Application of the Discrete-Time Structured Singular Value for DC-DC Converters

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Abstract — This paper considers the application of the discrete-time structured singular value to assess the robustness of systems subject to real parametric uncertainties. Since the associated uncertainty set is strictly real, the resulting \( \mu \) bounds may be discontinuous irrespective of the fineness of the frequency sweep. It is therefore necessary to seek other more reliable methods of test. One such method detailed is a discrete robust stability state-space test that is not dependent on a frequency sweep. It is shown that this is a discrete skewed structured singular value problem. For illustration purposes, a robustness analysis using both a frequency sweep and state-space transformation is performed on a forward converter with a real parameter uncertainty. The results are detailed and critically assessed.

Keywords — robust control; DC-DC converters.

I INTRODUCTION

This paper considers the application of the structured singular value to assess the robustness of discrete-time systems subject to real parametric uncertainties [1]. The introduction of uncertainty is easily justified and extends the bounds of linear analysis on nominal models. To date, application of the structured singular value, \( \mu \), has been predominantly used in the continuous-time domain. In this paper, it is applied in the discrete-time domain with the definition of the discrete-time structured singular value given. The application is particularly suited in the robust analysis of DC-DC converters as discretisation of the plant model and direct digital control law design methods are becoming more and more prevalent.

For ease of implementation, an explicit linear fractional transformation is included for a standard one-zero two-pole representation of a forward DC-DC converter with uncertainty [2]. Since this uncertainty set is strictly real, the resulting \( \mu \) bounds may be discontinuous irrespective of the fineness of the frequency sweep [3]. As an alternative method, a robust stability state-space test that is not dependent on a frequency sweep is detailed. It is shown that this is a skewed (or generalised) structured singular value problem [4]. To compare the reliability of both approaches, a robustness analysis using both a frequency sweep and the state-space transformation technique outlined is performed on a forward converter with specified real parameter uncertainty. The results are summarised and assessed.

This paper is outlined as follows, section II details the definition of the structured singular value and application to the discrete-time domain. Section III details the development of the perturbed plant model and the introduction of uncertainty. It also details the discrete-time state-space test and how the skewed structured singular value can be applied to this problem description. This is followed by a robust analysis performed on a forward converter control configuration in section IV. Concluding remarks and an outline of future work is given in section V.

II ROBUSTNESS ANALYSIS TECHNIQUES

The \( \mu \) approach for systems analysis is based on the observation that problems involving intercon-
Connections of linear time invariant (LTI) systems with uncertain parameters and unmodelled dynamics can be reduced to considering the constant matrix feedback interconnection in Figure 1. The uncertainty block $\Delta$ is structured where three non-negative integers $m_r$, $m_c$, and $m_C$ specify the number of uncertainty blocks of each type. The block structure $\mathcal{K}(m_r, m_c, m_C)$ is an $m$-tuple of positive integers.

$$\mathcal{K} = (k_1, \ldots, k_{m_r}, k_{m_r+1}, \ldots, k_{m_r+m_c}, k_{m_r+m_c+1}, \ldots, k_m) \quad (1)$$

with $m = m_r + m_c + m_C$. This $m$-tuple specifies the dimensions of the perturbation blocks, which determines the set of allowable perturbations, namely define

$$X_\mathcal{K} = \{ \Delta = \text{block diag}(\delta_1 I, \ldots, \delta_m I, \Delta), \ldots, \delta_1 I_{k_{m_r+m_c+1}}, \ldots, \delta_m I_{k_m} : \delta_i \in \mathbb{R}, \delta_i^c \in \mathbb{C}, \Delta^c \in \mathbb{C}^{k_{m_r+m_c+i} \times k_{m_r+m_c+i}} \}$$

Note that $X_\mathcal{K} \subset \mathbb{C}^{n \times n}$ (where $n = \sum_{i=1}^{m} k_i$) and that this block structure allows for repeated real scalars ($\delta_1 I$), repeated complex scalars ($\delta_i^c I$), and full complex blocks ($\Delta^c$). Noting this block structure, the following definition, taken from [1] is introduced.

**Definition 1** The structured singular value, $\mu_\mathcal{K}(M)$, of a matrix $M \in \mathbb{C}^{n \times n}$ with respect to a block structure $\mathcal{K}(m_r, m_c, m_C)$ is defined as

$$\mu_\mathcal{K}(M) = \frac{1}{\min_{\Delta \in X_\mathcal{K}} \{ \hat{\sigma}(\Delta) : \det(I_n - \Delta M) = 0 \}} \quad (2)$$

with $\mu_\mathcal{K}(M) = 0$ if no $\Delta \in X_\mathcal{K}$ solves $\det(I_n - \Delta M) = 0$.

