Numerical Analysis of Singularly Perturbed Nonlinear Reaction-Diffusion Equations

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Abstract

This work is concerned with finding accurate numerical approximations to nonlinear reaction-diffusion problems that exhibit layer phenomena. It considers three variations of problems of this type. These are; a two-dimensional steady state equation with Dirichlet boundary conditions exhibiting interior layer solutions; a time-dependent equation with singularly perturbed Neumann boundary conditions with boundary layer solutions; and a steady state equation with singularly perturbed Neumann boundary conditions exhibiting boundary layer solutions.

Asymptotic analysis is called upon from previous literature in order to obtain upper and lower solutions to the problems. The theory of Z-fields are then used along with discrete upper and lower solutions to prove existence of a discrete solution and obtain accuracy bounds. Discretisations in the finite difference method and the finite element method are presented on layer adapted meshes such as the Shishkin and Bakhvalov mesh. In cases where incorrect computed solutions are obtained from a conventional discretisation the stabilised method by Kopteva and Savescu [16] is employed, giving solutions of the correct form.

It is found that the problems have second-order convergence in space in the maximum norm, with a logarithmic factor for the Shishkin mesh, and, for the time-dependent problem, first order convergence in time in the maximum norm, again with a logarithmic factor for the Shishkin mesh. Finally, numerical examples are given to support the theoretical results.

Author's Declaration

This thesis is presented in fulfilment of the requirements for the degree of Doctorate of Philosophy. It is entirely my own work and has not been submitted to any other University or higher education institution, or for any other academic award in this University. Where use has been made of the work of other people it has been fully acknowledged and fully referenced.

Signature of Author: ____________________

Date: ____________________

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Contents

Chapter 1

Introduction

This work presents three variations of nonlinear singularly perturbed reaction-diffusion problems with a focus on finding reliable numerical methods for each problem. The problems are as follows: a two-dimensional steady state equation with Dirichlet boundary conditions exhibiting interior layer solutions; a time-dependent equation with singularly perturbed Neumann boundary conditions exhibiting boundary layer solutions; and a steady state equation with singularly perturbed Neumann boundary conditions also exhibiting boundary layer solutions. The purpose of this investigation is to prove existence of a computed solution and obtain accuracy results for each problem. These types of equations can be found in many biological and chemical processes and, as they are singularly perturbed, can be difficult to solve numerically.

The first of the three problems we consider is

$$-\varepsilon^2 \Delta u + b(x,u) = 0 \quad \text{for } x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \tag{1.0.1a}$$

$$u(x) = g(x) \quad \text{for } x = (x_1, x_2) \in \partial\Omega, \tag{1.0.1b}$$

where $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ and the small perturbation parameter, ε, is taken to satisfy $0 < \varepsilon \ll 1$. We look for interior layer solutions to (1.0.1) under

certain local assumptions, including that the reduced problem $b(x,u) = 0$, found by setting $\varepsilon = 0$, has multiple solutions. The solution to (1.0.1) switches dramatically between different stable solutions of the reduced problem in a narrow region in the interior of the domain. Although the diffusion term has a minor role to play in the majority of the domain, in this region it becomes important. Here the derivative terms grow as ε goes to zero and the area of rapid change occurs in the solution. For visualisation purposes a schematic of a one-dimensional interior layer solution is shown in Figure 1.1a. We aim to prove existence of a computed solution and obtain accuracy bounds for that solution.

The second problem we consider is

$$\mathcal{T}u := \varepsilon^2 \left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \right) + f(x,t,u) = 0, \tag{1.0.2a}$$

$$\text{for} \quad (x,t) \in \mathcal{D} := \{(x,t) \in [0,1] \times [0,T],\, T \in \mathbb{R}^+\},$$

$$\varepsilon \left. \frac{\partial u}{\partial x} \right|_{x=l} = g_l(t) \quad \text{for } l = 0,1,\ t \in [0,T] \tag{1.0.2b}$$

and

$$u(x,0) = \varphi(x) \quad \text{for } x \in [0,1]. \tag{1.0.2c}$$

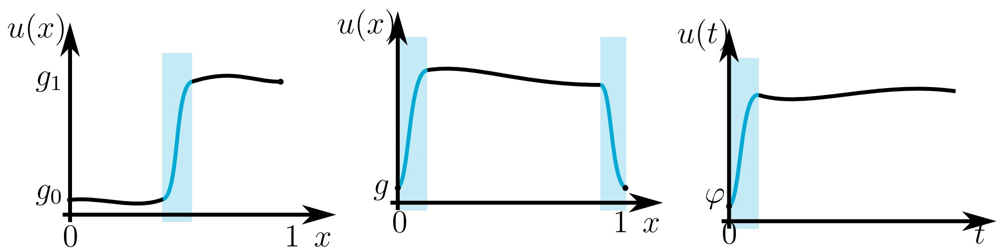
Boundary and initial layer solutions to (1.0.2) will be considered by making certain local assumptions on the problem. In this case the solution in the majority of the domain does not necessarily match the boundary or initial condition and hence the area of rapid change occurs near this condition. We show a one-dimensional boundary and initial layer solution in Figure 1.1b and Figure 1.1c, respectively. Corner layer solutions will be omitted by enforcing certain conditions at the corners of the domain. The aim is to prove existence and accuracy of a computed solution.

The third and final problem we present is

$$-\varepsilon^2 \Delta u + b(x,u) = 0 \quad \text{for} \quad x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \tag{1.0.3a}$$

$$\varepsilon \frac{\partial u}{\partial n}\bigg|_{x \in \partial\Omega} = g(x) \text{ for } \quad x \in \partial\Omega, \tag{1.0.3b}$$

Here we again look for boundary layer solutions to the problem. Much of the work of the previous two problems can be combined to obtain results for this problem.



(a) An interior layer solution in one dimension with boundary conditions $u(0) = g_0$ and $u(1) = g_1$.

(b) A boundary layer solution in one dimension with layers at $x = 0$ and $x = 1$ and boundary conditions $u(0) = u(1) = g$.

(c) An initial layer solution in one dimension with a layer at $t = 0$ and initial condition $u(0) = \varphi$.

Figure 1.1: Examples of possible layer solutions in one dimension.

We aim to find reliable numerical methods to solve these problems with a focus on efficiency and accuracy. Using numerical methods designed for standard problems gives errors that depend on the small parameter ε and thus are not acceptable methods for singularly perturbed problems. Due to this difficulty, we aim to find methods that converge ε-uniformly, i.e., methods that do not have a dependence on ε and have only a dependence on the number of mesh nodes.

In much of the numerical analysis literature it is assumed that the nonlinear function is monotonically increasing with respect to the solution, i.e., $b_u(x,u) > 0$ for (1.0.1) and (1.0.3) and $f_u(x,t,u) > 0$ for (1.0.2). With

this assumption the maximum and comparison principles [28] are applicable. However this is a restrictive criteria as many physical problems do not have this requirement. Many chemical and biological processes involve problems where the nonlinear term is non-monotone with respect to the solution, see [10, §2.3] and [23, §14.7]. Furthermore this criterion rules out the existence of interior layer solutions. Instead, here weaker local assumptions are employed in each problem.

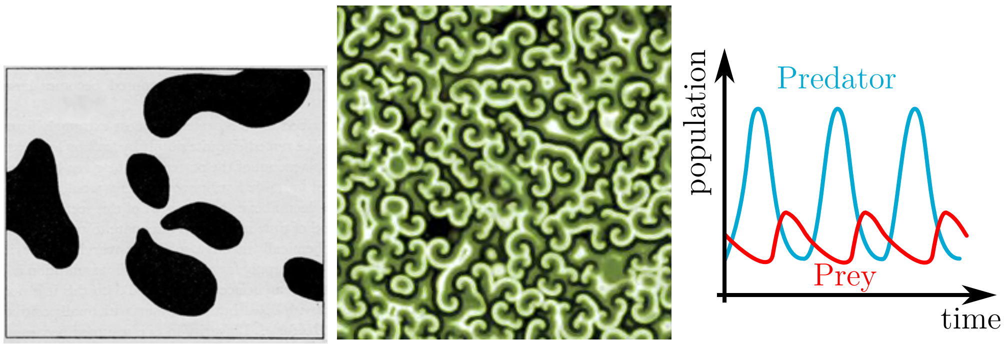
In the remainder of this chapter applications of singularly perturbed reaction-diffusion equations are discussed, important concepts are introduced that are central to the analysis in this thesis; upper and lower solutions, Z-fields and layer-adapted meshes, and a literature review of works that are central to the discussion is included.

1.0.1 Applications

Problems involving layer solutions arise in many areas of science and key components of reaction-diffusion equations are able to give further explanation to these systems. Three important examples of problems exhibiting layer phenomena are the modelling of pattern formation, chemical reactions and predator-prey interaction.

Discussion of problems involving layer phenomena greatly help the understanding of the pattern formation mechanism in nature. In *The Chemical Basis of Morphogenesis* [38], Turing considered such examples as reaction-diffusion systems giving rise to dappling pattern formation, whorled leaves and phyllotaxis. An image of dappling is represented in Figure 1.2a. Further discussion of layer phenomena in pattern formation can be found in [9], [10, §2.3], [7] and [25].

A chemical reaction is modelled in [36] by a coupled system of reaction-diffusion equations with a large parameter multiplying the reaction terms.



(a) An example of dappling from Turing's paper [38].

(b) Pattern formation in a Belousov-Zhabotinsky reaction.
Image source: [3]

(c) Schematic of a predator-prey solution showing change in population with time.

Figure 1.2: Examples of pattern formation in nature.

The chemical reaction involves two species A and B with a thin membrane separating the region of species A from species B. A third intermediate species is generated quickly when A and B coexist and is depleted quickly when reacting with A. Solutions are found to exhibit interior layers in space that move with time.

A more complicated example of a chemical reaction giving rise to interior layer solutions is the Belousov-Zhabotinsky reagent, [43] and [13]. This reagent again exhibits spatial interior layers that move with time. In the two-dimensional case these are typically a closed ring expanding away from a point of initial excitation while another possible pattern form is represented in Figure 1.2b.

Predator-prey models also often involve layer solutions. In these models two populations coexist, the predator and the prey. This is modelled by a reaction-diffusion system where the reaction term represents birth and death rates and the diffusion term represents spatial movement. The resulting model is a coupled system with a small parameter. A schematic of a

predator-prey solution over time is shown in Figure 1.2c where the prey population has a chance to grow significantly while the predator population is low. Models of such problems can be found in [10, §2.3] and in [23, §14.7] where the interaction between two tribes and the local deer population in Wisconsin, USA is discussed.

Further examples include the modelling of combustion processes in autocatalytic reactions, heat conduction in thin bodies, relaxation waves in the FitzHugh-Nagumo system modelling propagation of excitation in a nerve axon and Allen-Cahn equation modelling phase separation in iron alloys, [39, §4], [10, §3.4], [25]. It is clear that accurate solutions of singularly perturbed problems are important to many areas of science and help in understanding the underlying mechanisms in such systems.

1.0.2 Upper and Lower Solutions and Z-fields

The theory of upper and lower solutions is central to the arguments made for all three problems in this thesis. This theory can also be extended to the discrete space where, with the theory of Z-fields, existence and accuracy of a computed solution can be shown.

A full discussion on upper and lower solutions in various contexts can be found in [27]. For the purposes of this thesis we present upper and lower solutions in the context of a general time-dependent parabolic problem,

$$\frac{\partial u}{\partial t} - Lu = f(x,t,u) \quad \text{in } \Omega, \tag{1.0.4a}$$

$$Bu = h(x,t) \quad \text{on } \partial\Omega, \tag{1.0.4b}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{1.0.4c}$$

where

$$Lu := \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j \frac{\partial u}{\partial x_j}, \quad Bu := \alpha_0(x,t)\frac{\partial u}{\partial n} + \beta_0(x,t)u, \quad (1.0.5)$$

where a_{ij} and b_j are Hölder continuous in Ω, $\partial/\partial n$ is the outward normal derivative and $\alpha_0 \geqslant 0$, $\beta_0 \geqslant 0$ and $\alpha_0 + \beta_0 > 0$ on $\partial\Omega$.

Definition 1. [27, p. 57] A function $\tilde{u} \in C(\bar{\Omega}) \cap C^{1,2}(\Omega)$ is called an upper solution of (1.0.4) if it satisfies the inequalities

$$\frac{\partial \tilde{u}}{\partial t} - L\tilde{u} \geqslant f(x,t,\tilde{u}) \quad \text{in } \Omega, \quad (1.0.6a)$$

$$B\tilde{u} \geqslant h(x,t) \quad \text{on } \partial\Omega, \quad (1.0.6b)$$

$$\tilde{u}(x,0) \geqslant u_0(x) \quad \text{in } \Omega. \quad (1.0.6c)$$

Similarly, $\hat{u} \in C(\bar{\Omega}) \cap C^{1,2}(\Omega)$ is called a lower solution if it satisfies all the reversed inequalities in (1.0.6).

We consider the case where $\tilde{u}$ and $\hat{u}$ are ordered upper and lower solutions, that is $\tilde{u} \geqslant \hat{u}$. Assume f satisfies, for some bounded functions $\underline{c} \equiv \underline{c}(x,t)$ and $\bar{c} \equiv \bar{c}(x,t)$, the following condition

$$-\underline{c}(u_1 - u_2) \leqslant f(x,t,u_1) - f(x,t,u_2) \leqslant \bar{c}(u_1 - u_2) \quad \text{for } \hat{u} \leqslant u_2 \leqslant u_1 \leqslant \tilde{u}. \quad (1.0.7)$$

For some suitable initial iteration $u^{(0)}$ upper and lower sequences, $\{\bar{u}^{(k)}\}$ and $\{\underline{u}^{(k)}\}$, are constructed using the system

$$\frac{\partial u^{(k)}}{\partial t} - Lu^{(k)} + cu^{(k)} = F(x,t,u^{(k-1)}) \quad \text{in } \Omega, \quad (1.0.8a)$$

$$Bu^{(k)} = h(x,t) \quad \text{on } \partial\Omega, \quad (1.0.8b)$$

$$u^{(k)}(x,0) = u_0(x) \quad \text{in } \Omega, \quad (1.0.8c)$$

for $k = 1, 2, \ldots$ and $F(x, t, u) = cu(x, t) + f(x, t, u)$ with $c = \bar{c}$, $c = \underline{c}$ for the upper and lower sequences respectively. For the upper solution the initial iteration is taken as $\bar{u}^{(0)} = \tilde{u}$ and for the lower solution take $\underline{u}^{(0)} = \hat{u}$. The following theorem states that under condition (1.0.7) these converge to a unique solution of (1.0.4) and importantly for our analysis that a unique solution exists between the upper and lower solutions.

Theorem 1.0.1. *[27, Theorem 4.1, p.64] Let $\tilde{u}$, $\hat{u}$ be ordered upper and lower solutions of* (1.0.4) *respectively, and that f satisfies* (1.0.7)*. Then the sequence $\{\bar{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ converges monotonically to a unique solution u of* (1.0.4) *and*

$$\hat{u} \leqslant \underline{u}^{(k)} \leqslant \underline{u}^{(k+1)} \leqslant u \leqslant \bar{u}^{(k+1)} \leqslant \bar{u}^{(k)} \leqslant \tilde{u} \quad \textit{in } \bar{\Omega}. \tag{1.0.9}$$

For the discrete space a similar theory can be used to obtain discrete upper and lower solutions and prove existence of a computed solution. The discrete upper solution is obtained by considering the discrete operator for the system and a discrete version of (1.0.6). The discrete lower solution is obtained similarly where (1.0.6) has again got reversed inequalities.

The theory of Z-fields can now be used to prove existence of a discrete solution and obtain bounds for that solution. A description of Z-fields is given by Lorenz in the unpublished work [21] and also by Kopteva and Stynes in [17]. We present this description here.

Definition 2. [21, p. 6][17, § 3.2] An operator $H : \mathbb{R}^n \to \mathbb{R}^n$ is a Z-field if for all $i \neq j$ the mapping $x_j \mapsto (H(x_1, x_2, \ldots, x_n))_i$ is a monotonically decreasing function from $\mathbb{R}$ to $\mathbb{R}$ when $x_1, ..., x_{j-1}, x_{j+1}, \ldots, x_n$ are fixed.

Remark 1.0.2. If an operator H is differentiable, then H is a Z-field if and only if its Jacobian has non-positive off-diagonal entries, that is its Jacobian is a Z-matrix [21, p.7].

The following lemma provides existence of a computed solution and states that the discrete upper and lower solutions bound the computed solution.

Lemma 1.0.1. *[21, p. 7][17, Lemma 3.1] Let $H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be continuous and a Z-field. Let $r \in \mathbb{R}^{n+1}$ be given. Assume that there exists α, $\beta \in \mathbb{R}^{n+1}$ such that $\alpha \leqslant \beta$ and $H\alpha \leqslant r \leqslant H\beta$. Then the equation $Hy = r$ has a solution $y \in \mathbb{R}^{n+1}$ with $\alpha \leqslant y \leqslant \beta$. (The inequalities are understood to hold true componentwise.)*

Proof. This proof can be found in [21, p. 7] (unpublished) and can also be found in [17, Lemma 3.1]. It is included in Appendix A.2 for completeness. □

Discrete upper and lower solutions and the theory of Z-fields will be used throughout this thesis in order to obtain existence of a discrete solution and accuracy bounds for each problem.

1.0.3 Layer-adapted meshes

Due to the small parameter in (1.0.1), (1.0.2) and (1.0.3) causing regions of rapid change layer-adapted meshes are necessary to gain accuracy in the layer region. By using layer-adapted meshes the layer is resolved and high accuracy is obtained in the entire domain. Two important layer-adapted meshes are the Shishkin mesh and the Bakhvalov mesh which will be described in this section. For a general overview as well as discussions of the two meshes that are described here see [30, Part I §2.4], [29, Chapter I §2.4], [20, Chapter 2], [5, Chapter 3] and [22, Chapter 5 & 6].

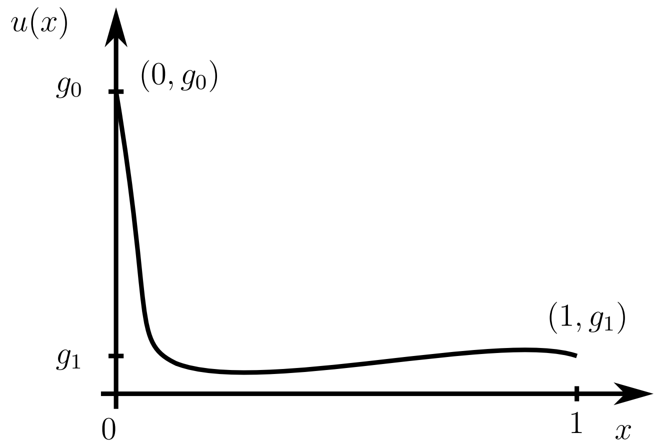


Figure 1.3: An example of a solution for a problem of type (1.0.10) *with a boundary layer at $x = 0$.*

Consider the one-dimensional problem

$$-\varepsilon^2 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + b(x, u) = 0 \quad \text{for } x \in (0, 1), \tag{1.0.10a}$$

$$u(0) = g_0, \quad u(1) = g_1, \tag{1.0.10b}$$

with a boundary layer at $x = 0$ and no boundary layer at $x = 1$. Assume $u_0(x)$ is a solution to the reduced problem, i.e., $b(x, u_0(x)) = 0$, and $b_u(x, u_0) > \beta^2 > 0$. In the majority of the domain the solution remains close to the solution of the reduced problem, $u_0(x)$, while in a narrow region near $x = 0$ the solution behaves like an exponential function changing rapidly between the boundary condition and the reduced solution. An example of the form of such a solution is given in Figure 1.3. Assume the computed solution of (1.0.10) is found by solving an appropriate finite difference approximation with order p.

Consider a mesh with the properties $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$ where $N + 1$ is the number of mesh points in the region $[0, 1]$. The mesh size

is given by $h_i := x_i - x_{i-1}$ and $h := \max_{i=1,\ldots,N} h_i$. A uniform mesh is simply defined as $x_i := i/N$. We now present the Shishkin mesh and the Bakhvalov mesh for problem (1.0.10).

The Shishkin Mesh [33, 34] (see also [15])

The Shishkin mesh first appeared in [33] and [34]. It is a piecewise uniform mesh in one dimension and a tensor product mesh in higher dimensions. It is the simpler of the two meshes considered here. A Shishkin mesh has a fine mesh inside the layer region and a coarse mesh elsewhere. This is represented in Figure 1.4.

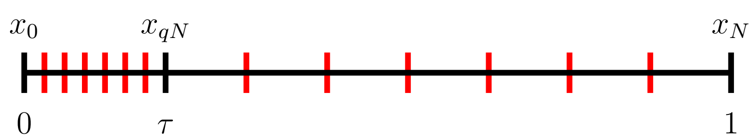


Figure 1.4: The Shishkin mesh for a one-dimensional problem with a boundary layer at $x = 0$ and mesh transition point τ.

The importance of the Shishkin mesh versus other piecewise uniform meshes is based on the choice of transition parameter, τ, the point at which the mesh changes size. The regions $[0, \tau]$ and $[\tau, 1]$ are divided into qN and $(1-q)N$ intervals respectively where N is the number of mesh intervals and q represents the portion of the mesh used to resolve the layer. Hence the mesh size in the first region is $h = \tau/(qN)$ and in the second is $h = (1-\tau)/(N-qN)$. The mesh transition parameter, τ, is chosen to ensure ε-uniform convergence in the discrete maximum norm. Two conditions are enforced;

$$\frac{h_1}{\varepsilon} \to 0 \quad \text{as} \quad N \to \infty, \tag{1.0.11a}$$

and

$$\frac{\tau}{\varepsilon} \to \infty \quad \text{as} \quad N \to \infty, \tag{1.0.11b}$$

where $h_1 := x_1 - x_0$. Now the mesh transition point is chosen such that $\dfrac{\tau}{\varepsilon q N}$ and $e^{-\beta\tau/\varepsilon}$ are approximately the same order, coming from the difference between the exact and computed solutions. This is done by choosing

$$\tau := \min\left\{C_\sigma \varepsilon \ln N, q\right\}, \tag{1.0.12}$$

where C_σ is a user-chosen positive constant such that $C_\sigma \geqslant p/\beta$ recalling p is the order of the method. This results in a mesh that is fine in the layer regions, near $x = 0$, i.e., on $[0, \tau]$, and coarse elsewhere, i.e., on $[\tau, 1]$, as illustrated in Figure 1.4. Note that when N is sufficiently large $C_\sigma \varepsilon \ln N > q$ and so the mesh returns to a uniform mesh throughout the domain.

The error in the discrete maximum norm on a Shishkin mesh is $O(N^{-p} \ln^p N)$ where p is the order of the method. Note [15] for a full description of the mesh and analysis of the importance of the mesh transition parameter.

The Bakhvalov Mesh [1]

The Bakhvalov mesh is a more complex mesh. It has a fine mesh in the layer region grading to a coarse mesh in the outer region. This is contrast to the Shishkin mesh which has a sudden transition from a fine to coarse mesh. We present the Bakhvalov mesh similar to the form given by [20], noting alternate forms in [29], [30] and [17] and the original in [1].

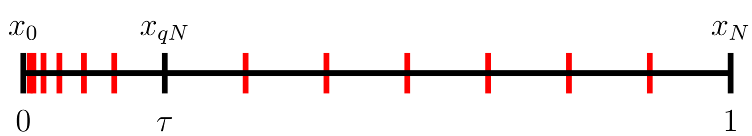


Figure 1.5: The Bakhvalov mesh.

The mesh is defined, close to $x = 0$, as

$$x(i/N) = -C_\sigma \varepsilon \ln\left(1 - \frac{i}{qN}\right) \quad \text{for } \frac{i}{N} \leqslant q - C_3\varepsilon, \tag{1.0.13}$$

and away from $x = 0$ the mesh becomes a uniform mesh with space step d/N where d is user chosen positive constant. The result is a graded mesh that is fine near $x = 0$, gradually becoming sparser as x increases. This is represented in Figure 1.5. We note that if $C_\sigma \varepsilon > q$ the Bakhvalov mesh reverts to a uniform mesh with step size N^{-1}.

The order of convergence for a method of order p on the Bakhvalov mesh is $O(N^{-p})$. The Bakhvalov mesh provides better accuracy compared to the Shishkin mesh. However, the Shishkin mesh is favoured in more complicated problems due to its simplicity.

1.1 Literature Review

A significant number of studies have been carried out on layer solutions of nonlinear reaction-diffusion equations. In this section a review of a number of papers that are central to the arguments in this thesis is included. A comprehensive study of the analytic and numerical behaviour of solutions to singularly perturbed problems exists is provided in [29] and [30]. We refer the reader to [39] for a discussion of boundary function methods including phase plane analysis that is used to prove existence of boundary layer solutions. We also refer the reader to works on numerical methods for singularly perturbed problems such as [22], [5] and [35]. For a discussion of the finite difference method and the finite element method in the context of PDE's we note *The Numerical Treatment of Partial Differential Equations* [11] which includes mesh generation for the finite element method.

1.1.1 Singularly Perturbed Elliptic Boundary Value Problems

Consider the nonlinear reaction-diffusion two point boundary value problem

$$-\varepsilon^2 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + b(x,u) = 0 \quad \text{for } x \in (0,1), \tag{1.1.1a}$$

$$u(0) = g_0, \quad u(1) = g_1, \tag{1.1.1b}$$

where $\varepsilon \ll 1$ and $b(x,u)$ is nonlinear. The reduced solution $u_0(x,t)$ is found by formally setting $\varepsilon = 0$, and solving the remaining equation. The condition $b_u(x,u) > 0$ is often assumed in numerical analysis literature but can be dropped for weaker local assumptions. In [4], [37], [17] and [6] the problem is considered with the following weaker local assumptions;

(i) there exists a stable reduced solution, i.e., there exists $u_0(x)$ such that

$$b_u(x, u_0(x)) > \gamma^2 > 0 \quad \forall x \in [0,1], \tag{1.1.2}$$

(ii) the boundary conditions satisfy

$$\int_{u_0(l)}^{v} b(l,s)ds > 0 \quad \forall v \in (u_0(l), g_l]', \quad l = 0,1, \tag{1.1.3}$$

where the notation $(a,b]'$ is defined as $(a,b]$ when $a < b$, $[b,a)$ when $b < a$ and $(a,b] = \emptyset$ when $a = b$.

These assumptions are necessary for the existence of zero-order boundary layer functions. Asymptotic analysis is carried out in all four papers. D'Annunzio [4] and Sun and Stynes [37] use upper and lower solutions with degree theory to prove existence and local uniqueness of a solution.

In [17] upper and lower solutions and dynamical systems are used to prove existence of the zero-order boundary layer solution and by doing this the

necessity of (1.1.3) is made clearly visible. Asymptotic analysis is carried out and upper and lower solutions are found. The problem is considered using a 3-point central difference scheme on two non-uniform meshes; the Shishkin mesh and the Bakhvalov mesh. It is assumed $\varepsilon \leqslant CN^{-1}$ for N the number of space steps. Using discrete upper and lower solutions the theory of Z-fields is used to prove existence of a discrete solution and accuracy bounds for it. It is found that the numerical method gives a second-order convergence rate in the maximum norm, with a logarithmic factor in the case of the Shishkin mesh.

This is extended by Kopteva in [14] to two dimensions, where the problem

$$-\varepsilon^2 \Delta u + b(x,u) = 0 \quad \text{for } x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \tag{1.1.4a}$$

$$u(x) = g(x) \quad x \in \partial\Omega, \tag{1.1.4b}$$

is considered with $\Delta := \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$, $\varepsilon \ll 1$ and $b(x,u)$ is nonlinear. In this paper Ω is a bounded two-dimensional domain where the boundary, $\partial\Omega$, is a sufficiently smooth closed curve. A curvilinear coordinate system is set up in the layer region, which is then rescaled to find the necessary boundary layer functions. Layer-adapted meshes are used to improve accuracy in the layer region. The discretisation of the domain is taken in two regions with an interface between. The finite difference method is used to solve the system in the layer region, the finite element method is used in the outer region and a fictitious Neumann boundary condition is used along the interface to match the solutions. Both sides of the interface curve are discretised with this fictitious Neumann condition and then combined eliminating the condition. Again, upper and lower solutions and Z-fields are used to obtain accuracy bounds and existence results. It is found that the system has second-order convergence with a logarithmic factor for the Shishkin mesh.

1.1.2 Singularly Perturbed Parabolic Boundary Value Problems

A stabilised method has been introduced by Kopteva and Savescu in [16]. In this paper an initial boundary value problem with a small parameter multiplying the derivative terms is considered;

$$\varepsilon^2\left[\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\right]+f(x,t,u)=0 \quad \text{for } (x,t)\in(0,1)\times(0,T], \tag{1.1.5a}$$

$$u(0,t)=g_0(t), \quad u(1,t)=g_1(t) \quad \text{for } t\in[0,T], \tag{1.1.5b}$$

$$u(x,0)=\varphi(x) \quad \text{for } x\in[0,1], \tag{1.1.5c}$$

where g_0, g_1 and f are sufficiently smooth, f is nonlinear and $t\in\mathbb{R}^+$. This equation exhibits initial and boundary layers. By numerical examples it is shown that conventional methods may give incorrect computed solutions for equations of type (1.1.5). To remedy this, a stabilised method is developed to give correct-looking computed solutions on any mesh and accuracy in the entire domain on layer-adapted meshes.

