attend the competition physically but they are unsure if there will be traces of their medication in their bloodstream, i.e. if they will pass the testing by WADA.

(i) Advise each patient as to whether they should compete in the competition on Monday 1st February or not, in your solution, provide all relevant data as to why they should or should not take part. Inform each athlete exactly how much of their medication will be in their system at 8:30pm on Sunday 31st January (both as a percentage and in mg).

Another event they both wish to compete in, is scheduled to take place on Sunday 21st February. The 1st drug test for this competition will occur on Saturday 20th February at exactly 8pm. The patient who received fluoxetine is slightly concerned that there might still be traces of fluoxetine in his bloodstream, i.e. if he will pass the testing by WADA.

(ii) Advise this patient as to whether he should compete in the competition on Monday 22nd February or not, in your solution, provide all relevant data as to why he should or should not compete. Inform the athlete exactly how much fluoxetine will be in his system at 8pm on Saturday 20th February (both as a percentage and in mg).

(iii) The patients’ coaches have asked you to draw up some graphs, showing how the concentration of each drug behaves as a function of time. You agree to do this and have decided to draw the graph for morphine showing how the concentration of morphine changes every 3 hours from the initial dose over a 24 hour period. For the fluoxetine, you want to draw the graph showing how the concentration of it changes at 8 time periods from the initial dose too, similar to what you did for the morphine graph. Draw both graphs and write the appropriate corresponding function on each one.

(iv) Explain one method of checking the appropriateness of the function you used to model the situation showing the decrease in the concentration of morphine as time passes. How does this method of checking show that the particular function you used fits the data with high accuracy?

Problem 35. A person who has diabetes needs insulin to help process the glucose in their body. To determine how quickly the insulin breaks down in a patient’s body, 20mg of insulin was injected into the body of a patient, and every

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56 Usiskin et al., 2003
10 minutes blood was drawn to determine the level of insulin in the body. The following is the data obtained for a particular patient. Although there can be slight variations in the rate at which insulin is broken down in diabetic patients, this data is representative for the majority of the patients with diabetes.

<table>
<thead>
<tr>
<th>Time elapsed (in mins)</th>
<th>Insulin (in mg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>10</td>
<td>9.5</td>
</tr>
<tr>
<td>20</td>
<td>3.6</td>
</tr>
<tr>
<td>30</td>
<td>1.3</td>
</tr>
<tr>
<td>40</td>
<td>0.2</td>
</tr>
<tr>
<td>50</td>
<td>0.1</td>
</tr>
<tr>
<td>60</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Table 59: Amount (mg) of Insulin in Body

(i) Use the collected data to formulate a mathematical model which describes the level of insulin in the body as a function of time.

(ii) Explain how you made sure that the model you formed in (i) was a good fit for the collected data.

These tasks adhere to the design principles (Doerr and Lesh, 2003) of an MEA as shown in Table 60
The author included some problems dealing with structural relationships and proportional reasoning as these were other areas which the sample of pre-service teachers demonstrated difficulty with. The following problem (problem 37) from the intervention is an example of a problem requiring the use of proportional reasoning:

Table 60: MEA Examples
Problem 37.

It took you six hours to cut grass on half of the land which has been set aside for silage. Your father did it last year and boasts that it only took him four hours to do the same amount, you decide to put him to the test and say that you will both work together to finish the half remaining. Assuming that your father was telling the truth and that you both have access to identical tractors and mowers, how long should it take you both working together to finish cutting the grass for silage. Are there any other assumptions you are making when solving this problem? If there are, please state them.

Examples of solutions provided by two different first year groups are as shown in Figure 24 and Figure 25.

Figure 24: Solution given by first years (group 2, class 5)
In considering the framework for teaching and assessing problem solving, the tasks were designed to also incorporate areas of interests of the particular sample of pre-service teachers. These were sports, music, farming, driving, and watching films.

**Step 2: Develop Assessment Instruments:**

Pre and post-tests were designed to assess mathematical thinking and mathematical proficiency. Similar to the design of the learning tasks, the problems on the assessments were chosen designed to elicit mathematical thinking while assessing various components of mathematical proficiency in the solving of them. The aim of developing the participants’ problem solving ability is examined through the solving of these problems (in addition to ongoing observation during the intervention). Therefore each problem included in the assessments was chosen with consideration of the combination of the problem solving skills and problem solving abilities in addition to the mathematical proficiency and mathematical thinking which were necessary to solve them (section 4.4.4). Scoring rubrics were developed to assign a score based on the learners’ performance on all relevant characteristics of problem solving ability (Appendix K and Appendix M).

**Step 3: Sequence Learning Tasks:**

Consideration of the sequencing of the learning tasks of the intervention was made in relation to:
• the nature of the tasks (from familiar context and domain to unfamiliar context and domain (O’Donoghue, 1997; Weaver, 1949)) and the factors that affect the complexity of the task (Heller et al., 1992; Stein et al., 1996);

• the progression in mathematical thinking as described in the van Hiele Model of Development of Mathematical Thought and in Tall’s Framework of the Development of Mathematical Thinking;

• the analysis of mental models described in the theories and models of mathematical understanding (section 2.6).

van-Merriërnboer and Kirschner (2007) include the following phases in the sequencing of the learning tasks:

1. **Identify the different conditions under which a task may be performed;**
   The conditions of the problems vary from a problem in a familiar context, with all information provided clearly and a clear end goal specified, to a problem in an unfamiliar context where all information is not clearly provided, along with the inclusion of surplus irrelevant information with or without a clear end goal, to proof requiring deductive skills and rigour from familiar to unfamiliar context/mathematics.

   Also the conditions under which a problem may be solved vary; solving the problem individually to solving the problem in a group, with or without the use of manipulatives.

2. **Identify the conditions that affect the complexity of the task;**

   Heller et al. (1992) identified six attributes which have an impact on the difficulty of problems in context.

   (a) **Context of the Problem:** Problems in which the context is familiar to the students (through experience) are easier than problems in which the context is not familiar.

   (b) **Problem Cues:** Problems that provide cues as to how the problem may be solved; (for example: form two equations and then solve the system of equations provides a cue to use simultaneous equations to solve the problem) are easier than problems in which such cues are not provided.

   (c) **Given information:** Problems with extra irrelevant or insufficient information are more difficult than problems with no unnecessary/superfluous information.
(d) **Explicitness of Question**: Problems that specifically ask for the unknown variable (e.g., at what distance will the second train be in line with the first train?) are easier than problems in which the unknown must be determined (e.g., explain possible reasons why the formula you determined confirms/does not confirm the number of reported cases).

(e) **Number of Approaches**: Problems that can be solved with one set of related principles are easier than problems that require the application of more than one set of related principles.

(f) **Memory Load**: Problems that require the solution of 5 or less equations are easier than problems that require the solution of more than 5 equations.

Stein et al’s. (1996) characteristics of mathematical tasks at each of the four levels of cognitive demand (Appendix V) was also utilised in the sequencing of the tasks for the items on the assessment as well as the items in the intervention.

3. **Provide values to the condition that affect the complexity of the task**;
   Easy - Difficult: this step was not utilised in this intervention as the author did not assign values to different problems.

4. **Define the first task class using the most simple conditions and the final task class using the most complex conditions**;

The analysis of mental models (step 6 of component 2 (supportive information)) in the development of mathematical understanding was utilised to inform the layout of the classes presented here. The analysis completed in step 6 of component 2, (along with the items discussed in Step 3) resulted in the following sequence of classes of learning tasks:

**Class 1 & 2: Mathematical Thinking, Sets, Area and Volume, and Patterns with use of Algebra (Visual, Descriptive and Relational):**

Mathematical thinking - problems in the presentation (Appendix U) require various types of mathematical thinking to solve them. This presentation is used to introduce the participants to the different types of mathematical thinking by asking them to solve problems which require specific types of mathematical thinking for each problem. After solving the problems which required use of the particular type of mathematical thinking, the pre-service teachers were then given information about that specific type of thinking and its use in problem solving. The following
ways of thinking mathematically (Katagiri (2004), Pólya (1945), Schoenfeld (1980), Tall (2013)) were introduced:

- Inductive - 1 problem;
- Deductive - 4 problems;
- Analogical - 1 problem;
- Abstract - information on, and advantages of abstract reasoning given;
- Specialising and generalising - 2 problems
- Thinking that expresses with numbers, symbols, quantities and figures - 3 problems;
- Dissecting and Recombining - 1 problem;
- Developmental - 1 problem.

In the first class, participants were also presented with information on mindset, metacognition and the problem solving models of Pólya (1945), Mason (2010) and Schoenfeld (1980) and the heuristics given by Schoenfeld (1980). The first booklet presented to the participants consisted of pages 1-23 (inclusive) and pages 48-81 (inclusive) of the course notes (Appendix W). The pages 1-23 cover problems on sets, area and volume, patterns and use of algebra.

**Problem 1** - This provides a gradual entry to problem solving and makes use of specialising and generalising and requires an explanation of the findings.

**Problem 2 and 3** - This requires an analysis of information presented verbally, mathematical representation, applying knowledge and skills to solve problems in familiar contexts and introductory probability (moving from 2 to 3 sets).

**Problem 4** - Specialising and generalising with introduction to more abstract symbols.

**Problem 5** - Solve a problem in unfamiliar context, representing in a similar way to what participants have been predisposed to may be more difficult to do for this problem.

**Problem 6** - Solve more abstract problem, using symbols.

**Problem 7** - Open-ended. Manipulatives (nets of cubes) and visualisation - ability to think logically in relation to spatial relationships, recognising
reflections and rotations of objects. Combinations.

**Problem 8** - Manipulatives (grids of 16 points) and visualisation - ability to think logically in relation to spatial relationships, recognising reflections and rotations of objects. Justification of solution. Solve a problem in unfamiliar context.

**Problem 9** - Conversion of units, recognising that volume = area times depth. Solve a problem in unfamiliar context. Understanding the structure of the problem - recognising irrelevant information.

**Problem 10** - Use of specialisation and generalisation. Seeing different perspectives, visualisation - ability to think logically in relation to spatial relationships, fractions.

**Problems 11-18**: Require recognition of patterns (Repetition of many problems of various contexts but similar structural/conceptual components - Diene - facilitate the individual perception of the underlying concept by each participant. Investigation of quadratic and exponential relationships in context.

**Problem 11** - Use of manipulatives (boxes of matches). Use of specialisation and generalisation. Visualisation - ability to think logically in relation to spatial relationships.

**Problem 12** - Specialisation and generalisation, developmental thinking. Representing information in words and then mathematically. Simultaneous equations.

**Problem 13** - Specialisation and generalisation, visualisation, developmental thinking. Representing information in words and then mathematically.

**Problem 14** - Use of manipulatives (drawing on the diagram provided and use of computer images). Specialisation and generalisation, visualisation developmental thinking. Representing information visually, in words and then mathematically.

**Problem 15** - Use of manipulatives (moving of the disks on computer). Logical thinking, specialisation and generalisation. Written explanation. Unfamiliar context.
Problem 16 - Specialisation and generalisation, developmental thinking, written, graphical and symbolic representation, ratios.

Problem 17 - Specialisation and generalisation, developmental thinking, written and symbolic representation. Quartic relationship, unfamiliar mathematics in formulating the equation.

Problem 18 - Use of co-ordinate geometry in an unfamiliar way, recognising the structure of the problem - recognising extra information. Developmental thinking.

Class 3: Algebra equations (Descriptive and Relational):

The second booklet presented to the participants (for class 3), consisted of pages 24-31 (inclusive) of the course notes (Appendix W). The pages 24-31 cover problems on linear, simultaneous, quadratic, and exponential relationships. Modelling of linear and quadratic relationships, understanding linear and quadratic relationships through mathematical thinking (developmental) and introduction to exponential relationships.

Problem 19 - Use of simultaneous equations in an unfamiliar way.

Problem 20 - Use of simultaneous equations in an unfamiliar way.

Problem 21 - Recognising the structure of the problem, unfamiliar context. Use of proportional reasoning.

Problem 22 - Solving a problem in 2 different ways, comparing graphical and algebraic representations.

Problem 23 - Understanding the relationship between graphical and algebraic representations. Determining the equation of the line from the graphical representation. Determining if the equation formed is a good fit for the data. Use of GeoGebra as a manipulative in graphing. Explanation of work and consideration of factors affecting decisions made in business.

Problem 24 - Formulation of a quadratic equation in unfamiliar context. Developmental thinking. Consideration of factors affecting the shape of quadratic curves.

Problem 25 - Formulation of a quadratic model to fit data, with explanation of work.
Problem 26 - Developmental thinking in the explanation about why a quadratic curve is u-shaped when the coefficient of the squared term is positive and n-shaped when the coefficient of the squared term is negative.

Problem 27 - Determining the terms of a quadratic equation in an unfamiliar context. Use of mathematical knowledge in an unfamiliar way.

Problem 28 - Use of mathematical knowledge of logarithms in a familiar way.

Problem 29 - Use of mathematical knowledge of logarithms in an unfamiliar way.

Problem 30 - Unfamiliar context problem in relation to proportions.

Problem 31 - Formulation of an exponential decay model to determine possible repeat dosage time of a drug.

Class 4: Algebra equations (Descriptive and Relational):

The third booklet presented to the participants (for class 3), consisted of pages 32-41 (inclusive) of the course notes (Appendix W). The pages 32-41 include problems on comparing linear, polynomial and exponential growth and decay. Matching symbolic, visual and written and graphical representations. Mathematical modelling; formulating mathematical models to represent data (exponential) and checking for appropriateness of choice of model. Distance, speed and time, proportional reasoning, understanding the structure of a problem, relevant and irrelevant data. Abstraction and unfamiliar contexts.

Problem 32 - Comparing graphical and algebraic representations of linear, quadratic, exponential and logistic functions. Providing an understanding of relationships between graphical and algebraic representations of these types of functions. Also developing understanding and recognition of both growth and decline models for each of the various types of functions.

Problem 33 - Matching written, symbolic and visual representations of mathematical relationships. Involving use of completing the square and understanding of quotients.

Problem 34 - Formulation of an exponential decay model in unfamiliar context to determine the decay constant of particular drugs. Assembling
all relevant calculations and information to provide a written report to advise athletes (patients) of possible detection of the drug in the bloodstream by the World Anti-Doping Agency (WADA) officials. Formulation of appropriate algebraic functions, with graphical representation to model how the concentration of each drug behaves as a function of time. Determining an appropriate test to check for goodness of fit of the model to the data.

**Problem 35** - Given a table of values for time elapsed and insulin remaining in the body, this problem requires the formulation of an appropriate function, to model the level of insulin in the body as a function of time. Explanation of how the model was checked for goodness of fit also required.

**Problem 36** - Proportion without number.

**Problem 37** - Proportional reasoning.

**Problem 38** - Problem solving requiring the introduction of variables, no numbers or variables given.

**Problem 39** - Linking ratios.

**Problem 40** - Linking ratios.

**Problem 41** - Proportional reasoning.

**Problem 42** - Problem solving requiring the introduction of variables, no numbers of variables given. Solving a quadratic in unfamiliar context and an unfamiliar way.

**Problem 43** - Functions, abstract and deductive reasoning.

**Problem 44** - Distance, speed and time, abstract reasoning.

**Problem 45** - Distance, speed and time. Recognising the structure of the problem, recognising irrelevant information.

**Problem 46** - Distance, speed and time.

**Problem 47** - Abstract proportional reasoning.
**Problem 48** - Using relative speed to solve the problem, or summing the series and seeing the benefit of one method of solving over the other.

**Problem 49** - Abstract algebraic and deductive thinking to solve a problem which seems to provide insufficient information.

**Class 5 & 6: Proofs (Deductive and Rigor):**

The fourth and final booklet presented to the participants consisted of pages 42-47 (inclusive) of the course notes (Appendix W). The pages 32-41 include problems on proofs. Proof by contradiction, counter example, deduction and induction, determining flaws in a proof and proofs in set theory are included in this booklet.

**Problems 50-52** - Proofs based on general form of odd and even integers, and proof of the multiplication rule for fractions.

**Problems 53-59** - Proofs based on palindromes, finding a flaw in a proof, roots of a general quadratic equation, algebraic identities, proof by contradiction, use of counter example in proofs involving prime numbers.

**Problems 60 - 65** - Proofs using induction.

**Problems 66 - 70** - Proofs on set theory.

**Problem 71** - Investigation into Goldbach’s conjecture.

5. **Add task classes in between in such a way that there is a gradual increase of complexity from one task class to another:**

The classes were sequenced to complete problems which developed mathematical thinking and understanding from classification, visualisation and description, to relational and deduction, and finally rigor in completing proofs. This sequence aligns with Tall’s framework for the development of mathematical thinking from conceptual embodiment to formal axiomatic.

**Component 2: Supportive Information**

There are three types of supportive information in the modified 4C-ID model:

- cognitive feedback (step 4).
  involves the use of reflection in facilitating the critical comparison of the
quality of a learners cognitive strategies (in their provision of a solution to a problem) with the cognitive strategies of an expert or their peers.

- systematic approaches to problem solving (SAP’s) (step 5); details the phases a problem solver passes through, in addition to the heuristics that may be helpful in order to successfully progress through these phases.
- domain models (step 6); describe how the learning domain is organised.

(van Merriënboer et al., 2007)

**Step 4: Design Supportive Information:**

The cognitive feedback aspect of supportive information was provided in the form of metacognitive questions (Appendix P) which the pre-service teachers completed after completing certain problems. The delivery of the intervention (Modified Moore Method - see section 6.4.2) also provided cognitive feedback in the form of problem solving processes reported by peers/instructor and the inquiry method at the heart of the modified Moore Method, which facilitated feedback by discovery.

**Step 5: Analyse Cognitive Structures:**

The analysis of cognitive structures provides information on how learners’ actions in the domain should be organised (van-Merriënboer and Kirschner, 2007). This facilitates the provision of systematic approaches in problem solving activities. These approaches were provided during the intervention by asking the students to utilise the perception, reflection, action, reflection processes of Tall’s framework, in addition to informing the pre-service teachers of the problem solving models of Pólya (1945), Mason (2010) and Schoenfeld (1980) and the heuristics given by Schoenfeld (1980), along with information on metacognition (Flavelle, 1979). These aspects were included in the Appendices of the course notes.

Pólya (1945) emphasises the processes of specialising, generalising, dissecting, connecting and recombining in developing an understanding of mathematics. In gaining an understanding of the problem situation where the information presented is unfamiliar to the student, Pólya recommends forming an analogy to compare with a similar situation/information one is familiar with. He stresses making connections of new information to knowledge which has been previously gained in one’s schema. The processes of dissecting complex problem situations into their component parts and recombining information to gain an understanding of the whole problem situation is encouraged in trying to make sense of complex mathematical problem situations. During the intervention the heuristics provided to the pre-service teachers, along with modeling of a solution(s) by the researcher, in addition to questioning provided a learning environment akin to that of cognitive apprenticeship.
Step 6: Analyse Mental Models:

The analysis of mental models provides information on how the learning domain is organised (van-Merriërnboer and Kirschner, 2007) i.e. the cognitive activity that occurs within the domain. The analysis of the theories and models of mathematical understanding completed in Chapter 2 provided the theoretical foundations for the decisions made in relation to the design of, inclusion of, and sequencing of the tasks and classes for the intervention (as described in step 3). The van Hiele model of the development of mathematical thought, based on developing visual, descriptive and theoretical thinking from concrete to abstract, in the formulation of a network of relationships in the development of mathematical insight was employed in this step of the 4C-ID model. Mathematical thinking as described by Katagiri (2004), in addition to Johnson-Laird’s (2004 p.179) analysis of “mental models” was also utilised here.

There are three types of “mental models” (van-Merriërnboer et al., 2002 p.48) included in this step of the 4C-ID model:

- **Conceptual Models (What is this?)**
  Mathematical thinking - pay attention to how pieces of mathematics are interrelated. This allows for the classification/description of mathematical objects according to their properties. These models include word models, picture models and causal loop diagrams. Generalisation and experience are useful in formulating these models (Danani, 2002). However the past experience of problem solvers may result in one-sidedness in their approaches, adapting a tunnel vision approach to solving certain problems (Danani, 2002);

- **Structural Models (How is this organised?)**
  These models focus on how mathematical information and thinking is organised - in particular how the mathematics and thinking fit together to form the overall cognitive network. Developmental questions (questions on what happens if or when a condition is changed) and questions which require the identification of patterns/building blocks facilitate the formulation of these models (van-Merriërnboer et al., 2002 p.48);

- **Causal Models (How does this work?)**
  These models are about how principles affect each other. These mental models help to make sense of processes, facilitate the provision of explanations and allow for predictions to be made. Exploratory learning and observing the outcomes and the associated probabilities (or relative frequencies) of various events facilitates the formulation of these models (Gopnik & Wellman, 2013). However, as in the development of conceptual models, prior learning can have an effect on the casual models formed (Gopnik & Wellman, 2013).

(van-Merriërnboer et al., 2002 p.48)
The combination of all three mental models provide a basis for the perception, learning, classification and generalising of new concepts (examples or problems) (Lake et al., in press 2016). This combination facilitates strong reasoning within a particular domain (van-Merriërnboer et al., 2002 p.48). The inclusion of problems requiring:

- the provision of word, diagram, graphical, symbolic or some representation of the problem situation;
- the provision of answers to developmental thinking questions;
- exploration, specialisation and generalisation in forming patterns and;
- the use of observed data to model relationships,

in the intervention, facilitated the building of conceptual, structural and causal models in the minds of the pre-service teachers.

APOS theory (models of cognitive activity) was also utilised to inform the design and choice of problems (both the mathematical content and the sequencing of these problems) for inclusion in the intervention. A genetic decomposition is a hypothetical model of the cognitive constructions (both the structures and mechanisms) which need to be built in a student’s mind in order to learn a specific mathematical concept (Arnon et al., 2014). This hypothesis is usually based on:

- the researcher’s experience in the learning and teaching of the concept;
- the researcher’s mathematical knowledge;
- the researcher’s knowledge of APOS theory;
- research currently available on the concept;
- how the concept developed historically.

(Arnon et al., 2014, p.28)

A genetic decomposition is a guide for the design of instruction that is in alignment with how students arrive at an understanding of a mathematical concept (Arnon et al., 2014). It is a description of the Actions that students need to carry out on existing mental phenomena, along with an explanation of how the interiorisation of these Actions to Processes occurs (Arnon et al., 2014). A concept may consist of many Actions, Processes and Objects. The coordination (recognition and organisation of relationships) of these conceptions result in the formation of a Schema for a particular concept. A genetic decomposition may also include a description of this organisation of relationships (Arnon et al., 2014) into a Schema. A description of how this Schema is arranged in themes to form an object (which can itself be examined and transformed by the
application of actions/processes/schemas) may also be included in the genetic decomposition. The mental mechanisms of reflective abstraction are essential in the development of the mathematical thinking of the students in their construction of schemas for use in problem solving situations. It stands to reason that students will experience problems with problem solving if these schemas have been incoherently constructed. Using the genetic decomposition of APOS theory should maximise the potential for the coherent building of schemas in the minds of learners. However, since each student is unique in how they make sense of mathematics and since the genetic decomposition formulated is subject to the researcher’s mathematical knowledge and experience, the schemas developed in the minds of the students may vary with respect to their coherency, as is the case for other methods of instruction. A genetic decomposition which has been well researched by a researcher with a high level of mathematical proficiency should facilitate the development of high quality instruction, which in turn should allow for the development of coherent schemas in the minds of the students.

The author employed APOS theory by including problems (e.g. problem 24 (designed by author to develop concept of quadratic relationships) in the book of course notes) in the intervention which facilitate the development of coherent building of schemas in the minds of learners.

Component 3: Procedural Information

This component is concerned with the compilation of knowledge (van-Merriërnboer et al., 2002). Procedural information for the intervention consisted of two types: “just in time procedural information” and “corrective feedback” van-Merriërnboer et al., 2002 p.151-152). van-Merriërnboer et al. (2002) state that procedural information is best presented “just in time” (i.e. exactly when learners need it). Corrective feedback indicates that an error has been made. It aids the learner in recovering from the error and provides a hint on how to continue (van-Merriërnboer et al., 2002).
Step 7: Design Procedural Information:

The “just in time procedural information” consisted of definitions, reminders, explanations, formulae, information on mathematical notation and algorithms. These were provided in the set of course notes and directly preceded particular tasks which necessitated the use of such information. Corrective feedback was provided by the peers of the pre-service teacher(s) and by the researcher, in the form of questioning after presentations of solutions, questioning during completion of tasks and self-questioning in the completion of the metacognitive journals.

Step 8: Analyse Cognitive Rules:

The inclusion of algorithms in the set of course notes provided this information to the pre-service teachers. Also, when the solution to a problem was not progressing correctly due to errors in carrying out procedures, questioning on the particular procedure (e.g. multiplication of fractions in problem 10 (ii)) aided the pre-service teachers in recovering from the errors and continuing with the problem.