With this definition, it is possible to check that the interconnection in Figure 1 is well posed for all $\Delta \in X_\mathcal{K}$ with $\hat{\sigma}(\Delta) < \frac{1}{\beta}$ if and only if $\mu_\mathcal{K}(M) \leq \beta$.

Linear Fractional Transformations (LFTs) are used to reorganise a perturbed problem with uncertainty into the feedback interconnection in Figure 1. In particular, if $M \in \mathbb{C}^{n \times n}$ is partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad (3)$$

with $M_{11} \in \mathbb{C}^{n_1 \times n_1}$, $M_{22} \in \mathbb{C}^{n_2 \times n_2}$, and $n = n_1 + n_2$. When an LFT is well-posed, it is defined to be the unique mapping from $\omega \to z$ (Figure 2), i.e. the vectors $z$ and $\omega$ satisfy $z = (\Delta \ast M)\omega$, where

$$\Delta \ast M = M_{22} + M_{21}(I_{n_1} - M_{11}\Delta)^{-1}M_{12}$$

LFTs provide a mechanism to rearrange general LTI robustness problems into a standard form. Using this framework, the robust stability question for discrete-time systems can now be addressed. With reference to Figure 1, $M$ now represents a linear shift-invariant stable discrete time $M(z)$, and $\Delta$ is a structured dynamic perturbation. The frequency sweep test for the robust stability problem follows

**Theorem 2** Suppose that $M(z)$ has all of its poles in the open unit disk (i.e. nominal stability), and let $\beta > 0$. Then for all $\Delta \in \mathcal{M}(X_\mathcal{K})$ with $\|\Delta\|_{\infty} < \beta$, the perturbed closed-loop system in Figure 1 is (well-posed and) stable if and only if

$$\sup_{\theta \in [0, 2\pi]} \mu_\mathcal{K}(M(e^{j\theta})) \leq \frac{1}{\beta} \quad (4)$$

From this theorem, the robust stability of a system can be addressed with repeated computation of a constant matrix $\mu$ problem.

III Uncertainty and LFT Descriptions

Figure 3 shows a typical DC-DC converter representation with digital voltage mode control. Without loss of generality a sensing gain of $H = 1$ is assumed. Both the constant-frequency leading-edge Digital Pulse-Width Modulator (DPWM) and the Analogue-to-Digital converter (A/D) are
Switching power converter

Fig. 3: Switching DC-DC converter with digital voltage-mode control.

designed with unity gain. The converter operates in continuous-conduction mode. In each state of the switch (1 or 2), the converter circuit is linear, time-invariant, with the corresponding state-space description [2]

\[
\begin{align*}
\dot{x} &= A_i x + b_i V_g \\
y &= C_i x + e_i V_g, \quad i = \{1, 2\}
\end{align*}
\]

where \( x \) is the vector of converter states (e.g. inductor current and capacitor voltage, \( x = [v_i] \)). It is assumed that the voltage \( V_g \) is constant, since the primary interest is in the control-to-output responses. The nominal state-space representation of the small-signal model is given by (all losses are neglected except for the dominant effect of the capacitor Equivalent Series Resistance (ESR), \( R_C \))

\[
A \approx \begin{bmatrix} -\frac{1}{R_C} & \frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}, \quad C \approx \begin{bmatrix} 1 & R_C \end{bmatrix}
\]

\[
D = 0, \quad x = [v_i], \quad y = v_{out}.
\]

To develop the perturbed model, uncertainty is introduced for each of the varying parameters \( L, C, R_C \) and \( R \). An example substitution is \( L = L_0(1 + W_L \delta_L) \) where \( L_0 \) is the nominal value, \( W_L \) is a weight that represents the percentage variation, and \( \delta_L \) is the uncertain parameter. With reference to Figure 4, a state-space representation of the perturbed plant \( P \) is (assumes disturbance \( d = 0 \))

\[
\begin{bmatrix}
\dot{x} \\
y_{\Delta} \\
y
\end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ u_{\Delta} \\ r \end{bmatrix}
\]

with exogenous inputs and outputs

\[
u_{\Delta} = [d_L, d_C, d_{R}, d_R]^T
\]

\[
y_{\Delta} = [e_L, e_C, e_{R}, e_R]^T
\]

associated with the uncertainty variables. An explicit form of the perturbed plant is now given

\[
P = \begin{bmatrix}
-\frac{1}{R_C C_0} & \frac{1}{C_0} & 0 & -W_C & 0 & \frac{W_R}{C_0} & 0 \\ 
-\frac{1}{L_0} & 0 & -W_L & 0 & 0 & 0 & \frac{1}{L_0} \\ 
-\frac{1}{R_C C_0} & \frac{1}{C_0} & 0 & -W_C & 0 & \frac{W_R}{C_0} & 0 \\ 
0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 
\frac{1}{R_0} & 0 & 0 & 0 & 0 & -W_R & 0 \\ 
1 & R_C C_0 & 0 & 0 & W_{R_C C_0} & 0 & 0
\end{bmatrix}
\]