The condition $f_u(x,t,u)>0$ is dropped and replaced by weaker local assumptions, the first two conditions remain as in §1.1.1, i.e., time-dependent versions of (1.1.2) and (1.1.3) hold, and the final condition is

(i) the initial condition is in the domain of attraction of the reduced solution $u_0(x,t)$, that is

$$sf(x,0,u_0(x,0)+s)>0, \quad s\in(0,\varphi(x)-u_0(x,0)]', \quad x\in[0,1]. \tag{1.1.6}$$

Compatibility conditions are enforced so that the solution is sufficiently smooth, i.e., $\varphi(l)=g_l(0)=u_0(l,0)$ where $l=0,1$ and $u_0(x,t)$ is the solution to the reduced solution.

The instability in which incorrect computed solutions occur is a result of $\varepsilon \ll 1$, in particular when $\varepsilon^2 \ll k_j$ where $k_j = t_j - t_{j-1}$ is the time step. In this case the size of the discrete time derivative term is $\varepsilon^2 \delta_t U_{ij} = O(\varepsilon^2/k_j)$ and so is negligible. In this situation the system solves a steady state discrete equation at each time step. As there are multiple solutions to the reduced equation the discrete system may jump between solutions to the reduced problem. To repair this the parameter in front of the time derivative is artificially strengthened by replacing ε^2 with a stabilisation parameter. This strengthens the time derivative adding artificial diffusion where necessary and ensures the system 'remembers' the solution at the previous time step.

The stabilised form of the discrete system is given by

$$[\hat{\varepsilon}^2(t_j)\delta_t - \varepsilon^2\delta_{xx}]\hat{U}_{ij} + f(x_i, t_j, \hat{U}_{ij}) = 0, \quad \hat{\varepsilon}^2(t_j) := \max\{\varepsilon^2, \widehat{C}k_j\}, \quad (1.1.7a)$$

$$\hat{U}_{0j} = g_0(t_j), \quad \hat{U}_{Nj} = g_1(t_j), \quad j = 1, \ldots, M, \quad (1.1.7b)$$

$$\hat{U}_{i0} = \varphi(x_i), \quad i = 0, \ldots, N, \quad (1.1.7c)$$

where δ_t and δ_{xx} are the discretisations of $\partial/\partial t$ and $\partial^2/\partial^2 x$, $\hat{U}_{ij}$ is the stabilised discrete solution and k_j is the time step. The parameter $\hat{C}$ is used to control the stabilisation and ensure there exists a unique computed solution. The method relies on the derivative of the nonlinear function with respect to the solution having a lower bound, that is the parameter is chosen such that $\widehat{C}^2 \leqslant -\min f_u(x, t, u)$. Taking $\widehat{C} = 0$ gives the standard method.

The change made by adding artificial stabilisation does not change the order of the method and Kopteva and Savescu show that with an appropriate choice of $\widehat{C}$ the method always gives at most one discrete solution. The method has second-order convergence in space and first-order convergence in time in the maximum norm, with a logarithmic factor for the case of the Shishkin mesh.

1.1.3 Singularly Perturbed Elliptic Problems Exhibiting Interior Layer Solutions

Kopteva and Stynes [18, 19] consider an equation of the form (1.1.1) with interior layer solutions,

$$-\varepsilon^2 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + b(x, u) = 0 \quad \text{for } x \in (0, 1), \tag{1.1.8a}$$

$$u(0) = g_0, \quad u(1) = g_1. \tag{1.1.8b}$$

They carry out asymptotic analysis and study a discretisation of (1.1.8) to find accuracy and existence of computed solutions.

Certain assumptions are given that intrinsically arise in the asymptotic analysis of such a problem. These assumptions give a unique solution to (1.1.8). They are

(i) the reduced equation has three solutions,

$$b(x, \varphi_k(x)) = 0 \quad \text{for } k = 0, 1, 2, \tag{1.1.9}$$

(ii) these solutions satisfy

$$\varphi_1(x) < \varphi_0(x) < \varphi_2(x), \tag{1.1.10}$$

and no other solution exists between $\varphi_1(x)$ and $\varphi_2(x)$,

(iii) the solutions $\varphi_1(x)$ and $\varphi_2(x)$ are stable reduced solutions, i.e.,

$$b_u(x, \varphi_i(x)) > 0 \quad \text{for } i = 1, 2,\ x \in \bar{\Omega}, \tag{1.1.11}$$

(iv) the solution $\varphi_0(x)$ is an unstable reduced solution, i.e.,

$$b_u(x, \varphi_0(x)) < 0 \quad \text{for } x \in \bar{\Omega}, \tag{1.1.12}$$

(v) the transition point t_0 is defined by

$$\int_{\varphi_1(t_0)}^{\varphi_2(t_0)} b(t_0, v)dv = 0 \tag{1.1.13}$$

with

$$\int_{\varphi_1(t_0)}^{\varphi_2(t_0)} b_x(t_0, v)dv = -C_I < 0, \tag{1.1.14}$$

(vi) there are no boundary layers,

$$\varphi_1(0) = g_0, \quad \varphi_2(1) = g_1, \quad \varphi_1''(0) = \varphi_2''(1) = 0. \tag{1.1.15}$$

Under these assumptions the solution lies in a neighbourhood of $\varphi_1(x)$ for a portion of the domain, in a neighbourhood of $\varphi_2(x)$ for another portion of the domain and has a rapid jump between the two regions at t_0.

Numerical results are given that have incorrect computed solutions, with the solution jumping many times between different solutions to the reduced problem. In another example it is shown that with different initial guesses, different solutions are obtained that at first glance appear correct but by carrying out asymptotic analysis it is found that the jump occurs in the wrong location in some of these cases.

The Bakhvalov mesh is omitted from this analysis due to the high difficulty of the problem and to avoid further complexity. Existence is proven and accuracy results are shown to be $O([N^{-1} \ln N]^{2-\sigma})$ with $\sigma \in [0, 2]$ inside the layer region and $O(N^{-2})$ outside. The stabilised method of [16] is employed to cure the numerical instability in the problem and is shown to have accuracy $O([N^{-1} \ln N]^{2-\sigma} + N^{-1})$ inside the layer region and $O(N^{-1})$ outside.

Existence of an interior layer solution for the continuous two-dimensional elliptic problem has been carried out by [24],

$$-\varepsilon^2 \Delta u + b(x, u, \varepsilon) = 0, \quad x \in \Omega \subset \mathbb{R}, \tag{1.1.16a}$$

$$u(x, \varepsilon) = g(x), \quad x \in \partial\Omega. \tag{1.1.16b}$$

In [24] interior layer solutions of (1.1.16) are analysed and an asymptotic expansion is carried out. From this, existence of a solution is proved and accuracy of the asymptotic expansion is found using similar methods to the previous papers discussed. In this paper both boundary and interior layers are considered. Similar assumptions are made in [24] to that in [18]. The first four of these resemble two-dimensional versions of (1.1.9)-(1.1.12) with $\varphi_0(x) = 0$. The transition point in one dimension becomes the transition curve C in two dimensions whose asymptotic expansion is given by C_λ with leading order term C_0. The curve C_0 is defined by a two-dimensional version of (1.1.13), that is

(i)

$$I(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} b(x, u, 0) du = 0 \quad \text{for } x \in C_0. \tag{1.1.17}$$

This is the leading order term in the asymptotic expansion of the curve along which the solution switches between solutions of the reduced equation. The two-dimensional version of (1.1.14) becomes

(ii)

$$\frac{\partial}{\partial n} I(x) < 0 \quad x \in C_0. \tag{1.1.18}$$

The assumption

(iii)

$$\int_{\varphi_1(x)}^{s} b(x, u, 0) du > 0 \quad \text{for } s \in (\varphi_1(x), \varphi_2(x)), x \in \bar{\Omega}, \tag{1.1.19}$$

is made as well as a two-dimensional version of (1.1.3) to give existence of boundary layer solutions. The true location of the transition curve is given by

$$u(x, \varepsilon) = \varphi_0(x) = 0, \quad x \in C. \tag{1.1.20}$$

It is found that the asymptotic expansion $u^n_{as}(x, \varepsilon) = \sum_i^n \varepsilon^i (u_i + v_i)$ where v_i are boundary layer functions, satisfies

$$\max_{\bar{\Omega}} |u(x, \varepsilon) - u^n_{as}(x, \varepsilon)| = O(\varepsilon^{n+1}). \tag{1.1.21}$$

Variations of the assumptions given in this section can be found in [42] for a one-dimensional time-periodic problem, and in [40], [41] and [8]. Fife and Greenlee [8] consider the interior layer solution of (1.1.16) with assumptions (1.1.9)-(1.1.14) and (1.1.17)-(1.1.18). They find that there exists an interior layer solution to (1.1.16) by using the implicit function theorem. This involves a decomposition of the space into two parts and hence a decomposition of the problem into two parts. Results given are not as strong as we propose as the technique used cannot be easily imitated in numerical analysis. A version of (1.1.17) can be found in [7, §V].

1.1.4 Parabolic Boundary Value Problems with Singularly Perturbed Neumann Boundary Conditions

The time periodic reaction-diffusion equation considered in [2] is

$$\varepsilon^2 \left[\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \right] + f(x, t, u, \varepsilon) = 0 \quad \text{for } (x, t) \in (-1, 1) \times \mathbb{R}, \tag{1.1.22a}$$

$$\varepsilon \left. \frac{\partial u}{\partial x} \right|_{x=-1} = u^{(-)}(t), \quad \varepsilon \left. \frac{\partial u}{\partial x} \right|_{x=1} = u^{(+)}(t) \quad \text{for } t \in \mathbb{R}, \tag{1.1.22b}$$

$$u(x, t, \varepsilon) = u(x, t + T, \varepsilon) \quad \text{for } x \in [-1, 1],\ t \in \mathbb{R}, \tag{1.1.22c}$$

where $u^{(\pm)}$ and f are sufficiently smooth and T-periodic in time, f is non-linear and $0 < \varepsilon \leqslant \varepsilon_0 \ll 1$. The solution of (1.1.22) exhibits boundary layers as the reduced solution does not necessarily meet the boundary conditions. In [2], Butuzov *et al.* carry out asymptotic analysis and give the necessary assumptions for existence of boundary layer functions. These assumptions include that (1.1.2) and

(i) $sf(l, t, u_0 + s, 0) > 0$ for $l = -1, 1$,

hold. This second assumption is required for the existence of boundary layer functions. A formal asymptotic analysis is carried out which is then modified to obtain upper and lower solutions to the problem. The main result is (1.1.21) for (1.1.22).

In [31, Chapter 3] a singularly perturbed time-dependent semilinear reaction-diffusion problem with homogeneous Neumann boundary conditions is discussed. Here a corner layer exists and so corner layer functions are considered and bounds are found for these as well as the initial and boundary layer functions. The stabilised method is again employed due to incorrect computed solutions with the conventional method. We also refer the reader to another examples of singularly perturbed Neumann boundary conditions in [12].

1.2 Outline of Thesis and Significance of Work

The purpose of this study is to obtain numerical results for three problems; interior layer solutions of a singularly perturbed elliptic equation with Dirichlet boundary conditions and singularly perturbed reaction-diffusion equations with singularly perturbed Neumann boundary conditions in two cases; the time-dependent case and the two-dimensional steady state case. The thesis has three chapters of analysis followed by a chapter of numerical results. Finally an appendix is included.

In *Chapter 2*, the largest and most significant chapter, we consider interior layer solutions of (1.0.1). We prove existence and accuracy bounds for discrete solutions of this problem. Under review of the literature there has been no satisfactory numerical analysis of a problem of this type found. The one-dimensional case has been considered by Kopteva and Stynes [18], as discussed in §1.1.3, and we extend this to two dimensions. An extension of this approach to two dimensions is not straightforward and we have to address a number of technical difficulties. We take a second order asymptotic expansion, as was the case in one dimension, but now also have to introduce curvilinear coordinates, and hence include curvature terms. We consider each region of the discrete system separately following ideas considered by Kopteva [14] in the simpler setting of a two-dimensional boundary layer problem. With the complexity of the interior layer problem this is a non-trivial task.

Notably, there are two areas of the extension that become particularly technical; finding the equations for the layer functions and order of the system and calculating the truncation error.

Equations for the interior layer functions are much more complex than in the one-dimensional case. We introduce curvilinear coordinates in the layer region and hence have curvature terms in the right hand sides of the layer equations. In order to obtain bounds for the layer functions and their deriva-

tives, we have to calculate higher order derivatives of the layer equations, some needing derivatives up to and including sixth order; this entails many more calculations with many more terms than previously needed in [14] or [18].

The truncation error analysis in this case becomes substantially more difficult than either of the papers mentioned above. We consider a second order asymptotic expansion, as mentioned, and the upper and lower solutions contain functions that are discontinuous across the transition curve and involve many terms. In the layer region we use the finite difference method in curvilinear coordinates which gives further terms that need to be carefully considered. All of this leads to a highly technical and complex analysis and hence the truncation error analysis is broken down to simpler settings to assist with the complexity of this problem.

The chapter is organised as follows. We will first make certain local assumptions that are required for the existence of interior layer solutions to such a problem and the restrictive global condition $b_u(x, u) > 0$ will not be enforced. The analysis here will be considered using the order one location of the transition curve allowing placement of a non-uniform mesh with relative ease; asymptotic analysis of the problem using the exact location of the transition curve has been done by [24]. Upper and lower solutions and the theory of Z-fields will be employed to prove existence and obtain accuracy results for a computed solution. Due to incorrect computed solutions on the standard discretisation we will consider the stabilised method of [16] for this problem. Technical properties of the asymptotic expansion and one-dimensional analogues of the truncation error analysis are deferred to the final section of the chapter. We also include a nomenclature for this chapter at the end of the thesis due to the extensive list of notation required for it.

In *Chapter 3* we look at a time-dependent one-dimensional reaction-diffusion equation with singularly perturbed Neumann boundary conditions. We con-

sider [2] for assumptions that are necessary for existence of a boundary layer solution, and aim to prove existence and find accuracy results for the computed solution. From review of the literature no numerical analysis of such a problem was found. To carry out the analysis upper and lower solutions will be employed as well as the theory of Z-fields as in *Chapter 2*. Solutions with boundary and initial layers will be considered and the existence of corner layer solutions will be discussed.

In *Chapter 4* a two-dimensional steady state reaction-diffusion equation with singularly perturbed Neumann boundary conditions will be considered. Problems of this type involving boundary layer solutions will be the topic of the chapter. Using similar methods to *Chapter 2* and *Chapter 3* we aim to prove existence and accuracy bounds for the problem.

In *Chapter 5* we will describe the implementation of the discretisation and the method for obtaining error bounds and computational rates. We will include numerical results for *Chapter 2* and *Chapter 3*. Error bounds and computational rates will be given for the standard and stabilised method and compared with the theoretical results for *Chapter 3*.

An appendix is included at the end including phase plane analysis for *Chapter 2* and auxiliary proofs.

1.2.1 Chapter Referencing and Notation

Throughout this thesis there are a number of notational conventions that we follow. Crucial assumptions that hold true throughout each chapter are denoted (A1), (A2), etc, in *Chapter 2*, (B1), (B2), etc, in *Chapter 3* and (C1), (C2), etc, in *Chapter 4*.

Let C be a generic positive constant that may take different values in different formulae. A subscripted C (e.g., C_1), is a positive fixed constant that is independent of ε and of the mesh. We define θ such that $\theta \in (0, 1)$. The notation $(a, b]'$ is defined as $(a, b]$ when $a \leqslant b$ and $(b, a]$ when $b \leqslant a$.

For any function $v \in C(\bar{\Omega})$, v_{ij} refers to $v(x_{ij})$, where $x_{ij} \in \bar{\Omega}$ are mesh nodes. We use $v(x)$, with no subscript, as an arbitrary function that is used in different contexts. The notation $f = O(z)$ means $|f| \leqslant Cz$ for some positive constant C. Finally when a normal to a curve or domain is considered it is assumed that it is the outward normal.

Chapter 2

Singularly Perturbed Nonlinear Elliptic Problem Exhibiting Interior Layer Solutions

In this section, we look at interior layer solutions of a singularly perturbed nonlinear elliptic reaction-diffusion problem

$$Fu(x) := -\varepsilon^2 \Delta u + b(x, u) = 0 \quad \text{for } x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \tag{2.0.1a}$$

$$u(x) = g(x) \quad \text{for } x = (x_1, x_2) \in \partial\Omega, \tag{2.0.1b}$$

where ε is a small parameter, $b(x, u)$ is a nonlinear function, $g(x)$ is a given boundary condition and $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. We denote Ω to be a bounded two-dimensional domain with sufficiently smooth boundary $\partial\Omega$. The functions $b(x, u)$ and $g(x)$ are sufficiently smooth. We are looking for solutions of this problem that exhibit interior layers. The reduced problem is found by formally setting $\varepsilon = 0$ in (2.0.1);

$$b(x, \varphi(x)) = 0, \quad \text{for } x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2. \tag{2.0.2}$$

$$u(x) = g(x) \quad \text{for } x = (x_1, x_2) \in \partial\Omega, \tag{2.0.3}$$

The outline of the chapter is as follows. In §2.1 we give an analysis of the numerical difficulties involved in solving (2.0.1). In §2.2 necessary assumptions are made for existence of an interior layer solution. We then present the main result of the chapter in §2.3. In §2.4 we consider asymptotic analysis and create upper and lower solutions for the system. A discretisation of the system is given in §2.5 and discrete upper and lower solutions are found as well as accuracy bounds for the system. A technical section, §2.7, is also included at the end of the chapter with phase plane analysis and truncation error analysis that are necessary but not central to our argument. As there is a substantial amount of notation contained in this chapter, an index of notation is included at the end of the thesis to assist the reader.

2.1 Numerical Difficulties of the Problem

We now show the challenges involved in solving equations of this type. We consider (2.0.1) with

$$b(x, u) = (u - \varphi_b)u(u + 1), \tag{2.1.1a}$$

where

$$\varphi_b(x) = 1.5 - \frac{\rho_b(x)}{\rho_b(x) + 1} \quad \text{and} \quad \rho_b(x) = \left(\frac{x_1}{0.5}\right)^2 + \left(\frac{x_2}{0.4}\right)^2, \tag{2.1.1b}$$

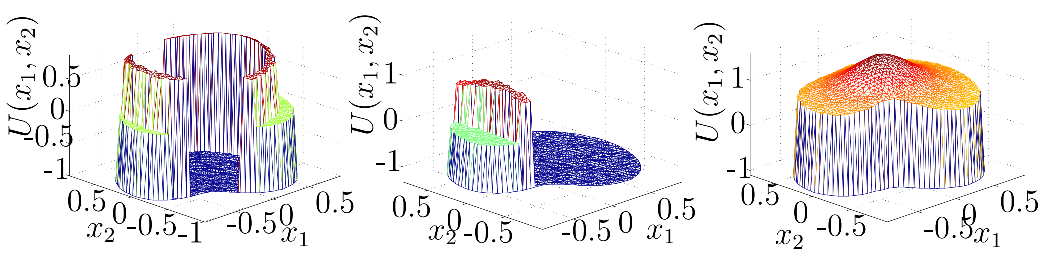
and the boundary condition,

$$u(x) = -1, \quad x \in \partial\Omega. \tag{2.1.1c}$$

We solve this on the domain Ω whose boundary, $\partial\Omega$, is represented by $x_1 = R\sin\theta$ and $x_2 = 1.5R\cos\theta$ with

$$R = R(l) = 0.4 + \frac{\cos^2(l)}{2}, \quad \theta = \theta(l) = l + e^{(l-5)/2}\sin(l/2)\sin l, \quad l \in [0, 2\pi]. \tag{2.1.2}$$

This problem has two solutions; the trivial solution $u(x) \approx -1$ and an interior layer solution. To solve this problem we use lumped mass finite elements on a quasiuniform Delaunay triangulation with Newton's method. A full description of the method used will be given in *Chapter 5*, however we show some results to illustrate the difficulty of such a problem here.



(a) Incorrect solution obtained using an initial guess of $x_1^2 + x_2^2 - 1$.

(b) Incorrect solution obtained using an initial guess of $-1 - x_1$.

(c) Incorrect solution obtained using an initial guess of 1.

Figure 2.1: Incorrect solutions found using the conventional method with $\varepsilon = 10^{-3}$. See §5.1 for larger images of the solutions.

Using initial guesses $u(x) = x_1^2 + x_2^2 - 1$, $u(x) = -1 - x_1$ and $u(x) = 1$ results in the three different solutions shown in Figure 2.1. Figure 2.1a and Figure 2.1b may be considered unreasonable enough to not warrant further investigation. However, Figure 2.1c appears to be a plausible solution to the system. This is a boundary layer solution. Without doing asymptotic analysis on this problem one could easily, and wrongly, assume this to be a correct solution to (2.0.1) with (2.1.1). We again refer the reader to *Chapter 5* where

asymptotic analysis is carried out showing there is no boundary layer solution to (2.0.1) with (2.1.1) and hence Figure 2.1c is also an incorrect solution.

As will be demonstrated in §2.2, an interior layer solution will have a jump in its solution occurring along a smooth closed curve in $\Omega\backslash\partial\Omega$. None of the above graphs have this property. Also from §2.2, we can say the solution will switch from $u \approx -1$ in the exterior of the domain to $u \approx \varphi_b(x)$ in the interior. The correct solution is represented in Figure 2.2.

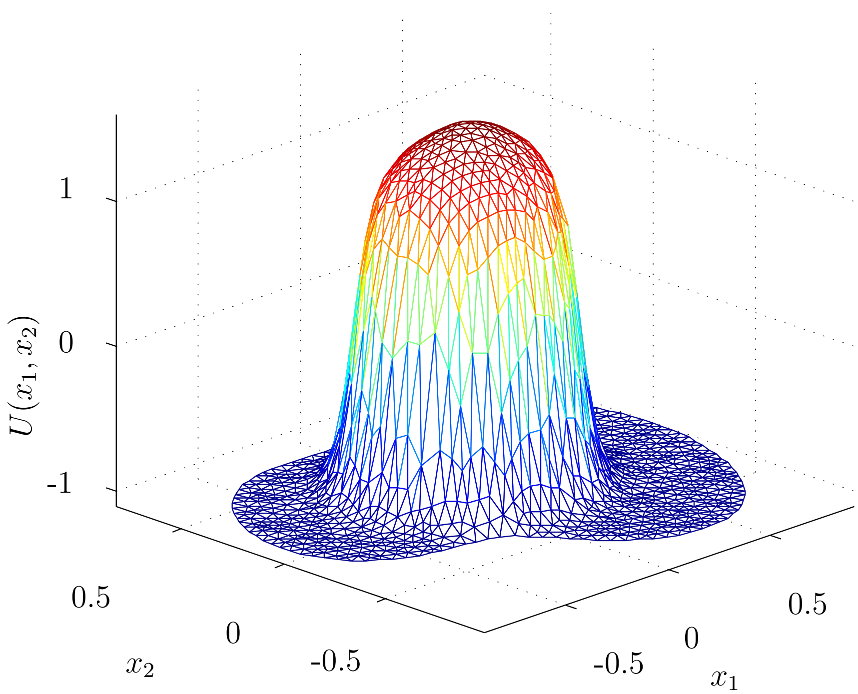


Figure 2.2: The correct solution of (2.0.1) *with* (2.1.1) *using the stabilised method of §2.1.1 with the scheme described in Chapter 5 and initial guess of* 1*. Other initial guesses give the trivial solution $u(x) = -1$ for all $x \in \bar{\Omega}$.*

In the one-dimensional case, [18], incorrect computed solutions are also found. The following system is considered

$$-\varepsilon^2 \frac{\partial^2 u}{\partial x^2} + u(u-1)(u-x-3/2)(u+x+3/2) = 0 \quad \text{for } x \in (0,1), \tag{2.1.3a}$$

with boundary conditions

$$u(0) = 0, \quad u(1) = 2.5. \tag{2.1.3b}$$

Three interior layer solutions are found by using three different initial guesses on an appropriate finite difference scheme with a uniform mesh. The solutions are shown in Figure 2.3. By asymptotic analysis the order one location of the jump in the solution is found to be T_0 and as this does not match up with the layer location of two of the computed solutions in Figure 2.3 these are found to be incorrect. The correct solution is represented by the central red dashed line.

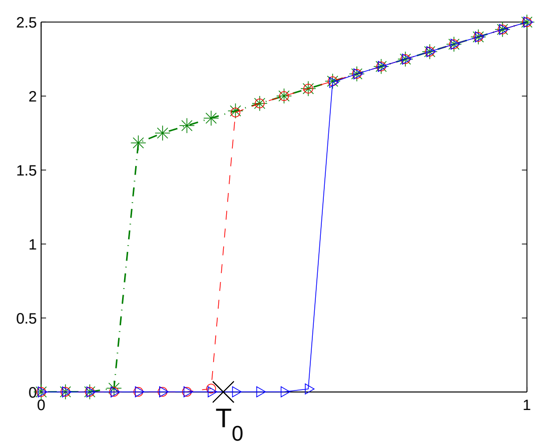


Figure 2.3: Incorrect computed solutions for the one-dimensional problem (2.1.3) *of [18]. Three different initial conditions giving three different answers. All three appear to be correct.*

In both one dimension and two dimensions, if ε is small, in particular, if $\varepsilon \ll N^{-1}$, then the discretisation of the Laplacian on an equidistant mesh, or on the coarse part of a non uniform mesh, is $O(\varepsilon^2 N^2)$ and hence the contribution from this term is negligible. Without these terms the system is left to solve the reduced problem, $b(x, u(x)) = 0$, and hence can give incorrect computed solutions. A worrying factor from this analysis is that some solutions appear to be correct if further analysis has not been carried out, i.e., the two incorrect solutions in the one-dimensional case in Figure 2.3 and the boundary layer solution in the two-dimensional problem represented in Figure 2.1.

To deal with this numerical instability we propose a stabilised method in §2.1.1 based on [18]. This gives solutions of the correct type that are not strongly dependent on the initial guess.

2.1.1 Standard and Stabilised Numerical Schemes

Due to the incorrect computed solutions obtained using conventional methods we employ a stabilised method of [16]. This stabilised method is obtained by replacing ε in the numerical scheme with a new parameter $\hat{\varepsilon}(x)$. This new parameter is picked up outside the layer region. It will artificially strengthen the diffusion term of the system where it is needed, i.e., away from the interior layer and in the case of a layer adapted mesh where the mesh is coarse. This stabilised scheme is presented in [16] for a time-dependent parabolic problem and in [18, 19] for the one-dimensional interior layer problem. We extend this to two dimensions.

We define the discrete stabilised problem with solution $\hat{U}_i$ for the set of all mesh nodes, $X_i \in \Omega$, as

$$\hat{F}\hat{U}_{ij} := -\hat{\varepsilon}^2(X_i)\Lambda^N \hat{U}_i + b(X_i, \hat{U}_i) = 0, \quad X_i \in \Omega \subset \mathbb{R}^2, \tag{2.1.4a}$$

$$\hat{U}_i = g(X_i), \quad X_i \in \partial\Omega, \tag{2.1.4b}$$

where $\hat{\varepsilon}(X_i)$ will be described in §2.5.2 and Λ^N is the discrete representation of the two-dimensional Laplacian operator.

2.2 Hypotheses for the Continuous Problem

In this section we describe a number of hypotheses that intrinsically arise with the interior layer problem. These are given as follows.

The reduced problem, (2.0.2), has three simple roots $\varphi = \varphi_k(x) \in C^\infty(\bar{\Omega})$ for $k = 0, 1, 2,$

$$b(x, \varphi_k(x)) = 0 \quad \text{for } k = 0, 1, 2 \text{ and } x = (x_1, x_2) \in \bar{\Omega}, \tag{A1}$$

where

$$\varphi_1(x) < \varphi_0(x) < \varphi_2(x) \quad \text{for } x = (x_1, x_2) \in \bar{\Omega}, \tag{A2}$$

and there are no other solutions between $\varphi_1(x)$ and $\varphi_2(x)$.

Assume that

$$b_u(x, \varphi_k(x)) > \gamma^2 > 0 \quad \text{for } k = 1, 2 \text{ and } x = (x_1, x_2) \in \bar{\Omega}, \tag{A3}$$

$$b_u(x, \varphi_0(x)) < 0 \quad \text{for } x = (x_1, x_2) \in \bar{\Omega}. \tag{A4}$$

Assumptions (A1)-(A4) describe a bistable equation. Assumptions (A3) and (A4) state that $\varphi_1(x)$ and $\varphi_2(x)$ are stable solutions to the reduced problem while $\varphi_0(x)$ is an unstable solution. From this it can be seen that on some subdomain of Ω, the solution of the full problem (2.0.1) will be very close to $\varphi_1(x)$, on another it will be very close to $\varphi_2(x)$, while it will not be close to $\varphi_0(x)$ on any subdomain.

We denote Γ is the transition curve along which

$$u(x)|_{x\in\Gamma} = \varphi_0(x)|_{x\in\Gamma} + O(\varepsilon). \tag{2.2.1}$$

This relationship is discussed further in Remark 2.4.1. The curve Γ is a smooth closed curve inside Ω and is not known in advance. It can be represented by an asymptotic expansion in ε, with leading order term Γ_0. The use of this definition of the transition curve is extremely beneficial to us. For the method we only require the order one location of the transition curve to be known. We will centre a layer adapted mesh around this location. For error analysis we will consider the asymptotic expansion including higher order terms, see (2.4.21) and Lemma 2.4.5. Consider the function

$$\mathcal{I}(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} b(x,v)dv. \tag{2.2.2}$$

Assume

$$\mathcal{I}(x) = 0 \quad \text{for } x = (x_1, x_2) \in \Gamma_0, \tag{A5a}$$

defines a smooth closed curve, Γ_0, in Ω.