Step 9: Analyse Prerequisite Knowledge:

The prerequisite knowledge was provided in steps 7 and 8 and this step 9 of component 3 requires analysis of the information provided in steps 7 and 8 and provides further elaboration of any words or concepts which may not be familiar to the learners.

Component 4: Part-Task Practice

Step 10: Design Part-Task Practice:

Repetition of problems with similar structural/conceptual components facilitated strengthening of the learning process. Also, where particular skills were causing impediments to progression in solving the problem, several simpler part-tasks involving routine aspects of the task were provided for completion. This facilitated practice of the skill, also these routine parts of task were supported in the provision of supportive information.

(van Merriërnboer and Kirschner 2007)

6.4.2 (2) Mode of Delivery of the Intervention

Taplin (2006) outlines the following seven characteristics which should feature in a problem solving approach to teaching mathematics:

1. interactions occurring between student/student and student/teacher (Van Zoest et al., 1994):
2. mathematical discourse between students with agreement reached between them on the validity of solutions/arguments presented (Van Zoest et al., 1994);

3. provision of problems with sufficient (minimum necessary) information given to ascertain the background/goal of the problem situation, by the teacher, with the interpretation and formulation of solution methods by students (Cobb et al., 1991);

4. the acceptance of both correct/incorrect solutions by the teacher in a non-judgemental way (Cobb et al., 1991);

5. teachers facilitate learning through insightful questioning and shared engagement in the process of solving problems (Lester et al., 1994);

6. teacher knowledge of appropriateness of when to offer help and when to refrain from helping in order to allow students to solve the problem themselves (Lester et al., 1994);

7. the encouragement of students’ use of the process of generalisation, with respect to concepts and rules (Evan and Lapin, 1994).

These characteristics all featured in this intervention, the first six characteristics are fundamental elements of the Moore method (section 6.4.2; Appendix W). Problems (including modelling problems) were provided that satisfied the third characteristic. The seventh characteristic was provided through the focus on developing mathematical thinking (Tall, van Hiele, Katagiri and Mason), with particular attention to developing specialisation and generalisation.

Taplin (2006) notes the challenge in teaching mathematics is the co-development of the process of mathematical thinking with the acquisition of mathematical knowledge. The genetic decomposition of concepts to be acquired in order to solve mathematical problems in the area of number and algebra facilitates the determination of the mathematical thinking required for the successful acquisition of this knowledge along with the construction of coherent structures to achieve coherent schemas for application to problem solving with mathematical proficiency. Hence the problems required to achieve both simultaneously can be identified and designed.

Lehman (1977) noted that to understand mathematics, one needs to show understanding in three types of knowledge: applications, meanings and logical relationships. In gaining an understanding of mathematics, students should be able to display their understanding by applying their knowledge appropriately to problem situations, be able to explain meanings of mathematics presented in their problem situations and also be able to explain the problem and their solution to another. In this explanation, they should be able to justify their reasons for choice of solution method and also justify the solution obtained where
Usiskin (2013) notes that there are five dimensions to understanding mathematics:

- skill-algorithm - shown by following an algorithm to get to the correct answer to a task. The greater the understanding the individual has of the procedures involved in obtaining the correct answer - the more alternative algorithms they can suggest to get this answer and the more efficient is their choice of algorithm used;

- property-proof - this involves knowing why their method of obtaining the correct answer worked. This involves an understanding of the mathematical properties involved in the algorithm used and also the ability to prove the general case of the algorithm using these properties;

- use-application - involves the recognition of situations where one can use the concept. Students can possess both skill-algorithm and understanding of a concept but may not know any use for it outside of calculations in mathematics class;

- representation-metaphor - representing the concept in some form - draw diagram or picture, graph, use metaphor, organisation of words and signs, use of concrete objects;

- history-culture - knowledge of how and why certain mathematics developed.

Lehman’s, Pólya’s and Usiskin’s perspectives of mathematical understanding, along with the authors’ F-TAPS in mathematics, were taken into account in the development of this intervention, both for the organisation (as just described) and mode of delivery. Connecting graphical, algebraic, numerical and verbal representations (as founded and recommended by the Harvard Calculus Consortium) of problems was also emphasized during the organisation of the intervention.

The mode of delivery for the intervention was decided based on its alignment in achieving the aims necessary to address both:

- the components of the framework for teaching and assessing problem solving in mathematics, and

- the design of the intervention to achieve complex learning.

The following Theory of Totally Integrated Education, along with the Modified Moore Method both aligned well with achieving these aims and were thus

\(^{57}\)The history-culture aspect of Usiskin’s dimensions of understanding was not emphasized in this research.
chosen as the mode of delivery for the intervention. The Theory of Totally Integrated Education notes the significant role that a learner’s emotion plays in the organisation of knowledge in a learner’s mind. The Modified Moore Method is a method of instruction in mathematics which is focused on developing the problem solving ability of the learners by requiring the learners to solve problems in groups and then present (explaining the reasoning behind their solution) and defend their solutions.

**The Theory of Totally Integrated Education: TIE theory**

According to Greenspan and Bendenly (1997, cited in Frick, 2015), emotion serves as the mind’s primary architect. Every experience has a corresponding affect or emotion associated with it. Both the experience and the emotion invoked by that experience are dually coded in the mind of an individual (Frick, 2015). Learning involves the reconstruction of existing neuronal connections, in addition to the formation of new neuronal connections in the building of increasingly complex knowledge arrangements. According to TIE theory, it is the individual’s emotion that organises this complex arrangement. TIE theory suggests connecting cognition, intention and emotion in the learning experience to maximise the strength of the mental schema formed in the minds of the learners and to thereby minimise the susceptibility of the mind to forgetting information (Frick, 2015). Nine specific learning outcomes are identified in TIE theory (Figure 27). TIE theory suggests connecting cognition, intention and emotion with these nine types of learning outcomes through the use of authentic learning tasks. This theory involves changing from a curriculum which is organised by topics to a curriculum which is organised from a task-centered approach. Resnick (1987) in an examination of numerous programs that claimed to teach thinking skills or higher order cognitive abilities identified three key features common to the most successful programs:

1. the work is shared in a group setting.

2. Hidden processes are made explicit that is, for example, solving a problem aloud while revealing all the thinking behind it, including the metacognitive prompts that the problem solver engages in. While revealing these hidden thought processes, student commentary and observation is encouraged.

3. program of learning is organised around subject matter and interpretation of the knowledge of this subject matter rather than ability.

The nine types of learning outcomes are as shown in Figure 27. Each type of knowing (knowing that one, knowing how, and knowing that) are displayed by responses to questions involving forming correct opinions in relation to properties and relations of individual mathematical objects/information, choosing appropriate knowledge/ combinations of different strategies to provide solutions and demonstrating flexible use of this knowledge, and being able to classify
mathematical objects/relations/functions of the same kind, explain relationships between mathematical objects and make judgements according to a norm (Frick, 2015).

Krutetskii (1976) stresses the importance of a person’s emotions in the development of ability in any activity they are engaged in, with mathematics being no exception. The joy one feels during and after the creation of some work, the satisfaction one feels after engaging in intense mental work and the level of enjoyment derived from this process are all factors involved in the perseverance and continued engagement by an individual in an activity, even in the face of ongoing difficulty. Krutetskii references Bogoyavlenskii and Menchinskaya in stating that the joy felt by making a small discovery is not only due to them experiencing success but also because of an awareness of having overcome difficulties and the satisfaction that one’s own efforts have contributed to their success. This is essentially the growth mindset to which Dweck (2007) refers. Krutetskii also notes the importance of the tasks and interest in the development of abilities in mathematics, making reference to Rubinstein in stating that in

Figure 27: The nine specific types of learning outcomes

(Frick, 2015 p.6)
order for an ability to formulate and develop, there needs to be a corresponding need for that ability in some (created) activity and referencing Myasishchev in stating that a positive attitude toward mathematics, along with a deep interest and enthusiasm for studying mathematics is the strongest motivating factor in the development of abilities in mathematics.

The connection of cognition, intention, and emotion was achieved in this intervention by formulating the intervention based on the author’s F-TAPS. The inclusion of the presentations on mindset, mathematical thinking and the use of the Modified Moore Method as the delivery approach were essential features which facilitated the connection between cognition, intention and emotion in this intervention.

As described in section 2.6.4 the practice of making thinking explicit facilitates the external representation of the internal “mental model” which may aid learners to build up their repertoire of interiorised “mental models” of more and more problem situations. This aspect of representation of the internal “mental model” involves working at the van Hiele level 0 (section 2.4.1) or “knowing that one” (recognition, acquaintive and appreciative) component of the TIE theory to facilitate the recognition and classification of mathematics for use in perceiving problem situations. Level 1 of the van Hiele theory (section 2.9.4) and the “knowing that” (instantial, relational, criterial) component of the TIE theory may also be utilised in forming a perception of the relationships presented in the problem (problems at this stage should not include proofs - problems on number, sets, area and volume - focus on operational and structural - verbal and visual representation). More advanced problems (e.g. the proofs in the course notes) facilitate the adaptive and creative “knowing how” component of the TIE theory.

In planning and solving the problem, the knowing how (imitative, adaptive, creative) component of TIE theory was adhered to in order to develop the flexibility of the student in their approaches to solving problems. This involved keeping a record of solution methods (pictures of all of the participants’ solutions were taken during every class of the intervention), which students could revisit again after some time to attempt to solve problems in different ways, as well as comparison of other students’ methods with their own at the time of solving (presenting solutions and resubmitting of solutions in the Moore Method case).

The next part of this section describes the Modified Moore Method (and the Moore Method) which was employed as the method of instruction in this intervention.

**Moore Method**

The Moore Method was developed by Robert Lee Moore, a mathematics professor at the University of Pennsylvania from 1911 to 1916 (Mahavier et al.,
Moore utilised this method throughout his teaching career (1916-1969) at the University of Texas (Mahavier et al., 2006). The fundamental aim of a course which utilises the Moore Method, is to attend to the mathematical content of the course in a way which develops the students’ self-reliant ability to examine and solve problems (Coppin et al., 2009). The Moore-method involves the provision of problems by the instructor to which students formulate written solutions (Mahavier et al., 2006). The students are also required to present their solutions to the class. The class decides on the validity of the solutions and arguments presented (Mahavier et al., 2006). Three main aims for students to achieve in a Moore method mathematics course include:

- independent formulation of a solution to a problem, with a corresponding reinforcing argument to defend their solution.
- communication of their solution in written and oral form, utilising the accompanying argument to explain their reasoning.
- defend (or adapt) their solution, in cases where their solution is considered (by the class) to be deficient in terms of validity.

(Mahavier et al., 2006)

In cases where the solution cannot be defended, the student begins again to formulate a new solution to the problem.

The Moore Method of instruction and learning mathematics typically leads to the development of ambition, competition and isolation among the mathematics students and has mainly been utilised in postgraduate mathematics education settings (Dancis and Davidson, 1970; Cohen, 1982; Mahavier et al., 2006). The implementation of the Moore method in undergraduate level often produced disappointing results (Cohen, 1982). As a result of this, the Moore Method has undergone modifications by several researchers (Davidson, 1973; Cohen, 1982; Chalice, 1995) to make it more appropriate for use in undergraduate level mathematics courses. Dancis and Davidson (1970) modified the Moore Method to the Small Group Discovery Method. The methods utilised in his course are capable of increasing students’ ability to:

- solve mathematical problems;
- prove theorems;
- make conjectures;
- present coherent arguments.

(Dancis and Davidson, 1970)

The abilities which are developed by the Moore Method and Modified Moore Method are also encapsulated in the Project Maths syllabus under the heading of Synthesis and Problem Solving. It is stated in the syllabus that students should be able to:
• apply knowledge and skills to solve problems in familiar and unfamiliar settings;
• devise, select and use appropriate mathematical models, formulae or techniques to process information and draw relevant conclusions;
• analyse information presented verbally and translate to mathematical form;
• explore patterns and formulate conjectures;
• explain findings and justify conclusions;
• communicate mathematics verbally and in written form;

(Project Maths Junior Certificate Syllabus, 2015, p.25)

The alignment which exists between the aims of the Project Maths syllabus and the Modified Moore Method made the Modified Moore Method a suitable choice of instruction for improving problem solving ability among the pre-service teachers, who will be teaching this Project Maths syllabus to students. In addition, the requirements of having to present an accompanying argument to explain the reasoning of their solution, along with the subsequent defence or adaptation of their solution by the Modified Moore Method of instruction has potential to minimise the possible permanent adoption of a ‘pseudostructural conception’ (Sfard, 1994, p.117) of mathematical concepts by students.

A study conducted by Horton (2013) involved using an adaptation of the Moore Method devised by Cohen (1992) which led to less competition among students than the original Moore Method. Horton (2013) implemented a statistics course with the aim of developing problem solving skills of students through the use of challenging problems and their solutions along with complex case studies. The original Moore Method focused on students proving difficult theorems. Cohen (1992) modified the Moore Method with a focus on problem solving by adhering to three principles:

• students discovering things for themselves leads to them understanding the information/concept better and remembering it for longer;
• when an individual has to explain/teach an idea/concept to another individual/group, they gain a more thorough understanding of it;
• clear thinking and competent writing are intricately linked.

The problems were chosen by Horton (2013) according to the following conditions (3 out of the original four presented here):

• easy to understand the problem;
• difficult enough so that it poses a real challenge to solve it;
• linked to applications.

The duration of the statistics course was three times a week for 80 minutes per class for thirteen weeks. The content to be taught on the course was broken down into lists of problems which were given every two weeks as problem sets to the students to solve in groups of three over the course. Feedback to an independent evaluator (20 minute focus group with the mathematics learner center in the college), from the students who participated in the course (a first course in statistics) included the following:

• students wanted more input from the lecturer on what the correct answers are and what the correct way of thinking is;
• uneasiness among students on undertaking independent learning;
• more awareness of their own responsibilities as students (with regard to preparing for classes).

Recommendations from the independent evaluator to Horton indicated the need to have a discussion about the pedagogy and the learning goals of utilising this instructional method. There was a statistically significant increase in the results between a pre and post-test completed by the students ($p = 0.01$), although since this was a first course, this result was to be expected. There was also an indication of larger improvement among the students with the lower pre-test results.

The author included a discussion on the Modified Moore Method in relation to the pedagogy and goals of the method in the first class of the intervention for this study. A detailed description of the instruction methods employed in the Modified Moore Method and for the participants of this study is provided on page 1 and 3 of the course notes used for the intervention in this study (Appendix W). The pre-service teachers completed the problems given in the presentation on mathematical thinking (Appendix U). By completing these problems the pre-service teachers experienced the use of the different types of mathematical thinking which were necessary to solve them. Upon completion of these problems, the author informed the pre-service teachers of the type of mathematical thinking they had just used by showing the pre-service teachers the appropriate slides of the presentation. The results of the author’s study also indicated a larger improvement among the pre-service teachers with the lower pre-test scores (chapter 7).

Another study utilised a modified Davidson’s small group discovery method with a group of ten second year pre-service second level mathematics teachers in a course on proofs in geometry (Salazar, 2012). Another group of ten students from the same year and program formed the control group. A pre and post-test of proof constructions was given to all students. There was no statistically significant difference in the mean pre-test results between the teachers in the experimental and control groups. There was a statistically significant difference in
the mean post-test results between the teachers in the experimental and control groups \((p = 0.0257)\) with the experimental group achieving the higher result. A van Hiele geometry pre and post-test was also given to both groups. There was no significant difference in the mean test scores (pre and post-tests) between the two groups. Qualitative findings showed increased self-confidence among the pre-service teachers who participated in the Modified Moore Method. This study demonstrates successful implementation of the Modified Moore Method with pre-service second level mathematics teachers.

The Modified Moore Method was employed in this study. The methods employed in this course can increase students' ability to engage successfully in problem solving situations, to formulate conjectures and to present well-reasoned proofs (Dancis and Davidson, 1970):

- The class was divided into groups of 3 or 4 members. A group of 5 or more is considered too big for collaborative problem solving. In groups of 5 or more, one or two students tend to take over and the others are less inclined to participate fully. Each group discusses the problems and solves or proves them together on the whiteboard during class. Each group had their own space at the whiteboard. The members of each group provided a group solution to each problem on the board;

- The participants were informed of the following: Each group member must understand a problem solution or proof before the group can move onto solving another problem. Members in each group can ask each other questions, it is the responsibility of the group members to provide satisfactory answers to these questions (Dancis and Davidson, 1970). The instructor can change some group members and give suggestions on how to improve the interactions within a group;

- In the case of a group working at a much slower rate than the other groups, the instructor should split this group and reassign its members suitably to the other groups;

- The instructor checks on the progress of each group, giving suggestions for improvement if necessary. If a group is experiencing difficulty with a particular problem, the instructor might ask a perceptive question or if absolutely necessary provide a small hint (Dancis and Davidson, 1970). Preferably, the use of questions should be used in the case of a group experiencing difficulty as to provide them with a hint may send the message to them that you believe they are not capable of solving the problem without your help (Dweck, 2008);

- Each problem is designed so that it is possible to be solved within one class period. Students work at the board on problems they have not seen previously. During class the students get to experience the role of acting as teachers and by doing so, they gain skills in presenting proofs
and explanations in a clear coherent way (Dancis and Davidson, 1970). Additional problems are assigned for individual completion outside of class time, to be submitted as homework.

The emphasis in the intervention for the author’s research is on the coherent building of cognitive structures in the minds of the participants through perception, reflection, action (with reason), and visualisation. In perceiving the problem situation, participants are encouraged to pay close attention in noticing the information presented. They are also advised to represent information using images, diagrams, symbols, words, and numbers. Vergnaud (2009, p.88) notes that the cognitive structures (schemas) contain ‘conceptual components’. If these ‘conceptual components’ have not been constructed in the minds of the learners then these structures are not adaptable to the variety of problem situations a student encounters in their study of mathematics (Vergnaud, 2009). The participants were given the following criteria to adhere to when engaging in solving each task (Tall, 2013a, p.2) to aid in providing ‘conceptual components’ in the cognitive structures of the participants:

- Perception: form an analogy, diagram, some representation.
- Reflection: what mathematics do I know that relates to this perceived problem?
- Action: how should I proceed? How will this be useful to solving this particular problem?
- Reflect on result:
  - what was achieved by the action I performed/took?
  - can I justify my solution?
  - is there another method for solving this problem?
  - which method is more efficient? why is it more efficient?
  - would these methods always work for certain types of problems?
  - what would happen if some condition/information changed?

### 6.5 Conclusion

This chapter discussed the intervention phase of this research. The aims of the intervention were presented. The development of the intervention was discussed in relation to the organisation of the intervention and the instructional methods employed to deliver it. A comprehensive detailed description of the design of the intervention in relation to the design and choice of mathematical problems, development of assessment components, sequencing of the mathematical problems, supportive information, analysis of cognitive structures and mental models, procedural information, and part-task practice was provided in this
chapter. Rationale was given for the choice of the Modified Moore Method as the mode of instruction for the intervention and a discussion of the method was included in this chapter. The integration of the frameworks employed by the author in the development of the intervention for this study forms the design principles as part of the Educational Research Design Approach. Chapter 7 presents the findings and results from the post-tests (relative to the pre-tests) through the use of the data-analysis that was conducted.
7 Post-Intervention Results and Findings

7.1 Introduction

The findings of the assessments after the intervention (post-tests) are discussed in this chapter. These findings are discussed relative to the pre-test findings for the pre-service teachers who participated in all three phases of this research study (pre-test, intervention and post-test phases). The findings for each phase (reading and understanding, planning and solving, and solution and checking) of the problem solving process are presented by:

- discussing the findings of the pre-test, post-test and the difference from pre-post test for each phase of the problem solving cycle;
- discussing the changes in the median type of response provided by the group of participants;
- discussing the changes in the total score (results) of the group of participants for each phase.

Examples of the solutions provided by some participants are included to show a particular type of increase/decrease in the type of response awarded to the participant. The examples also aid in explaining possible reasons for why a decrease may have occurred in the type of response awarded to a participant. The examples are also utilised to show the particular type of misinterpretation of a problem that may have occurred.

The changes that occurred in the median difficulty rating of the problems by the pre-service teachers are presented towards the end of the discussion (section 7.3.1 and 7.3.5) on the planning and solving phase.

The findings from the pre and post-test on mindset are discussed in section 7.6. After the discussion on mindset, each individual pre-service teacher’s results (pre and post-test) are presented (section 7.7). The participants’ opinions of the intervention are also presented in this chapter. The chapter concludes with an evaluation of the intervention.

7.2 Reading and Understanding: Pre-test to Post-test Findings

The percentage of the pre-service teachers\(^{58}\) at each type of response for the reading and understanding phase (pre-assessment) of the problem solving cycle is shown in Table 61\(^{59}\). The percentage of participants at each response type are:

\(^{58}\)The pre-service teachers who participated in all three phases of the research.
\(^{59}\)For problems 7, 8 and 9, participant (P13) results are not included as P13 did not complete these problems in the post-test due to having to leave post-test due to unexpected circumstances.
for reading, understanding and representing the problem situation in problem 4(a) is shown in Table 62.

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<td>0</td>
<td>8.3</td>
<td>0</td>
<td>50</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>91.7</td>
</tr>
<tr>
<td>Problem 9</td>
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<td>0</td>
<td>16.7</td>
<td>8.3</td>
<td>0</td>
<td>75</td>
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</tbody>
</table>

Table 61: Percentage of the Participants at each Response Type for the Reading and Understanding Phase (Pre-test).

<table>
<thead>
<tr>
<th></th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
<th>Type 5</th>
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<tbody>
<tr>
<td>Problem 4 a(i)</td>
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<tr>
<td>Problem 4 a(iii)</td>
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<td>7.7</td>
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<td>69.2</td>
</tr>
<tr>
<td>Problem 4 a(iv)</td>
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<td>0</td>
<td>0</td>
<td>38.5</td>
<td>23.1</td>
</tr>
</tbody>
</table>

Table 62: Percentage of the Participants at each Response Type for Reading, Understanding and Representing the Problem Situation in problem 4 (Pre-test).

This group of pre-service teachers demonstrated most difficulty in the reading and understanding of problems 1 and 5 with five of the thirteen responses for problem 1 demonstrating some substantial omissions/errors in the mathematical understanding of relations among the data. One of the thirteen responses to problem 5 demonstrated complete misinterpretation of problem 5, while no attempt was made by another one of the participants. This group showed least difficulty in the reading and understanding of problems 2, 7(i) and 8 with over 84% of the responses demonstrating complete understanding of the problem situation. The participants demonstrated good ability in the graphical representations of the problem situations in problem 4. The problem situation (4a(ii)) which required the participants to sketch a graph to model the motion of a ferris wheel, caused the most difficulty (of the four situations in problem 4) for the second year participants, with 40% demonstrating substantial error in providing the graphical representation of the situation. The problem situation
(4a(iv)) which required the participants to sketch a graph to model the depre-
ciation of a car, caused the most difficulty (of the four situations in problem 4) for
the first year group, with 50% of them using an inappropriate graph to
model the problem situation.

The percentage of the pre-service teachers at each type of response for the
reading and understanding phase (Post-assessment) of the problem solving cycle
is shown in Table 63. The percentage of the participants at each response type
for reading, understanding and representing the problem situation in problem
4a) is shown in Table 6460.

<table>
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<tr>
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<td>0</td>
<td>0</td>
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<td>83.3</td>
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<tr>
<td>Problem 9</td>
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<td>0</td>
<td>8.3</td>
<td>8.3</td>
<td>0</td>
<td>83.3</td>
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</table>

Table 63: Percentage of the Participants at each Response Type for the Reading
and Understanding Phase (Post-test).

<table>
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<tr>
<th>Post</th>
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<td>30.8</td>
<td>0</td>
<td>38.5</td>
<td>30.8</td>
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<td>Problem 4 a(iv)</td>
<td>7.7</td>
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<td>0</td>
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<td>46.2</td>
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Table 64: Percentage of the Participants at each Response Type for Reading,
Understanding and Representing the Problem Situation in problem 4 (Post-
test).

This group of pre-service teachers demonstrated most difficulty in the reading
and understanding of problems 3 with four of the thirteen responses for problem

60For problems 7, 8 and 9, participant (P13) results are not included as P13 did not com-
plete these problems in the post-test due to having to leave post-test due to unexpected
circumstances.
1 demonstrating some substantial omissions/errors in the mathematical understanding of relations among the data. This group showed least difficulty in the reading and understanding of problems 1, 2, 6, 7, and 8, with over 83% of the responses demonstrating complete understanding of the problem situation. The participants demonstrated good ability in the graphical representations of the problem situations in problem 4. The problem situation (4a(iv)) which required the participants to sketch a graph to model the cooling of a thermometer placed in a refrigerator caused the most difficulty (of the four situations in problem 4) for the second year participants, with 20% demonstrating substantial error in providing the graphical representation of the situation. The problem situation (4a(iii)) which required the participants to sketch a graph to model the sharing of the cost of hiring a bus, caused the most difficulty (of the four situations in problem 4) for the first year group, with 50% of them demonstrating substantial error in providing the graphical representation of the situation.