Note that for an upper LFT \( B_2 = B, C_2 = C \) and \( D_{22} = D \). The previous \( \mu \) test gives not only a number regarding the worst-case robustness of the system but also a \( \mu \) plot across frequency which details the frequency ranges where the system is sensitive to parameter changes. This information is very useful but in many cases a robustness test is only required. In this instance a frequency sweep may be avoided. Furthermore, in practice, an appropriate frequency sweep and the fineness of the grid has to be decided \textit{a priori}. More critically, is that there is the possibility of missing important points, especially as \( \mu \) may be discontinuous for real parametric uncertainty [3]. To avoid frequency sweeps of this nature, an alternative state-space test may be used. This approach provides a one shot computation of the worst-case \( \mu \) value across frequency.
Fig. 4: Perturbed Plant $P$.

For this test, the state-space representation of the transfer function is expressed as a LFT. Notably this is an LFT of a constant matrix on the frequency variable

$$M(z) = \hat{C}(zI_p - \hat{A})^{-1}\hat{B} + \hat{D} = \frac{1}{z}I_p \ast \hat{M}$$

where $\hat{M}$ is the constant matrix

$$\hat{M} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$$

and $p$ is the dimension of the state-space. Note that it is necessary to search over $\frac{1}{z}I_p$ inside the unit disk. This can be achieved by including $\frac{1}{z}I_p$ as one of the uncertainties whose worst-case is being sought with the substitution

$$z = \frac{1}{\delta}$$

Note that the frequency uncertainty variable $\delta I_p$ is bounded within the unit disk. The subdivision of the set of allowable perturbations, $X_{\hat{K}}$, into fixed and varying subsets is a skewed structured singular problem, $\mu^s$, problem [4].

Suppose $M \in \mathbb{C}^{n \times n}$ is partitioned as in (3) and two block structures are defined as $X_{\hat{K}1} \subset \mathbb{C}^{n_1 \times n_1}$, $X_{\hat{K}2} \subset \mathbb{C}^{n_2 \times n_2}$. Then the augmented block structure, $X_{\hat{K}} \in \mathbb{C}^{n \times n}$, is defined as

$$X_{\hat{K}} = \{\Delta = \text{block diag} (\Delta_f, \Delta_v) : \Delta_f \in X_{\hat{K}1}, \Delta_v \in X_{\hat{K}2}\}$$

where the closed unit ball, $B\mathbb{X}_{\hat{K}1}$, is defined as

$$B\mathbb{X}_{\hat{K}1} = \{\Delta_f \in \mathbb{X}_{\hat{K}1} : \sigma(\Delta_f) \leq 1\}$$

The definition of the skewed structured singular problem, or “skew $\mu$”, is the smallest structured singular value of a subset of perturbations that destabilises the system $M$ with the remainder of the perturbations being of fixed size.

**Definition 2** The skewed structured singular value, $\mu^s(M)$, of a matrix $M \in \mathbb{C}^{n \times n}$ with respect to a block structure, $\hat{K}(m_{r_1}, m_{c_1}, m_{r_2}, m_{c_2}, m_{c_3})$, is defined as

$$\mu^s_\hat{K}(M) = \min_{\Delta \in \mathbb{X}_{\hat{K}}} \left\{ \sigma(\Delta) : \det(I_n - \Delta M) = 0 \right\}$$

with $\mu^s_\hat{K}(M) = 0$ if no $\Delta \in \mathbb{X}_{\hat{K}}$ solves $\det(I_n - \Delta M) = 0$.