Also assume

$$0 < C^* \leqslant \partial_n \mathcal{I}(x), \quad \text{for } x = (x_1, x_2) \in \Gamma_0, \tag{A5b}$$

where ∂_n is the outward normal derivative to the curve Γ_0 and C^* is a positive constant.

Condition (A5b) states $\partial_n \mathcal{I}(x)$ is bounded away from zero. This means the root found in condition (A5a) is simple, as the derivative of the integral does not equal zero. The positive sign of the derivative corresponds to the Lyapunov stability of an interior layer solution of the time-dependent problem. This states the solution switches from $\varphi_1(x)$ near $\partial\Omega$ to $\varphi_2(x)$ in the

centre of Ω. A negative derivative implies the switch would be from $\varphi_2(x)$ near $\partial\Omega$ to $\varphi_1(x)$ in the centre.

In Lemma 2.4.5 we will require $\partial_n I(x) \neq 0$ so that higher order terms in the asymptotic expansion of Γ can be found, see (2.4.66). Furthermore in Lemma 2.4.5 we will require $\partial_n I(x) > 0$ so that we can prove necessary results needed for existence of upper and lower solutions, see (2.4.76). Assumption (A5) is made in other literature on interior layer solutions, i.e., [40], [7] and [8] and for a reaction-advection-diffusion equation see [26]. It has also been made in [18] and [24], as discussed in §1.1.3; we note an inward unit normal was used in these hence the sign of $\partial_n I(x)$ is switched.

In Nefedov [24] the full asymptotic expansion of the transition curve is used. The curve Γ is unknown a priori and dependent on ε. The order one location of this curve can be found a priori while higher order terms are very difficult to evaluate. To obtain accuracy results and use a non uniform mesh we cannot use the true location of the curve nor the full asymptotic location Γ as we cannot centre a non-uniform mesh around an unknown location. For this reason we consider the order one location Γ_0 in the following work.

Along with hypotheses (A1)-(A5), one further assumption is made to simplify the presentation,

$$\varphi_1(x) = g(x), \quad \Delta\varphi_1(x) = 0 \quad \text{ for } x = (x_1, x_2) \in \partial\Omega. \tag{A6}$$

It is not in general true that any of the reduced solutions will match the boundary condition but for the sake of this analysis we assume (A6) to avoid solutions involving boundary layers and focus on those involving interior layers only. The first condition implies $u_0(x) = g(x)$ for $x \in \partial\Omega$. The second condition comes from putting the reduced solution back into (2.0.1) and ensuring the $O(\varepsilon^2)$ terms match. For a problem involving boundary layers as well as interior layers this work can be combined with the two-dimensional

boundary layer problem by Kopteva [14].

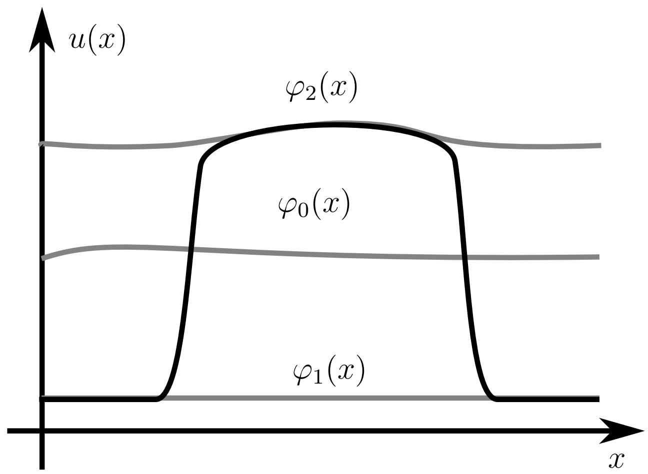


Figure 2.4: Sketch of a cross section of a solution to the two-dimensional problem.

Assumptions (A1)-(A6) imply there exists a solution lying in the neighbourhood of $\varphi_1(x)$ on the outer part of the domain. This solution rapidly switches to $\varphi_2(x)$ along the curve Γ_0 resulting in an interior layer of width $O(\varepsilon|\ln\varepsilon|)$. Figure 2.4 represents a cross section through the expected solution of the two-dimensional problem.

We consider problems where ε is small, i.e.,

$$\varepsilon \leqslant CN^{-1}, \tag{2.2.3}$$

where C is an arbitrary fixed constant. If ε was not small the analysis of this problem would be very different.

2.3 Existence and Accuracy of Discrete Solutions; Main Results

We now give the main results of the chapter. Considering the discretisation of the system (2.0.1) that will be given by the discrete systems (2.5.13), (2.5.80) and (2.5.102), the stabilised method of §2.5.2, and the Shishkin mesh of §2.5.1, the following theorems hold.

Theorem 2.3.1. *Let the mesh $\{r_i, l_j\}$ be the Shishkin mesh of §2.5.1. Set $C' = 4C_\tau/\bar{\gamma}$. Let N be sufficiently large and ε sufficiently small. For some $\varpi \in [0,2]$ we assume $c_0\varepsilon \geqslant (C'N^{-1}\ln N)^{2+\varpi}$,*

(i) If $C_\tau > 2$, then there is a discrete solution U of (2.5.13), (2.5.80), (2.5.102) *such that for N sufficiently large,*

$$|U(X_i) - u(X_i)| \leqslant C \begin{cases} (N^{-1}\ln N)^{2-\varpi} & \text{for } X_i \in \Omega_{(-\tau,\tau)} \\ N^{-2} & \text{for } X_i \in \bar{\Omega}^N \backslash \Omega_{(-\tau,\tau)}. \end{cases} \tag{2.3.1}$$

(ii) If $C_\tau > 3$, then there is a discrete solution $\hat{U}$ of (2.5.13), (2.5.80), (2.5.102) *with ε replaced by $\hat{\varepsilon}(X_i)$ from* (2.5.6) *such that for N sufficiently large,*

$$|\hat{U}(X_i) - u(X_i)| \leqslant C \begin{cases} (N^{-1}\ln N)^{2-\varpi} + N^{-1} & \text{for } X_i \in \Omega_{(-\tau,\tau)} \\ N^{-1} & \text{for } X_i \in \bar{\Omega}^N \backslash \Omega_{(-\tau,\tau)}. \end{cases} \tag{2.3.2}$$

The next theorem considers the case where the relationship between N and ε is stronger than (2.2.3), that is, $\varepsilon \leqslant CN^{-\varpi'}$ for some $\varpi' \geqslant 4 - \lambda$.

Theorem 2.3.2. *Let the mesh $\{r_i, l_j\}$ be the Shishkin mesh of §2.5.1. Fix $\lambda \in (0,1)$. Assume that $\varepsilon \leqslant CN^{-\varpi'}$ for some $\varpi' \geqslant 4-\lambda$ and $C > 0$, and N is sufficiently large independently of ε.*

(i) If $C_\tau > 2$, then there exists a solution U of the standard scheme (2.5.13), (2.5.80), (2.5.102) *such that*

$$|U(X_i) - u(X_i)| \leqslant CN^{-\min\{2,\varpi'-2\}} \leqslant CN^{-(2-\lambda)} \quad \text{for } X_i \in \mathring{\Omega}^N \cup \Gamma_{\pm\tau}. \tag{2.3.3}$$

(ii) If $C_\tau > 1$, then there exists a solution $\hat{U}$ of the stabilised scheme (2.5.13), (2.5.80), (2.5.102) *such that*

$$|\hat{U}(X_i) - u(X_i)| \leqslant CN^{-1} \quad \text{for } X_i \in \mathring{\Omega}^N \cup \Gamma_{\pm\tau}. \tag{2.3.4}$$

Theorem 2.3.1 and Theorem 2.3.2 state that the convergence rate is dependent on the relative sizes of ε and N. In practice, for a particular ε, N is chosen such that ϖ and ϖ' are minimised and the best possible convergence rate is obtained.

As Theorem 2.3.1 and Theorem 2.3.2 do not give ε-uniform accuracy in the entire domain, post-processing is used to obtain this for small values of ε, i.e., $\varepsilon \in (0, \bar{\varepsilon})$ with $\bar{\varepsilon} := CN^{-2}\ln^3 N$ for some positive constant N. The post-processed solution is defined as $\{\tilde{u}_i^N\}$. Remark 2.5.7 gives the results

$$|\tilde{u}_i^N - u(\tilde{X}_i)| \leqslant CN^{-2}\ln^4 N \quad \forall\, \tilde{X}_i \in \bar{\Omega}^N, \tag{2.3.5}$$

for the standard method and

$$|\tilde{u}_i^N - u(\tilde{X}_i)| \leqslant C\max\{N^{-2}\ln^4 N, N^{-1}\} \quad \forall\, \tilde{X}_i \in \bar{\Omega}^N, \tag{2.3.6}$$

for the stabilised method.

The results of Theorem 2.3.1, Theorem 2.3.2 and from post processing are consistent with the one-dimensional case by Kopteva and Stynes [18]. Away from the layer region the results for the standard method are also consistent with [14].

2.4 Analysis of the method

2.4.1 Local Curvilinear Coordinates

We create local curvilinear coordinates in the interior layer, i.e., the region near Γ_0, by the following procedure. Parameterise Γ_0 by

$$x_1 = q_1(l), \quad x_2 = q_2(l), \quad 0 \leqslant l \leqslant L, \tag{2.4.1}$$

with $(q_1(0), q_2(0)) = (q_1(L), q_2(L))$ and l increasing in the anticlockwise direction. We have

$$\tau = \tau(l) = \sqrt{q_1'^2 + q_2'^2}, \quad \kappa = \kappa(l) = \frac{q_1' q_2'' - q_2' q_1''}{\tau^3}, \tag{2.4.2}$$

where τ is the magnitude of the tangent vector $(q_1'(l), q_2'(l))$, with $\tau > 0$ and κ is the curvature of Γ_0 at $(q_1(l), q_2(l))$. In a narrow strip near the curve Γ_0 introduce curvilinear local coordinates (r, l) by

$$x_1 = q_1(l) + r n_1(l), \quad x_2 = q_2(l) + r n_2(l), \tag{2.4.3}$$

where r is the signed distance between a point and the curve Γ_0 along the outward unit normal vector $n = n(l)$ to Γ_0 at the point on the curve represented by $(q_1(l), q_2(l))$. It is orthogonal to the tangent vector $(q_1'(l), q_2'(l))$,

$$n = (n_1, n_2) \quad \text{with} \quad n_1 = \frac{q_2'}{\tau} \quad \text{and} \quad n_2 = \frac{-q_1'}{\tau}. \tag{2.4.4}$$

On Γ_0, $x(r,l) = (q_1(l), q_2(l))$, i.e., $r = 0$ on Γ_0.

The coordinates $x(r,l)$ are only well defined in the narrow region around the smooth curve Γ_0; r is not well defined outside of this region. We define a sufficiently small positive parameter c_1, which depends on the curvature of Γ_0 and ε, such that in the subdomain

$$\Omega_{[-c_1,c_1]} := \{x(r,l) \,:\, |r| \leqslant c_1\}, \tag{2.4.5}$$

the coordinates (r,l) are well defined. We will later, in §2.5.1, say that $c_1 \geqslant C\varepsilon \ln N = \tau$ where 2τ is the width of the fine portion of the mesh around the curve Γ_0. The normal vectors from different points of Γ_0 do not intersect and $x(r,l)$ is well defined as (x_1, x_2) and (r,l) correspond one to one and are invertible.

Lemma 2.4.1. *The curvilinear coordinates* (2.4.3) *are orthogonal and for the Laplace operator we have*

$$\Delta u = \eta^{-1}\frac{\partial}{\partial r}\left(\eta\frac{\partial u}{\partial r}\right) + \zeta\frac{\partial}{\partial l}\left(\zeta\frac{\partial u}{\partial l}\right), \tag{2.4.6}$$

where

$$\eta(r,l) := 1 + \kappa r, \quad \zeta(r,l) := (\tau\eta)^{-1}. \tag{2.4.7}$$

Proof. This follows from [14, Lemma 2.1] and the proof is included in Appendix A.3 for completeness. □

2.4.2 Regions

The curve Γ_0 splits Ω into two open regions, we denote the external region as $\Omega^{(1)}$, and the internal region as $\Omega^{(2)}$. This is represented in Figure 2.5. In

$\Omega^{(1)}$ the order one solution is $u_0(x) = \varphi_1(x)$ while in $\Omega^{(2)}$ it is $u_0(x) = \varphi_2(x)$. We note that $\bar{\Omega} = \bar{\Omega}^{(1)} \cup \bar{\Omega}^{(2)}$.

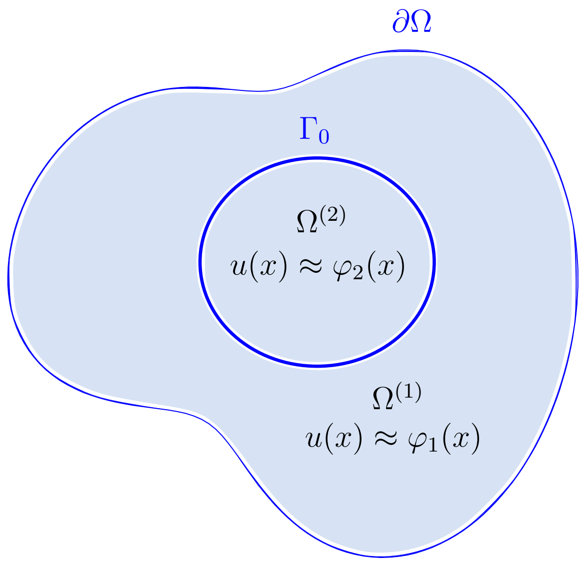


Figure 2.5: Regions of Ω showing $\Omega^{(1)}$, $\Omega^{(2)}$ and the curve Γ_0.

The smooth component $u_0(x) + \varepsilon^2 u_2(x)$ includes

$$u_0(x) := \begin{cases} \varphi_1(x), & x \in \Omega^{(1)} \\ \varphi_2(x), & x \in \Omega^{(2)} \end{cases} \qquad x \notin \Gamma_0, \tag{2.4.8}$$

which implies

$$u_2(x) := \frac{\Delta u_0}{b_u(x, u_0(x))}, \quad x \notin \Gamma_0. \tag{2.4.9}$$

We have

$$F[u_0 + \varepsilon^2 u_2] = O(\varepsilon^4) \quad \forall x \notin \Gamma_0. \tag{2.4.10}$$

The proof of this is included in Lemma 2.7.1.

We will use the notation $\Omega_{(a,b)} := \{x(r, l) \,:\, a < r < b\}$ to denote subdomains of Ω where the boundary of the subdomain is included when square brackets are used, i.e., in $\Omega_{[-c_1,c_1]}$ of (2.4.5). These two subdomains are then

divided again; the inner region and the outer region of $\Omega^{(1)}$ are defined as

$$\Omega_{(0,c_1)} := \left\{ x(r,l) \in \bar{\Omega} \,:\, 0 < r < c_1 \right\}, \quad \mathring{\Omega}^{(1)} := \Omega^{(1)} \backslash \Omega_{(0,c_1)}, \tag{2.4.11}$$

respectively and the inner and outer region of $\Omega^{(2)}$ are defined as

$$\Omega_{(-c_1,0)} := \left\{ x(r,l) \in \bar{\Omega} \,:\, -c_1 < r < 0 \right\}, \quad \mathring{\Omega}^{(2)} := \Omega^{(2)} \backslash \Omega_{(-c_1,0)}, \tag{2.4.12}$$

respectively. We note that $\Omega_{[-c_1,c_1]} = \overline{\Omega_{(0,c_1)}} \cup \overline{\Omega_{(-c_1,0)}}$ and includes the curve Γ_0. Notation for the outer regions combined is $\mathring{\Omega} := \mathring{\Omega}^{(1)} \cup \mathring{\Omega}^{(2)}$. Note this can also be written as $\mathring{\Omega} = \bar{\Omega} \backslash \Omega_{[-c_1,c_1]}$ and we may switch between notation depending on the context. These regions are represented in Figure 2.6.

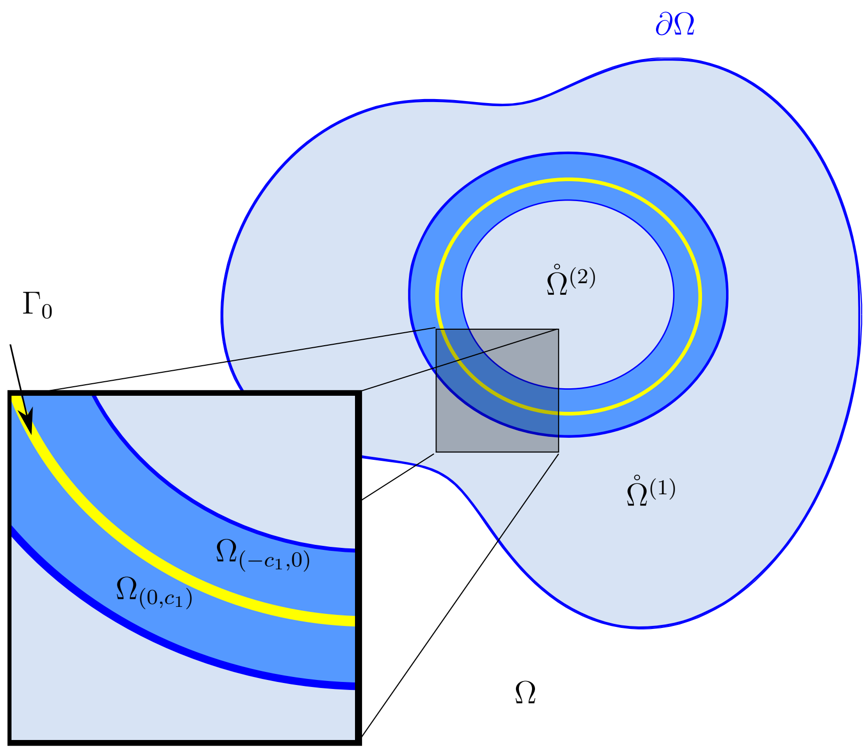


Figure 2.6: Regions of Ω showing the interior layer region, the outer regions and the transition curve.

We now denote $\Gamma_a := \{x(r,l) : r = a\}$. The boundaries of $\Omega_{(-c_1,c_1)}$ is now Γ_{-c_1} and Γ_{c_1}. This notation follows that of the transition curve Γ_0, i.e., where $\Gamma_0 = \{x(r,l) : r = 0\}$.

It will be convenient to introduce the notation $\bar{x}$ as $x(r,l)$ near Γ_0 as

$$\bar{x} = \bar{x}(r,l) := x(0^{\operatorname{sgn} r}, l) = \begin{cases} \lim\limits_{\rho \to 0^-} x(\rho, l), & r < 0 \\ \lim\limits_{\rho \to 0^+} x(\rho, l), & r > 0 \\ x(0,l), & r = 0 \end{cases}, \tag{2.4.13}$$

and also define

$$\bar{x}_0 := x(0,l). \tag{2.4.14}$$

This notation will be useful for functions that are discontinuous across the curve Γ_0. If a suitable function $v(x)$ is discontinuous across Γ_0 then we have $v(\bar{x}(r,l)) = v(x(0^-,l))$ for $r < 0$ and $v(\bar{x}(r,l)) = v(x(0^+,l))$ for $r > 0$. For example, $u_0(\bar{x}(r,l)) = \varphi_1(x)$ for $r > 0$ and $u_0(\bar{x}(r,l)) = \varphi_2(x)$ for $r < 0$ while $u_0(x(0,l))$ is not defined. If the function $v(x)$ is continuous on Γ_0 then $v(\bar{x}(0,l)) = v(x(0,l))$ and this notation is not necessary.

To deal with the jump across Γ_0 we define Γ_0^- as the interior side of the curve Γ_0 where $r < 0$ and Γ_0^+ as the exterior side of the curve Γ_0 where $r > 0$. If $r > 0$ then $\bar{x}(r,l)$ is the nearest point on Γ_0^+ to Γ_0. The curve Γ_0^+ describes the region in which x approaches Γ_0 in $\Omega_{(0,c_1)}$.

A more precise approximation of the transition curve Γ is

$$\Gamma = \Big\{(q_1(l), q_2(l)) + [\varepsilon t_1(l) + \varepsilon^2 t_2(l) + \ldots]\, n(l) \;:\; l \in [0, L]\Big\}. \tag{2.4.15}$$

This curve Γ_0, found in (A5a), is the leading order term of this asymptotic expansion in ε of Γ and is described by $(q_1(l), q_2(l))$. Distance is added to the curve Γ_0 by $[\varepsilon t_1(l) + \varepsilon^2 t_2(l) + \ldots]\, n(l)$ where $n(l)$ points in the outward normal direction. Variables $t_1(l)$ and $t_2(l)$ will be defined in Lemma 2.4.5.

Remark 2.4.1. We pause to discuss Γ_0 and its relationship with $u_0(x)$ and $\varphi_0(x)$. The solution $u_0(x)$ changes rapidly between Γ and Γ_0 due to the steepness of the interior layer. We note that $u_0(x)|_{\Gamma} \neq u_0(x)|_{\Gamma_0}$. Also $u_0(x)|_{\Gamma_0} \neq \varphi_0(x)|_{\Gamma_0}$ as $u_0(x)$ is the discontinuous function in (2.4.8). Taking a Taylor expansion of $\varphi(x)|_{\Gamma}$ about $\varepsilon = 0$ we have $\varphi_0(x)|_{\Gamma} = \varphi_0(x)|_{\Gamma_0} + \varepsilon t_1 \varphi_0'(x)|_{\Gamma_0}$ where $\varphi_0'(x)|_{\Gamma_0}$ is bounded and $\varepsilon t_1 \varphi_0'(x)|_{\Gamma_0} = O(\varepsilon)$. Using this we can transform $u(x)|_{\Gamma} = \varphi_0(x)|_{\Gamma} + O(\varepsilon)$ into $u(x)|_{\Gamma_0} = \varphi_0(x)|_{\Gamma_0} + O(\varepsilon)$.

2.4.3 Asymptotic Expansion

First we set up the full order one solution in the interior layer region, $V_0(\xi, l)$, and then modify this to find the purely interior layer part, $v_0(\xi, l)$.

We define the stretched variable

$$\xi := r/\varepsilon. \tag{2.4.16}$$

Recalling (2.4.14), the order one term $V_0(\xi, l)$ of the asymptotic expansion is now the autonomous differential equation

$$-\frac{\partial^2 V_0}{\partial \xi^2} + b(\bar{x}_0, V_0) = 0 \qquad \text{for } \xi \in \mathbb{R}, \tag{2.4.17a}$$

$$V_0(\infty, l) = \varphi_1(\bar{x}_0), \quad V_0(-\infty, l) = \varphi_2(\bar{x}_0). \tag{2.4.17b}$$

Due to the jump in the solution occurring only in the r direction, we rescale only in this direction. Therefore the second derivative in the l direction is not present at first order. This solution is not unique as $V_0(\xi \pm C, l)$ is also a solution for any constant C. For a specific solution we let

$$\hat{V}_0(0, l) = \varphi_0(\bar{x}_0). \tag{2.4.18}$$

Recall (2.2.1), as Γ_0 is $O(\varepsilon)$ away from Γ and $\varphi_0(x)$ is smooth we have

$$u(x)|_{x\in\Gamma} \approx u_{as}(x)|_{x\in\Gamma} = \varphi_0(x)|_{x\in\Gamma_0} + O(\varepsilon). \tag{2.4.19}$$

Taking the leading order terms of this equation we get (2.4.18). The interior layer is described by $\hat{V}_0(\xi - t_1(l) - \varepsilon t_2(l) - \ldots, l)$ where $t_1(l)$ and $t_2(l)$ from (2.4.15) are described in Lemma 2.4.5. We skip higher order terms and take the specific solution as

$$\hat{V}_0(\xi - \bar{t}_1, l) \quad \text{where } \bar{t}_1(l) := t_1(l) + \varepsilon t_2(l). \tag{2.4.20}$$

To construct upper and lower solutions for problem (2.0.1) we need a perturbed version of this; we perturb $\hat{V}_0(\xi - \bar{t}_1, l)$ by introducing the small positive parameter p such that $V_0(\xi, l)$ in (2.4.17) becomes

$$V_0(\xi, l; p) := \hat{V}_0(\xi - \bar{t}_1 + p, l). \tag{2.4.21}$$

With slight abuse of notation, throughout this chapter we may write (r, l) and (ξ, l) for $x(r, l)$ and $x(\xi, l)$ respectively.

Define $\bar{\gamma}$ as

$$\bar{\gamma} := \sqrt{\min_{\substack{k=1,2\\ x\in\Gamma_0}} b_u(x, \varphi_k(x))}. \tag{2.4.22}$$

Lemma 2.4.2. *Let* $\bar{\gamma}$ *be defined in* (2.4.22) *and* $0 < \gamma < \bar{\gamma}$ *from* (A3). *For* $\bar{t}_1(l) = t_1(l) + \varepsilon t_2(l)$ *sufficiently smooth and all* $|p| \leqslant p^*$, *there exist unique monotone solutions* $\hat{V}_0(\xi, l)$ *and* $V_0(\xi, l; p)$ *of* (2.4.17) *that satisfy* (2.4.18). *Furthermore,* $\hat{V}_0(\xi, l)$ *and* $V_0(\xi, l; p)$ *are in* $C^\infty(\mathbb{R} \times [0, L])$, *and*

$$\hat{\chi}(\xi, l) := \frac{\partial}{\partial \xi}\hat{V}_0(\xi, l) > 0, \quad \chi(\xi, l; p) := \frac{\partial}{\partial \xi} V_0(\xi, l; p) > 0, \tag{2.4.23}$$

for $\xi \in \mathbb{R}$ *and* $l \in [0, L]$. *For any arbitrarily small but fixed* $\lambda \in (0, \bar{\gamma})$, *there*

is a constant C_λ *such that*

$$\hat{\chi}(\xi, l) + \chi(\xi, l; p) \leqslant C_\lambda e^{-(\bar{\gamma}-\lambda)|\xi|} \quad \text{for } \xi \in \mathbb{R},\ l \in [0, L],\ |p| \leqslant p^*. \tag{2.4.24}$$

There are constants C' *and* C'' *such that for all* $|p| \leqslant p^*$ *one has*

$$C'\chi \leqslant V_0 - \varphi_1(\bar{x}) \leqslant C''\chi, \quad \text{for } \xi > 0, \quad C'\chi \leqslant \varphi_2(\bar{x}) - V_0 \leqslant C''\chi, \quad \text{for } \xi < 0. \tag{2.4.25}$$

Proof. This proof follows that of [19, Lemma 2.1] and is included in Appendix A.1 for completeness. □

Define the auxiliary function

$$B(x, s) := b(x, u_0(x) + s), \tag{2.4.26}$$

for some function $s(x)$, note $B(x, 0) = 0$. Recalling (A2) and as $\left.\dfrac{\partial B}{\partial r}\right|_{s=0} = 0$ for all $x \notin \Gamma_0$, we have

$$\left|\frac{\partial^m B}{\partial r^m}\right| \leqslant C|s| \qquad \text{for } x \in \overline{\Omega}\backslash\Gamma_0,\ m = 0, 1, 2,\ s \in \mathbb{R}^2. \tag{2.4.27}$$

Note that expanding (A.1) and rescaling r, for an arbitrary function $v(x)$ we have

$$\varepsilon^2 \Delta v := \frac{\partial^2 v}{\partial \xi^2} + \varepsilon\kappa \frac{\partial v}{\partial \xi} + \varepsilon^2 \zeta \frac{\partial}{\partial l}\left(\zeta \frac{\partial v}{\partial l}\right). \tag{2.4.28}$$

The asymptotic expansion of the solution with $x = (x_1, x_2)$ is

$$u_{as}(x; p) := u_0(x) + \varepsilon^2 u_2(x) + [v_0(\xi, l; p) + \varepsilon v_1(\xi, l; p) + \varepsilon^2 v_2(\xi, l; p)]\vartheta(x), \tag{2.4.29}$$

where $x = (x_1, x_2)$ and $\vartheta(x)$ is a smooth positive cut-off function. Although $v_0 + \varepsilon v_1 + \varepsilon^2 v_2$ are negligible outside the layer region, they do not equal

zero there. As $x(r,l)$, and its stretched coordinates $x(\xi,l)$, only exist in the neighbourhood of Γ_0 these terms must vanish away from Γ_0. We do this by including the function $\vartheta(x)$, which is a smooth positive cut-off function that takes values in $[0,1]$. The function $\vartheta(x)$ equals 1 when $|r| \leqslant \frac{c_1}{2}$ and vanishes in $\bar{\Omega}\backslash\Omega_{[-c_1,c_1]}$. This is represented in Figure 2.7.