The percentage of the participants demonstrating complete understanding of the problem situation increased for problems 1, 3, 5, 6, 7(ii) and 9. A response to problem 8 which had demonstrated complete misinterpretation of the problem situation (pre-test) demonstrated basic understanding of the problem situation in the post-test. The participant (P1) who made no attempt at problem 5 in the pre-test demonstrated complete understanding of problem 5 in the post-test. The participant (P2) who demonstrated complete misinterpretation of the problem situation in problem 5 (pre-test), demonstrated substantial understanding of the problem situation in problem 5 with minor errors/omissions in the mathematical understanding of relations among the data (post-test). The percentage of the participants who gave a response of type 0 to type 3 (making no attempt - demonstrating but with substantial errors or omissions) decreased for problems 1, 5, 7, and 9. These increases (↑) and decreases (↓) are summarised in Table 65.
Table 65: Change in the percentage of participants at each response type for the Reading and Understanding Phase (from Pre to Post-test). The number before the → is the percentage of participants at this type(s) in the pre-test and the number after the → is the percentage of participants at this type(s) in the post-test. The symbol − represents no change.

The responses to problem 1, given by participant 5 (P5) for both the pre and post-test are shown in Figures 28 and 29 respectively.
Figure 28: Response by P5 to Problem 1 in pre-test
Figure 29: Response by P5 to Problem 1 in post-test
For the pre-test, in the first equation, it seems that \( y \) actually represents 1 cage and then P5 is using \( a \) to represent the number of cages so P5 writes \( ay \). The introduction of a third variable is overcomplicating the problem. P5 is aware that there were two hamsters left without a cage (when one hamster is placed in each cage) so P5 writes one hamster times the number of them that are there, that is \( ax \) is equal to the number of cages \( ay + 2x \). This is an attempt to write that the number of hamsters = no of cages +2 (i.e. there are 2 less cages than hamsters). From P5’s attempt it is clear P5 has read and understood the information properly but is unable to accurately represent this information. P5 writes that the total cages = \( ay \) and the total number of hamsters = \( (2 + a)x \) (this time P5’s attempt at writing the number of hamsters = number of cages +2 is incorrectly attempted as writing the number of cages = no of hamsters +2). The second equation P5 writes to represent the two hamsters per cage situation is \( \frac{1}{2}(2 + a)(x) = ay + 2y \). This is an attempt to write the (number of hamsters)/2 = the no. of cages -2. So P5’s attempt is incorrect in reasoning by writing +2 instead of -2. P5’s equations are incorrect but there is clear evidence of understanding of the mathematics needed to solve this problem.

For the post-test, P5 attempted to form a set of equations and did actually write two correct equations (one of these subsequently crossed out) to represent the information. P5 also showed reasoning in stating that there must be an even number of cards. Used reasoning and trial and error to solve the problem correctly to obtain correct solution.

The responses to problem 5, given by participant 1 (P1) for both the pre and post-tests are shown in Figures 30 and 31 respectively.

The responses to problem 3, given by participant 4 (P4) for both the pre and post-tests are shown in Figures 32 and 33 respectively.

295
Problem 5

A library needs to bind some English, French, and German books whose number is in the ratio of 3:5:1. Three shops were contacted. The first shop could bind the books bound in 10 days, the second shop could bind the books bound in 15 days, and the third shop could bind the books bound in 18 days. In order to have the books bound as quickly as possible, it was decided to give the job to all three shops at once. In how many days will the shops do the job, working simultaneously? In your solution, have you made any assumption about this problem situation? If yes, please state these assumptions.

1st shop: 10 days
2nd shop: 15 days
3rd shop: 18 days

Not sure how to approach this.
Problem 6

A company wants some stone walls, paving areas and ponds built whose number is in the ratio of 5 : 2 : 1. Three stone masons, Jack, Null and Shane were contacted. Jack and Shane, jointly can complete the work in 12 days, Shane and Null, jointly can complete the work in 20 days and Jack and Null together can complete the work in 15 days. In order to have the work completed as quickly as possible, the company decided to use all three of the stone masons to work together. In how many days will the three stone masons complete the work, working simultaneously? In your solution, have you made any assumptions about this problem situation? If you have, please state these assumptions.

\[ \text{Walls:} \quad \text{paving areas:} \quad \text{ponds} \]

\[ 5 \quad 2 \quad 1 \]

\[ \frac{1}{12} \text{ done in one day, } \]

\[ \text{Jack and Shane} = 12 \text{ days} \]

\[ \frac{1}{20} \text{ done in one day, } \]

\[ \text{Shane and Null} = 20 \text{ days} \]

\[ \frac{1}{15} \text{ done in one day, } \]

\[ \text{Jack and Null} = 15 \text{ days} \]

\[ \begin{align*}
X + Y + Z &= 12 \\
0.1X + Y + Z &= 10 \\
X + 0.1Y + Z &= 15 \\
X - Y + 0.1Z &= 12 \\
0.1X - Y + Z &= 20 \\
0.1X + 0.1Y + Z &= 12 \\
0.1X + Y + Z &= 20 \\
0.1X + 0.1Y + Z &= 6 \\
Z &= 13
\end{align*} \]

\[ \left( \frac{1}{12} + \frac{1}{20} + \frac{1}{15} \right) = \frac{5}{12} = 1 \text{ day} \]

\[ \Rightarrow \text{they would complete it in 5 days}\]

Figure 31: Response by P1 to Problem 5 in post-test
Problem 3

216 identical little cubes have been stacked into a large box which is also a cube. The little cubes fit exactly into the large box, with no space remaining. The large box is see-through, with an open top, such that the little cubes can still be seen from outside the box.

(i) You view the large box from a point that allows you to see the maximum number of distinct little cubes. What is the maximum number of distinct little cubes that you can see from this point?

Now suppose that $n^3$ identical little cubes have been arranged into one larger cube. You view the larger cube from a point that allows you to see the maximum number of distinct little cubes.

(ii) Determine a formula for the maximum number of distinct little cubes that can be seen from this viewing point.

(i) \( \sqrt[3]{216} = 6 \)

Sides of cube \( n \times n \times n = 18 \) cubes total can be seen

(ii) \( 3n = \) no. of distinct cubes that can be seen
Problem 3

A wooden cube has sides which are 5 meters in length. Square holes with sides 1 meter in length and centered in each face are cut through to the opposite face with the edges of the holes parallel to the edges of the cube.

(i) What is the entire surface area, including the inside?

Now suppose a wooden cube has sides which are $a$ meters in length. Square holes with side $1$ meter in length and centered in each face are cut through to the opposite face with the edges of the holes parallel to the edges of the cube.

(ii) Determine a formula for the entire surface area, including the inside?

\[
\frac{1}{2} \text{ of inside: } (1^2) \times 6 = 8
\]

\[
\frac{a^2}{40} = \frac{4a^2}{10} + \frac{12a^2}{10} = \frac{16a^2}{10}
\]

\[
6(n^3 - 1) + 6(2(n-1))
\]

\[
= 6n^3 - 6 + 12n - 6
\]

\[
= 6n^3 - 12n - 8
\]

Figure 33: Response by P4 to Problem 3 in post-test
For the pre-test P4 correctly stated that 3 sides can be seen. Found the cube root of 216 and then multiplied this by 3 to give 18 as the answer to (i). This should be $6 \times 6 \times 3 = 108$, not $3 \times 6$. P4 interpreted this problem as being asked to find the number of faces of cubes that could be seen instead of the number of distinct cubes that could be seen.

For the post-test P4 drew out the cube and showed the piece that would be cut on the inside of the cube. Correctly interpreted the problem, obtained the total area before any removals, then correctly subtracted 6 from this to give $150 - 6 = 144m^2$. P4 calculated the surface area of each ‘half tunnel’ and then multiplied this by 6 for each of the 6 sides: $1 \times 2 \times 4 = 8 \times 6 = 48$. Then P4 added the 144 and the 48 to obtain the correct solution of $192m^2$.

The percentage of the second year participants demonstrating complete understanding of the problem situation decreased for problems 2 (participant P5 went from type 5 to type 4) and 7(i) (participant P5 went from type 5 to no attempt). One of the participants, P5 who had demonstrated complete understanding of problem 2 in the pre-test showed substantial understanding of the problem situation in problem 2 with minor errors/omissions in the mathematical understanding of relations among the data (post-test). On examining possible reasons for this, it emerged that P5 had seen a similar problem to problem 2 in the pre-test which was helpful in solving the problem, and rated the problem at a difficulty level of 3 (moderate). For problem 2 in the post-test P5 had not seen a similar problem to it before and rated it at a difficult level of 5 (very difficult). This participant also did not attempt problem 7(ii) in the post-test whereas this participant did attempt 7(ii) in the pre-test. In the pre-test P5 stated that she knew how to solve part (i) immediately after reading it and rated it at a difficulty level of 1 (very easy), with (ii) rated at a difficulty level of 3 (moderate). In the post-test P5 stated that she did not know how to solve part (i) immediately after reading it and rated it at a difficulty level of 3 (moderate), with (ii) rated at a difficulty level of 5 (very difficult).

The responses to problem 2, given by participant 5 (P5) for both the pre and post-tests are shown in Figures 34 and 35 respectively.

---

61 Problem 5 in the pre-test gave the individual time taken by each of three shops to complete a job. Problem 5 in the post-test gave the time taken by each of the possible pairs of three workers to complete a job. Both problems required calculating the time taken by all three shops/workers to complete job if all three shops/workers completed the job together.
Figure 34: Response by P5 to Problem 2 in pre-test

Problem 2

The first train passed the Eiffel Tower in Paris on route to Berlin at a constant speed of 60 km/hr. Two hours later, the second train passed the same point on the Eiffel Tower in Paris on its way to Berlin, at a constant speed of 80 km/hr on a parallel track. At what distance from the Eiffel Tower will the two trains be in line with (at same level as) the back of the first train, if the distance between the Eiffel Tower and Berlin is 1200 km and there are twice as much carbides on the first train as there are on the second?

Train 1: k = 0, 60 km/hr
Train 2: k = 2, 80 km/hr.

Eiffel Tower: 0 km

When T2 is at Eiffel Tower, T1 has traveled 96 km.

\[ T_2 \quad \text{Eiffel Tower} \quad \text{T1} \quad 96 \text{ km} \quad \rightarrow \quad D \]

In how many hours will they be equal? Multiplied by 56 km/hr gives answer.

\[ \text{Distance } D = T_1@56x \]

\[ \text{Distance } D + 96 \text{ km} = T_2@56x \]

\[ \begin{cases} D = 56x \\ D - 96 = 48x \end{cases} \]

\[ \begin{align*} D &= 56x \\ D &= 48x + 96 \end{align*} \]

They will be level 672 km from the Eiffel Tower.
Figure 35: Response by P5 to Problem 2 in post-test
For problem 2 in the post-test, P5 wrote that after 4 hours Shane had $\frac{2}{3}$ of his journey completed. Understanding the relationship between the fraction of the journey already completed by Shane and the fraction to be completed by Katie would allow for the solution to be found.

The percentage of the second year participants demonstrating complete understanding of the problem situation remained the same for problems 7(i), 8 and 9.

The percentage of the first year participants demonstrating complete understanding of the problem situation decreased for problem 8 (participant P9 went from type 5 to type 4 and P7 went from type 5 to type 3). P9 demonstrated only minor omissions - neglected to add the 16, and made computational errors. P7 used $n(n + 2)(n + 4)(n + 6)$ as the consecutive even numbers without stating that $n$ was even.

The percentage of the first year participants who gave a response of type 0 - type 3 increased for problem 3 (two students (P6 and P10) went from a type 4 to a type 3, and one student (P9) went from a type 5 to a type 3). All three students demonstrated understanding of the problem situation but with incomplete consideration given to some of the area remaining. A note by the panel of expert mathematicians who reviewed both the pre and post-tests was that problem 3 in the post-test, although deemed to be comparable to the pre-test and given the same difficulty rating, “uses similar thought processes to the pre-test, but is a bit trickier as one needs to visualise the inside surfaces of the “tunnels” cut into the cube”. The number of the first year participants showing complete understanding increased for this problem.

The median type of response given by all the participants (both first and second year groups) for each of the problems (1, 2, 3, 5, 6, 7(i), 7(ii), 8, and 9) in the pre-test is shown in the boxplots in Figure 36 with the summary of descriptive statistics shown in Figure 37.
Figure 36: Median Type of Response in Reading and Understanding pre-test

Figure 37: Summary of Descriptive Statistics for Reading and Understanding pre-test
The median type of response given by the participants for each of the problems (1, 2, 3, 5, 6, 7(i), 7(ii), 8, and 9) in the post-test is shown in the boxplots in Figure 38 with the summary of descriptive statistics shown in Figure 39.

Figure 38: Median Type of Response in Reading and Understanding post-test
The median type of response given by the participants in the reading and understanding phase either remained at the highest level of a type 5 response, increased from a type 4 to a type 5 response (problems 3 and 7(ii)) or increased from a type 3 to a type 5 response (problem 5). Also it is visible in the boxplot (Figure 36) that problems 5, 7(i) and 7(ii) had a minimum score of a type 0 response in the pre-test, with a type 1 response being the minimum response type to problem 8. Problem 9 had a minimum response of type 2. All other problems in the pre-test had a minimum response of type 3 or more. In the boxplot (Figure 38) it is visible that in the post-test, only problem 7(ii) had a minimum score of a type 0 response. Problems 5, 8 and 9 had a minimum score of a type 2 response in the post-test. All other problems in the post-test had a minimum response of type 3 or more.

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Problem 1</th>
<th>Problem 2</th>
<th>Problem 3</th>
<th>Problem 4</th>
<th>Problem 5</th>
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Figure 39: Summary of Descriptive Statistics for Reading and Understanding post-test
7.2.1 Changes in Total Score for the Reading and Understanding Phase for All Participants

The total score for the reading and understanding phase (TRU) of the problem solving cycle for each participant across the nine problems for both the pre and post-test was computed using SPSS 24.0 for Windows. Normality of the distribution of the differences (TRU diff) in TRU from pre to post-test for the sample of 12 pre-service participants was assessed using the Shapiro-Wilk test. The Shapiro-Wilk test returned the output shown in Table 66.

<table>
<thead>
<tr>
<th>Shapiro-Wilk</th>
<th>Statistic</th>
<th>df</th>
<th>Sig.(p-value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TRU diff</td>
<td>0.937</td>
<td>12</td>
<td>0.457</td>
</tr>
</tbody>
</table>

Table 66: Shapiro-Wilk Test of Normality for the distribution of the differences in TRU.

Since the p-value (sig. = 0.457), is greater than 0.05, the distribution of the differences from pre to post-test (TRU diff) is normally distributed.

A Wilcoxon signed-rank test was conducted to determine if there was a difference in the median TRU score between the pre and post-test. The test showed that there was a statistically significant increase ($Z = -2.317, p = 0.020$) in the median TRU score of the participants from 42 (7.5) to 46 (4.5).

This increase (effect size $r = 0.47$) is shown in the boxplots in Figure 40. Over 75% of the participants in the post-test scored higher than the median TRU score in the pre-test. Participants P2 and P6 showed the highest increase in the reading and understanding phase from pre to post-test, increasing 13 and 12 units respectively. Participant P5 showed the lowest increase, P5 actually decreased by 3 units. This slight decrease occurred in problem 2 (Figures 34 and 35), problem 4 and problem 7(ii). P5 made minor errors in some graphical representations in problem 4 and she did not attempt problem 7(ii) in the post-test.

---

$^{62}$P13 not included as only 6 problems completed. Note that for the 6 problems, the TRU for P13 increased from 26 to 28.

$^{63}$The value in the brackets is the interquartile range
7.3 Planning and Solving: Pre-test to Post-test Findings

The percentage of the pre-service teachers at each type of response for the planning and solving phase (pre-assessment) of the problem solving cycle is shown in Table 67\(^{64}\). The percentage of the pre-service teachers at each type of response for understanding the relationship between the symbolic representation of the problem situation and the problem situation and solving problem 4(b) is shown in Table 68.

\(^{64}\) For problems 7, 8 and 9, participant (P13) results are not included as P13 did not complete these problems in the post-test due to having to leave post-test due to unexpected circumstances.
This group of pre-service teachers demonstrated most difficulty in the planning and solving of problems 1, 5 and 8. Four of the thirteen responses for problem 1 demonstrated high error in solving. Five of the thirteen responses to problem 5 formed an inappropriate plan, no attempt was made by one of the participants, and the work written by one of the participants did not lead to any structured plan of action. For problem 8, two of the thirteen responses demonstrated high error in solving, no attempt was made by one of the participants, and one response formed an inappropriate plan. This group showed least difficulty in the planning and solving of problem 7(i), with 83.3% of the responses forming a correct plan free from error. The percentage of the pre-service teachers at each type of response for the planning and solving phase (post-assessment) of the problem solving cycle is shown in Table 69\textsuperscript{65}.  

\textsuperscript{65}For problems 7, 8 and 9, participant (P13) results are not included as P13 did not complete these problems in the post-test due to having to leave post-test due to unexpected circumstances.
Table 69: Percentage of the participants at each Response Type for the Planning and Solving Phase (Post-test).

<table>
<thead>
<tr>
<th>Post</th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
<th>Type 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>0</td>
<td>0</td>
<td>7.7</td>
<td>7.7</td>
<td>7.7</td>
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<tr>
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<td>0</td>
<td>7.7</td>
<td>0</td>
<td>7.7</td>
<td>0</td>
<td>84.6</td>
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<tr>
<td>Problem 3</td>
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<td>23.1</td>
<td>7.7</td>
<td>15.4</td>
<td>46.2</td>
</tr>
<tr>
<td>Problem 5</td>
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<td>38.5</td>
<td>0</td>
<td>0</td>
<td>7.7</td>
<td>53.8</td>
</tr>
<tr>
<td>Problem 6</td>
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<td>7.7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>92.3</td>
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<tr>
<td>Problem 7</td>
<td>0</td>
<td>0</td>
<td>8.3</td>
<td>8.3</td>
<td>0</td>
<td>83.3</td>
</tr>
<tr>
<td>Problem 7(ii)</td>
<td>16.7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>83.3</td>
</tr>
<tr>
<td>Problem 8</td>
<td>0</td>
<td>8.3</td>
<td>25</td>
<td>25</td>
<td>0</td>
<td>41.7</td>
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<tr>
<td>Problem 9</td>
<td>0</td>
<td>16.7</td>
<td>8.3</td>
<td>16.7</td>
<td>16.7</td>
<td>41.7</td>
</tr>
</tbody>
</table>

Table 70: Percentage of the participants at each Response Type for Understanding the Relationship between the Symbolic Representation of the Problem Situation and the Problem Situation and Solving (Post-test).

<table>
<thead>
<tr>
<th>Post</th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
<th>Type 5</th>
<th>Type 6</th>
<th>Type 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>30.8</td>
<td>15.4</td>
<td>38.5</td>
<td>15.4</td>
</tr>
</tbody>
</table>

This group of pre-service teachers demonstrated most difficulty in the planning and solving of problem 5 with five of the responses demonstrating an inappropriate plan (it was noted that some correct reasoning was evident in the plans formed by these responses). Both the pre and post-test responses by two of the participants (P2 and P5) are shown in Figures 41 and 42, and in Figures 43 and 44.

In the pre-test P5 found the number of days taken by each shop to complete 1 “unit”. P5 used the ratio of books $5 : 2 : 1$ to state that there were 8 “units” or crates of books. P5 was able to determine a number of days (2.5) such that the addition of the number of units bound by the three shops was a whole number (4) factor of the number of “units” in the problem. P5 used this result then to determine the correct solution. P5 has written that “with a more complex problem I am aware my approach would not work”. P5 attempted to try a similar approach to problem 5 in the post-test but was unable to work out the problem in the post-test. In the post-test P5 found the percentage of the job completed per day, by each pair but did not make use of these calculations in a productive way. P5’s rating of problem 5 went from moderate in the pre-test to difficult in the post-test. P5 stated that she had seen a similar problem which was helpful.
in solving the pre-test but which was not helpful in solving the post-test.66

P2 made a guess at the solution for the problem in the pre-test whereas in the post-test P2 attempted to express the work completed by one of the masons in the pair in terms of the other mason and days taken for that pair to complete the job. For example: writing $J + S = 12$, $S = 12 - J$ (but not expressing this as fractions). P2 stated that he had not seen a similar problem in the pre-test. P2 stated that he had seen a similar problem to problem 5 in the post-test but that it wasn’t not helpful in solving the problem. P2’s rating of problem 5 went from very difficult in the pre-test to difficult in the post-test. This group showed least difficulty in the planning and solving of problems 2, 6 and 7(i) and 7(ii) with over 83% of the responses forming a correct plan free from error.

66 Although the problems were similar in structure, the pre-test stated the amount of time taken by each shop to complete job and asked to determine the time taken to complete job if all worked together. The post-test stated the amount of time it took pairs of individuals (3 individuals - 3 pairs) to complete a job and asked to determine the time taken to complete job if all three worked together.
Problem 5

A library needs to bind some English, French, and German books whose number is in the ratio of 5:2:1. Three shops who bind books were contacted. The first said it could have the books bound in 10 days, the second said it would have the books bound in 30 days and the third could have the books bound in 15 days. In order to have the books bound as quickly as possible, it was decided to give the job to all three shops at once. In how many days will the shops do the job, working simultaneously? In your solution, have you made any assumptions about the problem situation? If you have, please state these assumptions.

Assume there are 8 "units" or crates of books, all of which contain the same amount of books.

1st step: 1.25 days per unit
2nd step: 3.75 days per unit
3rd step: 1.875 days per unit.

8 units in total in 25 days.

\[
\begin{align*}
1^\text{st} & : 2 \text{ units} \\
2^\text{nd} & : \frac{2}{3} \text{ unit} \\
3^\text{rd} & : 1 \frac{3}{4} \text{ units}
\end{align*}
\]

Total 4 units

If it takes 25 days for 4 units then it takes 5 days for 8 units:

\[
\begin{align*}
1^\text{st} & : \text{units} \\
2^\text{nd} & : \frac{4}{3} \text{ units} \\
3^\text{rd} & : 2 \frac{2}{3} \text{ units}
\end{align*}
\]

Total 4 units

Figure 41: Response by P5 to Problem 5 in pre-test
Figure 42: Response by P5 to Problem 5 in post-test
Figure 43: Response by P2 to Problem 5 in pre-test
Problem 5

A company wants to build stone walls, paving areas and ponds built whose number is in the ratio of 3 : 2 : 1. Three stone masons, Jack, Nall and Shane were asked to complete the work. Jack and Shane, jointly can complete the work in 12 days, Shane and Nall, jointly can complete the work in 20 days and Jack and Nall together can complete the work in 16 days. In order to have the work completed as quickly as possible, the company decided to ask all three of the stone masons to work together. In how many days will the three masons complete the work, working simultaneously? In your solution, have you made any assumptions about this problem situation? If you have, please state these assumptions.

Figure 44: Response by P2 to Problem 5 in post-test
The percentage of the participants formulating a correct plan (which was free from error) for the problem situation increased for problems 1, 2, 5, 6, 7(ii), and 8. The percentage of the participants who gave a response of type 0 to type 1 (no attempt-inappropriate plan) decreased for problems 1, 2, 5, 7(ii), 8, and 9. This decreases in the type 0 to type 1 response show that the number of participants making an attempt to form a plan to solve problems 1, 2, 5, 7(ii), 8, and 9 increased. In addition, it shows that the number of participants forming an appropriate plan for these problem has increased. These increases (↑) and decreases (↓) are summarised in Table 71.

<table>
<thead>
<tr>
<th>Pre to Post-test for All Participants</th>
<th>Type 0 - Type 1</th>
<th>Type 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>↓15.4 → 0</td>
<td>↑46.2 → 76.9</td>
</tr>
<tr>
<td>Problem 2</td>
<td>↓15.4 → 7.7</td>
<td>↑46.2 → 84.6</td>
</tr>
<tr>
<td>Problem 3</td>
<td>−</td>
<td>↓76.9 → 46.2</td>
</tr>
<tr>
<td>Problem 5</td>
<td>↓53.9 → 46.2</td>
<td>↑30.8 → 53.8</td>
</tr>
<tr>
<td>Problem 6</td>
<td>−</td>
<td>↑61.5 → 92.3</td>
</tr>
<tr>
<td>Problem 7</td>
<td>↓16.7 → 0</td>
<td>−</td>
</tr>
<tr>
<td>Problem 7(ii)</td>
<td>↓41.7 → 16.7</td>
<td>↑41.7 → 83.3</td>
</tr>
<tr>
<td>Problem 8</td>
<td>↓16.6 → 8.3</td>
<td>↑0 → 41.7</td>
</tr>
<tr>
<td>Problem 9</td>
<td>↓25 → 8.3</td>
<td>↓58.3 → 41.7</td>
</tr>
</tbody>
</table>

Table 71: Change in the percentage of the participants at each Response Type for the Planning and Solving Phase (from Pre to Post-test). The number before the → is the percentage of participants at this type(s) in the pre-test and the number after the → is the percentage of participants at this type(s) in the post-test. The symbol − represents no change.
The percentage of the second year participants forming a correct plan free from error decreased for problem 9 (participant P5 went from a type 5 to a type 4 response). P5 demonstrated only minor errors in solving - the sum of the rectangles was written as $= ((n-1)n)^2 + ((n-2)(n-1))^2 + ((n-3)(n-2))^2$. Instead of squaring here, these should be $\times 2$. A note by the panel of expert mathematicians who reviewed both the pre and post-tests was that problem 9 in the post-test, although deemed to be comparable to the pre-test and given the same difficulty rating, was “possibly more difficult than problem 9 in the pre test because there are so many more possibilities to account for in the rectangles case”. The response provided by P5 to problem 9 in the post-test shows clear evidence of complete understanding and reasoning in the use of a correct plan to solve the problem (but made minor error) whereas the provision of two correct lines of work in the pre-test showed the correct solution without displaying the work involved. Although the response type for P5 decreased slightly for problem 9, there is more substantial work shown in the post-test.