Skew $\mu$ is a generalisation of the structured singular value and can be directly applied to (10) where the frequency variable is contained within the closed unit ball. The robust stability theorem for discrete state-space skew $\mu$ follows [5]

**Theorem 3** Suppose that $M(z)$ has all of its poles in the open unit disk (i.e. nominal stability), and let $\beta > 0$. Let the minimal state-space representation for $M(z)$ be given as in (10). Given $X_{\hat{K}2}$ compatible with $M(z)$, then for all $\Delta_v \in \mathbb{M}(X_{\hat{K}2})$ with $\|\Delta_v\|_{\infty} < \beta$ and $\Delta_f = \delta I_p$ such that $\|\Delta_f\|_{\infty} \leq \beta$
\( B x_{\text{C}s}, \) the perturbed closed-loop system in Figure 1 is (well-posed and) stable if and only if

\[
\mu_K^\Delta \left( \begin{bmatrix} \hat{A} & \sqrt{\frac{1}{\beta} \hat{B}} \\ \sqrt{\frac{1}{\beta} \hat{C}} & \frac{1}{\beta} \hat{D} \end{bmatrix} \right) < 1
\]  \hspace{1cm} (12)

Using this theorem, the general robust stability problem reduces exactly to computing a single constant matrix \( \mu \) problem. The added benefit is that the new uncertainty structure \( x_{\text{C}s} \) is always mixed (contains real and complex uncertainty) and efficient computation of upper and lower bounds for “mixed” \( \mu \) are well developed and commercially available [6]. The authors also recommend using the Skew Mu Toolbox for use with Matlab that is freely downloadable [7].

IV RESULTS

As an illustrative example, consider the forward converter with \( L_0 = 10 \mu H, C_0 = 22 \mu F, R_{C} = 1 \text{m} \Omega, R_0 = 1.25 \Omega, V_p = 24 \text{V}, n = 3, f_s = 1/T_s = 1 \text{MHz}, f_{sw} = 1/T_{sw} = 1 \text{MHz}. \) The uncertainty variables are normalised ensuring that \( \beta \) in the previous expressions is simply unity. The percentage variation for the \( L, C, R_{C} \) and \( R \) are 30%, 40%, 20% and 50% respectively. With reference to the configuration in Figure 5, the pre-filter \( H_r \) is set to 1, the influences of disturbances \( d \) and noise \( n \) are ignored for the initial analysis and the perturbed plant \( P \) is determined using (7). The plant model was converted to the discrete domain using a Zero-Order-Hold transformation [8]. The digital controller \( K(z) \) was developed using a generalised predictive control method detailed in [9].

\[
K(z) = \frac{120.2075z^2(z^2 - 1.866z + 0.8757)}{z(z - 1)(z^2 + 0.4644z - 0.2314)}
\]

With this information and with reference to Figure 5, the \( M - \Delta \) interconnection structure in Figure 1 may be determined either analytically or using the Robust System Toolbox in Matlab [6]. For a robust stability analysis, only the influence of the exogenous inputs \( (u_\Delta) \) and outputs \( (y_\Delta) \) are considered. The \( \mu \) upper and lower bounds were calculated for (4) using the structured singular value \( \text{mu} \) function from the Robust Control Toolbox. A relatively dense sweep of 500 frequency points was used for a frequency range \( f \in [0, f_N] \) where \( f_N \) is the Nyquist frequency. The resulting plot is shown in Figure 6. Note that in most cases, the algorithm failed to determine a lower bound. This is not uncommon for a strictly real parametric uncertainty problem. The maximum value of \( \mu \) is 0.8364, indicating that the system is robustly stable for this level of uncertainty (since it is less than 1). However, applying the skewed structured singular value on the state-space formulation given in Theorem 3 delivers a contradictory result as the maximum value of \( \mu \) returned was 1.0304. It is important to highlight that this is an upper bound. It is not necessary to proceed with a robust performance test as this state-space skew \( \mu \) result verifies that the system is not robustly stable. In this instance, many choices are available to the design engineers. These include the reevaluation of the levels of uncertainty by assessing the \( \mu \) sensitivities associated with each uncertain parameter [10] and/or the redesign of the controller \( K(z) \), among others.

V CONCLUSIONS

In this paper, the application of the structured singular value for the robust analysis of discrete-time systems with real parametric uncertainty is consid-
ered. This is suitably applicable for modern day DC-DC power converters as more and more digital solutions are being sought and implemented. An explicit LFT expression necessary to conduct a $\mu$-analysis test for the perturbed converter model is detailed. As shown, the drawback of a frequency sweep robust analysis approach is that critical frequencies may be missed irrespective of the fineness of the grid for this type of uncertainty. As an alternative method, a discrete-time state-space formulation was detailed to assess robust stability. It was shown for a forward DC-DC converter model that this method is fully reliable and computationally efficient.

Future work will focus on determining a lower bound $\mu$ and skew $\mu$ solution in order to obtain candidate worst-case values for the governing parameters that will prove useful to the design engineers.

**REFERENCES**