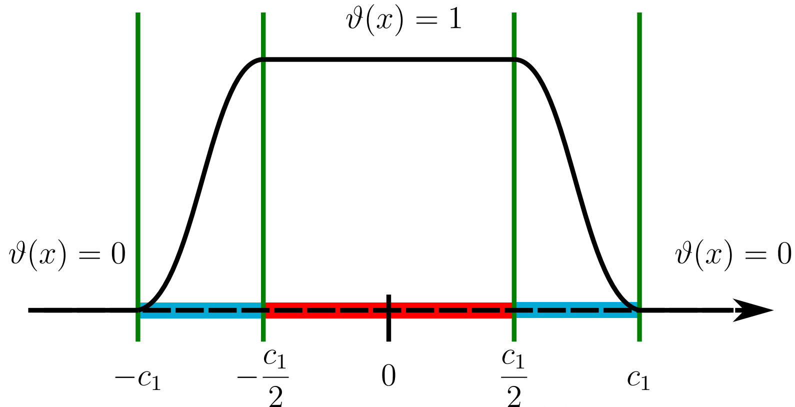


Figure 2.7: Sketch of $\vartheta(x)$.

We now define the interior layer functions, $v_0(\xi,l;p)$, $v_1(\xi,l;p)$ and $v_2(\xi,l;p)$, from (2.4.29). The function $v_0(\xi,l;p)$ is defined as

$$v_0(\xi,l;p) = V_0(\xi,l;p) - u_0(\bar{x}). \tag{2.4.30}$$

This is the order one interior layer function and is the solution of

$$-\frac{\partial^2 v_0}{\partial \xi^2} + B(\bar{x}, v_0) = 0, \tag{2.4.31a}$$

$$v_0(0^{\pm},l) = V_0(0,l;p) - u_0(0^{\pm},l), \quad v_0(\pm\infty,l) = 0, \tag{2.4.31b}$$

We linearise the operator in (2.4.31a) about $u_0 + v_0$ and define the differ-

ential operator $\mathcal{L}_\xi$, in the variable ξ, as

$$\mathcal{L}_\xi[v(\cdot)] := -\frac{\partial^2 v}{\partial \xi^2} + v \frac{\partial B}{\partial s}\bigg|_{s=v_0, x=\bar{x}}. \tag{2.4.32}$$

This is an important operator in the analysis.

Recalling $B := B(x, s)$ from (2.4.26), the $O(\varepsilon)$ boundary layer function, $v_1(\xi, l) = v_1(\xi, l; p)$ is given by

$$\mathcal{L}_\xi[v_1] = \psi_1(\xi, l), \tag{2.4.33a}$$

$$v_1(0, l) = v_1(\pm\infty, l) = 0, \tag{2.4.33b}$$

where

$$\psi_1(\xi, l) := \kappa \frac{\partial v_0}{\partial \xi} - \xi \frac{\partial B}{\partial r}\bigg|_{s=v_0, x=\bar{x}}, \tag{2.4.33c}$$

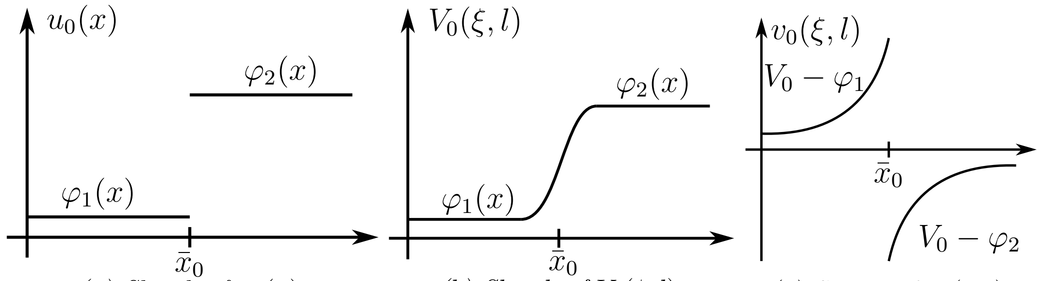
where the first term in ψ_1 (in blue) comes from the Laplace operator in curvilinear coordinates in (2.4.28). For the $O(\varepsilon^2)$ boundary layer function, $v_2(\xi, l) = v_2(\xi, l; p)$, we have

$$\mathcal{L}_\xi[v_2] = \psi_2(\xi, l), \quad v_2(0, l) = v_2(\pm\infty, l) = 0, \tag{2.4.34a}$$

where

$$\begin{aligned} \psi_2(\xi, l) :=\, & \kappa \frac{\partial v_1}{\partial \xi} + \zeta \frac{\partial}{\partial l}\left(\zeta \frac{\partial v_0}{\partial l}\right) \\ & - \left(\frac{\xi}{2}^2 \frac{\partial^2 B}{\partial r^2} + \xi v_1 \frac{\partial^2 B}{\partial r \partial s} + \frac{v_1^2}{2}\frac{\partial^2 B}{\partial s^2}\right)\bigg|_{s=v_0, x=\bar{x}} \\ & - u_2(\bar{x}) \frac{\partial B}{\partial s}\bigg|_{s=0, x=\bar{x}}^{s=v_0, x=\bar{x}}. \end{aligned} \tag{2.4.34b}$$

The first two terms of ψ_2 (in blue) again come from (2.4.28). The full calculation of (2.4.31), (2.4.33) and (2.4.34) is included in §2.7.1. The purely



(a) Sketch of $u_0(x)$. *(b) Sketch of $V_0(\xi, l)$.* *(c) Sketch of $v_0(\xi, l)$.*

Figure 2.8: Cross sectional sketch of solutions $u_0(x)$, $V_0(\xi, l)$ and $v_0(\xi, l)$ near $\bar{x}_0$.

interior layer component of $u_{as}(x; p)$ is $v_0 + \varepsilon v_1 + \varepsilon^2 v_2$. Figure 2.8 illustrates u_0, V_0 and v_0. The functions v_0 and v_2 are discontinuous at $\xi = 0$ but $u_0 + v_0$ and $u_2 + v_2$ are continuous at this point, therefore $u_{as}(x; p)$ is continuous.

In considering v_1 and v_2 across the transition layer we set up the following functional for any suitable function $v(x)$,

$$\Phi[v(\cdot)] := \varepsilon \frac{\partial v}{\partial r}\bigg|_{r=0^-, l} - \varepsilon \frac{\partial v}{\partial r}\bigg|_{r=0^+, l} = \frac{\partial v}{\partial \xi}\bigg|_{\xi=0^-, l} - \frac{\partial v}{\partial \xi}\bigg|_{\xi=0^+, l}. \tag{2.4.35}$$

Remark 2.4.2. Although $v_0(\xi, l)$ is discontinuous across Γ_0, its derivatives are continuous and so $\Phi[v_0] = \Phi[V_0] = 0$. Recalling (2.4.30), the function $v_0(\xi, l)$ is $V_0(\xi, l)$ separated on Γ_0 by subtracting $u_0(x)$ therefore the derivatives match across the jump giving $\Phi[v_0] = \Phi[V_0] = 0$.

Note v_1 and v_2 are special cases of

$$\mathcal{L}_\xi(\nu(\xi, l)) = \psi(\xi, l) \quad \text{ for } x \in \bar{\Omega} \backslash \Gamma_0, \tag{2.4.36a}$$

$$\nu(0^\pm, l) = \nu_0^\pm(l), \quad \nu(\pm\infty, l) = 0. \tag{2.4.36b}$$

Lemma 2.4.3. *Let* $|\psi(\xi,l)| \leqslant C(1+|\xi|^k)\chi(\xi,l)$. *Then there exists a solution* $\nu(\xi,l)$ *of the problem* (2.4.36), *which satisfies* $|\nu(\xi,l)| \leqslant C(1+|\xi|^{k+1})\chi(\xi,l)$ *and*

$$\begin{aligned}\Phi[\nu] &= \left.\frac{\partial\nu}{\partial\xi}\right|_{\xi=0^-,l} - \left.\frac{\partial\nu}{\partial\xi}\right|_{\xi=0^+,l}, \\ &= \frac{1}{\chi(0,l)}\left(-\int_{-\infty}^{\infty}\psi(\xi,l)\chi(\xi,l)d\xi + [\nu_0^-(l)-\nu_0^+(l)]\left.\frac{\partial\chi}{\partial\xi}\right|_{\xi=0}\right).\end{aligned} \tag{2.4.37}$$

Furthermore, if $\psi(\xi,l)\geqslant 0$ *and* $\nu^{\pm}(l)\geqslant 0$, *then* $\nu(\xi,l)\geqslant 0$ *for all* ξ *and* l.

Proof. The proof follows [19, Lemma 2.2].

Taking the derivative with respect to ξ of (2.4.17a) gives

$$\frac{\partial^2\chi}{\partial\xi^2} - \chi(\xi,l)B_s(\xi,l,v_0) = 0. \tag{2.4.38}$$

Multiplying (2.4.36) by $\chi(\xi,l)$ we have

$$-\chi(\xi,l)\frac{\partial^2\nu}{\partial\xi^2} + \chi(\xi,l)\nu(\xi,l)B_s(\xi,l,v_0) = \chi(\xi,l)\psi(\xi,l). \tag{2.4.39}$$

Using (2.4.38), (2.4.39) can be written as

$$-\frac{\partial}{\partial\xi}\left(\chi^2(\xi,l)\frac{\partial}{\partial\xi}\left(\frac{\nu(\xi,l)}{\chi(\xi,l)}\right)\right) = \chi(\xi,l)\psi(\xi,l). \tag{2.4.40}$$

Integrating once and using $\nu(\infty,l)=0$ from (2.4.36b), (2.4.40) becomes,

$$\frac{\partial}{\partial\xi}\left(\frac{\nu(\xi,l)}{\chi(\xi,l)}\right) = \chi^{-2}(\xi,l)\int_{\xi}^{\infty}\chi(t,l)\psi(t,l)dt \quad \text{for } \xi>0. \tag{2.4.41}$$

Integrating a second time gives the solution for $\nu(\xi, l)$ for $\xi > 0$ as

$$\nu(\xi, l) = \chi \int_0^\xi \chi^{-2}(\eta, l) \int_\eta^\infty \psi(t, l)\chi(t, l) dt d\eta + \frac{\nu(0^+, l)}{\chi(0, l)} \chi(\xi, l) \quad \text{for } \xi > 0. \tag{2.4.42}$$

Similarly we find

$$\nu(\xi, l) = \chi \int_\xi^0 \chi^{-2}(\eta, l) \int_{-\infty}^\eta \psi(t, l)\chi(t, l) dt d\eta + \frac{\nu(0^-, l)}{\chi(0, l)} \chi(\xi, l) \quad \text{for } \xi < 0. \tag{2.4.43}$$

If $\psi(\xi, l) \geqslant 0$ and $\nu^\pm(l) \geqslant 0$ then (2.4.42) and (2.4.43) are both positive, giving $\nu(\xi, l) \geqslant 0$ for all ξ and l.

To prove (2.4.37) we consider (2.4.41) and use the quotient rule to get

$$\frac{\partial \nu}{\partial \xi} \chi(\xi, l) - \frac{\partial \chi}{\partial \xi} \nu(\xi, l) = \int_\xi^\infty \chi(t, l)\psi(t, l) dt \quad \text{for } \xi > 0. \tag{2.4.44}$$

Letting ξ go to zero and rearranging terms we have

$$\left.\frac{\partial \nu}{\partial \xi}\right|_{\xi=0^+} = \frac{1}{\chi(0, l)} \left(\int_0^\infty \chi(t, l)\psi(t, l) dt + \nu_0^+(l) \left.\frac{\partial \chi}{\partial \xi}\right|_{\xi=0} \right). \tag{2.4.45}$$

For $\xi < 0$ we have an equation similar to (2.4.41), with altered limits of integration, and we get a similar result,

$$\left.\frac{\partial \nu}{\partial \xi}\right|_{\xi=0^-} = \frac{1}{\chi(0, l)} \left(-\int_{-\infty}^0 \chi(t, l)\psi(t, l) dt + \nu_0^-(l) \left.\frac{\partial \chi}{\partial \xi}\right|_{\xi=0} \right). \tag{2.4.46}$$

Finally subtracting (2.4.45) from (2.4.46) gives (2.4.37).

To prove $|\nu(\xi, l)| \leqslant C(1 + |\xi|^{k+1})\chi(\xi, l)$ we take the case where $\xi < 0$. As $\chi \leqslant C v_0$ by (2.4.25) and (2.4.30) we can write $C'\chi^2 dt \leqslant v_0 dv_0$ which can be rewritten as $\chi^2 dt \leqslant \frac{1}{2C'} dv_0^2$. Using $|\psi(\xi, l)| \leqslant C(1 + |\xi|^k)\chi(\xi, l)$ gives

$\psi(t,l)\chi(t,l)dt \leqslant C(1+|\xi|^k)dv_0^2$. Integrating by parts gives

$$\begin{aligned}\left|\int_{-\infty}^{\eta}\Psi(t)\chi(t)dt\right| \leqslant & Cv_0^2(\eta)(1+|\eta|^k) - C\lim_{t\to-\infty} v_0^2(t)(1+|t|^k)\\ & - C\int_{-\infty}^{\eta}|t|^{k-1}v_0^2(t)dt.\end{aligned} \tag{2.4.47}$$

As the second and third terms on the right hand side of (2.4.47) are negative and by (2.4.25) we can say

$$\left|\int_{-\infty}^{\eta}\psi(t,l)\chi(t,l)dt\right| \leqslant C\chi^2(1+|\eta|^k), \tag{2.4.48}$$

and putting this information into (2.4.43) gives the desired result.

□

So far no assumption on the higher order terms for the true location of the transition curve have been made, i.e., the terms $t_1(l)$ and $t_2(l)$. In the following two lemmas we prove existence of the solutions $v_0(\xi,l;p)$, $v_1(\xi,l;p)$ and $v_2(\xi,l;p)$ and find bounds for these. The order of the system, as well as existence of $t_1(l)$ and $t_2(l)$, i.e., the curves Γ_1 and Γ_2, are then proven.

Lemma 2.4.4. *For $t_1(l)$ and $t_2(l)$, independent of ξ, there exists solutions $v_0(\xi,l;p)$, $v_1(\xi,l;p)$ and $v_2(\xi,l;p)$ of problems* (2.4.31), (2.4.33) *and* (2.4.34). *The function $v_0(\xi,l) = v_0(\xi,l;p)$ satisfies*

$$(\operatorname{sgn}\xi)\cdot v_0(\xi,l) > 0 \quad \text{and} \quad |v_0(\xi,l)| \leqslant C''\chi(\xi,l) \quad \forall \xi\in\mathbb{R}\setminus\{0\} \text{ and } l\in[0,L]. \tag{2.4.49}$$

Furthermore, if $|t_1(l)|+|t_2(l)| \leqslant C$ and $p \leqslant p^*$, for any arbitrarily fixed, small constant $\lambda \in (0,\bar\gamma)$ there exists a constant C_λ such that

$$\left|\frac{\partial^{k+m}v_j}{\partial\xi^k\partial l^m}\right| + \left|\frac{\partial^{k'+m'}v_2}{\partial\xi^{k'}\partial l^{m'}}\right| \leqslant C_\lambda e^{-(\bar\gamma-\lambda)|\xi|}, \tag{2.4.50}$$

for $j = 0, 1,$ $k = 0, \dots, 6,$ $m = 0, \dots, 4$ *where* $k + m \leqslant 6$ *and* $k' = 0, 1, 2$ *and* $m' = 0, 1, 2$ *with* $k' + m' \leqslant 2.$

Proof. This follows from the one-dimensional case found in [18, Lemma 6.2] with further detail in [19, Lemma 2.3].

Existence of the solution $v_0(\xi, l)$ is found using Lemma 2.4.2 with (2.4.30). Equations for $v_1(\xi, l)$ and $v_2(\xi, l)$, (2.4.33) and (2.4.34), are of type (2.4.36) and the existence of a solution is found by applying Lemma 2.4.3 to these problems. The first inequality in (2.4.49) is found using (2.4.30),

$$(\operatorname{sgn}\xi) v_0(\xi, l) = \begin{cases} \varphi_2(r, l) - V_0 & \text{if } r < 0, \\ V_0 - \varphi_1(r, l) & \text{if } r > 0. \end{cases} \tag{2.4.51}$$

As $V_0(\xi, l) > \varphi_1(x)$ when $\xi > 0$ and $V_0(\xi, l) < \varphi_2(x)$ when $\xi < 0$ we get $(\operatorname{sgn}\xi) v_0(\xi, l) > 0$. The second inequality in (2.4.49) again uses (2.4.30), now with (2.4.25).

We are left to prove (2.4.50). Using bounds obtained in Lemma 2.4.3 and (2.4.24), $\left|\dfrac{\partial^k v_0}{\partial \xi^k}\right| \leqslant C_\lambda e^{-(\bar\gamma - \lambda)|\xi|}$ for $k = 0, 1$. As $v_1(\xi, l)$ and $v_2(\xi, l)$ are of type (2.4.36) we can use Lemma 2.4.3 for the case when $k = 0$. Using (2.4.27) we can estimate the right hand sides of the equations and see that $|\psi(\xi, l)| \leqslant C(1 + |\xi|^k)\chi(\xi)$ holds. As λ is arbitrarily small and C_λ is a generic positive constant, then $C_\lambda(1 + |\xi|^{k+1}) e^{-(\bar\gamma-\lambda)|\xi|} \leqslant C_\lambda e^{-(\bar\gamma-\lambda)|\xi|}$ and therefore,

$$\left|\frac{\partial^k v_j}{\partial \xi^k}\right| \leqslant C_\lambda(1 + |\xi|^{k+1}) e^{-(\bar\gamma-\lambda)|\xi|} \leqslant C_\lambda e^{-(\bar\gamma-\lambda)|\xi|}. \tag{2.4.52}$$

Equations for the higher order terms of v_0, v_1 and v_2 with respect to ξ and derivatives with respect to l are again of type (2.4.36) and their right hand sides can be estimated using (2.4.27) and (2.4.25) giving the required results.

For bounds for derivatives with respect to l and mixed derivatives we take derivatives with respect to l and ξ of the equations for v_j for $j = 0, 1, 2$. The

resulting equations are of type (2.4.36) and so we can apply Lemma 2.4.3 with (2.4.24) and get

$$\left|\frac{\partial^{k+m} v_j}{\partial r^k \partial l^m}\right| \leqslant C_\lambda e^{-(\bar{\gamma}-\lambda)|r_i|/\varepsilon}, \tag{2.4.53}$$

for $k = 0, ..., 6$, $m = 0, ..., 3$ and $j = 0, 1, 2$. We include higher derivatives here as the right hand sides of equations for the derivatives with respect to r, l and mixed derivatives of (2.4.33a) and (2.4.34a), and later (2.4.86), involve curvature terms and must be bounded to use Lemma 2.4.3. $\square$

Lemma 2.4.5. *For the asymptotic expansion* $u_{as}(x;p)$ *from* (2.4.29) *we have*

$$Fu_{as}(x;p) = O(\varepsilon^3) \quad \textit{for } x \in \Omega \setminus \Gamma_0. \tag{2.4.54a}$$

Furthermore, there exists values of $t_1(l)$ *and* $t_2(l)$ *in* (2.4.20)*, independent of* ε *and* p*, positive constants* C_1 *and* C_2*, and* $\varepsilon^* = \varepsilon^*(p^*)$ *such that for all* $\varepsilon \leqslant \varepsilon^*$ *and* $0 < |p| \leqslant p^*$ *we have*

$$(\operatorname{sgn} p) \cdot \Phi[u_{as}(x;p)] \geqslant C_1 \varepsilon |p| - C_2 \varepsilon^3. \tag{2.4.54b}$$

Proof. This proof follows [18, Lemma 6.3] and [19, Lemma 2.4]. We sketch it here for completeness. The proof of (2.4.54a) can be found in Lemma 2.7.3.

We take $\vartheta(x) = 1$. The function $\Phi[u_{as}(x;p)]$ can be represented as

$$\Phi[u_{as}] = \varepsilon \left(\frac{\partial \varphi_2}{\partial r} - \frac{\partial \varphi_1}{\partial r}\right)\bigg|_{x=\bar{x}} + \varepsilon \Phi[v_1] + \varepsilon^2 \Phi[v_2] + O(\varepsilon^3). \tag{2.4.55}$$

This is found by recalling (2.4.35) and (2.4.29) and observing that

$$\Phi[u_0] = \varepsilon \left(\frac{\partial \varphi_2}{\partial r} - \frac{\partial \varphi_1}{\partial r}\right)\bigg|_{x=\bar{x}}, \tag{2.4.56}$$

and

$$\Phi[\varepsilon^2 u_2] = \varepsilon^3 \frac{\partial u_2}{\partial r}\bigg|_{r=0^-} - \varepsilon^3 \frac{\partial u_2}{\partial r}\bigg|_{r=0^+} = O(\varepsilon^3). \tag{2.4.57}$$

We also note $\Phi[V_0] = \Phi[v_0] = 0$ as both V_0 and v_0 have continuous derivatives across $\xi = 0$.

For $\Phi[v_1]$, we apply (2.4.37) to (2.4.33) and get

$$\begin{aligned}\Phi[v_1] = &-\frac{1}{\chi(0,l)}\int_{-\infty}^{\infty}\left(\kappa\frac{\partial v_0}{\partial \xi} - \xi\frac{\partial B}{\partial r}\bigg|_{x=\bar{x},s=v_0}\right)\chi(\xi,l)d\xi \\ &+ \frac{1}{\chi(0,l)}[v_1(0^-,l) - v_1(0^+,l)]\frac{\partial \chi}{\partial \xi}\bigg|_{\xi=0},\end{aligned} \tag{2.4.58}$$

where the second term equals zero by (2.4.33b). Now as

$$\frac{\partial B}{\partial r}\bigg|_{x=\bar{x},s=v_0} = \frac{\partial b}{\partial r}\bigg|_{x=\bar{x},u=V_0} + \frac{\partial u_0}{\partial r}\bigg|_{x=\bar{x}}\frac{\partial b}{\partial u}\bigg|_{x=\bar{x},u=V_0}, \tag{2.4.59}$$

we get

$$\Phi[v_1] = \frac{1}{\chi(0,l)}\int_{-\infty}^{\infty}\left(-\kappa\frac{\partial v_0}{\partial \xi} + \xi\frac{\partial b}{\partial r}\bigg|_{x=\bar{x},u=V_0} + \xi\frac{\partial u_0}{\partial r}\bigg|_{x=\bar{x}}\frac{\partial b}{\partial u}\bigg|_{x=\bar{x},u=V_0}\right)\chi(\xi,l)d\xi. \tag{2.4.60}$$

Looking at the third term in this integral we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\xi\frac{\partial u_0}{\partial r}\bigg|_{x=\bar{x}}\frac{\partial b}{\partial u}\bigg|_{x=\bar{x},u=V_0}\chi(\xi,l)d\xi = \\ &\frac{\partial \varphi_2}{\partial r}\bigg|_{x=\bar{x}}\int_{-\infty}^{0}\xi\frac{\partial b}{\partial u}\bigg|_{x=\bar{x},u=V_0}\chi(\xi,l)d\xi + \frac{\partial \varphi_1}{\partial r}\bigg|_{x=\bar{x}}\int_{0}^{\infty}\xi\frac{\partial b}{\partial u}\bigg|_{x=\bar{x},u=V_0}\chi(\xi,l)d\xi.\end{aligned} \tag{2.4.61}$$

We consider the integral from $\xi = -\infty$ to $\xi = 0$ and use integration by parts

to get,

$$\int_{-\infty}^{0} \xi \frac{\partial b}{\partial u}\bigg|_{x=\bar{x},u=V_0} \chi(\xi,l)d\xi = \int_{-\infty}^{0} b(x,V_0)d\xi - [\xi b(x,V_0)]\,\Big|_{\xi=-\infty}^{\xi=0}. \tag{2.4.62}$$

Recalling (2.4.26), we have $[\xi b(x,V_0(\xi,l))]\,\Big|_{\xi=-\infty}^{\xi=0} = 0$ as $b(x,V_0(-\infty,l)) = b(x,u_0(x))$ and by (2.0.2) we have $b(x,u_0(x)) = 0$. We now transform this by using $\dfrac{\partial \chi}{\partial \xi} + b(x,V_0) = 0$, giving

$$\int_{-\infty}^{0} \xi \frac{\partial b}{\partial u}\bigg|_{x=\bar{x},u=V_0} \chi(\xi,l)d\xi = -\int_{-\infty}^{0} \frac{\partial}{\partial \xi}\chi(\xi,l)d\xi = -\chi(0,l)+\chi(-\infty,l) = -\chi(0,l). \tag{2.4.63}$$

A similar argument follows for the integral from 0 to ∞ and gives the solution $\chi(0,l)$. Therefore the third term of (2.4.60) is now

$$\int_{-\infty}^{\infty} \xi \frac{\partial u_0}{\partial r}\bigg|_{x=\bar{x}} \frac{\partial b}{\partial u}\bigg|_{x=\bar{x},s=V_0} \chi(\xi,l)d\xi = -\chi(0,l)\frac{\partial \varphi_2}{\partial r}\bigg|_{x=\bar{x}} + \chi(0,l)\frac{\partial \varphi_1}{\partial r}\bigg|_{x=\bar{x}}. \tag{2.4.64}$$

Putting this back into (2.4.60) gives

$$\begin{aligned}\Phi[v_1] =& \frac{1}{\chi(0,l)} \int_{-\infty}^{\infty} \left(-\kappa \frac{\partial v_0}{\partial \xi} + \xi \frac{\partial b}{\partial r}\bigg|_{x=\bar{x},u=V_0} \right) \chi(\xi,l)d\xi \\ &+ \left(\frac{\partial \varphi_1}{\partial r} - \frac{\partial \varphi_2}{\partial r} \right)\bigg|_{x=\bar{x}}.\end{aligned} \tag{2.4.65}$$

Recall (2.4.21), we use $\hat{\xi} = \xi - \bar{t}_1(l) + p$ with $\bar{t}_1(l) = t_1(l) + \varepsilon t_2(l)$ so the above integral now involves $\hat{V}_0(\hat{\xi},l)$ and $\hat{\chi}(\hat{\xi},l)$, i.e., the dependence on p is brought outside the functions. Rearranging (2.4.65) we now define $C_I(l)$ and

$C_{II}(l)$ by

$$\begin{aligned}\Phi[v_1] =& \frac{1}{\chi(0,l)} \underbrace{\int_{-\infty}^{\infty} \left(-\kappa \frac{\partial v_0}{\partial \hat{\xi}} + \hat{\xi} \frac{\partial b}{\partial r}\bigg|_{x=\bar{x}, u=\hat{V}_0} \right) \hat{\chi}(\hat{\xi}, l) d\hat{\xi}}_{C_{II}(l)} \\ &+ \frac{1}{\chi(0,l)} (\bar{t}_1(l) - p) \underbrace{\int_{-\infty}^{\infty} \frac{\partial b}{\partial r}\bigg|_{x=\bar{x}, u=\hat{V}_0} \hat{\chi}(\hat{\xi}, l) d\hat{\xi}}_{-C_I(l)} \\ &+ \left(\frac{\partial \varphi_1}{\partial r} - \frac{\partial \varphi_2}{\partial r} \right)\bigg|_{x=\bar{x}}. \end{aligned} \tag{2.4.66}$$

Here we have

$$C_I(l) = -\int_{-\infty}^{\infty} \frac{\partial b}{\partial r}\bigg|_{u=\hat{V}_0(\hat{\xi},l)} \hat{\chi}(\hat{\xi}, l) d\hat{\xi} = -\int_{\varphi_2(\bar{x}_0)}^{\varphi_1(\bar{x}_0)} \frac{\partial b}{\partial r} dv, \tag{2.4.67}$$

and from (A5b) we can write this as

$$C_I(l) = \partial_n \mathcal{I}(x(r,l)) \geqslant C_* > 0, \tag{2.4.68}$$

and $C_{II}(l)$ is

$$C_{II}(l) = \int_{-\infty}^{\infty} -\kappa \frac{\partial v_0}{\partial \hat{\xi}} \hat{\chi}(\hat{\xi}, l) d\hat{\xi} + \int_{-\infty}^{\infty} \hat{\xi} \; \frac{\partial b}{\partial r}\bigg|_{u=\hat{V}_0(\hat{\xi},l)} \hat{\chi}(\xi, l) d\hat{\xi}. \tag{2.4.69}$$

Rewriting (2.4.66) now gives

$$\Phi[v_1] = \frac{1}{\chi(0,l)} \left[C_{II}(l) + (t_1(l) + \varepsilon t_2(l) - p)(-C_I(l)) \right] + \left(\frac{\partial \varphi_1}{\partial r} - \frac{\partial \varphi_2}{\partial r} \right)\bigg|_{x=\bar{x}}. \tag{2.4.70}$$

Letting $t_1(l) := \frac{C_{II}(l)}{C_I(l)}$, which is the $O(\varepsilon^2)$ term for Γ in (2.4.15), gives

$$\Phi[v_1] = \frac{1}{\chi(0,l)} \left[(\varepsilon t_2(l) - p)(-C_I(l)) \right] + \left(\frac{\partial \varphi_1}{\partial r} - \frac{\partial \varphi_2}{\partial r} \right)\bigg|_{x=\bar{x}}. \tag{2.4.71}$$

The final term in (2.4.55) to consider is $\Phi[v_2]$. Applying (2.4.37) to (2.4.34), the equation for v_2, results in

$$\Phi[v_2] = -\frac{1}{\chi(0,l)} \int_{-\infty}^{\infty} \psi_2(\xi, l)\chi(\xi, l) d\xi, \tag{2.4.72}$$

where $\psi_2(\xi, l)$ is defined in (2.4.34b). Setting $p = 0$ and $\bar{t}_1(l) = t_1(l)$, i.e., eliminating $t_2(l)$ in (2.4.72), gives $C_{III}(l)$, which is independent of p and ε. Now, the term for v_2 becomes

$$\Phi[v_2] = \frac{1}{\chi(0,l)}[C_{III}(l) + O(p + \varepsilon|t_2(l)|)]. \tag{2.4.73}$$

Substituting (2.4.71) and (2.4.73) into (2.4.55) gives

$$\begin{aligned}\Phi[u_{as}] =& \varepsilon \left(\frac{\partial \varphi_2}{\partial r} - \frac{\partial \varphi_1}{\partial r} \right) \Big|_{x=\bar{x}} \\ &+ \frac{\varepsilon}{\chi(0,l)} \left[(\varepsilon t_2(l) - p)(-C_I(l)) \right] + \varepsilon \left(\frac{\partial \varphi_1}{\partial r} - \frac{\partial \varphi_2}{\partial r} \right) \Big|_{x=\bar{x}} \\ &- \frac{\varepsilon^2}{\chi(0,l)}[C_{III}(l) + O(p + \varepsilon|t_2(l)|)] + O(\varepsilon^3).\end{aligned} \tag{2.4.74}$$

Cancelling terms and choosing $t_2(l) := \dfrac{C_{III}(l)}{C_I(l)}$ gives

$$\Phi[u_{as}] = \frac{\varepsilon^2 C_{III}(l)}{\chi(0,l)} + \frac{\varepsilon p C_I(l)}{\chi(0,l)} - \frac{\varepsilon^2 C_{III}(l)}{\chi(0,l)} - \frac{\varepsilon^2}{\chi(0,l)} O(p + \varepsilon|t_2(l)|) + O(\varepsilon^3), \tag{2.4.75}$$

and choosing C_2 sufficiently large and with further simplification we get

$$\Phi[u_{as}] \geqslant \frac{\varepsilon p C_I(l)}{\chi(0,l)} - C_2 \varepsilon^3. \tag{2.4.76}$$

From (A5b), $C_I(l) \geqslant C^*$. We now choose

$$C_1 := \frac{1}{2}C^*C' \leqslant \frac{1}{2}C_I(l)C', \tag{2.4.77}$$

so that $\dfrac{\varepsilon}{\chi(0,l)}C_I(l) \geqslant 2\varepsilon C_1$ and hence we obtain (2.4.54b). □

Lemma 2.4.6. *Let c_1 be the small positive constant from §2.4.1 and $x = x(r,l)$. Then*

$$u_{as}(x;0) = \mathcal{U}(x,\varepsilon) + O(c_1 + \varepsilon^2), \tag{2.4.78}$$

for

$$\mathcal{U}(x,\varepsilon) := \begin{cases} V_0(x(\frac{r}{\varepsilon},l);0) & x \in \Omega_{[-c_1,c_1]} \\ u_0(x) & x \in \bar{\Omega}\backslash\Omega_{[-c_1,c_1]}. \end{cases} \tag{2.4.79}$$

Proof. We partially imitate the proof of [19, Lemma 2.5], however the inclusion of $\vartheta(x)$ in (2.4.29) simplifies our argument.