The percentage of the second year participants who gave a response of type 0 - type 1 increased for problem 3 (one student (P2) went from a type 5 to a type 1 response. P2’s plan was to subtract the area (gained) on the inside from the total original area of the cube, instead of adding the area gained. The number of the second year participants showing complete understanding increased for this problem. The percentage of the second year participants who gave a response of type 0 - type 1 also increased for problem 7(ii). One participant (P5) made no attempt at this part of the problem in the post-test whereas she had completed problem 7(ii) correctly in the pre-test.

The percentage of the first year participants forming a correct plan free from error decreased for problems 3 (participants P7, P8, P9, P10 and P12) and 9 (participants P8 and P10). Participants P7, P8, P10 and P12 had formed a correct plan for problem 3 in the pre-test, based on incorrect understanding of the problem situation (which resulted in oversimplification of the problem). Participants P7 and P8 formed a correct plan with low error in solving to problem 3 in the post-test (went from a type 5* to a type 4 response). Both P10 and P12 also formed a correct plan for problem 3 but had high and medium error in solving respectively. P9 formed a correct plan but failed to consider all the area remaining (high error in solving). The responses (pre and post-test) given by P8 to problem 3 are shown in Figures 45 and 46 (and Figure 47) respectively. The responses (pre and post-test) by P10 to problem 9 are shown in Figures 48 and 49 respectively.
Figure 45: Response by P8 to Problem 3 in pre-test

Problem 3

256 identical little cubes have been stacked into a large box which is also a cube. The little cubes fit exactly into the large box, with no space remaining. The large box is one through, with an open top, such that the little cubes can still be seen from outside the box.

(i) You view the large box from a point that allows you to see the maximum number of distinct little cubes. What is the maximum number of distinct little cubes that you can see from this point?

Now suppose that a 4 identical little cubes have been arranged into one larger cube. You view the larger cube from a point that allows you to see the maximum number of distinct little cubes.

(ii) Determine a formula for the maximum number of distinct little cubes that can be seen from this viewing point.

1) \( 216 = x^3 \)
\( x = 6 \)

\(((6 \times 6) + (6 \times 6) + (6 \times 6)) = 36 + 36 + 36 = 108 \text{ cubes} \)

2) \((n \times n) + (n \times n) + (n \times n) = n^2 + n^2 + n^2 = 3n^2 \)
Problem 3

A wooden cube has sides which are 5 meters in length. Square holes with sides 1 meter in length and centered in each face are cut through to the opposite face with the edges of the holes parallel to the edges of the cube. 

(i) What is the entire surface area, including the inside?

Now suppose a wooden cube has sides which are 5 meters in length. Square holes with sides 1 meter in length and centered in each face are cut through to the opposite face with the edge of the hole parallel to the edges of the cube.

(ii) Determine a formula for the entire surface area, including the holes?

---

1) Surface area of cube w/o holes

\[ 6(5 \times 5) = 6(25) = 150 \text{m}^2 - 6(1 \times 1) = 6 \text{m}^2 \]

Inside:

\[ 5A = 2(1 \times 1) + 4(5 \times 1) = 22 \text{m}^2 \times 3 = 66 \text{m}^2 \]

Area of cube:

\[ 1 \text{m} \times 6 = 6 \times 60 \text{m}^2 = 60 \text{m}^2 \]

Total:

\[ 144 + 60 = 204 \text{m}^2 \]

---

Figure 46: Response by P8 to Problem 3 in post-test
Figure 47: Response continued. by P8 to Problem 3 in post-test

\[
\frac{6}{6n^2 - 6}
\]

Holes: \(2(1\star 1) + 4(n\star n)\)
\[
= 2\quad + 4n
\]
\[
= 4n + 2
\]

\[
\text{(area of cube)}
\]

\[
\frac{-6}{4n - 1}
\]

Total:

\[
6n^2 + 4n + 1 - 6
\]

\[
= 6n^2 + 4n - 5
\]

\[
(4n + 2) \times 3 = 12n + 6
\]

\[
-1(6)
\]

\[
\frac{12n}{12n}
\]

Total:

\[
6n^2 + 12n - 6
\]

\[
= 6n^2 + 12n - 6
\]
Problem 9

If you counted all the possible squares of all sizes on a standard chessboard (from size 1 x 1 to size 8 x 8), you would obtain an answer of 204 squares. How many squares of all sizes (from size 1 x 1 to size n x n) would there be on an n x n chessboard?

\[ (1^2 + 2^2 + 3^2 + \ldots + n^2) \]

\[ 1 + 4 + 9 + 16 + \ldots + n^2 \]

Algebraic patterns

1 4 9 16

3 5 7

2 2

Figure 48: Response by P10 to Problem 9 in pre-test
Problem 9

If you counted all the possible rectangles (including squares as rectangles) of all sizes on an 8 x 8 squared board (from size 1 x 1, 1 x 2, 7 x 8 x 8), you would obtain a number of 2096 rectangles. How many rectangles of all sizes (from size 1 x 1, 1 x 2, (n-1) x (n, n x n) would there be on a n x n squared board?

\[ n^2 + (n-1)^2 + (n-2)^2 + \cdots + (n-n)^2 \]

Figure 49: Response by P10 to Problem 9 in post-test
In the pre-test P8 misinterpreted the problem, she determined the maximum number of faces of the little cubes that could be seen for part (i). She did this incorrect calculation correctly. The misinterpretation of the problem resulted in simplification of the problem, the solver did not need to consider cubes shared by common sides then. P8 formed a correct plan per her incorrect reading and understanding of the problem. P8 correctly solved the problem as per her incorrect understanding of the problem situation, and so was deemed to be a 5* type of response. The plan formed by P8 to problem 3 in the post-test was correct but P8 made low error in solving it, hence this response was of type 4. P8 rated both problems the same level of difficulty, however in the pre-test she stated that she knew how to solve the problem immediately whereas she stated she did not know how to solve the problem immediately in the post-test. P8 correctly generalised from part (i) to (ii) in both pre and post-tests.

In problem 9 in the post-test, P10 correctly drew out a table, showing the number of each type of square and each type of rectangle in the 8 x 8 case. She has correctly written out the number of squares for the n x n case and has written out the number of rectangles for the n x n but neglected to multiply this by 2. Her table shows consideration of this but in the formulation of her formula, this part is omitted. P10’s plan is perfect for this problem but there is minor error in solving. P10 rated problem 9 as difficult in the pre-test and very-difficult in the post-test.

The percentage of the first year participants who gave a response of type 0 - type 3 decreased for all problems except for problems 6 and 8, where there was no change.

The median type of response given by all the participants (both first and second year groups) for each of the problems (1, 2, 3, 5, 6, 7(i), 7(ii), 8, and 9) in the pre-test is shown in the boxplots in Figure 50 with the summary of descriptive statistics shown in Figure 51.

The median type of response given by the participants for each of the problems (1, 2, 3, 5, 6, 7(i), 7(ii), 8, and 9) in the post-test is shown in the boxplots in Figure 52 with the summary of descriptive statistics shown in Figure 53.
Figure 50: Median Type of Response in Planning and Solving pre-test

![Median Type of Response in Planning and Solving pre-test](image)

Figure 51: Summary of Descriptive Statistics for Planning and Solving pre-test

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Problem 1</th>
<th>Problem 2</th>
<th>Problem 3</th>
<th>Problem 4</th>
<th>Problem 5</th>
<th>Problem 6</th>
<th>Problem 7 part 1</th>
<th>Problem 7 part 2</th>
<th>Problem 8</th>
<th>Problem 9</th>
</tr>
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<tbody>
<tr>
<td>N</td>
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<tr>
<td>Median</td>
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<td>Std. Deviation</td>
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<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
</tr>
</tbody>
</table>
Figure 52: Median Type of Response in Planning and Solving post-test

Figure 53: Summary of Descriptive Statistics for Planning and Solving post-test
7.3.1 Changes in Response Type and Difficulty Rating for All Participants

The following changes (Table 72) occurred in the median difficulty rating\textsuperscript{67} of the problems given by the participants from the pre-test to the post-test. The changes that occurred in the median type of response given by the participants to the problems in the reading and understanding, and in the planning and solving phase from the pre-test to the post-test are also shown in Table 72.

\textsuperscript{67}In the table, the numbers in the brackets are the end points of the interquartile range $(Q_1 - Q_3)$
<table>
<thead>
<tr>
<th>Problem</th>
<th>Difficulty</th>
<th>R&amp;U</th>
<th>P&amp;S</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>(3 - 4.75)</td>
<td>(3 - 5)</td>
<td>(5 - 5)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(2.5 - 4)</td>
<td>(5 - 5)</td>
</tr>
<tr>
<td>P2</td>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>(2.5 - 4)</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(2.5 - 4)</td>
<td>(5 - 5)</td>
</tr>
<tr>
<td>P3</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>(2 - 3)</td>
<td>(3 - 5)</td>
<td>(5 - 5)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>(4 - 5)</td>
<td>(5 - 5)</td>
</tr>
<tr>
<td>P(3g)</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>(2 - 3)</td>
<td>(2.25 - 3.75)</td>
<td>(5 - 5)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(1.5 - 3)</td>
<td>(5 - 5)</td>
</tr>
<tr>
<td>P4a(i)</td>
<td>2</td>
<td>1.25</td>
<td>1.25</td>
</tr>
<tr>
<td></td>
<td>(2 - 3)</td>
<td>(2 - 3)</td>
<td>(2 - 3)</td>
</tr>
<tr>
<td>P4a(ii)</td>
<td>2</td>
<td>2.5</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>(2 - 3)</td>
<td>(2.25 - 4)</td>
<td>(2 - 3)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(4.5 - 5)</td>
<td>(5 - 5)</td>
</tr>
<tr>
<td>P4a(iii)</td>
<td>3</td>
<td>2.5</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>(2 - 4)</td>
<td>(2 - 3)</td>
<td>(2 - 3)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(4 - 5)</td>
<td>(5 - 5)</td>
</tr>
<tr>
<td>P4a(iv)</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>(2 - 3)</td>
<td>(2.25 - 3.75)</td>
<td>(1 - 4.5)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>(4 - 5)</td>
<td>(4 - 5)</td>
</tr>
<tr>
<td>P4b(i)</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>(2.25 - 4)</td>
<td>(2 - 3)</td>
<td>(2 - 3)</td>
</tr>
<tr>
<td>P4b(ii)</td>
<td>2</td>
<td>2.5</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>(2 - 3)</td>
<td>(2.25 - 4.75)</td>
<td>(2 - 3)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(4.5 - 5)</td>
<td>(5 - 5)</td>
</tr>
<tr>
<td>P4b(iii)</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>(2 - 4)</td>
<td>(2.25 - 3.75)</td>
<td>(2 - 3)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>(4 - 5)</td>
<td>(4 - 5)</td>
</tr>
<tr>
<td>P4b(iv)</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>(2.25 - 3.75)</td>
<td>(2 - 3)</td>
<td>(2 - 3)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>(4 - 5)</td>
<td>(4 - 5)</td>
</tr>
<tr>
<td>P5</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>(3 - 4.75)</td>
<td>(4 - 5)</td>
<td>(4 - 5)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(3 - 5)</td>
<td>(4 - 5)</td>
</tr>
<tr>
<td>P6(i)</td>
<td>3.5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>(2 - 4.75)</td>
<td>(2.75 - 5)</td>
<td>(2 - 3)</td>
</tr>
<tr>
<td>P6(ii)</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>(2 - 3.75)</td>
<td>(2.25 - 3.75)</td>
<td>(2 - 3.75)</td>
</tr>
<tr>
<td>P6(iii)</td>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>(3.25 - 4.75)</td>
<td>(3 - 3.75)</td>
<td>(3 - 3.75)</td>
</tr>
<tr>
<td>P7(i)</td>
<td>4</td>
<td>3.5</td>
<td>3.5</td>
</tr>
<tr>
<td></td>
<td>(1 - 3)</td>
<td>(2 - 3)</td>
<td>(2.5 - 3)</td>
</tr>
<tr>
<td>P7(ii)</td>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>(3 - 5)</td>
<td>(2 - 3)</td>
<td>(2 - 3)</td>
</tr>
<tr>
<td>P8</td>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>(4 - 5)</td>
<td>(5 - 5)</td>
<td>(4.25 - 5)</td>
</tr>
<tr>
<td>P9</td>
<td>4.5</td>
<td>4.5</td>
<td>4.5</td>
</tr>
<tr>
<td></td>
<td>(4 - 5)</td>
<td>(4.25 - 5)</td>
<td>(2.25 - 5)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(4 - 5)</td>
<td>(5 - 5)</td>
</tr>
</tbody>
</table>

Table 72: Changes (from pre to post-test) in the Median Difficulty Rating and in the Median Response Type, for the R&U and P&S phases of the Problem Solving Cycle
7.3.2 Changes in Total Score for the Planning and Solving Phase for All Participants

The total score for the planning and solving phase (TPS) of the problem solving cycle for each participant across the nine problems for both the pre and post-test was computed using SPSS 24.0 for Windows. Normality of the the distribution of the differences (TPS diff) in TPS from pre to post-test for the sample of 12 pre-service participants was assessed using the Shapiro-Wilk test. The Shapiro-Wilk test returned the output shown in Table 73.

<table>
<thead>
<tr>
<th>Shapiro-Wilk Statistic</th>
<th>df</th>
<th>Sig.(p-value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TPS diff</td>
<td>0.880</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 73: Shapiro-Wilk Test of Normality for TPS.

Since the p-value (sig. = 0.088), is greater than 0.05, the distribution of the differences from pre to post-test (TPS diff) is normally distributed. A Wilcoxon signed-rank test was conducted to determine if there was a difference in the median TPS score between the pre and post-test. The test \(Z = -1.926, p = 0.054\) showed that the increase in the median TPS score of the participants from median 24 (7.75)\(^69\) to median 29 (9.75) was not of statistical significance.

This increase (effect size \(r = 0.39\)) is shown in the boxplots in Figure 54. Participants P6 and P1 showed the highest increase in the planning and solving phase from pre to post-test, increasing 26 and 11 units respectively. Participant P5 showed the lowest increase, P5 actually decreased by 6 units. Possible reasons for this decrease were presented in section 7.2.

\(^{68}\)P13 not included as only 6 problems completed. Note that for the 6 problems, the TPS for P13 increased from 9 to 16.

\(^{69}\)The value in the brackets is the interquartile range
7.4 Solution and Checking: Pre-test to Post-test

The percentage of the pre-service teachers at each type of response for the solution and checking phase (Pre-assessment) of the problem solving cycle is shown in Table 7470.

This group of pre-service teachers demonstrated most difficulty in obtaining the solution to problems 5 and 8 with no participant obtaining the correct solution to problem 8 and 30.8% obtaining the correct solution to problems 5. This group had least difficulty in obtaining the correct solution to problem 7(i) with 83.4% the participants showing clear communication of a correct solution.

70For problems 7, 8 and 9, participant (P13) results are not included as P13 did not complete these problems in the post-test due to having to leave post-test for unexpected reasons.
Table 74: Percentage of the participants at each Response Type for the Solution and Checking Phase (Pre-test).

<table>
<thead>
<tr>
<th>Pre</th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>15.4</td>
<td>38.5</td>
<td>0</td>
<td>15.4</td>
<td>30.8</td>
</tr>
<tr>
<td>Problem 2</td>
<td>15.4</td>
<td>7.7</td>
<td>30.8</td>
<td>30.8</td>
<td>15.4</td>
</tr>
<tr>
<td>Problem 3</td>
<td>0</td>
<td>23.1</td>
<td>0</td>
<td>46.2</td>
<td>30.8</td>
</tr>
<tr>
<td>Problem 5</td>
<td>53.8</td>
<td>15.4</td>
<td>0</td>
<td>7.7</td>
<td>23.1</td>
</tr>
<tr>
<td>Problem 6</td>
<td>7.7</td>
<td>30.8</td>
<td>0</td>
<td>0</td>
<td>61.5</td>
</tr>
<tr>
<td>Problem 7</td>
<td>16.7</td>
<td>0</td>
<td>0</td>
<td>16.7</td>
<td>66.7</td>
</tr>
<tr>
<td>Problem 7(ii)</td>
<td>41.7</td>
<td>0</td>
<td>16.7</td>
<td>0</td>
<td>41.7</td>
</tr>
<tr>
<td>Problem 8</td>
<td>16.7</td>
<td>75</td>
<td>8.3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Problem 9</td>
<td>25</td>
<td>16.7</td>
<td>8.3</td>
<td>50</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 75: Percentage of the participants at each Response Type for the Solution and Checking Phase (Post-test).

<table>
<thead>
<tr>
<th>Post</th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>0</td>
<td>7.7</td>
<td>0</td>
<td>46.2</td>
<td>46.2</td>
</tr>
<tr>
<td>Problem 2</td>
<td>7.7</td>
<td>7.7</td>
<td>0</td>
<td>15.4</td>
<td>69.2</td>
</tr>
<tr>
<td>Problem 3</td>
<td>7.7</td>
<td>30.8</td>
<td>15.4</td>
<td>7.7</td>
<td>38.5</td>
</tr>
<tr>
<td>Problem 5</td>
<td>46.2</td>
<td>7.7</td>
<td>0</td>
<td>23.1</td>
<td>23.1</td>
</tr>
<tr>
<td>Problem 6</td>
<td>7.7</td>
<td>0</td>
<td>7.7</td>
<td>0</td>
<td>84.6</td>
</tr>
<tr>
<td>Problem 7</td>
<td>0</td>
<td>0</td>
<td>16.7</td>
<td>8.3</td>
<td>75</td>
</tr>
<tr>
<td>Problem 7(ii)</td>
<td>16.7</td>
<td>0</td>
<td>0</td>
<td>16.7</td>
<td>66.7</td>
</tr>
<tr>
<td>Problem 8</td>
<td>8.3</td>
<td>50</td>
<td>0</td>
<td>0</td>
<td>41.7</td>
</tr>
<tr>
<td>Problem 9</td>
<td>16.7</td>
<td>8.3</td>
<td>33.3</td>
<td>16.7</td>
<td>25</td>
</tr>
</tbody>
</table>

This group of pre-service teachers demonstrated most difficulty in obtaining the

---

71 For problems 7, 8 and 9, participant (P13) results are not included as P13 did not complete these problems in the post-test due to having to leave post-test for unexpected reasons.
solution to problems 8 and 9 with 41.7% obtaining the correct solution to these 2 problems. This group had least difficulty in obtaining the correct solution to problems 2 and 6, with 69.2% and 84.6% of the participants showing clear communication of a correct solution.

The percentage of the participants who gave a response of type 0 to type 1 decreased for problems 1, 2, 3, 5, 6, 7(i), 7(ii), 8 and 9. There was no change in the number of participants who gave a response of type 0 to type 1 for problems 5, 7(i), and (ii). The number of second year participants who formed a correct solution either clearly presented or justified increased for problems 1, 2, 3, 6, 8

<table>
<thead>
<tr>
<th>Pre to Post-test for All Participants</th>
<th>Type 0 - Type 1</th>
<th>Type 3</th>
<th>Type 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>↓ 53.9 → 7.7</td>
<td>↑ 15.4 → 46.2</td>
<td>↑ 30.8 → 46.2</td>
</tr>
<tr>
<td>Problem 2</td>
<td>↑ 23.1 → 15.4</td>
<td>↓ 30.8 → 15.4</td>
<td>↑ 15.4 → 69.2</td>
</tr>
<tr>
<td>Problem 3</td>
<td>↑ 23.1 → 38.5</td>
<td>↓ 46.2 → 7.7</td>
<td>↑ 30.8 → 38.5</td>
</tr>
<tr>
<td>Problem 5</td>
<td>↓ 69.2 → 53.9</td>
<td>↑ 7.7 → 23.1</td>
<td>–</td>
</tr>
<tr>
<td>Problem 6</td>
<td>↓ 38.5 → 7.7</td>
<td>–</td>
<td>↓ 61.5 → 84.6</td>
</tr>
<tr>
<td>Problem 7</td>
<td>↓ 16.7 → 0</td>
<td>↑ 16.7 → 8.3</td>
<td>↑ 66.7 → 75</td>
</tr>
<tr>
<td>Problem 7(ii)</td>
<td>↑ 41.7 → 16.7</td>
<td>↑ 0 → 16.7</td>
<td>↓ 41.7 → 66.7</td>
</tr>
<tr>
<td>Problem 8</td>
<td>↓ 91.7 → 58.3</td>
<td>–</td>
<td>↑ 0 → 41.7</td>
</tr>
<tr>
<td>Problem 9</td>
<td>↓ 41.7 → 25</td>
<td>↓ 50 → 16.7</td>
<td>↑ 0 → 25</td>
</tr>
</tbody>
</table>

Table 76: Change in the percentage of participants at each Response Type for the Solution and Checking Phase (from Pre to Post-test). The number before the → is the percentage of participants at this type(s) in the pre-test and the number after the → is the percentage of participants at this type(s) in the post-test. The symbol – represents no change.

The percentage of the second year participants who gave a response of type 0 to type 1 decreased for problems 1, 3, 5, 6, 8 and 9. There was no change in the number of participants who gave a response of type 0 to type 1 for problems 5, 7(i), and (ii). The number of second year participants who formed a correct solution either clearly presented or justified increased for problems 1, 2, 3, 6, 8
There was no change in the number of type 4 responses given to problem 7(i). There was a decrease in the number of type 4 responses to problems 5 and 7(ii), however there was no overall change in the number of participants who obtained the correct solution to problem 5 and 7(ii).

The number of first year participants who formed a correct solution either clearly presented or justified increased for problems 1, 2, 5, 6, 7(i), 7(ii), 8 and 9. An example of an increase in the type of response (in the solution and checking phase) from a type 2 to type 4 is shown for problem 2 (Figures 55 and 56 respectively). There was no change in the number of type 4 responses given to problem 3.

The responses (pre and post-test) by P11 to problem 2 are shown in Figures 55 and 56 respectively.

In the pre-test P11 equated times to find $t = 12$ hours correctly but did not use this result to obtain the distance.

In the post-test, P11 wrote that Katie takes 4 hours to move a distance of $j$ km at a speed of $z$ km/hr and expressed $j$ in terms of $z$. Similarly he wrote that Shane takes 4 hours to move a distance of $h$ km at a speed of $y$ km/hr and expressed $h$ in terms of $y$. Diagram also drawn, used the diagram and written work to write that Shane completes distance $j$ in 2 hours and therefore $j = 2y$ and then $h = 4y = 8z$ and since Katie remains at $z$ km/hr for distance $h$, this will take 8 hours. Correct solution obtained and P11 gave a general proof as an alternative way of solving the problem, thus verifying his solution.
7.4.1 Changes in Total Score for the Solution and Checking Phase for All Participants

The total score for the solution and checking phase (TSC) of the problem solving cycle for each participant\(^{72}\) across the nine problems for both the pre and P13 not included as only 6 problems completed. Note that for the 6 problems, the TSC for P13 increased from 4 to 12.

\(^{72}\)P13
post-test was computed using SPSS 24.0 for Windows. Normality of the the
distribution of the differences (TSC diff) in TSC from pre to post-test for the
sample of 12 pre-service participants was assessed using the Shapiro-Wilk test.
The Shapiro-Wilk test returned the output shown in Table 77.

<table>
<thead>
<tr>
<th>Shapiro-Wilk</th>
<th>Statistic</th>
<th>df</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>TSC diff</td>
<td>0.974</td>
<td>12</td>
<td>0.944</td>
</tr>
</tbody>
</table>

Table 77: Shapiro-Wilk Test of Normality for TSC.

Since the p-value (sig. = 0.944), is greater than 0.05, the distribution of
the differences from pre to post-test (TSC diff) is normally distributed.

A Wilcoxon signed-rank test was conducted to determine if there was a dif-
fERENCE in the median TSC score between the pre and post-test. The test
\( Z = -2.491, p = 0.013 \) showed that there was a statistically significant in-
crease in the median TSC score of the participants from median 12 (5.75)\(^73\) to
median 18 (7.5).