For $|r| > c_1$, i.e., outside the interior layer, $\vartheta(x) = 0$ and so $u_{as}(x;0) = u_0(x) + O(\varepsilon^2)$.

For $|r| \leqslant c_1$, we have

$$u_{as}(x;0) = u_0(x) + \vartheta(x)(V_0(\xi,l;0) - u_0(\bar{x})) + O(\varepsilon). \tag{2.4.80}$$

We note $\vartheta(x) = \vartheta(\bar{x}) + rn(l)\left.\dfrac{\partial\vartheta}{\partial r}\right|_{x=\bar{x}+rn(l)\theta}$. As $\vartheta(x)$ is a smooth function with $\vartheta(\bar{x}) = 1$ we have $\vartheta(x) = 1 + O(r)$ and as $u_0(x)$ and $V_0(\xi,l)$ are bounded (2.4.80) becomes

$$u_{as}(x;0) = u_0(x) - u_0(\bar{x}) + V_0(\xi,l;0) + O(r+\varepsilon). \tag{2.4.81}$$

Using the definition of x in (2.4.3) and (2.4.13) we have $|x(r,l) - \bar{x}(r,l)| = |r|$.

We also note that $u_0(x)$ and $\nabla u_0(x)$ are bounded and we get the result

$$|u_0(x) - u_0(\bar{x})| \leqslant C|r|, \tag{2.4.82}$$

and hence

$$u_{as}(x;0) = V_0(\xi, l; 0) + O(|r| + \varepsilon). \tag{2.4.83}$$

Finally as $|r| \leqslant c_1$, we obtain (2.4.78). $\square$

2.4.4 Perturbed Asymptotic Expansions

We now perturb the asymptotic expansion $u_{as}(x;p)$ to find sub and super solutions for the problem. Define β as

$$\beta(x) = \beta(x; p, p', \tilde{h}) := u_{as}(x;p) + p'[v_*(\xi, l; p)\vartheta(x) + C_0] + \hat{h}^2 z(\xi, l; p)\vartheta(x), \tag{2.4.84}$$

where $\hat{h}$ and p' are small positive parameters that will be defined in §2.5.3. As $v_*(\xi, l; p)$ and $z(\xi, l; p)$ are defined in curvilinear coordinates we add $\vartheta(x)$, the smooth cut off function defined after (2.4.29), to eliminate these functions outside the layer. The term $\hat{h}^2 z(\xi, l; p)$ represents the principal part of the truncation error while $p'[v_*(\xi, l; p) + C_0]$ is added to ensure $(\operatorname{sgn} p') \cdot F(u_{as} + p'[v_* + C_0]) \geqslant 0$.

Define the functions $v_*(\xi, l) = v_*(\xi, l; p)$ and $z(\xi, l) = z(\xi, l; p)$ by

$$\mathcal{L}_\xi(v_*) = |v_0|, \quad v_*(0, l) = v_*(\pm\infty, l) = 0, \tag{2.4.85}$$

$$\mathcal{L}_\xi(z) = \frac{1}{12}\frac{\partial^4 V_0}{\partial \xi^4}, \quad z(0, l) = z(\pm\infty, l) = 0. \tag{2.4.86}$$

Note, these equations could include higher order curvature terms coming from the two-dimensional Laplacian operator similar to those in (2.4.28). These terms are neglected from the above equations for v_* and z. In the case where

they are included they would be multiplied by p' and $\hat{h}^2$ respectively so these terms become $O(\varepsilon p')$ and $O(\varepsilon \hat{h}^2)$ and are negligible in the work below. We can prove existence of the solutions using (2.4.85) and (2.4.86) as they stand. If curvature terms are included, existence of solutions cannot necessarily be proven.

Remark 2.4.3. Nefedov, [24], considers similar upper and lower solutions to (2.4.84) with some modifications. The right hand side of (2.4.85) is added into the equation for $v_1(\xi, l)$, i.e., a version of $p'v_0$ is included in the right hand side of (2.4.33a), and a homogeneous version of (2.4.85) is included. Also the dependence of p on ε is pulled outside of the constant from the beginning, i.e., pC_0 is written as $\varepsilon^n \gamma$ where $\gamma = O(1)$ and n is the order of the asymptotic expansion. We carry the perturbation parameter p through the work and then later choose the value of this parameter, i.e., in Lemma 2.5.5 it is found that p depends on ε and $\hat{h}$. Finally there is no version of $z(\xi, l; p)$ in [24] as a discrete system is not considered.

We now give three lemmas in order to show that $\beta(x, \pm p, \pm p', \hat{h})$ are upper and lower solutions to our problem. We prove existence of the solutions v_* and z, give bounds for their derivatives and consider the jump of the normal derivative of β across Γ_0. We then give bounds for $F\beta$ and finally show that $\beta(x, \pm p, \pm p', \hat{h})$ are ordered solutions.

Lemma 2.4.7. *Assume that $|p| \leqslant p^*$ for some positive constant p^*. Then there exist solutions v_* and z of problems* (2.4.85) *and* (2.4.86) *respectively, and for any arbitrarily small but fixed $\lambda \in (0, \bar{\gamma})$, there is a constant C_λ such that*

$$v_* \geqslant 0, \qquad \left|\frac{\partial^{k+m} v_*}{\partial \xi^k \partial l^m}\right| + \left|\frac{\partial^{k+m} z}{\partial \xi^k \partial l^m}\right| \leqslant C_\lambda e^{-(\bar{\gamma}-\lambda)|\xi|}, \tag{2.4.87}$$

for $\xi \in \mathbb{R} \setminus \{0\}$, $k = 0, \dots, 2$ and $m = 0, \dots, 2$ with $k + m \leqslant 2$.

Furthermore, there exists C_1, C_2, C_3 and $\varepsilon^* = \varepsilon^*(p^*)$ such that for all

$\varepsilon \leqslant \varepsilon^*$ *and* $0 < |p| \leqslant p^*$ *we have*

$$(\operatorname{sgn} p) \cdot \Phi[\beta(x; p, p', \hat{h})] \geqslant C_1 \varepsilon |p| - C_2 \varepsilon^3 - C_3 |p'|. \tag{2.4.88}$$

Proof. This proof follows [18, Lemma 6.4] and [19, Lemma 3.1].

As $\vartheta(x)$ is bounded above by 1 we can take the upper bound and work with β with $\vartheta = 1$. For the existence of v_* and z, apply Lemma 2.4.3; these equations are of type (2.4.36), therefore a solution exists by applying this lemma. As the right hand side of (2.4.85) is non-negative, we have $\psi \geqslant 0$ and $v_*(0) = 0$. Using Lemma 2.4.3 we get $v_* \geqslant 0$.

Applying Lemma 2.4.3 in the same manner as in Lemma 2.4.4 to (2.4.85) and (2.4.86) and their derivatives, i.e., checking the right hand sides are bounded, gives the remainder of (2.4.87).

Now (2.4.88) is left to prove. We have

$$(\operatorname{sgn} p) \cdot \Phi[\beta(x; p, p', \hat{h})] = (\operatorname{sgn} p) \left(\Phi[u_{as}] + p' \Phi[v_*] + \hat{h}^2 \Phi[z] \right). \tag{2.4.89}$$

For $u_{as}(x; p)$ we use Lemma 2.4.5 and get

$$(\operatorname{sgn} p) \cdot \Phi[u_{as}] \geqslant C_1 \varepsilon |p| - C_2 \varepsilon^3. \tag{2.4.90}$$

For v_* we have

$$\begin{aligned} \Phi[v_*] &= \frac{\partial}{\partial \xi} v_*(0^-, l) - \frac{\partial}{\partial \xi} v_*(0^+, l) \\ &= \frac{1}{\chi(0, l)} \left(- \int_{-\infty}^{\infty} \psi \chi d\xi + [v_*^-(0, l) - v_*^+(0, l)] \frac{\partial \chi}{\partial \xi} \bigg|_{\xi = 0} \right). \end{aligned} \tag{2.4.91}$$

As v_* is continuous across the curve Γ_0, $v_*^-(0, l) - v_*^+(0, l) = 0$. From the

equation for v_* we have $\psi = |v_0|$ and so

$$\Phi[v_*] = -\frac{1}{\chi(0,l)} \int_{-\infty}^{\infty} |v_0|\chi d\xi. \tag{2.4.92}$$

Splitting the integral into two intervals, using $v_0\chi = \frac{1}{2}\frac{\partial}{\partial\xi}(v_0^2)$ and $v_0(\pm\infty) = 0$ gives

$$-\frac{1}{\chi(0,l)}\left[\int_{-\infty}^{0} v_0\chi d\xi - \int_{0}^{\infty} v_0\chi d\xi\right] = -\frac{1}{2\chi(0,l)}\left[(v_0(0^-))^2 + (v_0(0^+))^2\right] \geqslant -C_3. \tag{2.4.93}$$

Finally, for z,

$$\Phi[z] = \frac{\partial}{\partial\xi}z(0^-,l) - \frac{\partial}{\partial\xi}z(0^+,l) = \frac{1}{\chi(0,l)}\left(-\int_{-\infty}^{\infty}\psi\chi d\xi + [z_0^- - z_0^+]\frac{\partial\chi}{\partial\xi}\bigg|_{\xi=0}\right), \tag{2.4.94}$$

where

$$\frac{1}{\chi(0,l)}\left(-\int_{-\infty}^{\infty}\frac{1}{12}\frac{\partial^4 V_0}{\partial\xi^4}\chi d\xi + [z_0^- - z_0^+]\frac{\partial\chi}{\partial\xi}\bigg|_{\xi=0}\right) = -\frac{1}{12\chi(0,l)}\int_{-\infty}^{\infty}\frac{\partial^3\chi}{\partial\xi^3}\chi d\xi. \tag{2.4.95}$$

Using integration by parts and $\chi(\pm\infty,l) = 0$ we get

$$-\frac{1}{12\chi(0,l)}\frac{\partial^2\chi}{\partial\xi^2}\chi(\xi,l)\bigg|_{\xi=-\infty}^{\xi=\infty} - \frac{1}{12\chi(0,l)}\int_{-\infty}^{\infty}\frac{\partial^2\chi}{\partial\xi^2}\frac{\partial\chi}{\partial\xi}d\xi = -\frac{1}{24\chi(0,l)}\left(\frac{\partial\chi}{\partial\xi}\right)^2\bigg|_{-\infty}^{\infty}. \tag{2.4.96}$$

As $\frac{\partial\chi}{\partial\xi} = \frac{\partial^2 V_0}{\partial\xi^2} = b(\bar{x}_0, V_0)$ and $V_0(\pm\infty,l) = \varphi_i$ with $i = 1,2$ which are solutions to the reduced problem, i.e., $b(\bar{x}_0,\varphi_i) = 0$, (2.4.96) equals zero. Therefore $\Phi[z] = 0$. Putting this information together results in

$$\Phi[\beta(x;p,p',\hat{h})] = (\operatorname{sgn} p)\left(\Phi[u_{as}] + p'\Phi[v_*]\right) \geqslant C_1\varepsilon|p| - C_2\varepsilon^3 - |p'|C_3, \tag{2.4.97}$$

and we have obtained (2.4.88). □

Lemma 2.4.8. *There exists positive constants C_0, C_4, p'^* and ε^* such that for all $x \in \Omega \setminus \Gamma_0$, $\varepsilon \leqslant \varepsilon^*$, $|p| \leqslant p^*$, $0 < |p'| \leqslant p'^*$, the function $\beta(x;p)$ of* (2.4.84) *satisfies*

$$(\operatorname{sgn} p') \cdot \left[F\beta - \frac{\hat{h}^2}{12} \frac{\partial^4}{\partial \xi^4} V_0 \right] \geqslant \frac{1}{2} C_0 |p'| \gamma^2 - C_4(\varepsilon^3 + \varepsilon \hat{h}^2 + \hat{h}^4). \tag{2.4.98}$$

Proof. This proof is an extension of the one-dimensional case [18, Lemma 6.5] and [19, Lemma 3.2].

To calculate $F\beta$ we represent it in the following manner,

$$F\beta = F\Big|_{u_{as}}^{u_{as}+p'v_*+\hat{h}^2 z} + F\Big|_{u_{as}+p'v_*+\hat{h}^2 z}^{u_{as}+p'v_*+\hat{h}^2 z + C_0 p'} + F(u_{as}), \tag{2.4.99}$$

where we again use the notation $F\Big|_b^a := Fa - Fb$. This eases the calculation as the order of Fu_{as} is known and terms from this in the Laplacian will cancel with that of $F\beta$. From Lemma 2.4.5 we have $Fu_{as}(x;p) = O(\varepsilon^3)$.

Taking the first term in (2.4.99) we have

$$\begin{aligned} F\Big|_{u_{as}}^{u_{as}+p'v_*+\hat{h}^2 z} = -p' \frac{\partial^2 v_*}{\partial \xi^2} - \hat{h}^2 \frac{\partial^2 z}{\partial \xi^2} + \left(\varepsilon\kappa \frac{\partial}{\partial \xi} - \varepsilon^2 \zeta \frac{\partial}{\partial l} \left(\zeta \frac{\partial}{\partial l} \right) \right) (p'v_* + \hat{h}^2 z) \\ + b(x, u_{as} + p'v_* + \hat{h}^2 z) - b(x, u_{as}). \end{aligned} \tag{2.4.100}$$

Note that $\left(\varepsilon\kappa \frac{\partial}{\partial \xi} - \varepsilon^2 \zeta \frac{\partial}{\partial l} \left(\zeta \frac{\partial}{\partial l} \right) \right) (p'v_* + \hat{h}^2 z) = O(\varepsilon p' + \varepsilon \hat{h}^2)$. Doing a Taylor series expansion of $b(x, u_{as} + p'v_* + \hat{h}^2 z)$ about $p'v_* + \hat{h}^2 z$ gives

$$b(x, u_{as} + p'v_* + \hat{h}^2 z) - b(x, u_{as}) = (p'v_* + \hat{h}^2 z) b_u(x, u_{as}) + O((p'v_* + \hat{h}^2 z)^2). \tag{2.4.101}$$

For the derivative in (2.4.101), $b_u(x, u_{as}) = B_s(x, v_0) + O(\varepsilon)$ and the final

term is $[p'v_* + \hat{h}^2 z]O(p' + \hat{h}^2)$. We now have

$$F\Big|_{u_{as}}^{u_{as}+p'v_*+\hat{h}^2 z} = -\,p'\frac{\partial^2 v_*}{\partial\xi^2} - \hat{h}^2\frac{\partial^2 z}{\partial\xi^2} + (p'v_* + \hat{h}^2 z)[B_s(x,v_0) + O(\varepsilon + p' + \hat{h}^2)] + O(\varepsilon p' + \varepsilon\hat{h}^2). \tag{2.4.102}$$

Using the equations for v_* and z gives

$$F\Big|_{u_{as}}^{u_{as}+p'v_*+\hat{h}^2 z} = p'|v_0| + \hat{h}^2\frac{1}{12}\frac{\partial^4 V_0}{\partial\xi^4} + [p' + \hat{h}^2]O(\varepsilon + p' + \hat{h}^2) + O(\varepsilon p' + \varepsilon\hat{h}^2). \tag{2.4.103}$$

The second term in (2.4.99) is

$$F\Big|_{u_{as}+p'v_*+\hat{h}^2 z}^{u_{as}+p'v_*+\hat{h}^2 z+C_0p'} = C_0p'[b_u(x, u_{as} + p'v_* + \hat{h}^2 z) + p'C_0 b_{uu}(x, u_{as} + p'v_* + \hat{h}^2 z + C_0p'\theta)]. \tag{2.4.104}$$

As $b_{uu}(\cdot)$ is bounded, (2.4.104) becomes

$$F\Big|_{u_{as}+p'v_*+\hat{h}^2 z}^{u_{as}+p'v_*+\hat{h}^2 z+C_0p'} = C_0p'[b_u(x, u_{as} + p'v_* + \hat{h}^2 z) + O(p')], \tag{2.4.105}$$

and using $b_u(x, u_{as}) = B_s(x, V_0) + O(\varepsilon)$ we get

$$F\Big|_{u_{as}+p'v_*+\hat{h}^2 z}^{u_{as}+p'v_*+\hat{h}^2 z+C_0p'} = C_0p'[B_s(x,v_0) + O(\varepsilon + p' + \hat{h}^2)]. \tag{2.4.106}$$

We next use $B_s(x,v_0) = B_s(x,0) - \lambda(x)|v_0|$ and choose $\lambda(x) := -(\operatorname{sgn} v_0)B_{ss}(x, v_0\theta(x))$ where $|\lambda(x)| \leqslant C_5$ and get

$$F\Big|_{u_{as}+p'v_*+\hat{h}^2 z}^{u_{as}+p'v_*+\hat{h}^2 z+C_0p'} = C_0p'[B_s(x,0) - \lambda(x)|v_0| + O(\varepsilon + p' + \hat{h}^2)]. \tag{2.4.107}$$

Adding together (2.4.54a), (2.4.103) and (2.4.107), we get (2.4.99) to be

$$\begin{aligned} F\beta - \frac{\hat{h}^2}{12}\frac{\partial^4 V_0}{\partial \xi^4} =& p'|v_0| + [p' + \hat{h}^2]O(\varepsilon + p' + \hat{h}^2) + O(\varepsilon p' + \varepsilon \hat{h}^2) \\ &+ C_0 p'[B_s(x,0) - \lambda(x)|v_0| + O(\varepsilon + p' + \hat{h}^2)] \\ &+ O(\varepsilon^3). \end{aligned} \tag{2.4.108}$$

Recall $B_s(x,0) = b_u(x,u_0) \geqslant \gamma^2$ and choose $C_0 = C_5^{-1}$, i.e., $1 - C_0\lambda(x) \geqslant 0$, and (2.4.108) becomes

$$F\beta - \frac{\hat{h}^2}{12}\frac{\partial^4 V_0}{\partial \xi^4} \leqslant C_0\gamma^2|p'| + \frac{1}{2}C_0|p'|\gamma^2 + C_4(\varepsilon^3 + \varepsilon\hat{h}^2 + \hat{h}^4), \tag{2.4.109}$$

where we choose ε^* and p'^* sufficiently small such that $|O(\varepsilon p' + p'^2)| \leqslant \frac{1}{2}C_0|p'|\gamma^2$ and C_4 sufficiently large. □

Later we will require a result similar to those in Lemma 2.4.8 for the outer region of the domain. We give this in the following lemma.

Lemma 2.4.9. *There exists $C_0 > 0$ and p' from* (2.4.84) *such that away from the layer region we have*

$$\pm F[\beta(x;p,\pm p',\hat{h})] \geqslant C_0|p'|\gamma^2 + O(|p'|\varepsilon^2 + \varepsilon^4), \quad x \in \bar{\Omega}\backslash\Omega_{[-c_1,c_1]} \cup \Gamma_{\pm c_1}. \tag{2.4.110}$$

Proof. As $\vartheta(x) = 0$ in this region, $\beta = u_0 + \varepsilon^2 u_2 + p'C_0$. We take the case $F[\beta(x;p')]$; the other being similar. Recalling (2.4.10) we get

$$F[\beta(x;p')] = p'C_0 b_s(x,u_0) + p'C_0\varepsilon^2 u_2(x) b_{uu}(x,u_0) + O(\varepsilon^4). \tag{2.4.111}$$

Using (A3) and as $b_{uu}(x,u_0)$ is bounded we obtain the desired result. □

We now have the necessary requirements to show $\dfrac{\partial \beta}{\partial p} \geqslant 0$ and the following lemma shows $\beta(x; \pm p, \pm p', \hat{h})$ are ordered upper and lower solutions.

Lemma 2.4.10. *Let $p \geqslant 0$, $p' = C'\varepsilon p$ for some positive constant C', and $\hat{h}^2 \leqslant C\varepsilon^{\mu}$ for some fixed $\mu \in [0, 1]$. Then there exists $\varepsilon^* = \varepsilon^*(C', \mu)$ such that for the function β from* (2.4.84) *we have*

$$\beta(x; -p, -p', \hat{h}) \leqslant \beta(x; p, p', \hat{h}) \quad \textit{for } x \in \Omega, \quad \varepsilon \leqslant \varepsilon^*, \quad |p| \leqslant p^*. \tag{2.4.112}$$

Furthermore, for any arbitrarily small but fixed $\lambda \in (0, \bar{\gamma})$, there is a constant $C_\lambda > 0$ such that $u_{as}(x)$ from (2.4.29) *satisfies*

$$|\beta(x; \pm p, \pm p', \hat{h}) - u_{as}(x; 0)| \leqslant C_\lambda(|p| + \hat{h}^2)e^{-(\bar{\gamma}-\lambda)|\xi|} + C\varepsilon|p|. \tag{2.4.113}$$

Proof. This proof imitates that of [18, Lemma 6.6] and [19, Lemma 3.3].

Fix $\hat{h}$, and consider $\tilde{\beta}(x; p) := \beta(x; p, p', \hat{h})$. As $0 \leqslant \vartheta(x) \leqslant 1$ we can replace it with $\vartheta(x) = 1$. The function β is continuous so it suffices to show $\dfrac{\partial \beta}{\partial p} \geqslant 0$ for $x \notin \Gamma_0$. Differentiating β with respect to p and taking $p' = C'\varepsilon p$ gives

$$\frac{\partial \beta}{\partial p} = \frac{\partial v_0}{\partial p} + \varepsilon \frac{\partial v_1}{\partial p} + \varepsilon^2 \frac{\partial v_2}{\partial p} + C'\varepsilon v_* + C'\varepsilon p \frac{\partial v_*}{\partial p} + \hat{h}^2 \frac{\partial z}{\partial p} + C'\varepsilon C_0. \tag{2.4.114}$$

Taking the derivative with respect to p of (2.4.17a), that is

$$\frac{\partial}{\partial p}\left[-\frac{\partial^2}{\partial \xi^2} V_0(\xi, l; p) + b(x, V_0(\xi, l; p))\right] = 0. \tag{2.4.115}$$

Recalling (2.4.30) and tidying terms in (2.4.115) gives

$$-\frac{\partial^2}{\partial \xi^2}\frac{\partial v_0}{\partial p} + \frac{\partial v_0}{\partial p} b_s(x, V_0(\xi, l; p)) = 0, \tag{2.4.116}$$

which is of the form (2.4.36) with $\psi(\xi, l) = 0$ and therefore Lemma 2.4.3 can be applied with $\chi = \dfrac{\partial v_0}{\partial p}$. For the other terms v_1, v_2, v_* and z we have

$$-\frac{\partial^2}{\partial \xi^2}\frac{\partial v_j}{\partial p} + \frac{\partial v_j}{\partial p} b_s(x, V_0(\xi, l; p)) + v_j B_{ps}(x, V_0(\xi, l; p)) = \frac{\partial \psi_j}{\partial p}, \tag{2.4.117}$$

$$-\frac{\partial^2}{\partial \xi^2}\frac{\partial v_*}{\partial p} + \frac{\partial v_*}{\partial p} b_s(x, V_0(\xi, l; p)) + v_* B_{ps}(x, V_0(\xi, l; p)) = \frac{\partial |v_0|}{\partial p}, \tag{2.4.118}$$

and

$$-\frac{\partial^2}{\partial \xi^2}\frac{\partial z}{\partial p} + \frac{\partial z}{\partial p} b_s(x, V_0(\xi, l; p)) + z B_{ps}(x, V_0(\xi, l; p)) = \frac{1}{12}\frac{\partial^4}{\partial \xi^4}\frac{\partial v_0}{\partial p}. \tag{2.4.119}$$

We find bounds for v_j, v_* and z using the above equations and Lemma 2.4.3, i.e., $\left|\dfrac{\partial v_j}{\partial p}\right| \leqslant C(1+\xi^{2j})\chi$ for $j = 1, 2$ and $\left|\dfrac{\partial v_*}{\partial p}\right| + \left|\dfrac{\partial z}{\partial p}\right| \leqslant C(1+|\xi|)\chi$. Putting these together in (2.4.114), we get

$$\begin{aligned}\frac{\partial \tilde{\beta}}{\partial p} \geqslant\ & \chi + C_0 C'\varepsilon + v_* C'\varepsilon - \varepsilon C(1+\xi^2)\chi - \varepsilon^2 C(1+\xi^4)\chi \\ & - C(1+|\xi|)\chi C'\varepsilon p - \hat{h}^2 C(1+|\xi|)\chi.\end{aligned} \tag{2.4.120}$$

Choose $\mu \in [0, 1]$ and some positive constant C'' such that

$$\frac{\partial \tilde{\beta}}{\partial p} \geqslant \chi + C_0 C'\varepsilon + v_* C'\varepsilon - C''\varepsilon^\mu (1+\xi^4)\chi. \tag{2.4.121}$$

As $v_*(\xi, l) > 0$, we have

$$\frac{\partial \tilde{\beta}}{\partial p} \geqslant \chi + C_0 C'\varepsilon - C''\varepsilon^\mu (1+\xi^4)\chi. \tag{2.4.122}$$

Using (2.4.24) we choose $\bar{C}$ sufficiently large so that for $|\xi| \geqslant \xi^* := \bar{C}|\ln \varepsilon|$

we have

$$(1+\xi^4)\chi \leqslant \frac{C'C_0}{C''}\varepsilon^{1-\mu}. \tag{2.4.123}$$

Putting (2.4.123) into (2.4.122) gives $\dfrac{\partial\tilde{\beta}}{\partial p} \geqslant \chi$, and since $\chi > 0$ we have the desired result.

If instead $|\xi| \leqslant \bar{C}|\ln\varepsilon|$, then

$$C''\varepsilon^{\mu}(1+\xi^4) \leqslant C''\varepsilon^{\mu}(1+|\ln\varepsilon|^4) \leqslant 1, \tag{2.4.124}$$

and so

$$\frac{\partial\tilde{\beta}}{\partial p} \geqslant C_0 C'\varepsilon \geqslant 0. \tag{2.4.125}$$

□

The final corollary of this section brings together the results so far and states that a solution to the original problem exists and lies between the upper and lower solutions that we have found.

Corollary 2.4.1. *There is $\varepsilon^* > 0$ such that for all $\varepsilon \leqslant \varepsilon^*$ there exists a solution $u(x)$ of problem* (2.0.1) *that satisfies*

$$|u(x) - u_{as}(x;0)| \leqslant C\varepsilon^2 \quad \textit{for } x \in \Omega. \tag{2.4.126}$$

Proof. A similar result can be found in [24]. Our proof imitates that of [18, Corollary 6.7] and is extended here to two dimensions.

For this proof we set up upper and lower solutions, $\beta(x; \pm\bar{p}, \pm\bar{p}', 0)$, and call upon Lemma 1.0.1 to show there exists an exact solution between these solutions.

Firstly choose $\bar{p} \geqslant \dfrac{C_2\varepsilon^2}{2C_1}$ and

$$\bar{p}' := \frac{C_1\varepsilon\bar{p}}{2C_3}, \tag{2.4.127}$$

combining these with (2.4.88) gives

$$(\operatorname{sgn} p) \cdot \Phi[\beta(x; \bar{p}, \bar{p}', \hat{h})] \geqslant \frac{C_2\varepsilon^3}{4} \geqslant 0. \tag{2.4.128}$$

If we now take (2.4.98) and choose $\bar{p} = O(\varepsilon^2)$ so large that $\dfrac{1}{2}C_0\bar{p}'\gamma^2 \geqslant C_4\varepsilon^3$ we have

$$(\operatorname{sgn} \bar{p}')F\beta \geqslant \frac{1}{2}C_0|\bar{p}'|\gamma^2 - C_4\varepsilon^3 \geqslant 0. \tag{2.4.129}$$

As $\beta(x; -\bar{p}, -\bar{p}', 0) \leqslant \beta(x; \bar{p}, \bar{p}', 0)$ then $\beta(x; \pm\bar{p}, \pm\bar{p}', 0)$ are ordered upper and lower solutions of (2.0.1) away from the boundary. Noting (A6) it can be seen that at the boundary $\pm\,\beta(x; \pm\bar{p}, \pm\bar{p}', 0)|_{x\in\partial\Omega} \geqslant \pm g(x)$ and hence $\beta(x; \pm\bar{p}, \pm\bar{p}', 0)$ are upper and lower solutions in the entire domain.

Calling on [27, Theorem 4.1, p.64] described in Lemma 1.0.1 a solution $u(x)$ of (2.0.1) exists and lies between $\beta(x; -\bar{p}, -\bar{p}', 0)$ and $\beta(x; \bar{p}, \bar{p}', 0)$. Using (2.4.113) we have that this solution lies in an $O(\bar{p}) = O(\varepsilon^2)$ region from $u_{as}(x)$. □

2.5 Discrete Space: Analysis of the Numerical Method

Considering the discrete space, we present a numerical scheme for the solution of (2.0.1). We aim to prove existence of discrete upper and lower solutions and find accuracy bounds for the conventional and the stabilised numerical methods.

Employing the theory of Z-fields as described in §1.0.2, we will prove the existence of a solution between the discrete upper and lower solutions. In order to use the theory of Z-fields we have the following remark for the discrete system. Note an M-matrix is a matrix with non-positive off diagonal entries and where the real part of its eigenvalues are positive.

Remark 2.5.1. [14, Remark 3.8]
Let $X_1, X_2, \ldots, X_n$ be interior points of Ω, $\bar{X}_1, \bar{X}_2, ..., \bar{X}_m$ be on $\partial\Omega$ and $U \in \mathbb{R}^{n+m}$ be a discrete function defined at these points.
Suppose $F^N : \mathbb{R}^{n+m} \to \mathbb{R}^n$ has the form

$$F^N U = \varepsilon^2 \Lambda^N U + [b(X_i, U_i)]_{i=1}^n, \tag{2.5.1}$$

where Λ^N is an M-matrix discretisation of the operator $-\Delta$.
Then the mapping $(X_1..., X_n, \bar{X}_1..., \bar{X}_m) \mapsto (F^N U_1..., F^N U_n, g(\bar{X}_1)..., g(\bar{X}_m))$ is a Z-field.

From this we have two requirements on our system for the use of the theory of Z-fields;

(i) the discretisation of $-\Delta$ is an M-matrix,

(ii) the discretisation of $b(x, u)$ at any interior mesh point X_i uses only U_k with $k = i$.

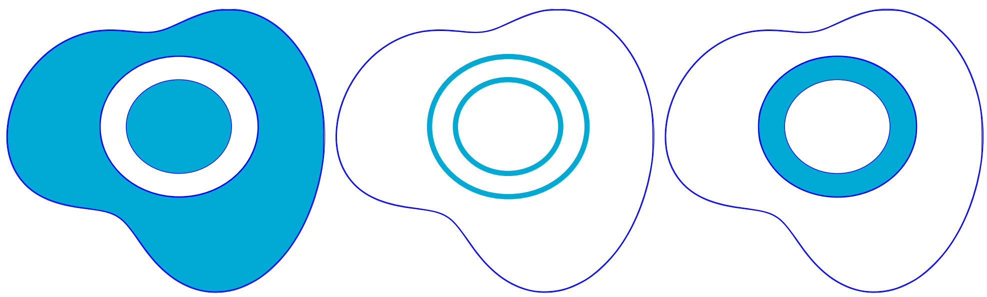
To ensure an M-matrix discretisation of the finite element method we

consider a quasiuniform Delaunay triangulation. Define h as the maximum side of any triangle, i.e., $h := \max_i h_i$ where h_i is the length of side i. A quasiuniform mesh is a mesh in which the area of any triangle is bounded below by Ch^2. A Delaunay triangulation is one in which the sum of the angles opposite to any edge is less than or equal to π, and any angle opposite to $\partial\Omega^h$ is less than or equal to $\frac{\pi}{2}$. Both of these requirements imply that we have a mesh whose triangles are essentially the same size and the stiffness matrix of $-\Delta$ is an M-matrix.

For a scheme where the discretisation of $b(x,u)$ at any interior mesh point X_i uses only U_k with $k = i$ we use lumped mass finite elements. This is done by diagonalising the matrix representation of $b(x,u)$, i.e., the mass matrix M is replaced by a diagonal matrix $\bar{M}$ where each non zero element $\bar{m}_{ii}$ is found by summing the ith row of M, i.e.,

$$\bar{m}_{ii} = \sum_k m_{ik} \quad \text{for each } i. \tag{2.5.2}$$

To solve the system in the discrete space Ω^h we would like to use finite element throughout the domain. Due to the above points this is not possible if we also want to refine the mesh. Refining the finite element mesh will mean we no longer have an M-matrix and therefore cannot use the theory of Z-fields. To refine the mesh in a region of the domain we use a finite difference discretisation in that region. This could be solved by bilinear finite elements, however going into polar coordinates in a finite elements discretisation would cause complications. To solve these issues we use a Delaunay triangulation with lumped mass linear finite elements in the outer regions and finite differences in curvilinear coordinates in the interior layer. On either side of the interface curve we use a fictitious Neumann boundary condition, these discretisations are then combined, eliminating the Neumann condition. These regions are represented in Figure 2.9.



(a) The outer regions including $\partial\Omega$ where finite elements are used. *(b) The interface curves $\Gamma_{\pm\tau}$ where the fictitious Neumann condition is used.* *(c) The interior layer region $\Omega_{(0,\tau)}$ where finite differences are used.*

Figure 2.9: The regions of Ω.

2.5.1 Layer Adapted Mesh

For the discrete domain we define τ to be a small positive parameter such that $\tau \leqslant c_1$ and the boundaries $\Gamma_{\pm\tau} := \{x(r,l) : r = \pm\tau\}$ are smooth closed curves in Ω that do not intersect with each other. The discretisation of $\Omega_{(-c_1,c_1)}$, the interior layer region, is the region denoted $\Omega_{(-\tau,\tau)} := \{x(r,l) : |r| < \tau\}$. Since this is a rectangle in curvilinear coordinates we can define a tensor product mesh.

Define the tensor product mesh to be $\{(r_i, l_j), i = -N, ..., N, j = -1, ..., N_l\}$ where $l_0 = 0$, $l_{N_l} = L$ and $l_1 = l_{N_l-1} - L$. The transition curve is located at $r_0 = 0$, i.e., $x_{0j} = x(r_0, l_j) \in \Gamma_0$ and $x_{\pm N,j}$ describes $\Gamma_{\pm\tau}$. Define $h_i := r_i - r_{i-1}$ and $h_i \leqslant h$. Let $\{l_j\}$ be a quasiuniform mesh with $C^{-1}N^{-1} \leqslant l_j - l_{j-1} \leqslant CN^{-1}$. The fine mesh $\{r_i\}$ is chosen in §2.5.1. Assume $r_i - r_{i-1} \leqslant h$ and

$$C^{-1}h^{-1} \leqslant N \leqslant Ch^{-1}. \tag{2.5.3}$$

In the outer regions, i.e., for $X_i \in \mathring{\Omega}^N$, we use piecewise linear finite elements on a quasiuniform Delaunay triangulation. We replace the original domain boundary with the polygonal boundary $\partial\Omega^N$. Along the boundary

of the domain we have the original boundary condition $u(X_i) = g(X_i)$ for $X_i \in \partial\Omega^N$. We also introduce the polygonal curve $\Gamma^N_{\pm\tau}$ for the interface in the outer region.

On the boundary interfaces between the interior layer regions and the outer regions, denoted $\Gamma_{\pm\tau}$, we introduce a fictitious Neumann boundary condition. We find bounds for this interface using the finite element and finite difference methods to the interior and exterior of $\Gamma_{\pm\tau}$, the solution can then be matched across this boundary and the fictitious boundary condition eliminated.

As in the continuous space, we may write $x_{ij} = (r_i, l_j)$ and $x_{ij} = (\xi_i, l_j)$ for $x_{ij} = x(r_i, l_j)$ and $x_{ij} = x(\xi_i, l_j)$ respectively in the discrete space.

The Shishkin Mesh [32]

Here we define τ, the mesh transition point, to be

$$\tau := \frac{C_\tau}{\bar{\gamma}} \varepsilon \ln N, \tag{2.5.4}$$

with $\bar{\gamma}$ from (2.4.22) and C_τ a sufficiently large user chosen constant. The constant C_τ will be later chosen in Lemma 2.5.5; for the standard method $C_\tau > 2$ and for the stabilised method $C_\tau > 3$. These choices will allow us to say that $\beta_*(x_{ij}; \pm p) \pm p'' w(x_{ij})$ are upper and lower solutions to the system where $w(x_{ij})$ and p'' will be defined in (2.5.11) and (2.5.12) respectively.

Introduce a uniform mesh $\{r_i\}_{i=-N}^{N}$ on $[-\tau, \tau]$, so

$$r_i - r_{i-1} = \tau/N = C_\tau \varepsilon N^{-1} \ln N/\bar{\gamma}. \tag{2.5.5}$$

We do not consider the Bakhvalov mesh as this will alter the truncation error part of our analysis and so would complicate the work further.

2.5.2 Cure for Numerical Instability: The Stabilised Method

For the stabilised system we replace ε, for some $x_{ij} \in \bar{\Omega}^N$, with

$$\hat{\varepsilon}(x_{ij}) = \max\left\{\varepsilon, \frac{\widehat{C}}{N} I(x_{ij})\right\}, \tag{2.5.6}$$

where

$$I(x_{ij}) := \begin{cases} 1 & x_{ij} \in \overline{\Omega} \backslash \Omega_{(-\tau,\tau)}, \\ 0 & x_{ij} \in \Omega_{(-\tau,\tau)}, \end{cases} \tag{2.5.7}$$

and $\widehat{C}$ is some user chosen constant satisfying

$$\widehat{C}^2 \geqslant - \min_{v \in [\varphi_1, \varphi_2], x \in \bar{\Omega}^N} b_u(x, u). \tag{2.5.8}$$

Compared with the standard method, this stabilisation will add artificial diffusion in $\overline{\Omega} \backslash \Omega_{(-\tau,\tau)}$ if $\varepsilon N < \widehat{C}$ and will remain unchanged elsewhere. This alters the system in the outer regions and strengthens the Laplacian term.

We denote the standard numerical scheme as F^N and the stabilised numerical scheme as $\hat{F}^N$. Let N be sufficiently large. For $X_i \in \overline{\Omega} \backslash \Omega_{(-\tau,\tau)}$ we have $\hat{F}^N - F^N = -(\hat{\varepsilon}^2(x_{ij}) - \varepsilon^2)\Lambda^N$ and for $X_i \in \Omega_{(-\tau,\tau)}$ we have $\hat{F}^N = F^N$. Note $\Omega_{(-\tau,\tau)} \cap \Gamma_{\pm\tau} = \emptyset$ therefore $\hat{F}^N - F^N = -(\hat{\varepsilon}^2(x_{ij}) - \varepsilon^2)\Lambda^N$ for $X_i \in \Gamma_{\pm\tau}$.

2.5.3 Truncation error

We want to examine the truncation error of the problem for F^N and $\hat{F}^N$ applied to the Shishkin mesh of §2.5.1 using a particular β defined by

$$\beta_*(x) := \beta_*(x; p) = \beta(x; p, p', \hat{h}), \tag{2.5.9}$$

where

$$\hat{h} := \varepsilon^{-1} \min_j h_j = CN^{-1} \ln N, \quad p' := \varepsilon \frac{C_1}{2C_3} p, \tag{2.5.10}$$

and p is such that $|p| \leqslant p^*$ for some fixed p^*.

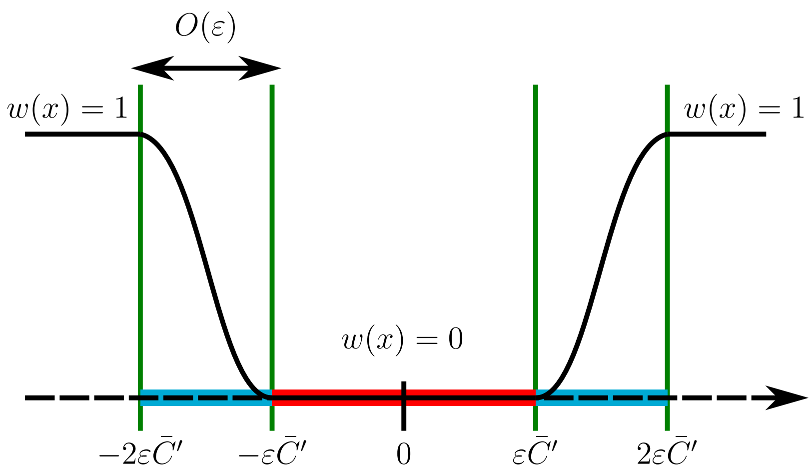


Figure 2.10: Sketch of $w(x_{ij})$.

Throughout this section we shall use $\beta_*(x_{ij}; p) \pm p'' w(x_{ij})$ as our upper and lower solutions. The term $p'' w(x_{ij})$ is introduced to later help dominate terms from the truncation error in the outer regions. We define

$$w(x_{ij}) := \begin{cases} 1 & X_i \in \overline{\Omega} \backslash \Omega_{(-\tau,\tau)}, \\ \omega(x_{ij}) e^{-(\bar{\gamma}-\lambda)(\tau - |r_i|)/\varepsilon} & x_{ij} \in \Omega_{(-\tau,\tau)}, \end{cases} \tag{2.5.11}$$

where $\omega(x_{ij})$ is a smooth cut-off function that takes values in $[0, 1]$, equals 1 when $|r_i| \geqslant 2\varepsilon\bar{C}'$ and vanishes when $|r_i| \leqslant \varepsilon\bar{C}'$. The constant $\bar{C}'$ is a sufficiently large positive constant and λ is a sufficiently small positive constant. The function $w(x_{ij})$ is represented in Figure 2.10.

For $\beta_*(x_{ij}; p) \pm p'' w(x_{ij})$ with $x_{ij} \in \bar{\Omega}^N$ and $w(x_{ij})$ from (2.5.11) we define p'' as

$$p'' := \bar{C}' N^{-\alpha}, \tag{2.5.12}$$

with $\alpha := 2$ for the discretisation of the standard method (2.0.1) and $\alpha := 1$ for the stabilised method (2.1.4) where $\hat{\varepsilon}(x_{ij})$ is defined in (2.5.6). We choose $\bar{C}'$ sufficiently large. The motivation for p'' will be explained for the standard method in §2.5.7 and for the stabilised method in §2.5.8.

To obtain global accuracy we look at each region and gain pointwise error bounds. We will then combine these to get global accuracy. We will consider three regions of the domain separately and combine the results to obtain global accuracy results. The three regions are the layer region, $\Omega_{(-\tau,\tau)}$, the outer regions, $\bar{\Omega}^N \backslash \Omega_{[-\tau,\tau]}$ and the interface between these, $\Gamma_{\pm\tau}$. We consider both sides of the curves $\Gamma_{\pm\tau}$. To distinguish between the sides of these curves we us the notation $\Gamma_{\pm\tau} \cap \partial\Omega_{[-\tau,\tau]}$ and $\Gamma_{\pm\tau} \cap \partial\mathring{\Omega}^N$, combining these to get the approximation along $\Gamma_{\pm\tau}$.

We now present lemmas necessary to gain pointwise error bounds in each region and then combine these to give results for the problem as a whole.

2.5.4 Discretisation of the interior layer regions, $\Omega_{(-\tau,\tau)}$

We are looking at the interior layer region $\Omega_{(-\tau,\tau)} = \{x(r_i, l_j) : |r_i| < \tau\}$ represented in Figure 2.9c. We note $x_{\pm N,j}$ describes the curves $\Gamma_{\pm\tau}$. In the layer region we are using finite differences in curvilinear coordinates similar to the two-dimensional boundary layer problem [14]. In this region $\hat{\varepsilon}(x_{ij}) = \varepsilon$ therefore the standard and stabilised numerical schemes are equal, i.e., $\hat{F}^N = F^N$.

For $i = -N+1, ..., N-1$ and $j = 0, ..., N_l - 1$, we have

$$F^N U_{ij} := -\varepsilon^2 \eta_{ij}^{-1} D_r \left[\tilde{\eta}_{ij} D_r^- U_{ij}\right] - \varepsilon^2 \zeta_{ij} D_l \left[\tilde{\zeta}_{ij} D_l^- U_{ij}\right] + b\left(x_{ij}, U_{ij}\right) = 0 \tag{2.5.13a}$$

$$U_{i,N_l} = U_{i,0}, \quad U_{i,-1} = U_{i,N_l-1}, \tag{2.5.13b}$$

where ζ and η are described in (A.2) and

$$x_{ij} := x(r_i, l_j), \quad \eta_{ij} := \eta(r_i, l_j), \quad \zeta_{ij} := \zeta(r_i, l_j), \tag{2.5.13c}$$

$$\tilde{\eta}_{ij} := \eta(r_{i-1/2}, l_j), \quad \tilde{\zeta}_{ij} := \zeta(r_{i-1/2}, l_j), \tag{2.5.13d}$$

$$D_r^- v_{ij} := \frac{v_{ij} - v_{i-1,j}}{r_i - r_{i-1}}, \quad D_r v_{ij} := \frac{v_{i+1,j} - v_{ij}}{(r_{i+1} - r_{i-1})/2}, \tag{2.5.13e}$$

$$D_l^- v_{ij} := \frac{v_{ij} - v_{ij-1}}{l_j - l_{j-1}}, \quad D_l v_{ij} := \frac{v_{ij+1} - v_{ij}}{(l_{j+1} - l_{j-1})/2}. \tag{2.5.13f}$$

Note that in the interior layer region $\hat{\varepsilon}(x_{ij}) = \varepsilon$ and so $\hat{F}^N = F^N$.

On any line r, with l being constant, this region behaves like the one-dimensional interior layer case [18]. Therefore we follow the outline of [18], extending it to two dimensions. We want to set up discrete sub and super solutions for this region and we do this in the following three lemmas.

To calculate the truncation error of our system we first look at the one-dimensional truncation error of the Laplace operator. We consider this for a uniform mesh and for a non uniform mesh. We will then combine these results to obtain the truncation error of our system (2.5.13).

Let $w(t)$ be some function with a discontinuity across $t = 0$. We consider

$$Mw := -q(t)(p(t)w'(t))', \tag{2.5.14}$$

where p and q are sufficiently smooth functions, and want to evaluate the truncation error $R_i := M^N w_i - (Mw)_i$.

For the case with a non uniform mesh the discrete analogue of (2.7.32) is

$$M^N w_i^N := -q_i D\left(\tilde{p}_i D^- w_i^N\right) \quad \text{for } i = 1, ..., N-1, \tag{2.5.15a}$$

where

$$D^- w_i := (w_i - w_{i-1})/k_i, \quad Dw_i := (w_{i+1} - w_i)/\hbar_i, \tag{2.5.15b}$$

$$\hbar_i := (k_i + k_{i+1})/2, \quad k_i := t_i - t_{i-1} \quad \text{and} \quad \tilde{p}_i := p_{i-\frac{1}{2}}. \tag{2.5.15c}$$

For a uniform mesh we have the following result for the truncation error.

Lemma 2.5.1. *Let $|w^{(j)}| \leqslant C$, $|p^{(j)}| \leqslant C$ for $j = 0, ..., 6$, and $k_i = k$ and $p(t)$ and $q(t)$ be sufficiently smooth functions. The truncation error $R_i := M^N w_i - (Mw)_i$ of (2.5.15) with $t \neq 0$ is*

$$R_i = -\alpha_1 k^2 q_i \left((pw''')' + (pw')'''\right)_i + O(k^4), \tag{2.5.16}$$

with $\alpha_1 = \dfrac{1}{2^2 3!}$.

Proof. The proof of this is included in §2.7.2. □

Remark 2.5.2. Considering Lemma 2.5.1 with the additional condition that $|w^{(j)}| \leqslant Ca$ for $j = 0, ..., 4$ we have

$$R_i = O(k^2 a). \tag{2.5.17}$$

Let us now look at the truncation error of (2.7.32) on a non uniform mesh. For this case we do not need to consider the truncation error to an accuracy as high as in the previous lemma. This lemma will be used to obtain the truncation error of derivatives with respect to l in (2.5.13).

Lemma 2.5.2. *Let* $|w^{(j)}| \leqslant C$ *and* $|p^{(j)}| \leqslant C$ *for* $j = 0, ..., 3$. *For a non uniform mesh the truncation error* $R_i := M^N w_i - (Mw)_i$ *of* (2.5.15) *is*

$$R_i := O(k), \tag{2.5.18}$$

where $k = \max k_i$.

Proof. The proof of this is included in §2.7.2. □

Corollary 2.5.1. *For* $i \neq 0$ *the truncation error of the system* (2.5.13) *for the mesh described in §2.5.1 is*

$$R_{ij}[\beta] := F^N \beta_{ij} - F\beta(x_{ij}) = -\frac{\hat{h}^2}{12} \frac{\partial^4 V_0}{\partial \xi^4}\bigg|_{ij} + O(\varepsilon \hat{h}^2 + \hat{h}^4). \tag{2.5.19}$$

Proof. We note

$$\left|\frac{\partial^{j+k}\beta}{\partial \xi^j \partial l^k}\right| \leqslant C, \quad \left|\frac{\partial^{j+k}\eta}{\partial r^j \partial l^k}\right| \leqslant C \text{ for } j = 0, ..., 6, \ k = 0, ..., 3, \tag{2.5.20}$$

due to the bounds (2.4.50) and (2.4.87) for terms in β, as well as (2.5.10) and (2.4.84).

We rescale $r = \varepsilon\xi$ in (2.5.13) and write $R_{ij}[\beta] = R_{ij}^{(1)}[\beta] + R_{ij}^{(2)}[\beta]$ with

$$R_{ij}^{(1)}[\beta] = -\eta_{ij}^{-1}\left[D_\xi\left(\tilde{\eta}_{ij} D_\xi^- \beta_{ij}\right) - \frac{\partial}{\partial \xi}\left(\eta \frac{\partial \beta}{\partial \xi}\right)_{ij}\right], \tag{2.5.21}$$

$$R_{ij}^{(2)}[\beta] = -\varepsilon^2 \zeta_{ij}\left[D_l\left(\tilde{\zeta}_{ij} D_l^- \beta_{ij}\right) - \frac{\partial}{\partial l}\left(\zeta \frac{\partial \beta}{\partial l}\right)_{ij}\right]. \tag{2.5.22}$$

For $R_{ij}^{(1)}[\beta]$ we employ Lemma 2.5.1 with $w = \beta$, $p = \eta$, $q = \eta^{-1}$, $t = \xi$, and

$k = \xi_i - \xi_{i-1}$, i.e., $k = \hat{h}$, and get

$$R^{(1)}_{ij}[\beta] = -\alpha_1 \hat{h}^2 \eta^{-1}_{ij} \left[\frac{\partial}{\partial \xi} \left(\eta \frac{\partial^3 \beta}{\partial \xi^3} \right) + \frac{\partial^3}{\partial \xi^3} \left(\eta \frac{\partial \beta}{\partial \xi} \right) \right]_{ij} + O(\hat{h}^4). \qquad (2.5.23)$$

For $R^{(2)}_{ij}[\beta]$ we use Lemma 2.5.2 with $w = \beta$, $p = \zeta$, $q = \zeta$, $t = l$ and $k_i = l_i - l_{i-1}$, i.e., $k_i \leqslant CN^{-1}$, and get

$$R^{(2)}_{ij}[\beta] = O(\varepsilon^2 N^{-1}). \qquad (2.5.24)$$

We note $N^{-1} \leqslant C\hat{h}$ using (2.5.10). Combining this information we find

$$R_{ij}[\beta] = -\alpha_1 \hat{h}^2 \eta^{-1}_{ij} \left[\frac{\partial}{\partial \xi} \left(\eta \frac{\partial^3 \beta}{\partial \xi^3} \right) + \frac{\partial^3}{\partial \xi^3} \left(\eta \frac{\partial \beta}{\partial \xi} \right) \right]_{ij} + O(\hat{h}^4 + \varepsilon^2 \hat{h}). \qquad (2.5.25)$$

Expanding terms gives

$$\begin{aligned} R_{ij}[\beta] = -\alpha_1 \hat{h}^2 \eta^{-1}_{ij} \left(2\eta \frac{\partial^4 \beta}{\partial \xi^4} + 4 \frac{\partial \eta}{\partial \xi} \frac{\partial^3 \beta}{\partial \xi^3} + 3 \frac{\partial^2 \eta}{\partial \xi^2} \frac{\partial^2 \beta}{\partial \xi^2} + \frac{\partial^3 \eta}{\partial \xi^3} \frac{\partial \beta}{\partial \xi} \right)_{ij} \\ + O(\hat{h}^4 + \varepsilon^2 \hat{h}). \end{aligned} \qquad (2.5.26)$$

As we are considering the layer region, we take $\vartheta(x) = 1$ in (2.4.84). Again using the bounds for β and η in (2.5.20) we have

$$R_{ij}[\beta] = -\alpha_1 \hat{h}^2 \eta^{-1}_{ij} \left(2\eta \frac{\partial^4 \beta}{\partial \xi^4} + O(\varepsilon) \right)_{ij} + O(\hat{h}^4 + \varepsilon^2 \hat{h}). \qquad (2.5.27)$$

We note $\dfrac{\partial^4 \beta}{\partial \xi^4} = \dfrac{\partial^4 V_0}{\partial \xi^4} + O(\varepsilon + \hat{h}^2)$ and $(\eta^{-1}\eta)_{ij} = 1$. Finally as $\varepsilon \leqslant CN^{-1}$ from (2.2.3) and $N^{-1} \leqslant C\hat{h}$ by (2.5.10) we have $\varepsilon^2 \hat{h} \leqslant C\varepsilon \hat{h}^2$ giving the order as $\varepsilon \hat{h}^2 + \hat{h}^4$ and obtaining (2.5.19).

□

Due to the discontinuity along Γ_0 of derivatives of the functions involved in β, we now consider $q(pw')'$ along the discontinuity. As the discontinuity occurs along $r = 0$ and our mesh is uniform in the r direction we consider this lemma on a uniform mesh.

Lemma 2.5.3. *Let $|w^{(j)}| \leqslant Ca$ and $|p^{(j)}| \leqslant C$ for $j = 0, ..., 3$ and some positive constant a. For the case with discontinuous derivatives across $t = 0$, the truncation error of* (2.7.32) *on a uniform mesh is*

$$\begin{aligned} M^N w(0) - \frac{1}{2}\left\{Mw(0^+) + Mw(0^-)\right\}_i = \\ -q_i \frac{1}{k}\left\{pw^{+\prime} - pw^{-\prime}\right\}_i + O(ka), \end{aligned} \tag{2.5.28}$$

where $w^\pm = w(0^\pm)$.