This increase (effect size \( r = 0.51 \)) is shown in the boxplots in Figure 57. Over
75% of the participants in the post-test scored higher than the median TSC
score in the pre-test. Participants P6 and P1 showed the highest increase in the
solution and checking phase from pre to post-test, increasing 16 and 10 units
respectively. Participant P5 showed the lowest increase, P5 actually decreased
by 4 units. The possible reasons for this slight decrease have been discussed.

\(^73\)The value in the brackets is the interquartile range

Figure 56: Response by P11 to Problem 2 in post-test
7.5 Summary of Post-Test Findings

Note\textsuperscript{74}

\textsuperscript{74}The numbers in the brackets are the end points of the interquartile range ($Q_1 - Q_3$)
Table 78: Changes (from pre to post-test) in the Median Difficulty Rating and in the Median Response Type, for each Phase of the Problem Solving Cycle

As seen in Table 78, where the median difficulty rating of a problem increased (from pre-test to post-test) the median response types given by the participants increased.

For problems 3 and 9, the problems were given the same median difficulty rating by the participants, the median response types given in the solution and checking phase decreased. The interquartile range for the median difficulty rating for
problem 3 changed from $2 - 3$ to $3 - 4$. This indicated that this problem was a little more difficult in the post-test. Both problems 3 and 9 were noted by the expert panel of mathematicians as being “a bit trickier” and “possibly more difficult” in the post-test.

As seen in Table 78, where the median difficulty rating of any other problem remained the same or decreased (from pre-test to post-test) the median response types given by the participants increased.

### 7.6 Mindset: Pre-test to Post-test

The effect of informing students about the growth mindset, on their level of perseverance and achievement in mathematics has been discussed in chapter 2 (section 2.4.5). A mindset questionnaire was given to the participants at the time of the pre-test, before the intervention. The same questionnaire was given to the participants after the intervention, when they completed the post-test. The participants were shown a presentation about mindset (Appendix U) in the first of the intervention classes. The results given in the mindset questionnaires were analysed using SPSS 24.0 for Windows. The results for the thirteen participants who completed the pre-test, intervention and post-test are shown in Table 79:

<table>
<thead>
<tr>
<th>Participant</th>
<th>Pre-intervention</th>
<th>Post-intervention</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>71</td>
<td>85</td>
</tr>
<tr>
<td>P2</td>
<td>61</td>
<td>66</td>
</tr>
<tr>
<td>P3</td>
<td>86</td>
<td>81</td>
</tr>
<tr>
<td>P4</td>
<td>72</td>
<td>80</td>
</tr>
<tr>
<td>P5</td>
<td>67</td>
<td>71</td>
</tr>
<tr>
<td>P6</td>
<td>74</td>
<td>77</td>
</tr>
<tr>
<td>P7</td>
<td>64</td>
<td>62</td>
</tr>
<tr>
<td>P8</td>
<td>73</td>
<td>75</td>
</tr>
<tr>
<td>P9</td>
<td>66</td>
<td>68</td>
</tr>
<tr>
<td>P10</td>
<td>72</td>
<td>81</td>
</tr>
<tr>
<td>P11</td>
<td>63</td>
<td>61</td>
</tr>
<tr>
<td>P12</td>
<td>72</td>
<td>87</td>
</tr>
<tr>
<td>P13</td>
<td>65</td>
<td>66</td>
</tr>
</tbody>
</table>

Table 79: Changes in the result of the mindset questionnaire for participants

The score of each individual is compared to the following scale\(^{75}\) to determine their mindset:

- **Strong Growth Mindset** $= 75 - 100$ points
- **Growth Mindset with some Fixed ideas** $= 56 - 74$ points

\(^{75}\)Adapted from mereworth.kent.sch.uk/wp-content/uploads/2015/04/Mindset-Quiz.pdf

337
• Fixed Mindset with some Growth ideas = 35 – 55 points
• Strong Growth Mindset = 0 – 34 points

The participants P1, P2, P4, P6, P8, P10 and P12 moved from a growth mindset with some fixed ideas, to a strong growth mindset. It was noted that P3, P7 and P11 decreased in the points scored, although remaining in the same mindset. All other participants increased in the points scored and remained in the same mindset.

Normality of the distributions (pre and post-test) of the scores for mindset, as well as the distribution of the differences (Mindset diff) in mindset from pre to post-test for the sample of 13 pre-service participants was assessed using the Shapiro-Wilk test. The Shapiro-Wilk test returned the output shown in Table 80.

<table>
<thead>
<tr>
<th>Shapiro-Wilk</th>
<th>Statistic</th>
<th>df</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mindset pre</td>
<td>0.896</td>
<td>13</td>
<td>0.117</td>
</tr>
<tr>
<td>Mindset post</td>
<td>0.940</td>
<td>13</td>
<td>0.458</td>
</tr>
<tr>
<td>Mindset diff</td>
<td>0.952</td>
<td>13</td>
<td>0.633</td>
</tr>
</tbody>
</table>

Table 80: Shapiro-Wilk Test of Normality for Mindset.

Since the p-values (Sig. = 0.117, 0.458 and 0.633), are greater than 0.05, the distributions of the scores for mindset (pre-intervention, post-intervention and difference from pre to post-intervention) are normally distributed.

A Wilcoxon signed-rank test was conducted to determine if there was a difference in the median mindset score between the pre and post-test. The test \(Z = -2.104, p = 0.035\) showed that there was a statistically significant increase in the median mindset score of the participants from median 71 (8)\(^{76}\) to median 75 (15).

This increase is shown in the boxplots in Figure 58.

\(^{76}\)The value in the brackets is the interquartile range
Figure 58: Total Mindset pre and post-test

The median response type per statement\textsuperscript{77} given by the participants to the mindset questionnaire (pre and post intervention) is shown in the bar chart in Figure 59.

There was a change in the median response type given to the following statements in the mindset questionnaire:

- Q8 Mathematics is much easier to learn if you are male or maybe come from a culture who value mathematics. Median changed from 2 (2)\textsuperscript{78} to 1 (0.5);
- Q9 The harder you work at something, the better you will be at it. Median changed from 5 (1) to 6 (0.5);
- Q11 Trying new things is stressful for me and I avoid it. Median changed from 3 (1) to 2 (1.5);
- Q12 Some people are good and kind, and some are not, it is not often that people change. Median changed from 3 (1.5) to 2 (2);
- Q15 All human beings without a brain injury or birth defect are capable of the same amount of learning. Median changed from 3 (1.5) to 4 (1);

\textsuperscript{77}F = Fixed Mindset Statement, G = Growth Mindset Statement, \textsuperscript{1}= Strongly Disagree - \textsuperscript{6}= Strongly Agree

\textsuperscript{78}the number in brackets is the interquartile range
• Q17 You can do things differently, but the important parts of who you are can’t really be changed. Median changed from 4 (2) to 3 (1);

• Q18 Human beings are basically good, but sometimes make terrible decisions. Median changed from 4 (1) to 5 (1);

• Q19 An important reason why I do my college work is that I like to learn new things. Median changed from 5 (1) to 4 (1).

The median response to all other statements remained the same.
7.7 Individual Results for each Participant

This section discusses the results of each individual participant in both the pre and post-test. The total score (TS) for each participant in each of the assessments is:

\[
TS = R\&U + P\&S + P\&S \text{ (for Problem 4)}
\]

The maximum score that can be achieved is 52 + 36 + 6 = 94. The score and percentage for each of the participants is shown in Table 81.

<table>
<thead>
<tr>
<th>Participant</th>
<th>Pre-intervention</th>
<th>Post-intervention</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total Score (TS)</td>
<td>Percentage</td>
</tr>
<tr>
<td>P1</td>
<td>43 + 20 + 2 = 65</td>
<td>69%</td>
</tr>
<tr>
<td>P2</td>
<td>31 + 21 + 4 = 56</td>
<td>60%</td>
</tr>
<tr>
<td>P3</td>
<td>51 + 29 + 5 = 85</td>
<td>90%</td>
</tr>
<tr>
<td>P4</td>
<td>48 + 28 + 5 = 81</td>
<td>86%</td>
</tr>
<tr>
<td>P5</td>
<td>47 + 28 + 3 = 78</td>
<td>83%</td>
</tr>
<tr>
<td>P6</td>
<td>34 + 1 + 3 = 38</td>
<td>40%</td>
</tr>
<tr>
<td>P7</td>
<td>42 + 22 + 4 = 68</td>
<td>72%</td>
</tr>
<tr>
<td>P8</td>
<td>42 + 20 + 5 = 67</td>
<td>71%</td>
</tr>
<tr>
<td>P9</td>
<td>41 + 25 + 6 = 72</td>
<td>77%</td>
</tr>
<tr>
<td>P10</td>
<td>40 + 26 + 5 = 71</td>
<td>76%</td>
</tr>
<tr>
<td>P11</td>
<td>48 + 29 + 5 = 82</td>
<td>87%</td>
</tr>
<tr>
<td>P12</td>
<td>42 + 23 + 4 = 69</td>
<td>73%</td>
</tr>
<tr>
<td>P13</td>
<td>26 + 9 + 4 = 39</td>
<td>63%</td>
</tr>
</tbody>
</table>

Table 81: Changes in the result of the problem solving assessment for participants

There was an increase in the total score of each participant, except for P5, and P10, both of the total scores for P5 and P10 decreased.

---

79P1 = Participant 1; Gender: 0 = Male, 1 = Female; L.C = Leaving Certificate grade achieved: HC1 = Higher Level C1 grade. Time = Time (in minutes) spent on the assessment, by the participant. R&U = Read and Understand, P&S = Plan and Solve, S&C = Solution and Checking, G(i-ii) = Generalising from part (i) to part (ii) for Q3. E(iii) = Explanation for part (iii) in Q6. Q4* = For Q4: R&U = Understand and Represent the problem situation, P&S = Understand the relationship between the symbolic representation of the problem situation and the problem situation, and Solve. See appendix (K AND L) for description of type of solution.

80It should be noted that the author originally checked the sum across each of the three phases and the results (including the total score for solution and checking (TSC)) were the same in terms of statistical significance as the results (not including the total score for solution and checking (TSC)).

81For P13, the total score is on the first six problems. The maximum score that can be achieved is 36 + 20 + 6 = 62
The total score (TS) for all three phases of the problem solving cycle for each participant across the nine problems for both the pre and post-test was computed using SPSS 24.0 for Windows. Normality of the distribution of the differences (TS diff) in TS from pre to post-test for the sample of 12 pre-service participants was assessed using the Shapiro-Wilk test. The Shapiro-Wilk test returned the output shown in Table 82.

<table>
<thead>
<tr>
<th>Shapiro-Wilk</th>
<th>Statistic</th>
<th>df</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>TS diff</td>
<td>0.875</td>
<td>12</td>
<td>0.075</td>
</tr>
</tbody>
</table>

Table 82: Shapiro-Wilk Test of Normality for TS.

Since the p-value (sig. = 0.075), is greater than 0.05, the distribution of the differences from pre to post-test (TSC diff) is normally distributed.

A Wilcoxon signed-rank test was conducted to determine if there was a difference in the median TS score between the pre and post-test. The test \( Z = -2.473, p = 0.013 \) showed that there was a statistically significant increase in the median TS score of the participants from median 74.5 (15.75)\(^83\) to median 85 (14.25).

This increase (effect size \( r = 0.50 \)) is shown in the boxplots in Figure 60. Over 75% of the participants in the post-test scored higher than the median TS score in the pre-test.

The actual response types of each participant are presented in section 7.7.2 and section 7.7.3. Table 98 summarises the changes in the understanding/planning/solving of the participant. For clarification on any of the response types, see the scoring frameworks (Appendix K and M).

\(^{82}\)P13 not included as only 6 problems completed. Note that for the 6 problems, the TS for P13 increased from 39 to 50.

\(^{83}\)The value in the brackets is the interquartile range.
Correlation between Changes in Mindset Score and Changes in Problem Solving Score from Pre to Post-Test

The author checked if there was a correlation between the changes in the mindset scores and the changes in the problem solving scores from pre to post-test. There was no significant correlation between these changes (Spearman’s rho\textsuperscript{84} $r_s = 0.158$, p-value = 0.623). However it was noted that of the top three improvements in the total score in problem solving from pre to post-test, two of these (participants P1 and P12) showed the highest increase in mindset scores, while all three (participants P1, P6 and P12) increased from a growth mindset with some fixed ideas to a strong growth mindset.

\textsuperscript{84}Spearman’s rank correlation coefficient is a statistical measure of the strength of a monotonic relationship between paired data. The closer $r$ is to ±1, the stronger the monotonic relationship. Spearman’s rank correlation is a non-parametric test.
7.7.1 Time: Pre-test to Post-test

The pre and post-tests in problem solving were not subject to a time con-
straint. The time (in minutes) that each participant spent working on the pre and post-tests is shown in Table 83.

<table>
<thead>
<tr>
<th>Participant</th>
<th>Pre-test</th>
<th>Post-test</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>111</td>
<td>178</td>
</tr>
<tr>
<td>P2</td>
<td>98</td>
<td>168</td>
</tr>
<tr>
<td>P3</td>
<td>103</td>
<td>168</td>
</tr>
<tr>
<td>P4</td>
<td>122</td>
<td>148</td>
</tr>
<tr>
<td>P5</td>
<td>59</td>
<td>148</td>
</tr>
<tr>
<td>P6</td>
<td>87</td>
<td>115</td>
</tr>
<tr>
<td>P7</td>
<td>87</td>
<td>95</td>
</tr>
<tr>
<td>P8</td>
<td>87</td>
<td>135</td>
</tr>
<tr>
<td>P9</td>
<td>87</td>
<td>115</td>
</tr>
<tr>
<td>P10</td>
<td>87</td>
<td>135</td>
</tr>
<tr>
<td>P11</td>
<td>92</td>
<td>145</td>
</tr>
<tr>
<td>P12</td>
<td>115</td>
<td>145</td>
</tr>
</tbody>
</table>

Table 83: Changes in the time (in minutes) spent by participants working on the pre and post-tests.

All of the participants spent more time engaged in the post-test than they did in the pre-test. The increase in the time (from the pre to post-test) spent engaged on the tests suggests an increase in perseverance among the participants.

Normality of the distributions (pre and post-test) of the time spent by participants completing the tests, as well as the distribution of the differences (diff) in the time spent by the participants from pre to post-test for the sample of 12 pre-service participants was assessed using the Shapiro-Wilk test. The Shapiro-Wilk test returned the output shown in Table 84.

<table>
<thead>
<tr>
<th>Shapiro-Wilk</th>
<th>Statistic</th>
<th>df</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time pre</td>
<td>0.917</td>
<td>12</td>
<td>0.259</td>
</tr>
<tr>
<td>Time post</td>
<td>0.955</td>
<td>12</td>
<td>0.709</td>
</tr>
<tr>
<td>Time diff</td>
<td>0.962</td>
<td>12</td>
<td>0.806</td>
</tr>
</tbody>
</table>

Table 84: Shapiro-Wilk Test of Normality for Time.

Since the p-values (Sig. = 0.259, 0.709 and 0.806), are greater than 0.05, the distributions of the (pre-intervention, post-intervention and difference from pre

---

85 There was no time limit given for completing these tests (section 3.8.2).
86 P13 not included as only 6 problems completed in the post-test. Note that P13 spent 95 minutes working on the problems he completed in the post-test and 87 minutes working on the pre-test.

344
A paired t-test was conducted to determine if there was a difference in the mean score in the time the participants spent engaged in the pre and post-tests. The results \( t(12) = -6.906, p = 0.000 \) show a statistically significant increase in the mean time the participants spent engaged in the pre-test to post-test from 94.58 ± 16.74 minutes to 141.25 ± 24.3 minutes.

This increase is shown in the boxplots in Figure 61.

Figure 61: Total Time pre and post-test

The time each of the participants spent engaged in the post-test exceeded the median time that this group of participants spent engaged in the pre-test.
Correlation between Changes in Mindset Score and Changes in Time spent on Pre to Post-Test

The author checked if there was a correlation between the changes in the mindset scores and the changes in the time spent by participants engaged in the problem solving tests (from pre to post-test). There was no significant correlation between these changes (Spearman's rho\textsuperscript{87} \( r_s = 0.109 \), p-value = 0.736). However it was noted that of the top three increases in the time spent by participants engaged in the problem solving tests (from pre to post-test), one of these (participants P1) showed the second highest increase in mindset score.

Perseverance

The increase in the time spent working on the problems from the pre to post-test indicates an increase in the level of perseverance among the participants. The participants confirmed this increase in perseverance through their behavior, and comments they made while participating in the intervention and the focus group. During the intervention, it was noted that the participants' levels of enjoyment and interest were high and that perseverance levels seemed to be increasing (Researcher Journal). Changes were noted from the second class of the intervention. The participants left the first class at the time of finishing the class, whereas in subsequent classes it was noted that the participants remained in the class (still working on the problems) after the time for the class was over (Researcher Journal).

At the end of class three, participant P10 stated that she “always nearly forgets about time” when she’s in the class solving problems, that “it’s hard to stop doing them” (solving the problems) and that she “just wants to keep going with them” (Researcher Journal). Also at the end of the third class, the participants P10, P11 and P12 revealed that they learned a lot from solving problem 18 which they all found very difficult. The participants were given some problems to complete during each class and were also asked to choose a problem to complete. P10 choose to work on problem 16 towards the end of class 3. She discussed the difference in the growth of the exponential and linear relationships in the problem and revealed that she had seen the movie and found the problem interesting (Researcher Journal).

P10 stated that what she liked best about the intervention was “when you’re stuck on a really hard question for ages and then something just clicks and then you just get it”. P12 made note of the sense of “fulfillment” that comes from having “been stuck on a problem for like fifty minutes and then you actually get it and everyone has this massive epiphany moment” (Focus Group). P6 noted that “it wouldn’t have mattered as much, you wouldn’t have gotten the sense

\textsuperscript{87}Spearman’s rank correlation coefficient is a statistical measure of the strength of a monotonic relationship between paired data. The closer \( r \) is to \( \pm 1 \), the stronger the monotonic relationship. Spearman’s rank correlation is a non-parametric test.
of achievement” if someone had given her the answer, with P12 agreeing “yeah there isn’t as much fulfillment for when you’re given the answer” (Focus Group).

P8 stated that participating in the intervention showed her that “if you stayed going that we would kind of finally get an answer” and made reference to the fact that while at school she was encouraged to move onto a question that she could get marks in and not to stay going. The act of continuing on at the problems in the intervention even when she (P8) was stuck facilitated the development of her perseverance. The fact that this behavior of perseverance was expected in the class meant that the participants simple engaged in what was expected, (working in their groups also reinforced this expectation of perseverance) (Researcher Journal).

P1 mentioned that participating in the intervention was “the most enjoyable part of maths that I do” (Focus Group). He “enjoyed actually doing out the problems” (on the white board and in groups) and there “was no pressure in it because you were able to trouble shoot and bounce things off other people, I was learning from P5 and P4 and it definitely helped me” (Focus Group). He also stated that the intervention allowed him to get practice at problem solving, including “practice with not getting the problems correct and being ok with not getting them correct and then working on that” to improve (Focus Group). P12 linked confidence and perseverance saying that “persistence is actually part of it, like you’re more confident in staying with it (the problem) past a certain amount of time just because you know you’ll probably get it out eventually” (Focus Group).

From the behavior and comments of the participants, it seems that interest, enjoyment, the sense of individual and shared achievement that comes from persevering on very difficult problems, and the high expectations of both the tutor and the participants of the learning environment facilitated the participants’ growth in perseverance. The lack of pressure mentioned by P1, in addition to the linking of confidence and perseverance by P12 are important comments. There was mutual respect between the tutor and the participants and between the participants themselves. This facilitated a learning environment conducive to growth in confidence and perseverance in problem solving.

The next section presents the types of responses given by each participant to both the pre and post assessments. The author first presents the types of responses for the five second year pre-service teachers (coded P1-P5). The author then presents the types of responses of the eight first year pre-service teachers (coded P6-P13). The pre and post-test were not identical tests but were judged

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88 Not explicitly stating that this was in an exam situation but this was implied by the context in which this comment was made.
89 The author used the rubrics and template (Appendix S) to write a detailed qualitative summary of the main increases/decreases in problem solving performance from the pre to post-test for each participant. The detailed results are available in the author’s notes.
by an expert panel of mathematicians to be comparable on the type of thinking needed to solve the problems and on the difficulty levels of the tasks. The author presents a summary (Table 98) of the problem solving skills and abilities which the participants have developed (from pre to post-test), in addition to showing improvements/regressions in the participants’ performance in the phases of the problem solving cycle from the pre to post-test.

### 7.7.2 Second year pre-service teachers 2015

#### Participant 1

<table>
<thead>
<tr>
<th>Participant</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4*</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S7-P1 (Pre)</td>
<td>R&amp;U</td>
<td>5</td>
<td>5</td>
<td>5,5,5,4</td>
<td>0</td>
<td>4</td>
<td>5,5</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>5,4</td>
<td>1</td>
</tr>
<tr>
<td>L.C = HB1</td>
<td>S&amp;C</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>4,2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Time = 111</td>
<td>G(i-ii)</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>E(iii)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S7-P1 (Post)</td>
<td>R&amp;U</td>
<td>5</td>
<td>5</td>
<td>5,5,5,5</td>
<td>5</td>
<td>5</td>
<td>5,5</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S</td>
<td>5</td>
<td>5</td>
<td>6a</td>
<td>4</td>
<td>5</td>
<td>5,5</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>L.C = HB1</td>
<td>S&amp;C</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>4,3</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Time = 178</td>
<td>G(i-ii)</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>E(iii)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 85: Pre and Post-test Response Types given by Participant 1 (P1)

The overall score for P1 in the post-test was 20% higher than his pre-test score. There was no change in the response type given to problems 1, 3 and 9 from the pre to the post-test, however changes were noted in the solution method provided to problems 1 and 9. All changes are summarised in Table 98.

#### Participant 2

<table>
<thead>
<tr>
<th>Participant</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4*</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S3-P2 (Pre)</td>
<td>R&amp;U</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>5,3,1(a),1(a)</td>
<td>1</td>
<td>5</td>
<td>5,5</td>
<td>1</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S</td>
<td>2</td>
<td>4</td>
<td>5*</td>
<td>5</td>
<td>0</td>
<td>5</td>
<td>5,5</td>
<td>0</td>
</tr>
<tr>
<td>L.C = HD2</td>
<td>S&amp;C</td>
<td>1</td>
<td>2</td>
<td>3*</td>
<td>0</td>
<td>4</td>
<td>4,4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Time = 98</td>
<td>G(i-ii)</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>E(iii)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S3-P2 (Post)</td>
<td>R&amp;U</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>5,5,5,5</td>
<td>4</td>
<td>5</td>
<td>5,5</td>
<td>2</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>6(a)</td>
<td>1(a)</td>
<td>5</td>
<td>5,5</td>
<td>1</td>
</tr>
<tr>
<td>L.C = HD2</td>
<td>S&amp;C</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4,4</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Time = 168</td>
<td>G(i-ii)</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>E(iii)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 86: Pre and Post-test Response Types given by Participant 2 (P2)
The overall score for P2 in the post-test was 12% higher than his pre-test score. There was no change in the response type given to problems 1, 6 and 7 from the pre to the post-test, however in problem 7, P2 used pictures in the post-test whereas he didn’t in the pre-test, and changes were also noted in the solution provided to problem 1. All changes are summarised in Table 98.

Participant 3

<table>
<thead>
<tr>
<th>Participant</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4⁺</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S2-P3 (Pre)</td>
<td>R&amp;U</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5,5,5,4,</td>
<td>5</td>
<td>5</td>
<td>5,5</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 1</td>
<td>P&amp;S</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>6(a)</td>
<td>2</td>
<td>3</td>
<td>5,5</td>
<td>3</td>
</tr>
<tr>
<td>L.C = HA2</td>
<td>S&amp;C</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>1(e)</td>
<td>4,4</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Time = 103</td>
<td>G(i-ii)</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>E(iii)</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| S2-P3 (Post)| R&U | 5  | 5  | 5  | 5,4,4,4, | 5  | 5  | 5,5 | 5  | 5  |
| Gender = 1  | P&S | 5  | 5  | 5  | 4   | 5  | 5  | 5,5 | 5  | 5  |
| L.C = HA2   | S&C | 3  | 4  | 3  | 3   | 4  | 4,4 | 4  | 4  | 4  |
| Time = 168  | G(i-ii) | 3 | 3  | 3  | 3   | 3  | 3  |
|             | E(iii) | 3 | 3  | 3  | 3   | 3  | 3  |

Table 87: Pre and Post-test Response Types given by Participant 3 (P3)

The overall score for P3 in the post-test was 4% higher than her pre-test score. There was no change in the response type given to problems 1, 2 and 7 from the pre to the post-test, however changes were noted in the solution provided to problem 1. All changes are summarised in Table 98.