Proof. The proof of this is included in §2.7.2. ☐

Corollary 2.5.2. *The truncation error of the system* (2.5.13) *at $r = 0$ is*

$$\begin{aligned} F^N\beta(0, l_j) - \frac{1}{2}\Big(F\beta(0^+, l_j) + F\beta(0^-, l_j)\Big) = \\ -\frac{\hat{h}^2}{12}\left.\frac{\partial^4 V_0}{\partial \xi^4}\right|_{r=0} + \frac{1}{\hat{h}}\Phi[\beta] + O(\varepsilon\hat{h} + \hat{h}^3). \end{aligned} \tag{2.5.29}$$

Proof. Firstly, we have $\frac{1}{2}\left(FV_0(0^+, l) + FV_0(0^-, l)\right) = FV_0(0, l)$ as V_0 is continuous across $r = 0$. Hence we revert to the previous continuous case for V_0 using Corollary 2.5.1 and get

$$F^N V_0(0, l) - \frac{1}{2}\Big(FV_0(0^+, l) + FV_0(0^-, l)\Big)_{ij} = -\frac{\hat{h}^2}{12}\left.\frac{\partial^4 V_0}{\partial \xi^4}\right|_{r=0} + O(\varepsilon\hat{h}^2 + \varepsilon^2\hat{h} + \hat{h}^4). \tag{2.5.30}$$

We are now left to consider $\tilde{\beta} = \beta - V_0$. We stretch the variable r, $r = \varepsilon\xi$, and use Lemma 2.5.3 with $w = \tilde{\beta}$, $q = \eta^{-1}$, $p = \eta$, $t = \xi$ and $k = \xi_i - \xi_{i-1}$,

i.e., $k = \hat{h}$. We take $a = \varepsilon + \hat{h}^2$ as $\left|\dfrac{\partial^j \tilde{\beta}}{\partial \xi^j}\right| = O(\varepsilon + \hat{h}^2)$ for $j = 0, ..., 6$. We also use Lemma 2.5.2 with $w = \tilde{\beta}$, $q = \zeta^{-1}$, $p = \zeta$, $t = l$ and $k_i = l_i - l_{i-1}$, i.e., $k_i \leqslant CN^{-1}$ and again note $N^{-1} \leqslant C\hat{h}$. Combining this information we get

$$\begin{aligned} F^N \tilde{\beta}(0, l_j) - \frac{1}{2}\left(F\tilde{\beta}(0^+, l_j) + F\tilde{\beta}(0^-, l_j)\right) = \\ -\eta^{-1}\frac{1}{\hat{h}}\left\{\eta\frac{\partial\tilde{\beta}}{\partial\xi}^+ - \eta\frac{\partial\tilde{\beta}}{\partial\xi}^-\right\}_{ij} + O(\hat{h}(\varepsilon + \hat{h}^2) + \varepsilon^2\hat{h}). \end{aligned} \tag{2.5.31}$$

We recall the notation $\Phi[\beta]$ from (2.4.35). As V_0 has continuous derivatives across $r = 0$ then $\Phi[\tilde{\beta}] = \Phi[\beta]$ and we get the desired result. □

Remark 2.5.3. For the finite difference scheme (2.5.13) in the layer region we have $\hat{\varepsilon} = \varepsilon$ and so in this region

$$|(\hat{F}^N - F^N)\beta_*(x_{ij})| = 0. \tag{2.5.32}$$

We now want to apply Lemma 1.0.1 in this region. To do this we must find discrete upper and lower solutions β and α such that $\alpha \leqslant \beta$ and $H\alpha \leqslant 0 \leqslant H\beta$. In our case the discrete upper and lower solutions are $\beta(x_{ij}; p) \pm p'' w(x_{ij})$.

Lemma 2.5.4. *Let N be sufficiently large and $p'' > 0$ sufficiently small and independent of ε. Let $C'_\tau < C_\tau$ where C_τ is a sufficiently large positive constant. There exists $w(x_{ij})$ such that $0 \leqslant w(x_{ij}) \leqslant 1$ and for all $|p| \leqslant p^*$ the function $\beta_*(x_{ij}; p)$ from* (2.4.84) *satisfies*

$$\pm\{F^N[\beta_*(x_{ij}; p) \pm p'' w(x_{ij})] - F^N \beta_*(x_{ij}; p)\} \geqslant -p'' C N^{-C'_\tau}, \tag{2.5.33}$$

for $x_{ij} \in \Omega_{(-\tau,\tau)}$.

Furthermore, this inequality holds true with F^N replaced by $\hat{F}^N$.

Proof. We extend the proof of [18, Lemma 6.9] to two dimensions.

For the region $|r_i| < \varepsilon\bar{C}'$ we have $w(x_{ij}) = 0$ from (2.5.11) and we immediately have $F^N[\beta_*(x_{ij}; p) \pm p''w(x_{ij})] - F^N\beta_*(x_{ij}; p) = 0$.

For the rest of the proof we deal with $\varepsilon\bar{C}' \leqslant |r_i| \leqslant \tau$ and consider the + case of $\pm$ in (2.5.33), the other case being similar. Choose λ sufficiently small such that

$$C_\tau(1 - \lambda/\bar{\gamma}) \geqslant C'_\tau. \tag{2.5.34}$$

We next want a bound for $b_u(x, \beta_*(x) + s)$ for $\varepsilon\bar{C}' \leqslant |r_i| < \tau$ and $|s| \leqslant c'$ where c' is a sufficiently small positive constant. We take the case where $r < 0$, the other case being similar. Firstly we have

$$b_u(x, \beta_*(x) + s) = b_u(x, \varphi_2(x)) + O(\beta_* - \varphi_2) + O(s). \tag{2.5.35}$$

As $|s| \leqslant c'$ with c' sufficiently small, we can make s as small as needed by making c' sufficiently small. For the remaining term we have $\beta_*(x) - \varphi_2(x) = v_0 + O(N^{-1})$. By (2.4.50) for $\varepsilon\bar{C}' \leqslant |r_i| < \tau$ we have $\beta_*(x) - \varphi_2(x) \leqslant e^{-(\bar{\gamma}-\lambda)\bar{C}'} + O(N^{-1})$. This can be made sufficiently small by choosing N and $\bar{C}'$ sufficiently large. Now for the first term in (2.5.35) we note $x = \bar{x} + O(\tau)$ with τ defined in (2.5.4) and $b_u(x, v(x))$ and $\varphi_2(x)$ are sufficiently smooth for any smooth $v(x)$, and we get

$$b_u(x, \varphi_2(x)) \geqslant b_u(\bar{x}, \varphi_2(\bar{x})) - C\tau. \tag{2.5.36}$$

As $\tau = C\varepsilon \ln N$, we recall (2.2.3) and hence we can make τ sufficiently small by again choosing N sufficiently large. We note (2.4.22) and choose λ sufficiently small from (2.5.34) such that we can now write

$$b_u(x, \beta_*(x) + s) \geqslant (\bar{\gamma} - \lambda/2)^2 \quad \text{for } \varepsilon\bar{C}' \leqslant |r_i| < \tau,\ |s| \leqslant c'. \tag{2.5.37}$$

Recall $w(x_{ij}) = \omega(x_{ij})e^{-(\bar{\gamma}-\lambda)(\tau-|r_i|)/\varepsilon}$ for $x_{ij} \in \Omega_{(-\tau,\tau)}$ from (2.5.11) and choose $0 < p'' \leqslant c'$. We have $-\varepsilon^2\zeta_{ij}D_l\left[\tilde{\zeta}_{ij}D_l^- w(x_{ij})\right] = 0$ as $w(x_{ij})$ has no dependence on l. We use a Taylor series expansion of $b(x, \beta_* + p''w(x_{ij}))$ and the bounds in (2.5.37) to get

$$\begin{aligned} F^N[\beta_*(x_{ij};p) + p''w(x_{ij})] - F^N\beta_*(x_{ij};p) \geqslant \\ - p''\varepsilon^2\eta_{ij}^{-1}D_r\left[\tilde{\eta}_{ij}D_r^- w(x_{ij})\right] + p''(\bar{\gamma}-\lambda/2)^2 w(x_{ij}). \end{aligned} \tag{2.5.38}$$

We now look for a bound for $w(x_{ij})$ and its derivatives. Firstly $|\omega(x)| \leqslant 1$ and $\left|\dfrac{\partial^l\omega}{\partial\xi^l}\right| \leqslant C$ for $l = 0, ..., 6$. Now the exponential term in $w(x_{ij})$ is left to be considered. Due to the differential properties of the exponential function and as $\bar{\gamma} - \lambda \leqslant C$, we can say $\left|\dfrac{\partial^l}{\partial\xi^l}\left(e^{-(\bar{\gamma}-\lambda)(\tau-|r_i|)/\varepsilon}\right)\right| \leqslant Ce^{-(\bar{\gamma}-\lambda)(\tau-|r_i|)/\varepsilon}$. Putting this together with the bound for $\omega(x_{ij})$ we have

$$\left|\frac{\partial^l w}{\partial\xi^l}\right| \leqslant Ce^{-(\bar{\gamma}-\lambda)(\tau-|r_i|)/\varepsilon} \qquad \text{for } l = 0, ..., 6. \tag{2.5.39}$$

To get a bound for (2.5.39) for $\varepsilon\bar{C}' \leqslant |r_i| \leqslant 2\varepsilon\bar{C}'$ we can write

$$e^{-(\bar{\gamma}-\lambda)(\tau-|r_i|)/\varepsilon} \leqslant e^{-(\bar{\gamma}-\lambda)(\tau-2\varepsilon\bar{C}')/\varepsilon}. \tag{2.5.40}$$

Using the definition of τ from (2.5.4) we get

$$e^{-(\bar{\gamma}-\lambda)(\tau-|r_i|)/\varepsilon} \leqslant e^{-(\bar{\gamma}-\lambda)(C_\tau/\bar{\gamma}\ln N-2\bar{C}')}. \tag{2.5.41}$$

We replace $e^{2\bar{C}'(\bar{\gamma}-\lambda)}$ by some constant C, i.e.,

$$e^{-(\bar{\gamma}-\lambda)(\tau-|r_i|)/\varepsilon} \leqslant Ce^{-(\bar{\gamma}-\lambda)C_\tau/\bar{\gamma}\ln N}. \tag{2.5.42}$$

Using the choice of λ from (2.5.34) we arrive at

$$e^{-(\bar{\gamma}-\lambda)(\tau-|r_i|)/\varepsilon} \leqslant CN^{-C'_\tau}, \tag{2.5.43}$$

and hence (2.5.39) becomes

$$\left|\frac{\partial^l w}{\partial \xi^l}\right| \leqslant CN^{-C'_\tau} \quad \text{ for } \varepsilon\bar{C}' \leqslant |r_i| \leqslant 2\varepsilon\bar{C}', \tag{2.5.44}$$

for $l = 0, ..., 6$. Recall (2.4.28) and that η^{-1} and $\tilde{\eta}$ are sufficiently smooth. We apply Lemma 2.5.1 along with Remark 2.5.2 with (2.5.44) and get

$$\varepsilon^2 \eta_{ij}^{-1} D_r \left[\tilde{\eta}_{ij} D_r^- w(x_{ij})\right] = \frac{\partial^2 w}{\partial \xi^2} + \kappa\varepsilon\frac{\partial w}{\partial \xi} + O(\hat{h}^2 N^{-C'_\tau}). \tag{2.5.45}$$

We can again use (2.5.44) to simplify further and get

$$\varepsilon^2 \eta_{ij}^{-1} D_r \left[\tilde{\eta}_{ij} D_r^- w(x_{ij})\right] = O((1+\varepsilon+\hat{h}^2)N^{-C'_\tau}), \tag{2.5.46}$$

for $\varepsilon\bar{C}' \leqslant |r_i| \leqslant 2\varepsilon\bar{C}'$. Putting (2.5.46) back into (2.5.38) and as $\varepsilon \ll 1$ and $\hat{h} \ll 1$, we have

$$F^N[\beta_*(x_{ij};p) + p''w(x_{ij})] - F^N\beta_*(x_{ij};p) \geqslant p''(\bar{\gamma}-\lambda/2)^2 w(x_{ij}) - Cp''N^{-C'_\tau}. \tag{2.5.47}$$

As $p''(\bar{\gamma}-\lambda/2)^2 w(x_{ij}) \geqslant 0$ we get the desired result, (2.5.33), for $\varepsilon\bar{C}' \leqslant |r_i| \leqslant 2\varepsilon\bar{C}'$.

For $2\varepsilon\bar{C}' \leqslant |r_i| < \tau$, $w(x_{ij}) = e^{-(\bar{\gamma}-\lambda)(\tau-|r_i|)/\varepsilon}$ as $\omega(x_{ij}) = 1$ and we have

$$-\varepsilon^2 \eta_{ij}^{-1} D_r \left[\tilde{\eta}_{ij} D_r^- w(x_{ij})\right] = -(\bar{\gamma}-\lambda)^2 w(x_{ij}) - \varepsilon(\bar{\gamma}-\lambda)w(x_{ij}) + O(\hat{h}^2 w(x_{ij})), \tag{2.5.48}$$

and hence,

$$\begin{aligned} F^N[\beta_*(x_{ij};p)+p''w(x_{ij})]-F^N\beta_*(x_{ij};p) \geqslant &-(\bar\gamma-\lambda)^2 w(x_{ij}) \\ &+(\bar\gamma-\lambda/2)^2 w(x_{ij}) \\ &+O(\varepsilon+\hat h^2). \end{aligned} \tag{2.5.49}$$

As $\lambda \in (0,\bar\gamma)$ we have $(\bar\gamma-\lambda/2)^2 \geqslant (\bar\gamma-\lambda)^2$ and as $0 \leqslant w(x_{ij}) \leqslant 1$ we can say $(\bar\gamma-\lambda/2)^2 w(x_{ij}) \geqslant (\bar\gamma-\lambda)^2 w(x_{ij})$. Therefore, $(\bar\gamma-\lambda/2)^2 w(x_{ij}) \geqslant 0$ dominates the other terms on the right-hand side of (2.5.49). Hence,

$$F^N[\beta_*(x_{ij};p)+p''w(x_{ij})]-F^N\beta_*(x_{ij};p) \geqslant 0 \quad \text{for } 2\varepsilon\bar C' \leqslant |r_i| < \tau, \tag{2.5.50}$$

and we obtain (2.5.33) for $|r_i| < \tau$.

For the stabilised method, $\hat\varepsilon(x_{ij}) = \varepsilon$ in this region so the above result holds. □

The following lemma gives the necessary condition that $H\alpha \leqslant r \leqslant H\beta$.

Lemma 2.5.5. *Let $C_\tau > 2$ and $0 \leqslant w(x_{ij}) \leqslant 1$. There exists sufficiently large positive constants $\bar C$, $\bar C'$ and sufficiently small positive constants $\varepsilon^* = \varepsilon^*(p^*)$, $c_0 = c_0(p^*)$ and $c_2 = c_2(p^*)$ such that if $\varepsilon \leqslant \varepsilon^*$, $N \geqslant c_2^{-1}$ and $\hat h^4 \leqslant c_0\varepsilon$, then $\bar p := \bar C(\varepsilon^2+\hat h^2+\hat h^4/\varepsilon) \leqslant p^*$, and $\beta_*(x;p)$ from* (2.5.9) *with $p = \pm\bar p$ satisfies*

$$F^N[\beta_*(x_{ij};-\bar p)-\bar C' N^{-2} w(x_{ij})] \leqslant 0 \leqslant F^N[\beta_*(x_{ij};\bar p)+\bar C' N^{-2} w(x_{ij})], \tag{2.5.51a}$$

for $x_{ij} \in \Omega_{(-\tau,\tau)}$.

Furthermore, if $C_\tau > 3$, then

$$\hat F^N[\beta_*(x_{ij};-\bar p)-\bar C' N^{-1} w(x_{ij})] \leqslant 0 \leqslant \hat F^N[\beta_*(x_{ij};\bar p)+\bar C' N^{-1} w(x_{ij})], \tag{2.5.51b}$$

for $x_{ij} \in \Omega_{(-\tau,\tau)}$.

Proof. We extend [18, Lemma 6.10] to two dimensions. We note $\operatorname{sgn} p = \operatorname{sgn} p'$ from (2.5.10).

We first take $x_{ij} \in \Omega_{(-\tau,\tau)} \backslash \Gamma_0$. Combining (2.5.19) with Lemma 2.4.8 gives the estimate of $(\operatorname{sgn} p) F^N \beta_*(x_{ij}; p)$ for $x_{ij} \in \Omega_{(-\tau,\tau)} \backslash \Gamma_0$ to be

$$(\operatorname{sgn} p) F^N \beta_*(x_{ij}; p) \geqslant \frac{1}{2} C_0 |p'| \gamma^2 - C_4' \left(\varepsilon^3 + \varepsilon \hat{h}^2 + \hat{h}^4\right), \tag{2.5.52}$$

for some $C_4' > C_4$. Using p'' defined in (2.5.12) and $\bar{C}' N^{-2} \geqslant (C_4'/\gamma^2) N^{-2}$, we choose

$$C_\tau' := 2 < C_\tau, \tag{2.5.53}$$

for the standard method. We now apply Lemma 2.5.4 giving

$$\pm \left\{ F^N[\beta_*(x_{ij}; \bar{p}) \pm \bar{C}' N^{-2} w(x_{ij})] - F^N \beta_*(x_{ij}; p) \right\} \geqslant -\bar{C}' N^{-2} N^{-C_\tau'} \geqslant -C_5 N^{-4}, \tag{2.5.54}$$

for $x_{ij} \in \Omega_{(-\tau,\tau)} \backslash \Gamma_0$ and some $C_5 > 0$. We eliminate $F^N \beta_*(x_{ij}; p)$ by using (2.5.54) with (2.5.52) to get

$$\pm F^N[\beta_*(x_{ij}; p) \pm \bar{C}' N^{-2} w(x_{ij})] \geqslant \frac{1}{2} C_0 |p'| \gamma^2 - C_4' \left(\varepsilon^3 + \varepsilon \hat{h}^2 + \hat{h}^4\right) - C_5 N^{-4}. \tag{2.5.55}$$

We choose N sufficiently large by choosing c_2 sufficiently small and saying $N \geqslant c_2^{-1}$. We also choose $\hat{h}$ such that $\hat{h}^4 \leqslant c_0 \varepsilon$ for c_0 sufficiently small, i.e., $N^{-4} \ln^4(N) \leqslant c_0 N^{-1}$. We have $\bar{C}(\varepsilon^2 + \hat{h}^2 + \hat{h}^4/\varepsilon) \leqslant p^*$. By choosing $\bar{p}$ such that $\bar{C}(\varepsilon^3 + \varepsilon \hat{h}^2 + \hat{h}^4) = \varepsilon \bar{p}$ and $C_5 N^{-4} \leqslant C \hat{h}^4$ we can obtain the following inequality for some $\bar{p} \in (0, p^*]$ and $\bar{p}' := \varepsilon \frac{C_1}{2C_3} \bar{p}$,

$$\frac{1}{2} C_0 \bar{p}' \gamma^2 \geqslant C_4'(\varepsilon^3 + \varepsilon \hat{h}^2 + \hat{h}^4) + C_5 N^{-4}, \tag{2.5.56}$$

and using this with (2.5.55) we have (2.5.51a) for $x_{ij} \in \Omega_{(-\tau,\tau)} \backslash \Gamma_0$.

For the case with $x_{ij} \in \Gamma_0$ we use Lemma 2.4.8 now with $r = 0^\pm$ and

again combine this with (2.5.29), noting $\dfrac{\partial^4 V_0}{\partial \xi^4}$ is continuous at $r=0$, to get

$$\begin{aligned}(\operatorname{sgn} p)F^N\beta_*(x_{ij};p) \geqslant & \frac{1}{2}C_0|p'|\gamma^2 - C_4\left(\varepsilon^3+\varepsilon\hat{h}^2+\hat{h}^4\right) \\ & + (\operatorname{sgn} p')\,\hat{h}^{-1}\left[\Phi[\beta]+O\left(\varepsilon\hat{h}^2+\hat{h}^4\right)\right].\end{aligned} \tag{2.5.57}$$

Using (2.4.88) and the definition of p' in (2.5.10) gives

$$(\operatorname{sgn} p)\Phi[\beta] \geqslant \frac{1}{2}C_1\varepsilon|p| - C_2\varepsilon^3. \tag{2.5.58}$$

Putting this into (2.5.57) we get

$$\begin{aligned}(\operatorname{sgn} p)F^N\beta_*(x_{ij};p) \geqslant & \frac{1}{2}C_0|p'|\gamma^2 - C_4\left(\varepsilon^3+\varepsilon\hat{h}^2+\hat{h}^4\right) \\ & + \hat{h}^{-1}\left\{\frac{1}{2}C_1\varepsilon|p| - C_2\varepsilon^3 - C_3'(\varepsilon\hat{h}^2+\hat{h}^4)\right\},\end{aligned} \tag{2.5.59}$$

for some C_3' and $C_4' > C_4$.

We again eliminate $F^N\beta_*(x_{ij};p)$ by using (2.5.54) now with (2.5.59) giving

$$\begin{aligned}\pm F^N[\beta_*(x_{ij};p) \pm \bar{C}'N^{-2}w(x_{ij})] \geqslant & - C_5N^{-4} + \frac{1}{2}C_0|p'|\gamma^2 \\ & - C_4'\left(\varepsilon^3+\varepsilon\hat{h}^2+\hat{h}^4\right) \\ & + \hat{h}^{-1}\left\{\frac{1}{2}C_1\varepsilon|p'| - C_2\varepsilon^3 + C_3'(\varepsilon\hat{h}^2+\hat{h}^4)\right\},\end{aligned} \tag{2.5.60}$$

for $x_{ij} \in \Gamma_0$. From the choice of N, $\hat{h}$ and $\bar{p}$ above, we can also obtain the inequality

$$\frac{1}{2}C_1\varepsilon\bar{p} \geqslant C_2\varepsilon^3 + C_3'(\varepsilon\hat{h}^2+\hat{h}^4). \tag{2.5.61}$$

Using this with (2.5.56) we obtain (2.5.51a) for $x_{ij} \in \Gamma_0$.

For the stabilised scheme we have $p'' := \bar{C}'N^{-1}$ from (2.5.12) and we

choose

$$C'_\tau := 3 < C_\tau, \tag{2.5.62}$$

such that $-Cp''N^{-C'_\tau} \geqslant -C_5N^{-4}$ still holds and the result follows as before giving (2.5.51b). □

As we have $F^N[\beta_*(x_{ij};-\bar{p})-p''w(x_{ij})] \leqslant 0 \leqslant F^N[\beta_*(x_{ij};\bar{p})+p''w(x_{ij})]$ and $\beta(x_{ij};-\bar{p})-p''w(x_{ij}) \leqslant \beta(x_{ij};\bar{p})+p''w(x_{ij})$ for $x_{ij} \in \Omega_{(-\tau,\tau)}$ with p'' defined in (2.5.12) and $\bar{p} \geqslant \dfrac{C_2\varepsilon^2}{2C_1}$, in the next sections we obtain similar results for the remaining regions. We then combine these and use the theory of Z-fields from §1.0.2 to prove the existence of a solution U satisfying $\beta(x_{ij};-\bar{p})-p''w(x_{ij}) \leqslant U \leqslant \beta(x_{ij};\bar{p}) + p''w(x_{ij})$.

2.5.5 Fictitious Neumann Condition along $\Gamma_{\pm\tau}$

We introduce the fictitious Neumann boundary condition, $\phi(x)$, to deal with the jump across Γ_0. This condition is introduced on the curves $\Gamma_{\pm\tau}$, i.e., the boundaries of $\Omega_{(-\tau,\tau)}$ in the interior regions and the outer regions. These approximations are then added together to get the system along $\Gamma_{\pm\tau}$ and the Neumann boundary condition is eliminated. The function $\phi(x)$ is arbitrary and when the approximations for the two sides of the curve are combined this will drop out of the calculations.

We define the fictitious Neumann condition for the outer regions as

$$\frac{\partial u}{\partial n} = \phi(x) \quad \text{for } x \in \Gamma_{\pm\tau} \cap \partial\mathring{\Omega}, \tag{2.5.63}$$

and for the layer regions as

$$\frac{\partial u}{\partial r} = \mp\phi(x) \quad \text{for } x \in \Gamma_{\pm\tau} \cap \partial\Omega_{[-\tau,\tau]}, \tag{2.5.64}$$

where the notation $\Gamma_{\pm\tau}\cap\partial\mathring{\Omega}$ and $\Gamma_{\pm\tau}\cap\partial\Omega_{[-\tau,\tau]}$ is used to distinguish between

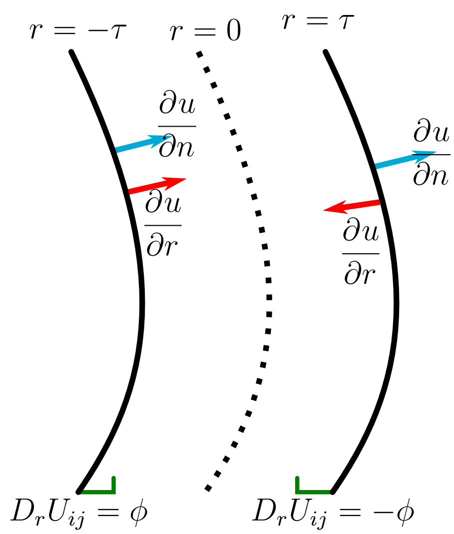


Figure 2.11: Representation of the choice of fictitious Neumann boundary conditions.

the two sides of each of the curves $\Gamma_{\pm\tau}$. These conditions are represented in Figure 2.11.

2.5.6 Discretisation of the Interface Boundary of the Interior Layer Region, $\Gamma_{\pm\tau} \cap \partial\Omega_{[-\tau,\tau]}$

In this section, we look at the interior layer interface. This consists of two curves, $\Gamma_{\pm\tau} \cap \partial\Omega_{[-\tau,\tau]}$. In this region the interface boundary occurs along the curve $|r_i| = \tau$, i.e., distance τ from Γ_0, and refers to Figure 2.9b. Here $i = \pm N$ and $j = 0, ..., N_l$. With the fictitious Neumann boundary condition (2.5.64), we use the following second-order finite difference discretisation,

$$F^N_{\Omega_{[-\tau,\tau]}} U_{\pm N,j} := -\varepsilon^2 \delta_r^2 U_{\pm N,j} - \varepsilon^2 \zeta_{\pm N,j} D_l [\tilde{\zeta}_{\pm N,j} D_l^- U_{\pm N,j}] + b(x_{\pm N,j}, U_{\pm N,j}) = 0, \tag{2.5.65a}$$

$$U_{\pm N,N_l} = U_{\pm N,0}, \quad U_{\pm N,-1} = U_{\pm N,N_l-1}, \tag{2.5.65b}$$

where

$$\delta_r^2 U_{\pm N,j} := \eta_{\pm N,j}^{-1} \frac{-\tilde{\eta}_{\{(\pm N+1/2)\pm 1/2,j\}}\phi_j \mp \eta_{\pm N,j} D_r^{\mp} U_{\pm N,j}}{h_{\pm N}/2}, \tag{2.5.65c}$$

which can be simplified to

$$\delta_r^2 U_{\pm N,j} := -\frac{2}{h_{\pm N}}\phi_j \mp \eta_{\pm N,j}^{-1}\kappa\phi_j \mp \frac{2}{h_{\pm N}} D_r^{\mp} U_{\pm N,j}, \tag{2.5.65d}$$

$$D_r^{\pm} v_{ij} := \frac{v_{i\pm 1,j} - v_{ij}}{r_{i\pm 1} - r_i}, \quad \phi_j := \phi(x_{\pm N,j}), \tag{2.5.65e}$$

where η^{-1} and $\tilde{\eta}$ are defined in (2.5.13). Define fictitious points r_{-N-1} and r_{N+1} such that $r_{-N-1} = r_{-N} - h_{-N+1}$ and $r_{N+1} = r_N + h_N$ respectively and define

$$h_{-N} := h_{-N+1}, \quad h_{N+1} := h_N. \tag{2.5.65f}$$

We recall the definition of $\hat{\varepsilon}(x_{ij})$ in (2.5.6) and see that in this region $\hat{\varepsilon}(x_{ij}) = \varepsilon$ and so $\hat{F}^N_{\Omega_{[-\tau,\tau]}} = F^N_{\Omega_{[-\tau,\tau]}}$.

Note (2.5.65d) is found by writing (2.5.65c) as

$$\delta_r^2 U_{\pm N,j} := \frac{-\eta_{\pm N,j}^{-1}\tilde{\eta}_{\{(\pm N+1/2)\pm 1/2,j\}}\phi_j \mp D_r^{\mp} U_{\pm N,j}}{h_{\pm N}/2}. \tag{2.5.66}$$

Taking Taylor expansions of $\tilde{\eta}$ gives

$$\tilde{\eta}_{(\pm N+1/2)\pm 1/2,j} = \eta_{\pm N} \pm \frac{h_{\pm N}}{2}\eta'_N. \tag{2.5.67}$$

Recalling (A.2) we find $\tilde{\eta}_{(\pm N+1/2)\pm 1/2,j} = \eta_{\pm N} \pm h_{\pm N}\kappa/2$, and so (2.5.66) becomes (2.5.65d) .

Remark 2.5.4. The discretisation (2.5.65) is obtained by considering the central difference discretisation of the Neumann condition $\dfrac{\partial u}{\partial r} = \mp\phi(x_{\pm N,j})$ for

$x_{\pm N,j} \in \Gamma_{\pm\tau}$, that is

$$\frac{U_{\pm N+1,j} - U_{\pm N-1,j}}{2\bar{h}} = \mp\phi_j \quad \text{for } j = 1, ..., M, \tag{2.5.68}$$

with ghost points U_{N+1} and U_{-N-1}. Putting (2.5.68) into (2.5.13) and eliminating U_{N+1} and U_{-N-1} gives (2.5.65).

Lemma 2.5.6. *Let $\tilde{\Omega} \supset \bar{\Omega}\backslash\Omega_{[-\tau+h_{-N},\tau-h_N]}$ be the exterior and interior of the curves $r = \tau - h_N$ and $r = -\tau + h_{-N}$ respectively, with τ chosen in* (2.5.4)*. Then for $\beta_*(x;p)$ from* (2.4.84) *with $r_i \neq 0$ we have*

$$\|\beta_*\|_{C^2(\tilde{\Omega})} \leqslant C\left(1 + \frac{\hat{h}^2}{\varepsilon^2}\right). \tag{2.5.69}$$

Proof. This proof imitates that of [14, Lemma 3.12] but is more complicated due to the existence of v_* and z in β_*.