Participant 4

<table>
<thead>
<tr>
<th>Participant</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4⁺</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S8-P4 (Pre)</td>
<td>R&amp;U</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>5,5,5,4,</td>
<td>5</td>
<td>5</td>
<td>5,5</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S</td>
<td>2</td>
<td>5</td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>5</td>
<td>5,5</td>
<td>3</td>
</tr>
<tr>
<td>L.C = HB2</td>
<td>S&amp;C</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>4,4</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Time = 122</td>
<td>G(i-ii)</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>E(iii)</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| S8-P4 (Post)| R&U | 3  | 5  | 5  | 5,5,4,4, | 5  | 5  | 5,5 | 5  | 5  |
| Gender = 0  | P&S | 3  | 5  | 5  | 4   | 5  | 5  | 5,5 | 3  | 5  |
| L.C = HB2   | S&C | 3  | 4  | 4  | 3   | 4  | 4,4 | 1  | 3  | 3  |
| Time = 148  | G(i-ii) | 3 | 3  | 3  | 3   | 3  | 3  |
|             | E(iii) | 3 | 3  | 3  | 3   | 3  | 3  |

Table 88: Pre and Post-test Response Types given by Participant 4 (P4)

The overall score for P4 in the post-test was 2% higher than his pre-test score. There was no change in the response type given to problems 7, 8 and 9 from the pre to the post-test. All changes are summarised in Table 98.
Participant 5

<table>
<thead>
<tr>
<th>Participant</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4*</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1-P5 (Pre)</td>
<td>3</td>
<td>5</td>
<td>5</td>
<td>4,3,5,5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 1</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>5(b)</td>
</tr>
<tr>
<td>L.C = HB3</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4,4</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Time = 59</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P&amp;S</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>5(b)</td>
</tr>
<tr>
<td>S&amp;C</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>G(i-ii)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E(iii)</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S1-P5 (Post)</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>5,4,3</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 1</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>6(b)</td>
<td>1(a)</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>L.C = HB3</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>4,0</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Time = 148</td>
<td></td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P&amp;S</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>6(b)</td>
<td>1(a)</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>S&amp;C</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>4,0</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>G(i-ii)</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<td></td>
<td></td>
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<tr>
<td>E(iii)</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 89: Pre and Post-test Response Types given by Participant 5 (P5)

The overall score for P5 in the post-test was 7% lower than her pre-test score. There was no change in the response type given to problem 6 from the pre to the post-test. It was noted that part (iii) was rated difficult in the pre-test and moderate in the post-test. All changes are summarised in Table 98.
7.7.3 First year pre-service teachers 2016

Participant 6

<table>
<thead>
<tr>
<th>Participant</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4*</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>P6</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5,5,5,4</td>
<td>3</td>
<td>4</td>
<td>2.0</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Gender = 1</td>
<td></td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>L.C = HC1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time = 87</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P6</td>
<td>5</td>
<td>5*</td>
<td>3</td>
<td>5,5,5,5</td>
<td>5</td>
<td>5</td>
<td>3(b),5</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Gender = 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L.C = HC1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Time = 115</td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Table 90: Pre and Post-test Response Types given by Participant 6 (P6)

The overall score for P6 in the post-test was 45% higher than her pre-test score. There was no change in the response type given to problem 9 from the pre to the post-test. P6 rated problem 9 in both pre and post-test as very difficult. All changes are summarised in Table 98.

Participant 7

<table>
<thead>
<tr>
<th>Participant</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4*</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>P7</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>5,5,4,1</td>
<td>5</td>
<td>5</td>
<td>5.0</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L.C = HA1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time = 87</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P7</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5,5,2,4</td>
<td>4</td>
<td>5</td>
<td>5.5</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 1</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>L.C = HA1</td>
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<td></td>
</tr>
</tbody>
</table>

Table 91: Pre and Post-test Response Types given by Participant 7 (P7)

The overall score for P7 in the post-test was 10% higher than her pre-test score. There was no change in the response type given to problems 1 and 7(i) from the pre to the post-test. There was a slight increase in problem 4. There was an increase in the types of responses provided to problems 2, 3, 6, 7(ii) and 9. All changes are summarised in Table 98.
Participant 8

<table>
<thead>
<tr>
<th>Participant</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4*</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>P8</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5,5,5,5</td>
<td>3</td>
<td>5</td>
<td>5,0</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L.C = HB1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Time = 87</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P8</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>5,5,4</td>
<td>3</td>
<td>5</td>
<td>5,5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<tr>
<td>L.C = HB1</td>
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</tr>
<tr>
<td>Time = 135</td>
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<td></td>
</tr>
</tbody>
</table>

Table 92: Pre and Post-test Response Types given by Participant 8 (P8)

The overall score for P8 in the post-test was 3% higher than her pre-test score. All changes are summarised in Table 98.

Participant 9

<table>
<thead>
<tr>
<th>Student</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4*</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>P9</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>2,5*,5*,1(a)</td>
<td>3</td>
<td>5</td>
<td>5,3(b)</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L.C = HB1</td>
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<td></td>
<td></td>
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<tr>
<td>Time = 87</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P9</td>
<td>5</td>
<td>5</td>
<td>3</td>
<td>5,5*,2,5*</td>
<td>5</td>
<td>5</td>
<td>5,5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 0</td>
<td></td>
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</tr>
<tr>
<td>L.C = HB1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time = 115</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 93: Pre and Post-test Response Types given by Participant 9 (P9)

The overall score for P9 in the post-test was 10% higher than his pre-test score. There was no change in the response types given to problem 6 from the pre to the post-test, except in the explanation provided which was better in the post-test. There were increases in the response type given in the R&U phase for problems 4, 5, 7(ii), in the P&S phase for problems 1, 2 and 8 and in the S&C phase of problems 8 and 9. All changes are summarised in Table 98.
Participant 10

Table 94: Pre and Post-test Response Types given by Participant 10 (P10)

The overall score for P10 in the post-test was 2% lower than her pre-test score. There was no change in the response types given to problem 6 from the pre to the post-test, except in the explanation provided which was better in the post-test. There was no change in the response type given to problem 1 from the pre to the post-test, however P10 used pictures in the post-test as well as solving the equations correctly to show that her solution to the equations was correct. She formed and solved the equations correctly for both the pre and post-test. All changes are summarised in Table 98.

Participant 11

Table 95: Pre and Post-test Response Types given by Participant 11 (P11)

The overall score for P11 in the post-test was 6% higher than his pre-test score. There was no change in the response type given to problems 1, 3, 5, 6, and 7 from the pre to the post-test, however changes were noted in the solution methods provided. All changes are summarised in Table 98.
Participant 12

<table>
<thead>
<tr>
<th>Student</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4*</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>P12</td>
<td>R&amp;U</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>5,4,5,5</td>
<td>5</td>
<td>5</td>
<td>0,0</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S</td>
<td>5</td>
<td>5</td>
<td>5*</td>
<td>5</td>
<td>2</td>
<td>5</td>
<td>0,0</td>
<td>3</td>
</tr>
<tr>
<td>L.C = HB1</td>
<td>S&amp;C</td>
<td>4</td>
<td>3</td>
<td>3*</td>
<td>1</td>
<td>4</td>
<td>0,0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Time = 115</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 96: Pre and Post-test Response Types given by Participant 12 (P12)

The overall score for P12 in the post-test was 15% higher than his pre-test score. There was no change in the response type given to problems 1 and 2 from the pre to the post-test, however changes were noted in the solution methods provided. All changes are summarised in Table 98.

Participant 13

<table>
<thead>
<tr>
<th>Student</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4*</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>P13</td>
<td>R&amp;U</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5,5,5,5</td>
<td>2</td>
<td>5</td>
<td>5,5</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S</td>
<td>5</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>0</td>
<td>5</td>
<td>5,5</td>
<td>5</td>
</tr>
<tr>
<td>L.C = HC1</td>
<td>S&amp;C</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>4,4</td>
<td>4*</td>
<td>4</td>
</tr>
<tr>
<td>Time = 87</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 97: Pre and Post-test Response Types given by Participant 13 (P13)

The overall score for P13 in the post-test was 18% higher than his pre-test score. There were increases in the R&U phase of problems 5 and 6 and in the P&S of problem 1. All changes are summarised in Table 98.

Table 98 summarises the problem solving skills and abilities which the participants have developed (from pre to post-test), in addition to showing improvements/regressions in the participants’ performance in the phases of the problems.

Based on scores to problems 1 to 6 only as P13 had to leave post-test early only got time.
problem solving cycle from the pre to post-test. Where abbreviations occur in the columns of Table 98, they are coded as follows:

- R&U - Read and understand.
- P&S - Plan and solve.
- S&C - Solution and checking.
- Math Proficiency - Mathematical Proficiency.
  - CU - Conceptual understanding
  - AR - Adaptive reasoning.
  - SC - Strategic competence.
  - PF - Procedural fluency.
- Math thinking - Mathematical thinking
- Commun. - Communication

The numbers presented in the R&U, P&S, and S&C cells of Table 98 show the percentage increase/decrease in the participants’ performance in these phases of the problem solving cycle. The positive + sign represents an increase, whereas the negative – sign represents a decrease, the number written under the sign shows the value of the percentage change from the pre to the post-test. For example, in the P&S cell for participant 2 (Pa2), the –, + with 5.6 and 16.7 underneath represents a decrease of 5.6% in the planning and solving phase from the pre to post-test for problems 1 – 3 and 5 – 9 collectively, and represents an increase of 16.67% in the planning and solving phase from the pre to post-test for problem 4.

The positive + signs in the cells for the other columns in Table 98 represent an improvement by that specific participant, in that particular competency. The writing under the + sign in these cells show the problems for which this improvement was evident. For example, participant 11 (Pa11) showed an improvement in mathematical thinking (from the pre-test to the post-test) in problems 2, 8 and 9.

The star * signs in the cells in Table 98 represent where further development is required by that specific participant, in that particular competency. The writing under the * sign in these cells show the problems for which this requirement was evident. For example, participant 4 (Pa4) demonstrated in his solutions to problems 1 and 8 that further development in his adaptive reasoning was required.

Any blank cells represent no change, also blanks within cells represent no change. For example participant 7 (Pa7) the (–,) with 16.7, * underneath represents a
decrease of 16.7% in the planning and solving phase from the pre to post-test for problems 1 – 3 and 5 – 9 collectively, and represents no change in the planning and solving phase from the pre to post-test for problem 4.

It can be seen in Table 98 that common areas of improvement from pre to post-test occurred in the following:

- Mathematical thinking;
- Problem solving skills;
- Problem solving ability;
- Use of heuristics;
- Procedural fluency;
- Adaptive reasoning;
- Conceptual understanding;
- Solution and checking phase;
- Reading and understanding phase.

It is evident in Table 98 that areas showing some improvement from pre to post-test include the following:

- Planning and solving phase;
- Strategic competence;
- Communication;
- Rigor;
- Efficiency of Solution method.
| Pa |  |  |  |  |  |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pa1 | + | + | + | + | + | + | + | + | + | + | + | + |
| Pa2 | + | - | - | + | + | + | + | + | + | + | + | + |
| Pa3 | - | + | - | + | + | + | + | + | + | + | + | + |
| Pa4 | + | - | + | 18.5 | 11.1, 33.3 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 |
| Pa5 | - | + | - | 14.8 | 16.7, 33.3 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 |
| Pa6 | + | + | + | 23.1 | 72.2, 33.3 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 |
| Pa7 | + | + | + | 5.8 | 16.7, * | 25.9 | + | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 |
| Pa8 | + | - | - | 7.7 | 2.8, * | 7.4 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 |
| Pa9 | + | + | + | 9.6 | 13.9, * | 33.3 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 |
| Pa10 | - | - | - | 6.9 | 11.1, 33.3 | + | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 |
| Pa11 | + | + | + | 5.8 | 11.1, 33.3 | 18.5 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 |
| Pa12 | + | + | + | 13.5 | 21.9, * | 22.2 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 |
| Pa13 | + | + | + | 5.6 | 35.3, 33.3 | 53.3 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 | P1,8 |

Table 98: Changes from Pre to Post-Test for each Participant (Pa = Participant, P = problem. *Denotes the percentage change for problem 4.)
7.8 Results of Post-Intervention Assessment Without Participation in Intervention

The participants F1, F3 and F4 completed both pre and post-tests but did not take part in the intervention classes (participation was voluntary). The results from the post-tests completed by these three pre-service teachers show that:

- the scores for all three over each of the phases of problem solving decreased;
- the ability to demonstrate correct generalisation and to provide an explanation decreased for two of the three pre-service teachers;
- the time spent engaged in problem solving decreased for all three (suggesting a decrease in perseverance).

Participant F1

<table>
<thead>
<tr>
<th>Student</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4*</th>
<th>Q5</th>
<th>Q6</th>
<th>Q7</th>
<th>Q8</th>
<th>Q9</th>
</tr>
</thead>
<tbody>
<tr>
<td>F1</td>
<td>R&amp;U</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>5,5,5,5,5*</td>
<td>3</td>
<td>5</td>
<td>5,5</td>
<td>5</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S</td>
<td>2</td>
<td>5</td>
<td>5*</td>
<td>4</td>
<td>1(a)</td>
<td>5</td>
<td>5,5</td>
<td>2</td>
</tr>
<tr>
<td>L.C = HC2</td>
<td>S&amp;C</td>
<td>1</td>
<td>4</td>
<td>4*</td>
<td>0</td>
<td>4</td>
<td>4,4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Time = 99</td>
<td>G(i-ii)</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F1</td>
<td>R&amp;U</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>5,5,5,2,2*</td>
<td>1</td>
<td>5</td>
<td>0,0</td>
<td>1</td>
</tr>
<tr>
<td>Gender = 0</td>
<td>P&amp;S</td>
<td>5</td>
<td>0</td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>5</td>
<td>0,0</td>
<td>0</td>
</tr>
<tr>
<td>L.C = HC2</td>
<td>S&amp;C</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>4*</td>
<td>0,0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Time = 73</td>
<td>G(i-ii)</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E(iii)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 99: Pre and Post-test Response Types given by Participant F1

Participant F1 achieved a score of $44 + 22 + 3 = 69$ marks in the pre-test which was 73%. In the post-test F1 achieved a score of $29 + 18 + 2 = 49$ marks in the post-test which was 52%. The result for F1 went down 21% from the pre to the post-test. F1 spent 26 mins less on the assessment in the post-test.
Participant F3

Table 100: Pre and Post-test Response Types given by Participant F3

Participant F3 achieved a score of $47 + 24 + 5 = 76$ marks in the pre-test which was 81%. In the post-test F3 achieved a score of $32 + 4 + 4 = 40$ marks in the post-test which was 42%. The result for F3 went down 39% from the pre to the post-test. F1 spent 25 minutes less on the assessment in the post-test.

Participant F4

Table 101: Pre and Post-test Response Types given by Participant F4

Participant F4 achieved a score of $46 + 22 + 5 = 73$ marks in the pre-test which was 78%. In the post-test F4 achieved a score of $31 + 18 + 1 = 50$ marks in the post-test which was 53%. The result for F4 went down 25% from the pre to the post-test. F1 spent 25 mins less on the assessment in the post-test.

7.9 Intervention Aims Achieved

The aims of the intervention are discussed in this section in relation to whether the developed and implemented intervention was successful in achieving these aims.

The aims of this intervention were to:
• immerse the pre-service teachers in a problem solving environment through the use of the Modified Moore method;

• increase the pre-service teachers’ awareness of metacognition, mindset and problem solving models;

• develop the mathematical thinking of the participants;

• develop the problem solving ability of participants;

• increase the perserverence levels of the participants in solving mathematical problems;

• develop the participants’ ability to teach and assess mathematical problem solving;

• provide the pre-service teachers with a resource pack which may be of use to them in their own teaching;

7.9.1 Problem Solving Environment

The Modified Moore Method integrated with the author’s F-TAPS provides a rich problem solving learning and teaching environment. The sections on metacognition, mindset and problem solving models, pre and post problem solving test results and comments made in the focus group all reflect the provision of a problem solving environment.

7.9.2 Increasing Awareness of Metacognition, Mindset and Problem Solving Models

Evident in the discussion on the solutions provided by the participants in both pre and post-tests, there has been an increase noted in the metacognition of the participants (evident in the increase in rigor (monitoring and regulation of the implementation of thought process)) in problem 8. The use of reflection by some of the participants on their own understanding, and the awareness of utilising reflection as a means of gaining understanding was seen to increase (participant P1 in class 5, stood looking at his solution to problem 33. When asked what he was doing, he replied that he just wanted to learn from his work so that he might be able to understand it better, in order to be able to use the knowledge and skill gained from solving the problem in the future (Researcher Journal)). Also P1 stated that he has “a better knowledge of where to apply certain skills”, while being aware that “it’s not where it needs to be but it has definitely improved a good bit” (Focus Group). The participants also displayed metacognitive awareness in the completion of the metacognitive journals.

The results of the mindset questionnaire from pre to post-test showed a statistically significant increase in the growth mindset of the participants (section 7.6). In class 1 with the first year group, a comment made by P6 (stating that
the instructor was only saying that the participants were capable of solving the homework set for them (without having to google the information) to make them feel better) was met with a response by another participant stating “that’s the fixed mindset now P6” (Researcher Journal). This comment demonstrates a correct awareness of the two mindsets.

From the individual analysis of the individual results of the participants in the pre and post-tests, it was seen that the use of heuristics and mathematical thinking improved in general from the pre to the post-test.

7.9.3 Development of Mathematical Thinking and Problem Solving Ability

The results of the assessments show a statistically significant increase in the demonstration of problem solving ability of the participants (section 7.7).

7.9.4 Increasing Perseverance

All participants showed an increase in the amount of time they spent solving the problems (section 7.7.1), with comments in the focus group reflecting this increase in perseverance, which participants in the first year group linked to an increase in confidence. Participant P12 stated that the confidence is part of the persistence, that “you’re more confident in staying with a problem past a certain amount of time because you know you’ll probably get it out eventually” (Focus Group). P6 stated that she had no prior experience (before the intervention) of “staying going” (Focus Group). Also noted in the classes (for both groups) was a tendency to continue working even after the class had finished. This tendency developed as the intervention progressed. In the third year group, it was noted that at the beginning of the classes (participants P1 and P3) in particular would be discussing their solutions or partial solutions (more so when they had spent considerable time on the problem and were not sure if the solution was correct or they knew they had not obtained the solution) to the homework problems.

7.9.5 Development of Ability to Teach and Assess Mathematical Problem Solving

Participants mentioned in the focus group (see section 7.10.2) that:

- guided questioning is good to assess for understanding;
- not giving the answers allowed them (the participants) to really think about what they knew and the approaches they were already aware of;
- presenting and explaining forces everyone to get involved and fully understand and is good for assessing understanding;
- survey of interests at the start of the year would be something they would do with their class to aid in using problems related to the students’ interests;
the use of specialising aids in generalising with more understanding, and
the use of practical tasks aids in developing understanding;

visualisation, using white boards and group work is something the third
year pre-service teachers would like to use in their own teaching;

allowing sufficient time for students to complete the tasks is important;

taking photos of the different groups’ approaches to the problems can build
up a bank of resources to aid with revision;

seeing different approaches to the same problem gives a wider perspective
to solving any problem, allowing the use of different lenses to view the
problem through.

This shows development of an ability to teach and assess problem solving. Participant P5 stated that she had already started using visual representations and specialising in her teaching to help students understand. This was a result of her not knowing where formulas derived from (stating that the intervention has helped herself), so expecting a student to know without specialising or using visual representations didn’t seem right to her now. The comments made by P6 shows an increase in awareness of the difficulties associated with certain aspects of problem solving and a development of the use of appropriate techniques to aid in the teaching of these aspects.

7.9.6 Provision of Resources

All students received booklets of the course notes and fully worked out solutions
to each problem (both those completed during the intervention and those that
were not undertaken during the intervention). Some of these solutions were the
participants’ own work which was referenced. The participants also received
the presentations on mathematical thinking and mindset. Copies of the pre and
post-tests with fully worked out solutions were also given to the participants
upon completion of all three phases of the research. These resources are a
valuable asset which can be used by the participants in their future teaching
and assessing of problem solving in mathematics. Both the problems and the
development of understanding of concepts revealed in the detailed solutions to
these problems are a useful resource for their future teaching careers.

7.10 Participants’ Response to Intervention

A focus group (Appendix I) was conducted with the participants after com-
pletion of the intervention. This was to produce data and insights from the
participants themselves through use of the interaction between the group (Mog-
gan, 1997). Participants P1, P2, P3 and P5 took part in the third year focus
group (An additional question was asked in a second focus group to which all
5 participants took part in). Participants P6, P8, P10 and P12 took part in
the first year focus group. These codes for the participants are the same as the
codes used to represent their pre and post-tests. The participants were chosen at random from the participants who completed the intervention for the first year group and all participants were asked to take part in the focus group for the third year group (due to the smaller number of third year participants). QSR NVivo 11 was used to code the data from the focus groups (which had been transcribed by the researcher) for analysis. The participants who had taken part in all three phases (pre-test, intervention and post-test) provided the data, which was objective. A source of potential bias was that the researcher conducted the focus group. Thematic analysis (section 3.10.3) of the data produced the following nodes:

- Participants’ opinion of Intervention (good or bad);
- Use in their own teaching and assessing (intervention resources, pedagogical approaches from intervention);
- Own learning (confidence, persistence, mathematics knowledge and understanding, problem solving);
- University course;
- Relevance of intervention to participants’ needs.

7.10.1 Participants’ Opinion of Intervention

The node ‘opinion of intervention’ was divided into two child nodes ‘good’ and ‘bad’. The ‘good’ node was further divided into 4 child nodes: ‘white boards’, ‘practice’, ‘group work’ and ‘questioning’ for the third year focus group as these were the themes that emerged from the third year focus group. The ‘good’ node was further divided into 5 child nodes ‘white boards’, ‘group work’, ‘questioning’, ‘different perspectives’ and ‘presenting and explaining’ for the first year focus group as these themes emerged from the first year focus group.

The opinions which reflected a positive experience of the intervention (11.39% and 7.38% of the first year and third year focus group discussions respectively) were:

---

91 Note that some of the comments made by the participants were coded in QSR NVivo 11 under more than one node where the comment was applicable to each node it was coded to.

92 All percentages used in these sections (discussing the nodes and child notes) refer to the percentage of time that the conversation in the focus group was focused on a particular aspect.

363
<table>
<thead>
<tr>
<th>Year</th>
<th>Comments</th>
</tr>
</thead>
</table>
| First | Problem solving in groups was helpful (4 references), the group work and presenting was helpful in gaining different perspectives and approaches to solving problems. The fulfillment gained from solving a difficult problem (5 references) was mentioned as a good point with P10 saying ‘I think the best part of it is when you’re stuck on like a really hard question for ages and then it’s just like something clicks and you get it’.

The presenting and explaining (6 references) was mentioned as being useful in developing understanding, seeing different perspectives and approaches to solving a problem, and promoting involvement in the group work. Questioning (4 references) was noted as being effective at helping to come out of a ‘tunnel vision’ approach, promoting self-questioning about the details of the problem, possible alternative approaches and also as being a good way to check for understanding.

This group also mentioned that they liked working on the white boards (2 references). This group mentioned that the intervention was enjoyable, definitely worth doing (with respect to gaining understanding, becoming better at mathematics and being helpful to their teaching practice) and was applicable to their needs. |
| Third | Working out the problems on the white board instead of a sheet of paper was noted as a good point by this group (2 references). It helped in the solving of the problem by being able to stand back to look at the work (Researcher’s journal) and they liked that if they made a mistake they could just rub it out as opposed to scribbling it out if they were working on paper. Working on the boards was also noted as being good practice for teaching (1 reference).

The questioning approach adopted by the researcher was noted as a good point of the intervention (2 references) with P5 mentioning that “you never gave us the answers which was very frustrating, but at the same time when we were really stuck you gave us enough guidance that we were able to do all the problems. The fact that you didn’t just hand us the answers and solutions made us really think about what we knew and the approaches we were already aware of”. The group work was noted as being helpful in gaining different perspectives and approaches to solving problems (3 references). |

No opinions that reflected a negative experience of the intervention were given
by the first year participants with P12 mentioning that “we all just had really high opinions of it” and P10 saying “it was really good, so I don’t know what to change”. An opinion by the third year group which reflected a negative experience of the intervention was they would like more individual problems to solve during the classes (4.27% of discussion (4 references)). The author used the Modified Moore Method as research has found that this method works better for undergraduate students than the Moore Method (which works well with postgraduate students). The third year group are closer to graduation than the first year group and so a slightly less modified Moore Method approach may work (more individual problem solving along with the group work). The homework aspect of the intervention was also noted as difficult given the pressure of their university course (third year group 1.7% of discussion (1 reference)).

It is evident that the good opinions of the intervention outweigh the bad. Another node which was coded in NVivo 11 was the use the participants may make of the intervention in their own teaching and assessing of problem solving.

7.10.2 Use in their own Teaching and Assessing

The node ‘Use in their own teaching and assessing’ (5.29% and 18.12% of the first year and third year focus group discussions respectively) was divided into two child nodes ‘intervention resources’ and ‘pedagogical approaches from intervention’. The possible/intended use of pedagogical approaches include:
<table>
<thead>
<tr>
<th>Year</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>The use of the questioning approach incorporated during the intervention (2 references), the group work (1 reference) and the presenting (2 references). P6 stated that “the group work and presenting, definitely would be good because that forces everyone to get involved, and understand whereas when I was in school, in maths class I didn’t really, I didn’t get involved. I just kind of sat there”.</td>
</tr>
<tr>
<td>Third</td>
<td>The presenting as a means of assessment for learning (1 reference), the specialising and generalising approach (with visual aids - tables and patterns) to generate formulae (1 reference) and the use of white boards with group work to help students who may be discouraged (2 references). Another pedagogical approach from the intervention which was mentioned was the use of a survey to gather information on the interests of their students so they could design problems to incorporate these interests (1 reference) which they feel would help with motivating their students to engage in problem solving. P1 also intends on using the idea (from the intervention) of taking pictures of the problem solving work completed by students in class to create a ‘bank of resources’ which would be helpful for revision and for seeing the different approaches taken to solving the same problems. P3 and P5 both mentioned that they would have been very quick to give students the answer and have now have learned to “hold back and not give the answer” as they see this as beneficial to the development of students’ understanding with P5 saying she would combine this “standing back” with the use of the questioning approach used in the intervention.</td>
</tr>
</tbody>
</table>

The possible/intended use of the intervention resources include:
<table>
<thead>
<tr>
<th>Year</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>Use of the problems provided in the course notes of the intervention. P10 mentioned that she liked the way the equations for the problems had to be formulated as part of solving the problem and not provided with the cue of saying ‘solve the simultaneous equations’. P10 found this “more applicable to the real world”.</td>
</tr>
<tr>
<td>Third</td>
<td>Use of the “real life problems” in addition to the use of a survey to determine interests which may help with motivation so that “it’s not just x and y, there’s a meaning behind the x and y” (P1). Another resource from the intervention that was mentioned was the use of little practical activities to help with solving problems. P3 liked the problem which made use of matchsticks to form the patterns in a sequence because she got to “do it rather than just looking at a page”.</td>
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</table>

The learning that occurred in respect to the teaching and assessing of problem solving is evident in the way in which the participants described the use they are intending of making of the resources and pedagogical approaches they experienced as a learner during the intervention. The next section looks at the impact the intervention had on the participants’ own learning in terms of confidence, perseverance, mathematics knowledge and understanding, and problem solving.
7.10.3 Own Learning

The node ‘Own Learning’ (13.2% and 20.91% of the first year and third year focus group discussions respectively) was divided into four child nodes ‘confidence’, ‘perseverance’, ‘mathematics knowledge and understanding’, and ‘problem solving’.

<table>
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<tr>
<th>Year</th>
<th>Comments on Confidence</th>
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<tr>
<td>First</td>
<td>All the first year participants said their confidence in problem solving in mathematics had increased with P6 saying “I wouldn’t panic now if I saw a question with all the words and stuff, like that with all those questions. I’d maybe sit there for a minute and read it and be like ok let’s break this down and do this whereas previously I would have been panicked and like oh shoot, let’s move onto the next question”. P12 felt that the increase in perseverance was a part of the increase in confidence stating that “you’re more confident in staying with it, past what may, like past a certain amount of time just because you know you’ll probably get it out eventually”. P8 made reference to being more confident in “being able to get the information out of the question, being able to put it from English into maths” as a result of the practice she had experienced in this during the intervention.</td>
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<tr>
<td>Third</td>
<td>The participants reported an increase in their confidence in problem solving (2 references) with P1 saying his confidence had increased overall (in all aspects of problem solving). The comment made by P1 was “It’s overall like, it’s just like problem solving for me just wouldn’t have been a strong point definitely for me before, like I’d have just been so procedure like, everything was just like I know a formula, I plug it into the formula like but when it comes to like actually having to do a bit of thinking with regards to like, I wouldn’t have been the best at it so from that point of view it’s been really really helpful”</td>
</tr>
<tr>
<td>Year</td>
<td>Comments on Perseverance</td>
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</table>
| First | All participants reported an increase in perseverance. A comment by P8 was “I became more like, what’s the word?, like is it persistent? Is that the word, like to staying going at problems?” also stating that “in the classes like it showed that if you stayed going that we would kind of finally get an answer, whereas the first time (the pre-test) we’d no experience of, well I’d no experience anyway of staying going”.

P10 agreed saying that in the post-test “my head was killing me but we were just, like you just really wanted to do it”. P12 agreed with P8 and P10 adding the comment made in relating perseverance to confidence “you’re more confident in staying with it, past what may, like past a certain amount of time just because you know you’ll probably get it out eventually”.

P1 said “you wouldn’t be as inclined to be worried about being frustrated with not getting the answer so you can persevere with it more”.

P5 agreed with P1 saying that “with problems I wouldn’t be as confident with, I’ve no problem sticking at a problem when I know how to do it but beforehand (before the intervention) I would have left the other problems (the problems she didn’t know how to solve) whereas now I’ll stick with them longer”.

P4 came to realise that “it is possible to spend ages looking at a problem and then figure it out after a while” whereas “before I might have thought that if I hadn’t solved it after 10 minutes then I’m never going to solve it”.

P2 reported that “probably I’d try more methods, before one thing would come into my head and if it didn’t work I’d just leave it. Now I’d try some other approaches”. P3 also made reference to trying more approaches.

Third | All participants reported an increase in perseverance, P1 said “you wouldn’t be as inclined to be worried about being frustrated with not getting the answer so you can persevere with it more”.

P5 agreed with P1 saying that “with problems I wouldn’t be as confident with, I’ve no problem sticking at a problem when I know how to do it but beforehand (before the intervention) I would have left the other problems (the problems she didn’t know how to solve) whereas now I’ll stick with them longer”.

P4 came to realise that “it is possible to spend ages looking at a problem and then figure it out after a while” whereas “before I might have thought that if I hadn’t solved it after 10 minutes then I’m never going to solve it”.

P2 reported that “probably I’d try more methods, before one thing would come into my head and if it didn’t work I’d just leave it. Now I’d try some other approaches”. P3 also made reference to trying more approaches. |
<table>
<thead>
<tr>
<th>Year</th>
<th>Comments on Mathematics Knowledge and Understanding</th>
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<tbody>
<tr>
<td>First</td>
<td>P8 felt her mathematics understanding had increased with respect to making equations that is “being able to get the information out of the question”. P10 made reference to the presentation on mathematical thinking being helpful in extracting the relevant information from a problem stating that “it’s easier to pick out what’s important, just from experience than you know if there’s any additional information whereas when you’re writing it out like sometimes you might think I need to use this, I need to use this but like it mightn’t be relevant to getting the answer at all”. P10 also stated that the intervention helped with her knowledge and understanding of Algebra. P6 said the intervention had helped with “reasoning out and stuff”.</td>
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<tr>
<td>Third</td>
<td>P5 said her knowledge of sequences and series had improved, the visual aspect of drawing out patterns and filling in tables had been very helpful to her in particular. Also the use of specialising and generalising has helped her to generate a formula or equation stating that she now uses this aspect to help her own students “you can see where the equation is really derived from rather than expecting a student to know how to derive that because I certainly didn’t and I’ve still, only from this (intervention) now has even helped me myself”. P1 realised the difficulty of trying to explain the solution to a problem “there’s a struggle to say what you’re actually doing, say I’ve come across a problem before of that I haven’t seen before and I solve it, it’s articulating how I solved it then is quite difficult like you’re trying to go back over it then and like how did I actually do it?”. This was an aspect he found surprising. P1 stated that his ability to make connections between the different areas of mathematics has improved saying “it just became a small bit more clear like when you’re problem solving”. P3 reported an increase in her understanding of mathematics in relation to her ability to view the individual components of a mathematics problem/proof as well as viewing the mathematics problem/proof in its entirety (this comment was made in relation to problem 54 in the course notes (finding a flaw in a proof)). P3 also reported an increase in her understanding of the concept of generality by having to show an equation is true for all cases as opposed to showing true for one case only.</td>
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</table>
### Year | Comments on Problem Solving
--- | ---
**First** | The comment made by P10 in the comments on mathematics knowledge and understanding reveal an improvement in problem solving. The comment made by P8 in the comments on mathematics knowledge and understanding on “being able to get the information out of the question” also show an improvement in problem solving in mathematics. P12 made this comment “I’ve gotten better at problem solving probably, I did the assessment and I felt like I did, I felt like it was easier to reason out some things in it than when I did it the first time anyways, just because I was able to, I asked myself, I guess I asked myself more questions”.

**Third** | A common theme for the third years was that the group work and the presenting of solutions facilitated the development of a “wider perspective coming into any problem now that you have different lenses to kind of come through” (P5) so that “if you’re not getting it one way, to look at a different approach” (P5). P1 stated that he was learning from the two other members of his group and that it “definitely helped me massively”. P5 agreed with P1 saying that “I just found I learned a lot from their approaches like, they took approaches sometimes that I’d never even dream of taking”. P1 stated that his ability to “actually having to do a bit of thinking” has improved and that his “skills set in problem solving is that bit better now, I’ve a better knowledge of where to apply certain skills, it’s not where it needs to be but it’s definitely improved a good bit.”. P3 made reference to having improved her knowledge of and application of the phases of problem solving and heuristics in “I wouldn’t have even thought about checking back like, working backwards”. The comments made in the comments on perseverance (on taking different approaches) also reveal an improvement in problem solving.

It is evident form the participants’ comments that an increase in confidence and perseverance, in addition to an improvement in mathematical knowledge and understanding, and in problem solving skills and abilities has occurred. The comment made by P1 in respect to an increase in perseverance “you wouldn’t be as inclined to be worried about being frustrated with not getting the answer so you can persevere with it more” suggests a shift away from some fixed mindset ideas to more of a growth mindset. Checking this in relation to the pre and post mindset score revealed that the change in the mindset score from pre to post-test for P1 was from a growth mindset with some fixed ideas to a strong growth mindset (section 7.6). The next node looks at whether this intervention
could have any impact on the participants’ university course.

7.10.4 University Course

<table>
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<tr>
<th>Year</th>
<th>Comments</th>
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<tbody>
<tr>
<td>First</td>
<td>The participants suggested having this intervention as an module or integrated as part of a slightly bigger module (P10 and P12 respectively). P8 stated that this intervention “is definitely more relevant than some of the stuff we’re doing”. P10 noted the relevance of this intervention (problems and finding own equations) to the Leaving Certificate course.</td>
</tr>
<tr>
<td>Third</td>
<td>The participants suggested having this intervention as an module as part of their university course (P3, with P1, P2 and P5 in immediate agreement). P1 said “this is us actually doing maths on a board like”. P5 stated that “I think it’s been one of the most beneficial pedagogy maths modules, this has really got us hands on, this is what we’re going to be doing this time next year like, we’re out teaching, we need to be practising that and working on approaches and if this was implemented as a module with broadening it out a bit more to hit more topics it’s a lot more hands on and beneficial I think, it leaves us a bit more prepared going out in my opinion anyway”.</td>
</tr>
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</table>

It is evident from the comments made by the participants that they think the intervention could have an impact on their course in university and feel it should be included as a module or part of a module. They have expressed the relevance that the intervention has to teaching the Leaving Certificate course and have stated their opinions on the benefits that this intervention has in relation to their teaching preparation. The next node looks at the relevance of the intervention to the participants’ needs.

7.10.5 Relevance of Intervention to Participants’ Needs

The comments made in relation to the confidence, perseverance, mathematical knowledge and understanding and problem solving, in addition to the comments made about the intended use of the resources and pedagogical approaches employed in the intervention suggest that the intervention has met the needs of the participants. Further comments made by the first year group was that this intervention is “far more applicable than a pure mathematics module” (P12). P10 stated that the problems were “a bit higher than Leaving Certificate” and that she could see some of the problems as potentially being on the Leaving Certificate examination paper as part of Section B (questions 7-9 usually which assesses context-based applications). P10 also stated that problem solving was becoming “more prominent in the Leaving Certificate, it’s just like not what
we’re doing in college (university)”. Further comments made by the third year group was that the need to be able to explain the solution to a problem is exactly what they need to be able to do for their students, they need to be able to explain the reason behind each part of the solution (P1). P5 (in relation to if this intervention was a compulsory module as part of their university course) stated that they “would be able to see the relevance and therefore there would be an awful lot more interest on our part seeing that this is actually really helpful and it’s preparing us to go out to teach”.

The next section uses this data in addition to the evaluation framework by Shapiro (1987) to evaluate the intervention.

7.11 Evaluation of the Intervention

Shapiro (1987) notes four critical components which need to be examined in order to evaluate intervention research, these components are:

- treatment effectiveness;
- treatment integrity;
- social validity;
- treatment acceptability.

Each of these four components were employed in the evaluation of the intervention aspect of this research. These components are discussed in sections 7.11.1-7.11.4.

7.11.1 Intervention Effectiveness

The effectiveness of an intervention is the “amount of change” or improvement in some aspect (which was the basis for the intervention), “evident among the participants after the intervention has been initiated, compared to a control group (group who have not participated in the intervention)” (Shapiro, 1987 p.290). The results of the pre and post assessments in problem solving as well as the pre and post assessments on mindset reveal that an improvement has taken place among the group who participated in the intervention. Intervention effectiveness also includes the speed at which the change occurs and the strength of this change (Shapiro, 1987). The effect sizes given for the changes that occurred indicate the strength of the changes. Changes were noted from the second class of the intervention. The participants left the first class at the time of finishing the class, whereas in subsequent classes it was noted that the participants remained in the class (still working on the problems) after the time for the class was over. The intervention was effective in:

- developing the problem solving ability of the participants (section 7.7). The effect size of the improvement was 0.50;
• improving the mathematical thinking of the participants - improvement in flexibility of deductive thinking (proof in problem 8), use of analogical reasoning evident in the post-test which wasn’t evident in the pre-test, improvement in specialising and generalising (problems 3, 6 and 8). Awareness also of dissecting and recombining - it was mentioned by participant P3 in the third year focus group that it was an area that had improved her understanding of mathematics;

• developing the ability of the participant in the teaching and assessing ability of problem solving - this was noted by their responses in the focus group - the third year group in particular (section 7.10.2) discussed ideas for teaching and assessing and also one participant (P5) stated that she was already using aspects of the intervention in her teaching;

• increasing the perseverance levels of the participants - the time spent solving problems increased for all of the participants, this was also confirmed in the focus groups;

• increasing the visualisation skills of participants - noticed in the intervention classes, post-tests and also mentioned in the focus group;

• increasing the growth mindset of the participants (Z = −2.104, p = 0.035 effect size=0.41);

• increasing the participants awareness of metacognition and problem solving models - noticed in the post-tests.

Ideally this change would be compared relative to a control group. A limitation for this study is that there were only three pre-service teachers who formed the control group. In comparison to the experimental group, it was noted that there were decreases in perseverance and problem solving ability of the limited control group (section 7.8).

7.11.2 Intervention Integrity

Intervention integrity is the “extent to which an intervention is delivered in adherence to its design features” (Zirkel & Thomas, 2010, cited in Kovaleski et al. 2017). Conclusions on the impact that an intervention has had on particular outcomes can only be made if the researcher shows that the intervention was implemented as intended (Gresham et al., 2000). The provision of the presentations (Appendix U) and class notes (Appendix W) used in the intervention classes, along with the results of the pre and post assessments on problem solving (Appendix K:L ), mindset (Appendix U) and the transcribed focus groups (Appendix I:R) demonstrate that the classes were run in adherence to the design features of the intervention. The pictures of students work also shows evidence of the implementation of the intervention as intended in its design.
Intervention integrity is important for the reproduction of the research study by others, with the documentation of the study being an important aspect in all research (Shapiro, 1987). The documentation of all aspects of this research study facilitates the reproduction of the intervention by other researchers. The detailed description of the organisation and delivery of the intervention in chapter six as well as the notes to instructors and participants in the book of course notes provided in Appendix W makes this study highly replicable by other researchers.

### 7.11.3 Intervention Acceptability

Intervention acceptability is the degree to which the participants themselves judge the intervention as being worthwhile/beneficial to them. That is how well was the intervention received by those who participated in it, how well did the intervention meet the needs of the participants (Shapiro, 1987). Key components involved in ensuring the acceptability of the intervention involved:

- an in depth review of existing literature and pre-assessment of the target participant group;
- time and cost of the intervention;
- method of delivery of the intervention;
- reproducibility of the intervention;
- integrity and effectiveness of the intervention.

The extensive review of literature informed the components of the framework for teaching and assessing problem solving. The pre-assessments demonstrated the needs of the target group; their problem solving ability, interests and current mindset. The interests of the target group was considered in order to tailor some of the problems to relate to their interests, and to comply with the author’s developed framework.

The time of the intervention was considered carefully. The researcher considered the time of similar interventions and looked at the changes that occurred in particular time frames for those interventions. Significant changes were found to have occurred in six weeks (Woodward et al., 2012). The researcher decided to conduct the intervention over the course of six weeks and to let the participants decide the time of the week that suited them best. The cost of the intervention to the participants was in terms of their time only as the intervention took place in the university they were attending. The cost to the researcher was the printing of the booklets, resources (manipulatives), a dictaphone, CDs to store the resources for the participants and refreshments for the participants (depending
on class time). These costs were minimal\footnote{The time in developing the resources was not minimal}. The method of delivery of the intervention was in adherence with the modified Moore Method as this was identified through research as being highly appropriate for developing problem solving ability and also its aims are in coherence with the Project Maths syllabus that the participants will be teaching on completion of their university course.

Reproducibility of this intervention by others is easily achievable due to the detailed documentation of all aspects of the intervention. This aspect has already been described in intervention integrity (section 7.11.2).

The effectiveness and integrity of the intervention were discussed in sections (7.11.1-7.11.2) and have been appropriately considered in ensuring acceptability of the intervention.

The researcher considered each of these key components in the design, structuring and delivery of the intervention to ensure acceptability. The adherence by the researcher to the researcher’s developed framework with the goal of developing mathematical proficiency of the participants and ensuring that they were exposed to an environment which was an accurate representation of a problem solving environment in mathematics was a strong point of this intervention. The focus group conducted with the participants revealed their acceptability of the intervention.

\section*{7.11.4 Social Validity}

Social validity refers to the degree to which the significance of the goals of the intervention, the appropriateness of the methods used in the intervention and the outcomes achieved by the intervention are deemed to be acceptable, relevant and useful by the participants (Shapiro 1987). The goals of the intervention included improving the participants mathematical thinking and overall proficiency in mathematics by engaging in a problem solving environment aimed also at developing their ability to teach and assess mathematical problem solving.

The comments made in the focus group revealed that the participants found the intervention relevant to their development as mathematics teachers. The references made in relation to the use they are making/intend to make of the resources and methods used in the intervention demonstrate their evaluation of the intervention as useful. The suggestions of including this intervention as part of their studies indicates the acceptability of the intervention by the participants.
7.12 Conclusion

This chapter provided the analysis of the findings and results from the post-tests. The results and findings showed that:

There was a statistically significant increase \((Z = -2.317, p = 0.020)\) in the median TRU score of the participants from 42 (7.5)\(^{94}\) to 46 (4.5) \((\text{effect size } r = 0.47)\). Over 75% of the participants in the post-test scored higher than the median TRU score in the pre-test.

There was an increase \((Z = -1.926, p = 0.054)\) in the median TPS score of the participants from median 24 (7.75)\(^{95}\) to median 29 (9.75) \((\text{effect size } r = 0.39)\) which was not of statistical significance.

There was a statistically significant increase \((Z = -2.491, p = 0.013)\) in the median TSC score of the participants from median 12 (5.75)\(^{96}\) to median 18 (7.5) \((\text{effect size } r = 0.51)\). Over 75% of the participants in the post-test scored higher than the median TSC score in the pre-test.

There was a statistically significant increase \((Z = -2.473, p = 0.013)\) in the median TS score of the participants from median 75 (15.75)\(^{97}\) to median 85 (14.25) \((\text{effect size } r = 0.50)\).

There was a statistically significant increase \((Z = -2.104, p = 0.035)\) in the median mindset score of the participants from median 71 (8)\(^{98}\) to median 75 (15) \((\text{effect size } r = 0.41)\).

The comments made in the focus group (section 7.10.5) in relation to the confidence, perseverance, mathematical knowledge and understanding, and problem solving, in addition to the comments made about the intended use of the resources and pedagogical approaches employed in the intervention reveal that the intervention was successful in meeting the needs of the participants as pre-service mathematics teachers.

The proof of concept has been demonstrated in this chapter. The problem solving ability, mathematical thinking, mindset, confidence, and perseverance of the pre-service mathematics teachers has improved. There is some indication of a larger gain in the problem solving ability among the participants with the lower pre-test scores. Evidence of the learning that occurred in respect to the teaching and assessing of problem solving is shown by the way the participants described the use they are intending of making of the resources and pedagogical approaches they experienced as a learner during the intervention. Although

\(^{94}\)The value in the brackets is the interquartile range
\(^{95}\)The value in the brackets is the interquartile range
\(^{96}\)The value in the brackets is the interquartile range
\(^{97}\)The value in the brackets is the interquartile range
\(^{98}\)The value in the brackets is the interquartile range
there were some decreases shown among two of the participants (P5 and P10 decreased by 7% and 2% respectively) the qualitative findings for both these participants showed increases in aspects of problem solving ability also, and both participants showed improvements in mindset, confidence, perseverance and demonstrated learning in relation to the teaching and assessing of problem solving. Three pre-service teachers who completed the pre test and post-test but did not participate in the intervention (F1, F3 and F4) showed decreases in problem solving ability (−21%, −39% and −25% respectively).

The findings and results of the analysis presented in this research demonstrates the feasibility of the F-TAPS and intervention in practice which gives credence to the F-TAPS and intervention also working on a larger scale.

A summary of the research conducted in this study is presented in the final chapter. A discussion of the conclusions are presented and the research questions that were posed at the beginning of this study are addressed. The contributions of this study to mathematics education are outlined and the author presents possible directions for future work.
8 Thesis Contributions and Future Work

8.1 Introduction
The research conducted during this study is summarised in this chapter. The research questions raised during each phase of the research are addressed systematically. Proof of concept for the author’s F-TAPS and intervention is established. Design principles used to develop the author’s F-TAPS in mathematics are presented. The author concludes the chapter by making recommendations based on the findings of this study, by discussing the contributions of this study to mathematics education, and by outlining possible directions for future work.

8.2 Summary
The main aim of this research was to develop a framework for teaching and assessing problem solving in mathematics which takes the cognitive and emotional factors of learners into account and which leads to the development of mathematical and problem solving proficiency.

The literature reviewed revealed the difficulties experienced among students and teachers with problem solving (section 2.3). Although general improvements in the performance of Irish students in the TIMSS and PISA assessments have been noted, there is persistent difficulty in assessment items that require the application of knowledge and problem solving skills in both mathematics and science (STEM Education Review Group, 2016). Students in Ireland are weaker than expected at formulating and representing, and planning and solving problem situations. The PISA results (OECD, 2014) showed that there are a significant number of 15 year old students who do not have the basic necessary problem solving skills in order to engage with unfamiliar problems, or to think even one step in advance. These basic skills are considered a requirement for successful participation in today’s world (OECD, 2014). Evidence from the wider literature on pre-service secondary teachers’ (n = 30) performance on two non-routine problem solving tasks (3.72 on a scale of (1 - 7)) and experience with solving such problems (12/30 stated they did not have enough practice with these types of problems), led Felmer and Diaz (2016) to recommend that action be taken on pre-service mathematics teacher education course. Felmer and Diaz (2016) also recommend professional development courses for teachers.

The author developed the F-TAPS in mathematics to improve problem solving among learners. The F-TAPS was integrated with the Modified Moore Method using the modified 4C-ID model to create an intervention that was implemented with 13 pre-service second-level mathematics teachers. The author decided to work with pre-service teachers in order to contribute to the research on “teachers as problem solvers” at secondary level and to aid in the possible translation of research in problem solving into improving school practice. The author’s review of the literature identified these areas as requiring further attention (Felmer and
The author’s F-TAPS in mathematics and its associated intervention was effective at improving the pre-service teachers’ problem solving ability and growth mindset. Findings from the focus groups and the increase in perseverance demonstrated by the pre-service teachers indicate that in general the intervention (and hence the author’s F-TAPS in mathematics on which it was based) was also effective in improving the pre-service teachers’ perseverance and their ability to teach and assess problem solving in mathematics. This demonstrated the functionality of the author’s F-TAPS in mathematics and intervention as a proof of concept by testing the performance of the F-TAPS, assessments and intervention on a small scale naturalistic setting (section 3.6.1). The next section revisits the research questions which were posed at the beginning of this study and the author now addresses those questions using findings gained through the completion of this research study.

8.3 Research Questions Revisited
The research questions which were posed during each phase of the research process are addressed in this section. The research questions were motivated by the various phases of the study and keyed to each phase.

8.3.1 Phases 1 and 2 Literature Review and the Formulation of the Framework

Q1. How should mathematics be taught and assessed in order to facilitate the development of learners’ problem solving proficiency in mathematics?

Resnick (1989, p.58) notes that the acquisition of mathematical habits such as interpretation and application of a rational thought process, play an equal if not more important role to the acquisition of knowledge, skills and strategies in the development of mathematical thinking. Krutetskii (1976) also agrees with Resnick in that the acquisition of this “mathematical cast of mind” is equally as important if not more than the acquisition of mathematical knowledge, skills and strategies. Krutetskii describes this “mathematical cast of mind” as the interpretation of the environment in a mathematical way; that is viewing phenomena from the perspective of logical and mathematical relationships. Individuals who possess this mathematical cast of mind have high levels of interest, concentration and perseverance in and for mathematics. An individual who possesses the combination of this mathematical cast of mind with the ability to obtain process and retain mathematical information is defined by Krutetskii to be mathematically gifted or very capable in mathematics. An individual who has the ability to obtain process and retain mathematical information but does not have this mathematical cast of mind is defined as being capable in mathe-
ematics or mathematically promising. Krutetskii (1976) notes the following as aspects involved in the processing of mathematical information:

- generalisation;
- flexibility;
- reversibility;
- logic;
- curtailing or compression of knowledge and;
- elegance of solution.

Krutetskii (1976) also stresses the importance of the learners’ emotions in the development of ability in any activity they are engaged in. He notes the significant role that the mathematical tasks play in the development of ability in mathematics. Good mathematical tasks are those that broaden students’ mathematical knowledge in a connected manner and which provoke questions (Hiebert and Wearne, 1993). Swan and Burkhardt (2012) classify a good mathematical task as one which assesses the mathematical proficiency that you want students to develop. Both mathematical content knowledge and problem solving processes should be assessed in the one task (Swan and Burkhardt, 2012).

The author created a framework for teaching and assessing problem solving in mathematics, which addresses the complexity of problem solving in mathematics, by integrating theories in relation to the discussed significant knowledge, and affective factors, which affect the growth of problem solving proficiency in individuals (Chapter 2). The proposed structure of the framework in Figure 17 and Figure 18 (Chapter 4) integrates knowledge and affective factors in the design and choice of mathematical problems (to be used in instruction and assessment), which are aimed at provoking mathematical thinking with the goal of developing mathematical proficiency in problem solving situations. Mathematical thinking (Tall 2001; Mason et al. 2010) is central in this framework. The author’s F-TAPS comprises 6 integrated components:

- knowledge;
- intrinsic motivation;
- mathematical thinking;
- mathematical proficiency;
- assessment;
- teaching strategies.
The author’s F-TAPS addresses the aspects involved in the processing of mathematical information through the central component of mathematical thinking. The author’s framework gives due attention to the different types of mathematical thinking, which facilitates learners’ recognition of mathematics in different areas. This encourages the development of the “mathematical cast of mind” (Krutetskii 1976).

Jung et al. (2007) state that when students’ intuitions are respected and appreciated and when they are encouraged to explain their solutions methods, as well as listening carefully to the solution paths of others, they naturally absorb more advanced methods of solving problems. Kaminski et al. (2009) found that although asking questions in context is a viable instrument for testing how well students have learned mathematics, teaching the concept using abstract symbols is more likely to result in transfer of mathematical knowledge in different problems and contexts. Kaminski et al. (2009) suggest that if learning concepts by employing contextual mathematics, then the ‘deep structural information’ of the mathematics problem should be made explicitly visible to students. The extra contextual information, which is not mathematically relevant may inhibit the learner from recognising relational similarities between analogous mathematical problems. In studies on self explanation, it has been shown that, regardless of the type of instruction, the use of self-explanation improved transfer among mathematics students (Atkinson et al., 2003; Wong et al., 2002). Encouraging students to self-explain while engaged in a discovery learning environment can improve procedural transfer on novel/unfamiliar problems (Siegler, 2002). Metacognition in the form of self explanation improved transfer among mathematics students (Atkinson et al., 2003; Wong et al., 2002). The author’s F-TAPS gives due consideration to intuition, metacognition, and the solving of contextual problems with similar structural information (chapter 4).

8.3.2 Phases 3-5 Designing the Assessment, Data Analysis (Pre-test), and Designing the Intervention

Q2(a). Who should the researcher implement the intervention with to best facilitate the transfer of research into practice?

The factors which can affect students’ ability to engage in problem solving were discussed in this research. These included both teacher and student knowledge ‘Teacher quality is believed to be one of the most important factors affecting students’ learning’ (Ní Riordáin and Hannigan, 2011, p.291). Foster et al. (2014) found that the teachers’ knowledge and ability to employ ‘effective strategies for teaching problem solving processes was particularly underdeveloped’ (p.7). Teachers’ understanding of process skills and their conceptualisation of what constitutes advancement in learning processes is ‘currently significantly under-developed’ (Foster et al., 2014, p.7). Evidence from the wider literature on pre-service secondary teachers’ (n = 30) performance on two non-routine problem solving tasks (3.72 on a scale of (1 - 7)) and experience with solving such
problems (12/30 stated they did not have enough practice with these types of problems), led Felner and Diaz (2016) to recommend that action be taken on pre-service mathematics teacher education course. The author noticed common factors between the aspects of the maths syllabus that the teachers found challenging and the deficiencies which were evident in students’ work (section 2.1) and chose to address the difficulties that students experience in problem solving through action on pre-service mathematics teacher education course in order to best facilitate the transfer of research into practice.

**Q2(b). What activities and teaching approaches should be used in the intervention?**

The author identified mathematical thinking as central to developing proficiency in problem solving. Resnick (1989 p.58) notes that the acquisition of ‘mathematical habits such as interpretation and application of a rational thought process’, play an equal if not more important role to the acquisition of knowledge, skills and strategies in the development of mathematical thinking. Resnick (1987) in an examination of numerous programs that claimed to teach thinking skills or higher order cognitive abilities identified three key features common to the most successful programs:

1. the work is shared in a group setting.

2. Hidden processes are made explicit, that is for example, solving a problem aloud while revealing all the thinking behind it, including the metacognitive prompts that the problem solver engages in. While revealing these hidden thought processes, student commentary and observation is encouraged.

3. Program of learning is organised around subject matter and interpretation of the knowledge of this subject matter rather than ability.

The researcher made note of the research findings of Resnick and organised the program of learning for the intervention around subject matter using the theories and models of the development of mathematical understanding in the sequencing of the problems. The author developed an intervention where the pre-service teachers worked on problems (solving them together aloud) in groups of mixed abilities. A presentation on the different types of mathematical thinking was given in the first class and the pre-service teachers worked through the problems in this presentation which necessitated different forms of mathematical thinking to solve them, including application of a rational thought process.

The author identified the Modified Moore Method as a teaching approach that would best enable the pre-service teachers to develop proficiency in problem solving (chapter 6). The author integrated her F-TAPS in mathematics with the Modified Moore Method to facilitate the development of the components necessary for proficiency in problem solving. The design and choice of the problems
utilised during the intervention are based on the requirement and development of mathematical thinking and proficiency in the acquisition of knowledge, while also catering for the emotional engagement of the learners. Activities involving the use of manipulatives, problems of different contexts (some in relation to the pre-service teachers’ interests) but with similar structural components, and some modelling eliciting activities were included in order to develop the pre-service teachers’ proficiency in problem solving.

8.3.3 Phases 6 and 7 Data-Analysis (During Intervention and Post-test)

Q3. What evidence is there to show the effectiveness of the framework employed in the intervention?

There was a statistically significant increase ($Z = -2.317, p = 0.020$) in the median TRU score of the participants from 42 (7.5)\(^{99}\) to 46 (4.5) (effect size $r = 0.47$). Over 75% of the participants in the post-test scored higher than the median TRU score in the pre-test.

There was an increase ($Z = -1.926, p = 0.054$) in the median TPS score of the participants from median 24 (7.75)\(^{100}\) to median 29 (9.75) (effect size $r = 0.39$) which was not of statistical significance.

There was a statistically significant increase ($Z = -2.491, p = 0.013$) in the median TSC score of the participants from median 12 (5.75)\(^{101}\) to median 18 (7.5) (effect size $r = 0.51$). Over 75% of the participants in the post-test scored higher than the median TSC score in the pre-test.

There was a statistically significant increase ($Z = -2.473, p = 0.013$) in the median TS score of the participants from median 74.5 (15.75)\(^{102}\) to median 85 (14.25) (effect size $r = 0.50$).

There was a statistically significant increase $Z = -2.104, p = 0.035$ in the median mindset score of the participants from median 71 (8)\(^{103}\) to median 75 (15) (effect size $r = 0.41$).

These findings support the conclusion that the intervention which was developed based on the author’s F-TAPS in mathematics was effective in achieving improvements in the pre-service teachers’ problem solving ability and mindset. These findings in addition to the findings from the focus groups, completion

\(^{99}\)The value in the brackets is the interquartile range
\(^{100}\)The value in the brackets is the interquartile range
\(^{101}\)The value in the brackets is the interquartile range
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\(^{103}\)The value in the brackets is the interquartile range
of the reflective journals (at the end of some classes) by the pre-service teachers, and the increase in perseverance demonstrated by the pre-service teachers indicate that in general the intervention (and hence the author’s F-TAPS in mathematics on which it was based) was effective in achieving the aims of improving the pre-service teachers’ problem solving ability, mindset, perseverance and their ability to teach and assess problem solving in mathematics.

Q4. *Is the framework and the developed intervention transferrable to a mathematics class in a secondary level school?*

The author’s F-TAPS in mathematics and the developed intervention are transferrable to a mathematics class in a secondary level school. The knowledge, intrinsic motivation, mathematical thinking and proficiency components of the author’s F-TAPS in mathematics are applicable to the secondary level mathematics curriculum. Almost all of the problems in the developed intervention are problems which are at the level of Junior Certificate to Leaving Certificate level, with only a small number being at a higher level than the secondary level syllabus. Fully worked out solutions to all of the problems in the intervention are available on request by the author. The resources required for the manipulative based activities are easily available and links to internet based activities are included where applicable in the course notes. The mathematical thinking and mindset presentations are at an appropriate level for use in a secondary school setting and are included (Appendix U). One limitation is the time that some of the problems may take to solve, however a lot of the problems are solvable within a class period, where some of the problems in the intervention take a longer time to solve, they could be given as homework or given out during a double period. Another limitation is the availability of white boards. However an alternative is laminated white card which is easily available. While a detailed description of the Modified Moore Method facilitates the transfer of the developed intervention to a mathematics class in a secondary school, this approach may take some practice to develop. The questioning approach employed by the author during the intervention may also be a skill that takes practice to develop (by some teachers).

### 8.4 Proof of Concept

The proof of concept approach (section 3.6.2) is a test to demonstrate that the author’s F-TAPS in mathematics with its integrated intervention and assessments when implemented in practice works as it was theoretically designed to do. The author’s F-TAPS in mathematics with its integrated intervention performed in practice as the author had intended by design (sections 7.5, 7.6, 7.7, 7.9 and 7.10.1). The intervention (and the integrated F-TAPS) had a moderate effect on the mindset of the participants and a strong effect on the problem solving ability of the participants.
The proof of concept approach provides evidence that the F-TAPS, pre and post-tests, and intervention work in practice and performs as intended (by design). This gives credence to the concept of these research components also working on a larger scale.

8.5 Design Principles

During the design and development phase of the research process of Educational Design Research (which was the methodological approach in this research):

- the theories and models of developing mathematical thought, the problem solving models for the phases of problem solving, the theories and models on metacognition, mindset, interest development, intuition and perseverance were noted as essential contributors to the development of problem solving ability in mathematics;

- the overall goal of developing mathematical proficiency, in addition to the essential contributors in the development of problem solving skills and ability were integrated in the development of the author’s F-TAPS as a possible solution;

- the author’s F-TAPS was utilised in conjunction with analysis on problems (from literature) to develop a pre-test, the results of which informed the design and development of the teaching intervention;

- the author’s F-TAPS was then employed to inform the design and implementation of a teaching intervention with pre-service mathematics teachers. The design principles of the Modified 4C-ID Model combined with the theoretical and practical principles of the Modified Moore Method also informed the organisation, sequencing of tasks and implementation of the intervention.

Based on the work involved during the design and development of the F-TAPS in mathematics, pre and post-tests, and the implementation of the intervention, the author offers the following design principles which may be generalised to similar studies:

1. Integration of the overall goal of developing proficiency in a particular domain (mathematics e.g. Problem solving in Number, Algebra) with the essential contributors in the development of skills and abilities in that domain should be completed in the development of a framework as a possible means of teaching and assessing this particular domain (note that what is meant by proficiency in the domain should be clearly stated in the framework);
2. The clear definition of proficiency of the domain (mathematics) included in the framework in conjunction with the key essential contributor to the development of skills and abilities in the gaining of proficiency in the domain, allows for the development of pre and post-tests (and accompanying scoring rubrics) aimed at evaluating this proficiency. Consultation with the content of appropriate syllabi of the particular domain should be made in addition to research on rating the difficulty of items included in the pre and post-tests;

3. The results and findings from the pre-test should be integrated with the developed framework to inform the design of an intervention aimed at developing the teaching and assessing of the particular domain through the process of developing proficiency in the domain.

4. The construction of the intervention requires decisions to be made in relation to organisation, sequencing and delivery. The tasks/content to be included in the intervention should address the deficiencies noted in the pre-tests and should facilitate the development of proficiency (including adapting the tasks to address the essential contributors stated in 1 above) in the domain. The sequencing of the tasks/content should be aligned with the theories and models which exist in the literature for the coherent building of understanding of the domain (noting pre-requisite knowledge). The choice of delivery of the course should facilitate gains in proficiency of the domain by learners while aligning with the framework and assessment components.

5. A detailed description of this entire process (including all developed material) should be recorded for future use.

8.6 Recommendations

The author makes the following recommendations based on the findings of this research:

- The F-TAPS in mathematics and its intervention could be included as a dedicated module (or part of) of pre-service mathematics teacher education. The F-TAPS in mathematics and its intervention demonstrated through proof of concept that in practice they work as they were designed to do. This could be implemented by including it as part of a mathematics module aimed at improving the problem solving ability of the pre-service teachers. The module by design also facilitates the pedagogical development of the pre-service teachers and allows for practice with explaining (presenting solutions) and assessing (asking questions). The pre-test should be completed before the pre-service teachers begin the module and the post-test should be completed upon finishing the module.
• Use of the presentations on mindset and mathematical thinking in second-level mathematics classes. The significant role that mindset plays in the achievement levels of students in mathematics has been extensively researched and proven by Dweck (2008), yet it is not a feature in second-level mathematics education. Inclusion of the presentation on mathematical thinking may aid in the provision of alternative ways of thinking for the students. Also the use of metacognitive and reflective journals at second level may help students to become more aware of their own learning.

• Careful consideration should be given to the design of problems so that learning of the concept is included in some of the problems to check understanding and to develop understanding where it is found to be deficient. Problems requiring multiple representations of the same mathematical situation are important for developing students’ flexibility. Problems requiring the use of specialisation and generalisation are significant in developing mathematical thinking. A survey of students’ interests should be taken at the beginning of the school year so that problems may be designed to aid in maintaining students interest and perseverance levels in mathematics.

• In relation to the above point on the design of problems, the author recommends that problem posing and design should feature in pre-service mathematics teacher education and also should be offered as a continuous professional development course. At the beginning of such courses, problems/exercises from current mathematics books could be used as a base upon which to design adapted problems. This could build up to teachers creating problems which connect between the various strands of the syllabus. A general guideline for this activity would be as follows:

  – Choose idea within a strand to base the problem on (foundation).
  – Think of the possible connections you are aware of to other mathematics within the same strand. What information from this strand will be required to complete the problem?
  – Think of the possible connections you are aware of to other mathematics from different strands.
  – Choose another strand(s) to incorporate onto the base strand (branch)
  – Consider the context of the problem as a linking tool between the strands.
  – Problem solving activity: connect, explain, specialise and generalise, justify solution.

For example (Figures 62 and 63\textsuperscript{104}):

\textsuperscript{104}Note image not to scale here.
8.7 Contributions to Mathematics Education

The author’s contribution to knowledge and practice in mathematics education is discussed in this section.

8.7.1. Extensive Research

The author carried out an extensive piece of developmental research on teaching problem solving which integrated several theories and models. This research:

a) is in support of national priorities in mathematics education in Ireland;

b) adds to knowledge for mathematics education generally (literature of pre-service teachers as problem solvers in particular);

c) makes a contribution to school practice.

8.7.2. Compact Comprehensive Analysis of Problem Solving

The author has created a compact analysis of problem solving in the review of literature as part of this study. Specifically, the literature has been reviewed in a highly integrated manner so that the relationships between the various elements which affect the teaching and learning of problem solving have been made explicitly clear. The review of the models and theories of mathematical understanding with the aim of integrating the findings with significant knowledge and affective factors to develop mathematical proficiency has provided a very useful resource which has addressed significant issues in problem solving.
8.7.3. Framework for Teaching and Assessing Problem Solving

The author created a framework F-TAPS in mathematics by reviewing the relevant literature and integrating it coherently and systematically. While many problem solving models and frameworks exist, the author’s framework is unique in its integrative features. The difficulties that students and teachers experience in problem solving in mathematics have been extensively reported and the details of the reports on problem solving in the last decade are similar to reports of previous decades. There is a significant need to bring research into practice. The central component of mathematical thinking in the author’s F-TAPS in mathematics is essential in the development of problem solving proficiency. The author’s framework focuses particular attention on the processes involved in problem solving. The consideration of the author’s framework to the overall development of mathematical proficiency from the acquisition of mathematical knowledge and a mathematical “cast of mind” (Krutetskii, 1976), to assessment, in both the forward and reverse direction, facilitates a logical coherent construction of mathematics in the minds of learners. In a similar way to how reflecting on the solution from a procedure is vital in condensing that procedure into an ‘object’, with further reflection organising related ‘actions’, ‘processes’ and ‘objects’ into ‘schemas’. The reflection on the end result of developing proficiency
in mathematics condensed the many aspects of problem solving into an ‘object’, while further reflection organised the related knowledge and affective ‘actions’, ‘processes’ and ‘objects’ into a problem solving ‘schema’ in the mind of the author. As the author’s understanding of the interconnecting components involved in the teaching, learning and assessing of problem solving advanced from understanding the set of individual interconnecting components required for problem solving in mathematics, to understanding the interconnected nature of problem solving as an overall process, her understanding of and development of a framework to cater for the complex nature of problem solving increased in fitness for purpose. The development of this understanding by the author was from both a mathematical and pedagogical perspective. Frameworks for mathematics programmes exist where problem solving is the central component, the author’s framework differs in that mathematical thinking is the fundamental permeating component in the mathematics programme for teaching and assessing problem solving. Mathematical thinking is developed through the understanding and appreciation of the co-existence and co-significance of cognition and emotion in developing proficiency in problem solving.

As identified in the review of literature it is emotion which organises the complex knowledge arrangements in our minds. However the emotion will organise whatever knowledge it receives. It is imperative to aid in the coherent construction of this knowledge while also taking emotional factors into consideration. The author’s F-TAPS in mathematics integrates significant knowledge and affective factors with mathematical thinking in order to develop proficiency in problem solving in mathematics through the coherent construction of meaningful knowledge.

8.7.4. Created a problem solving learning/teaching environment and pedagogical approach using the Modified Moore Method

The author integrated her F-TAPS in mathematics with the Modified Moore Method using the Modified 4C-ID model to aid with a systematic and comprehensive approach in the design of instruction. By recognising problem solving in mathematics as a complex phenomenon and thus focussing on the overall constitution of mathematical problem solving, the author was able to consider the individual components and their relationships within and between mathematics problem solving. This was key in developing the author’s F-TAPS and intervention. The Modified Moore Method integrated with the author’s F-TAPS provides a rich problem solving learning and teaching environment.

8.7.5. Created a problem solving learning/teaching environment which contributed to increased perseverance in problem solving among pre-service teachers.

An important finding of the research was the increase noted in perseverance levels in problem solving among the pre-service teachers. From the behavior and comments of the participants, the following:
• interest;
• enjoyment;
• experiencing both individual and shared achievement after persevering on very difficult problems;
• the high expectations of both the tutor and the participants of the learning environment;
• the lack of pressure;
• mutual respect between the tutor and the participants and between the participants themselves;

facilitated a learning environment conducive to growth in confidence and perseverance in problem solving.

8.7.6. Created assessment rubrics, template and scoring rubric.
The author created assessment rubrics and templates which allows for a learner’s problem solving proficiency to be examined under each of the phases of the problem solving cycle. Employing the assessment rubric to evaluate a learner's problem solution allows their competency in the phases of problem solving to be determined. This allows a teacher to know exactly where in the problem solving cycle a learner is having difficulty. Simultaneously employing the template to further evaluate a learner’s solution facilitates the determination of a learner’s ability and performance in mathematical processes, skills, abilities, mathematical thinking, and heuristics. This allows a teacher to know exactly what the difficulty is. Knowing which phase of the problem solving cycle is causing difficulty in addition to knowing whether it is mathematical thinking, problem solving skills or mathematical proficiency is at the core of the difficulty is useful in providing specific instruction for improvement.

8.7.7. Created an intervention which is portable by providing a detailed record of all aspects.
The author’s book of course notes (including a note to both instructors and students on how the Modified Moore Method is implemented), fully worked out solutions to all problems in the notes, metacognitive journals, reflective journals and design principles utilised in the development of the framework and intervention make this intervention portable for use by others.

8.7.8. Created a list of generalisable design principles for use by other researchers who wish to conduct similar studies.
The design principles that were identified and employed during this research are generalisable for the creation of frameworks, models or interventions in an area of mathematics or a subject within the STEM domain (e.g. Physics). The design principles are available for use by other researchers who wish to conduct similar studies.
8.8 Future Work

Based on the findings of this research, the author would like to continue research into problem solving in mathematics education. Some future work envisaged by the researcher includes the following:

Extend the intervention to include problem posing and design tasks by the pre-service teachers themselves (section 8.6).

Explore the longevity of the changes shown in the problem solving ability and perseverance by the pre-service teachers in this study through a longitudinal study. The use of the framework/template in assessing the pre-service teachers’ teaching and assessing of mathematics would also allow the researcher to determine if the teachers are using any of the resources/teaching approaches from the intervention in their own teaching. Further tests of problem solving, level of fixed/growth mindset and observing and evaluating their teaching practice for evidence of teaching and assessing of problem solving (5 visits) could be repeated in 1 year, 2 years and 5 years to explore the longevity of any changes shown in problem solving/perseverance/teaching practice by the pre-service teachers.

Further analyse and revise the intervention if necessary so that the F-TAPS in mathematics and its intervention can be employed with a larger sample of pre-service teachers. A longitudinal study could be undertaken with a larger sample of pre-service mathematics teachers. The pre-service teachers in each year (1-4) of their university programme could be randomly assigned to a control and experimental group. Each participant would complete the pre-test in problem solving, the pre-test in determining their level of fixed/growth mindset and their teaching practices would be assessed for evidence of teaching and assessing of problem solving (5 visits to observe and evaluate their teaching practice). Participants from each of the years (1-4) who form the experimental group would then participate in the intervention (to be completed over the course of a semester), while the participants forming the control group would not complete the intervention. After the intervention with the experimental group is complete each participant would complete the post-test in problem solving, the post-test in determining their level of fixed/growth mindset and their teaching practices would be assessed for evidence of teaching and assessing of problem solving (5 visits to observe and evaluate their teaching practice). Further tests of problem solving, level of fixed/growth mindset and observing and evaluating their teaching practice for evidence of teaching and assessing of problem solving (5 visits) could be repeated in 1 year, 2 years and 5 years to explore the longevity of any changes shown in problem solving/perseverance/teaching practice by the pre-service teachers.

Further analyse and revise the intervention if necessary so that the F-TAPS in mathematics and its intervention can be employed with secondary school students. A study could be undertaken where a sample of pre-service mathematics
teachers (and/or newly qualified mathematics teachers) who participated in the intervention, implement the intervention initially with transition year pupils (fourth year). The researcher would be available to support the teachers. The transition year students would complete pre and post-tests in problem solving ability and fixed/growth mind-set before and after the intervention. Focus groups would also be held with these transition year students to evaluate their response to the intervention. Focus groups would also be held with the teachers to evaluate their response to implementing it with their students. Reflective journals would be given to both the students and the teachers to allow for evaluation of their progression with participating/implementing the intervention.

Extend the book of course notes to include problems starting from the first year of second level education to the sixth (final) year of second-level education. This book of problems would integrate across the syllabus and would proceed from conceptual embodiment to formal axiomatic. The number of problems would be minimised to allow for time to engage with them but would facilitate the learning of all key concepts and processes in a logical coherent way.

Extend the use of the framework across more domains in mathematics e.g. geometry, calculus etc., and in STEM disciplines e.g. Physics by making adaptations from the mathematical content to apply to the specific domain/discipline.
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