Recalling (2.4.8), $u_0(x)$ is bounded by some constant as $\varphi_1(x)$ and $\varphi_2(x)$ are smooth functions. Using $b(x, u_0(x)) = 0$, by the implicit function theorem we can calculate derivatives of $u_0(x)$ that are dependent only on $b(x, u(x))$ and its derivatives, and hence we see the right hand sides are bounded as $b(x, u(x))$ is sufficiently smooth. For the purposes of calculating $\|u_2\|_{C^2(\bar{\Omega})}$ and $\|z\|_{C^2(\bar{\Omega})}$ we require $\|u_0\|_{C^6(\bar{\Omega})}$, hence we calculate sufficient derivatives for this and obtain $\|u_0\|_{C^6(\bar{\Omega})} \leqslant C$.

We look at (2.4.8) and firstly note that u_2 is bounded as, again, $b(x, u(x))$ is smooth and we have established that $\|u_0\|_{C^2(\bar{\Omega})} \leqslant C$. We can take derivatives of (2.4.9) with respect to r and l and see that we have the result $\|u_2\|_{C^2(\bar{\Omega})} \leqslant C$.

Recall Lemma 2.4.4 and Lemma 2.4.7 for the necessary bounds and bounds of derivatives with respect to ξ and l for v_j for $j = 0, 1, 2$, v_* and z. Now

noting $r = \varepsilon\xi$ with (2.4.50) and (2.4.87) implies

$$\left|\frac{\partial^{k+m} v_j}{\partial r^k \partial l^m}\right| + \left|\frac{\partial^{k+m} v_*}{\partial r^k \partial l^m}\right| + \left|\frac{\partial^{k+m} z}{\partial r^k \partial l^m}\right| \leqslant C_\lambda \varepsilon^{-k} e^{-(\bar{\gamma}-\lambda)|r_i|/\varepsilon}, \tag{2.5.70}$$

for $k = 0, 1, 2$, $m = 0, 1, 2$ with $k + m \leqslant 2$ and $j = 0, 1, 2$.

Finally we are considering the region in which $r = \tau - h_N$, i.e., $r = C_\tau \varepsilon \ln N \bar{\gamma}^{-1}(1 - N^{-1})$ and noting $\bar{\gamma} - \lambda < \bar{\gamma}$ with $\lambda \in (0, \bar{\gamma})$ we have

$$\|v_i\|_{C^2(\bar{\Omega})} + \|v_*\|_{C^2(\bar{\Omega})} + \|z\|_{C^2(\bar{\Omega})} \leqslant C\varepsilon^{-2} e^{-|C_\tau \ln N(1-N^{-1})|}, \tag{2.5.71}$$

for $i = 0, 1, 2$, which can be simplified to

$$\|v_i\|_{C^2(\bar{\Omega})} + \|v_*\|_{C^2(\bar{\Omega})} + \|z\|_{C^2(\bar{\Omega})} \leqslant C\varepsilon^{-2} N^{-C_\tau(1-N^{-1})}. \tag{2.5.72}$$

We have $N^{-C_\tau N^{-1}} < 1$ and $C_\tau > 2$, thus

$$\|v_i\|_{C^2(\bar{\Omega})} + \|v_*\|_{C^2(\bar{\Omega})} + \|z\|_{C^2(\bar{\Omega})} \leqslant C_\lambda \varepsilon^{-2} N^{-2} \leqslant C\varepsilon^{-2}\hat{h}^2. \tag{2.5.73}$$

Putting this together with the results for u_0 and u_2 we obtain the result. $\square$

Lemma 2.5.7. *Let $\beta(x;p)$ be defined by (2.4.84), and the mesh $\{r_i\}_{i=0}^N$ be the Shishkin mesh of §2.5.1. Then for all $|p| \leqslant p_0$ we have*

$$F^N_{\Omega_{[-\tau,\tau]}} \beta_{*\pm N,j} - F\beta_*(x_{\pm N,j}) = \frac{2\varepsilon^2}{h_{\pm N}} \left(\pm \frac{\partial \beta_*}{\partial r} + \phi_j \right)\bigg|_{x_{\pm N,j}} + O(\hat{h}^2), \tag{2.5.74}$$

at all interface-boundary mesh nodes $x_{\pm N,j} \in \Gamma_{\pm\tau} \cap \partial\Omega_{[-\tau,\tau]}$.

Furthermore this statement holds true with F^N and ε replaced by $\hat{F}^N$ and $\hat{\varepsilon}(x_{ij})$, respectively.

Proof. This follows [14, Lemma 3.13]. We want to estimate $F^N_{\Omega_{[-\tau,\tau]}}\beta_* - F\beta_*$, i.e.,

$$\begin{aligned} F^N_{\Omega_{[-\tau,\tau]}}\beta_{*\pm N,j} - F\beta_*(x_{\pm N,j}) = \frac{2\varepsilon^2\phi_j}{h_{\pm N}} \pm \varepsilon^2\eta^{-1}_{\pm N,j}\kappa\phi_j \\ -\varepsilon^2\left(\mp\frac{2}{h_{\pm N}}D_r^{\mp}\beta_{*\pm N,j} + \zeta_{\pm N,j}D_l[\tilde{\zeta}_{\pm N,j}D_l^{-}\beta_{*\pm N,j}]\right) \\ +b(x_{\pm N,j},\beta_*(x_{\pm N,j})) + \varepsilon^2\Delta\beta_* - b(x_{\pm N,j},\beta_*(x_{\pm N,j})). \end{aligned} \tag{2.5.75}$$

As $\Delta\beta_* \leqslant \|\beta_*\|_{C^2(\bar{\Omega})}$, $\phi_j \leqslant \|\beta_*\|_{C^2(\bar{\Omega})}$ by (2.5.64) and η^{-1} and κ are bounded, we have

$$\begin{aligned} F^N_{\Omega_{[-\tau,\tau]}}\beta_{*\pm N,j} - F\beta_*(x_{\pm N,j}) =& \frac{2\varepsilon^2\phi_j}{h_{\pm N}} \pm \varepsilon^2\frac{2}{h_{\pm N}}D_r^{\mp}\beta_{*\pm N,j} \\ &+ O\left(\varepsilon^2\|\beta_*\|_{C^2(\bar{\Omega})}\right). \end{aligned} \tag{2.5.76}$$

We now introduce $\dfrac{\partial\beta_*}{\partial r}$ to assist later when the interface discretisations are brought together, i.e.,

$$\begin{aligned} F^N_{\Omega_{[-\tau,\tau]}}\beta_{*\pm N,j} - F\beta_*(x_{\pm N,j}) = \frac{2\varepsilon^2}{h_{\pm N}}\left(\phi_j \pm \left.\frac{\partial\beta_*}{\partial r}\right|_{x_{\pm Nj}}\right) \\ -\varepsilon^2\frac{2}{h_{\pm N}}\left(\pm\left.\frac{\partial\beta_*}{\partial r}\right|_{x_{\pm Nj}} \mp D_r^{\mp}\beta_{*\pm N,j}\right) + O\left(\varepsilon^2\|\beta_*\|_{C^2(\bar{\Omega})}\right). \end{aligned} \tag{2.5.77}$$

As $\dfrac{2}{h_{\pm N}}\left(\pm\left.\dfrac{\partial\beta_*}{\partial r}\right|_{x_{\pm Nj}} \mp D_r^{\mp}\beta_{*\pm N,j}\right)$ is bounded by $\|\beta_*\|_{C^2(\bar{\Omega})}$, we use (2.5.69) along with (2.2.3) to get the desired result. □

2.5.7 Discretisation of the Outer Regions, $\mathring{\Omega}$

For the outer regions, $\mathring{\Omega}^N := \bar{\Omega}^N \backslash \Omega_{(-\tau,\tau)}$ represented in Figure 2.9a, we apply the standard finite element method to our equation (2.0.1). The standard weak form of this is the following; we want to find $u \in W_2^1(\mathring{\Omega})$ such that $\forall\, v \in W_2^1(\mathring{\Omega})$ with $v(x) = 0$ for $x \in \partial\Omega$ we have

$$\varepsilon^2(\nabla u, \nabla v) + (b(x,u), v) = \varepsilon^2 \oint_{\Gamma_{\pm\tau}} \phi v ds \quad \text{ for } x \in \mathring{\Omega}, \tag{2.5.78}$$

where the term on the right-hand side comes from the fictitious Neumann boundary condition (2.5.63) and $(\cdot,\cdot)$ is the inner product in $L_2(\mathring{\Omega})$. We use $\dfrac{\partial u}{\partial n} = \phi(x)$ for $x \in \Gamma_{\pm} \cap \partial\mathring{\Omega}$ from (2.5.63). We must also impose the boundary condition (2.0.1b), $u(x) = g(x)$ for $x \in \partial\Omega$, for the region $\mathring{\Omega}^{(1)}$.

We next set up nodal basis functions $\chi_i \in S^N$ where $S^N \in W_2^1(\mathring{\Omega}^N)$ is the standard finite element space of continuous functions that are linear on each triangle of our mesh in $\mathring{\Omega}^N$. The nodal basis functions have the property $\chi_i(X_j)$ equals 1 if $i = j$ and 0 otherwise, with X_j being a mesh node in $\mathring{\Omega}^N$. In this region we may write w_i for $w(X_i)$.

In $\mathring{\Omega}^N$ we define the approximate solution $U \in S^N$ by

$$\varepsilon^2(\nabla U, \nabla \chi_i) + b(X_i, U_i)(1, \chi_i) = \varepsilon^2 \phi_i \oint_{\Gamma_{\pm\tau}} \chi_i ds, \quad \forall\, \chi_i \in S^N, \tag{2.5.79a}$$

$$U_i = g(X_i), \quad \forall X_i \in \partial\Omega^N, \tag{2.5.79b}$$

where $U_i = U(X_i)$, $\phi_i = \phi(X_i)$, $v(X_i) = 0$ for $X_i \in \partial\Omega^N$ and $\Gamma^N_{\pm\tau}$ is the polygonal boundary of the domain $\mathring{\Omega}^N$, not including $\partial\Omega^N$. We use lumped mass discretisation for the integral involving the nonlinear function $b(x, u(x))$ as well as that involving the interface integral.

At interior mesh nodes, $X_i \in \mathring{\Omega}^N$, the term on the right hand side of (2.5.79a) is not present and rewriting it in a similar form to the finite difference

operator we have

$$F^N U_i := \frac{\varepsilon^2}{(1,\chi_i)}(\nabla U, \nabla \chi_i) + b(X_i, U_i) = 0, \quad \forall\, \chi_i \in \mathring{\Omega}^N. \tag{2.5.80}$$

Similarly the stabilised version of (2.5.80) is

$$\hat{F}^N \hat{U}_i := \frac{\hat{\varepsilon}^2(X_i)}{(1,\chi_i)}(\nabla \hat{U}, \nabla \chi_i) + b(X_i, \hat{U}_i) = 0 \quad \forall\, \chi_i \in \mathring{\Omega}^N, \tag{2.5.81}$$

where $\hat{U}_i = \hat{U}(X_i)$ is the stabilised approximate solution. As we have used a Delaunay triangulation, the Laplacian term gives an M-matrix and so equations (2.5.80) and (2.5.81) are of type (2.5.1).

Lemma 2.5.8. *Let $\beta_*(x;p)$ be defined by* (2.4.84)*, and $\beta^I \in S^N$ be its piecewise linear interpolant such that ${\beta_*}^I(X_i) = \beta_*(X_i)$ at all mesh nodes $X_i \in \mathring{\Omega}^N$. Furthermore, let τ be chosen as in §2.5.1. Then for all $|p| \leqslant p_0$ we have*

$$|F^N {\beta_*}_i^I - F\beta_*(X_i)| \leqslant CN^{-2} \quad \forall X_i \in \mathring{\Omega}^N. \tag{2.5.82a}$$

Furthermore this holds true with the standard method replaced by the stabilised method.

Proof. This proof combines [14, Lemma 3.15] with [18, Lemma 6.8].

We want to estimate $F^N \beta_*^I - F\beta_*(X_j)$, i.e.,

$$F^N \beta_*^I - F\beta_*(X_j) = \frac{\varepsilon^2}{(1,\chi_i)}(\nabla \beta_*^I, \nabla \chi_i) + b(X_i, \beta_*^I) + \varepsilon^2 \Delta \beta_*|_{X_j} - b(X_j, \beta_*(X_j)), \tag{2.5.83}$$

which can be simplified to

$$F^N \beta_*^I - F\beta_*(X_j) = \frac{\varepsilon^2}{(1,\chi_i)}(\nabla \beta_*^I, \nabla \chi_i) + \varepsilon^2 \Delta \beta_*|_{X_j}. \tag{2.5.84}$$

Using integration by parts, we have

$$(\nabla\beta_*^I, \nabla\chi_i) = (\nabla[\beta_*^I - \beta_*], \nabla\chi_i) - (\Delta\beta_*, \chi_i) + \oint_{\Gamma_{\pm\tau}} \chi_i \frac{\partial\beta_*}{\partial n} ds, \tag{2.5.85}$$

where the final term is zero as X_i is in the interior of $\mathring{\Omega}^N$ and so $\chi_i = 0$ here. Using the interpolation error estimate $|\nabla[\beta^I - \beta]| \leqslant CN^{-1}\|\beta\|_{C^2(\bar{\Omega})}$ and the standard quasiuniform mesh properties $(1, |\nabla\chi_i|) \leqslant CN^{-1}$ and $(1, \chi_i) \geqslant CN^{-2}$ along with (2.5.84) gives

$$|F^N\beta_*^I - F\beta_*(X_j)| \leqslant C\varepsilon^2\|\beta_*\|_{C^2(\bar{\Omega})}. \tag{2.5.86}$$

Finally using the bound for $\|\beta_*\|_{C^2(\bar{\Omega})}$ from (2.5.69) along with (2.2.3) gives the desired result.

This argument also holds for the stabilised method noting $\hat{\varepsilon}(X_i) \leqslant CN^{-1}$.

□

Lemma 2.5.9. *Let N and $\bar{C}'$ be sufficiently large. Let $C'_\tau < C_\tau$. Then there exists $w(X_i)$ such that $0 \leqslant w(X_i) \leqslant 1$ and for $\bar{p} := \bar{C}(\varepsilon^2 + \hat{h}^2 + \hat{h}^4/\varepsilon) \leqslant p^*$ the function $\beta_*(X_i; p)$ from (2.4.84) satisfies*

$$\pm\{F^N[\beta_*(X_i; \bar{p}) \pm \bar{C}'N^{-2}w(X_i)] - F^N\beta_*(X_i; \bar{p})\} \geqslant \bar{C}'N^{-2}\gamma^2, \tag{2.5.87a}$$

and

$$F^N[\beta_*(X_i; -\bar{p}) - \bar{C}'N^{-2}w(X_i)] \leqslant 0 \leqslant F^N[\beta_*(X_i; \bar{p}) + \bar{C}'N^{-2}w(X_i)]. \tag{2.5.87b}$$

These statements hold true for F^N and $\bar{C}'N^{-2}$ replaced by $\hat{F}^N$ and $\bar{C}'N^{-1}$ respectively.

Proof. This proof follows that of [18, Lemma 6.9] and [18, Lemma 6.10].

Using (2.5.80), and as $w(X_i) = 1$ implies $\nabla w(X_i) = 0$, we have

$$\pm\{F^N[\beta_*(X_i;p) \pm p''w(X_i)] - F^N\beta_*(X_i;p)\} = \pm\{b(X_i, \beta_* \pm p'') - b(X_i, \beta_*)\}. \tag{2.5.88}$$

Taking a Taylor expansion,

$$\pm\{F^N[\beta_*(X_i;p) \pm p''w(X_i)] - F^N\beta_*(X_i;p)\} = p''b_u(X_i, \beta_* \pm p''\theta), \tag{2.5.89}$$

for $\theta \in (0,1)$. We consider the $+$ case of $\pm$ in the following work. We require a lower bound for $b_u(X_i, \beta_* + s)$ with $s \leqslant c'$ and c' sufficiently small and follow the method used to calculate (2.5.91). We have

$$b_u(x;\beta_* + s) = b_u(x;\varphi_2(x)) + O(\beta - \varphi_2(x)) + O(s), \tag{2.5.90}$$

where, for $x \in \bar{\Omega}^N \backslash \Omega_{[-\varepsilon\bar{C}',\varepsilon\bar{C}']}$, $\beta - \varphi_2(x) = v_0 + O(N^{-1})$ by (2.4.84) and (2.5.10). Recalling (2.4.50), we can make $\bar{C}'$ sufficiently large and so v_0 is bounded by a sufficiently small constant. Also by making N sufficiently large and c' sufficiently small $\beta - \varphi_2(x)$ and s are sufficiently small. Finally, noting (A3), we obtain

$$b_u(x, \beta_*(x) + s) \geqslant \gamma^2 \quad \text{for } x \in \bar{\Omega}^N \backslash \Omega_{[-\varepsilon\bar{C}',\varepsilon\bar{C}']},\ |s| \leqslant c'. \tag{2.5.91}$$

Putting this into (2.5.89) gives (2.5.87a).

Taking the positive case of (2.5.82a), we use (2.4.110) and (2.5.10) to get

$$F^N\beta(X_i;p) \geqslant \frac{C_0C_1}{2C_3}|p|\varepsilon\gamma^2 - CN^{-2} + O(\varepsilon^3). \tag{2.5.92}$$

Now putting this into (2.5.87a) gives

$$\pm F^N[\beta_*(X_i;p) + p''w_i] \geqslant p''\gamma^2 - CN^{-2} + \frac{C_0C_1}{2C_3}|p|\varepsilon\gamma^2 + O(\varepsilon^3). \tag{2.5.93}$$

We want CN^{-2} to be dominated by the term $p''\gamma^2$. For this reason we choose $p'' = \bar{C}'N^{-2}$ for the standard method in (2.5.12) where $\bar{C}'$ sufficiently large with $\bar{C}'N^{-2} \geqslant C_4'N^{-2}/\gamma^2$. As $\dfrac{C_0C_1}{2C_3}|p|\varepsilon\gamma^2 > 0$ and dominates the $O(\varepsilon^3)$ term, and as we have set up p'' such that CN^{-2} is dominated we obtain (2.5.87b).

For the stabilised method this proof remains the same but with the small alteration that $p'' = \bar{C}'N^{-1}$ for this method and $p''\gamma^2$ now dominates the CN^{-2} term. The explanation of the choice of p'' for the stabilised method will be given in §2.5.8. □

2.5.8 Discretisation of the Interface Boundary of the Outer Region, $\Gamma^N_{\pm\tau} \cap \partial\mathring{\Omega}^N$

We now consider the interface boundaries $\Gamma^N_{\pm\tau} \cap \partial\mathring{\Omega}^N$. This refers to Figure 2.9b. Here we have the fictitious Neumann boundary condition described in (2.5.63). The discrete version of this is

$$F^N_{\mathring{\Omega}}U_j := \frac{\varepsilon^2}{(1,\chi_j)}(\nabla U, \nabla\chi_j) + b(X_j, U_j) = \varepsilon^2 a_j\phi_j \quad \forall X_j \in \Gamma^N_{\pm\tau} \cap \partial\mathring{\Omega}^N, \tag{2.5.94a}$$

where

$$a_j = \frac{1}{(1,\chi_j)}\oint_{\Gamma^N_{\pm\tau}} \chi_j ds, \quad \phi_j := \phi(X_j). \tag{2.5.94b}$$

We note for a_j we have

$$a_j = O(N). \tag{2.5.95}$$

Similarly for the stabilised method we have

$$\hat{F}^N_{\mathring{\Omega}}\hat{U}_j := \frac{\hat{\varepsilon}^2(X_j)}{(1,\chi_j)}(\nabla\hat{U}, \nabla\chi_j) + b(X_j, \hat{U}_j) = \hat{\varepsilon}^2(X_j)a_j\phi_j \quad \forall X_j \in \Gamma^N_{\pm\tau} \cap \partial\mathring{\Omega}^N. \tag{2.5.96}$$

Lemma 2.5.10. *Under the conditions of Lemma 2.5.8, for all* $|p| \leqslant p_0$ *we have*

$$F^N_{\mathring{\Omega}} \beta_{*j}^{\,I} - F\beta_*(X_j) = a_j \varepsilon^2 \left(\mp \frac{\partial \beta_*}{\partial r} - \phi \right)\Bigg|_{X_j} + O(N^{-2}) \quad \forall X_j \in \Gamma^N_{\pm\tau} \cap \partial\mathring{\Omega}^N, \tag{2.5.97}$$

with $a_j = O(N)$ *from* (2.5.95).

Furthermore this holds true for $F^N_{\mathring{\Omega}}$ *and* ε *replaced by* $\hat{F}^N_{\mathring{\Omega}}$ *and* $\hat{\varepsilon}(X_j)$ *respectively.*

Proof. This proof resembles [14, Lemma 3.16]. We first note

$$F^N_{\mathring{\Omega}} \beta_{*j}^{\,I} - F\beta_*(X_j) = \frac{\varepsilon^2}{(1,\chi_j)} (\nabla \beta^I_*, \nabla \chi_j) - \varepsilon^2 a_j \phi_j + \varepsilon^2 \Delta \beta_*|_{X_j}. \tag{2.5.98}$$

Putting (2.5.85) into (2.5.98) and cancelling terms we get

$$F^N_{\mathring{\Omega}} \beta_{*j}^{\,I} - F\beta_*(X_j) = \frac{\varepsilon^2}{(1,\chi_j)} \left\{ (\nabla[\beta^I_* - \beta_*], \nabla\chi_j) + \oint_{\Gamma^N_{\pm\tau}} \chi_j \frac{\partial \beta_*}{\partial n} ds \right\} - \varepsilon^2 a_j \phi_j. \tag{2.5.99}$$

Note for (2.5.85), the integral along $\Gamma^N_{\pm\tau}$ is non zero and survives on the interface. As in the outer region we use $|\nabla[\beta^I - \beta]| \leqslant CN^{-1}\|\beta\|_{C^2(\bar{\Omega})}$, $(1, |\nabla\chi_i|) \leqslant CN^{-1}$ and $(1,\chi_i) \geqslant CN^{-2}$. Looking at $\dfrac{\partial \beta}{\partial n}$, the outward normal derivative, is calculated at the points within $O(N^{-1})$ of X_j, and we recall the smoothness of $\Gamma_{\pm\tau}$ to get

$$\frac{\partial \beta_*}{\partial n} = \mp \frac{\partial \beta_*}{\partial r}\bigg|_{X_j} + O\left(N^{-1} \|\beta\|_{C^2(\bar{\Omega})} \right), \tag{2.5.100}$$

for $X_j \in \Gamma_{\pm\tau} \cap \mathring{\Omega}^N$. We now have

$$F^N_{\mathring{\Omega}} \beta^I_j - F\beta(X_j) \leqslant \mp \, \varepsilon^2 \frac{\partial \beta_*}{\partial r}\bigg|_{X_j} \frac{1}{(1, \chi_j)} \oint_{\Gamma^N_{\pm\tau}} \chi_j ds - \varepsilon^2 a_j \phi_j \\ + C\varepsilon^2 \|\beta\|_{C^2(\bar{\Omega})}. \tag{2.5.101}$$

Using the definition of a_j, (2.5.94b), and bounds for $\|\beta_*\|_{C^2(\bar{\Omega})}$ and $\varepsilon \leqslant CN^{-1}$ we get the result (2.5.97). As $\hat{\varepsilon}(X_j) \leqslant CN^{-1}$ the result holds for $F^N_{\mathring{\Omega}}$ and ε replaced by $\hat{F}^N_{\mathring{\Omega}}$ and $\hat{\varepsilon}(X_j)$ respectively. □

We can now combine the discretisations for $F^N_{\Omega_{[-\tau,\tau]}}$ and $F^N_{\mathring{\Omega}}$ to get the full discretisation on the interface boundary $\Gamma_{\pm\tau}$;

$$F^N U_j := \frac{(h_{\pm N}/2) F^N_{\Omega_{[-\tau,\tau]}} U_j + (1/a_j) F^N_{\mathring{\Omega}} U_j}{h_{\pm N}/2 + 1/a_j} \quad \forall X_j \in \Gamma_{\pm\tau}. \tag{2.5.102}$$

and combine $\hat{F}^N_{\mathring{\Omega}}$ and $\hat{F}^N_{\Omega_{[-\tau,\tau]}}$ to get the full stabilised discretisation on the interface boundary $\Gamma_{\pm\tau}$;

$$\hat{F}^N \hat{U}_j := \frac{(h_{\pm N}/2) \hat{F}^N_{\Omega_{[-\tau,\tau]}} \hat{U}_j + (1/a_j) \hat{F}^N_{\mathring{\Omega}} \hat{U}_j}{h_{\pm N}/2 + 1/a_j} \quad \forall X_j \in \Gamma_{\pm\tau}. \tag{2.5.103}$$

In doing this we eliminate the auxiliary unknown function $\phi(x)$ coming from the fictitious Neumann boundary condition.

Lemma 2.5.11. *Under the conditions of Lemma 2.5.7, for F^N of* (2.5.102) *we have*

$$|F^N \beta_*(X_j) - F\beta_*(X_j)| \leqslant CN^{-2} \quad \forall X_j \in \Gamma_{\pm\tau}, \tag{2.5.104a}$$

and for $\hat{F}^N$ of (2.5.102) *we have*

$$|(\hat{F}^N - F^N)\beta_*(X_j)| \leqslant CN^{-1} \quad \forall X_j \in \Gamma_{\pm\tau}. \tag{2.5.104b}$$

Proof. We combine the bounds (2.5.74) and (2.5.97) by substituting them into (2.5.102) for $F^N_{\Omega_{[-\tau,\tau]}}$ and $F^N_{\mathring{\Omega}}$, i.e.,

$$\begin{aligned} F^N\beta_*(X_j) - F\beta_*(X_j) = {} & \frac{h_{\pm N}/2}{h_{\pm N}/2 + 1/a_j}\left[\frac{2\varepsilon^2}{h_{\pm N}}\left(\pm\frac{\partial\beta_*}{\partial r} + \phi\right)\Bigg|_{x_{\pm N,j}} + O(h^2)\right] \\ & + \frac{1/a_j}{h_{\pm N}/2 + 1/a_j}\left[\varepsilon^2 a_j\left(\mp\frac{\partial\beta_*}{\partial r} - \phi\right)\Bigg|_{X_j} + O(N^{-2})\right]. \end{aligned} \tag{2.5.105}$$

Simplifying this we get

$$|F^N\beta_*(X_j) - F\beta_*(X_j)| = O(N^{-2}). \tag{2.5.106}$$

For the stabilised method we recall that we have (2.5.74) and (2.5.97) for the interfaces $\Gamma_{\pm\tau}$ with ε, $F^N_{\Omega_{[-\tau,\tau]}}$ and $F^N_{\mathring{\Omega}}$ replaced by $\hat{\varepsilon}(X_j)$, $\hat{F}^N_{\Omega_{[-\tau,\tau]}}$ and $\hat{F}^N_{\mathring{\Omega}}$. Combining these in (2.5.103) and noting $h = C\varepsilon\hat{h}$ from (2.5.10) and $a_j = O(N)$ from (2.5.95) we have

$$|(\hat{F}^N - F^N)\beta_*(X_j)| = \frac{\varepsilon^2 - \hat{\varepsilon}^2(X_i)}{h_{\pm N}/2 + 1/a_j}\left(\pm\frac{\partial\beta}{\partial r} + \phi\right) + O\left(\frac{\varepsilon\hat{h}(\hat{h}^2 + \varepsilon^2\hat{h}) + N^{-2}}{h + N^{-1}}\right). \tag{2.5.107}$$

Recalling the definition of $\hat{h}$ in (2.5.10) and noting that $\dfrac{\partial\beta}{\partial r}$ and $\phi(x)$ are bounded we have

$$|(\hat{F}^N - F^N)\beta_*(X_j)| \leqslant \frac{C(\varepsilon^2 - \hat{\varepsilon}^2(X_i))}{h_{\pm N}/2 + 1/a_j} + O\left(N^{-1}\right). \tag{2.5.108}$$

Hence we can get the desired result by writing $\varepsilon^2 - \hat{\varepsilon}^2(X_i) = O(N^{-2})$ as we have (2.2.3) and (2.5.6). $\square$

Note that (2.5.104a) and (2.5.104b) imply

$$|(\hat{F}^N - F)\beta_*(X_i)| \leqslant CN^{-1} \quad \forall X_i \in \Gamma_{\pm\tau}. \tag{2.5.109}$$

We can now set up sub and super solutions for the interface.

Lemma 2.5.12. *Let N and $\bar{C}'$ be sufficiently large. Let $C'_\tau < C_\tau$. Then there exists $w(X_j)$ such that $0 \leqslant w(X_j) \leqslant 1$ and for all $|p| \leqslant p^*$ the function $\beta_*(X_j;p)$ from* (2.4.84) *satisfies*

$$\pm\{F^N[\beta_*(X_j;\bar{p}) \pm \bar{C}'N^{-2}w(X_j)] - F^N\beta_*(X_j;\bar{p})\} \geqslant \bar{C}'N^{-2}\gamma^2, \tag{2.5.110a}$$

and

$$F^N[\beta_*(X_j;-\bar{p}) - \bar{C}'N^{-2}(X_j)] \leqslant 0 \leqslant F^N[\beta_*(X_j;\bar{p}) + \bar{C}'N^{-2}(X_j)]. \tag{2.5.110b}$$

These statements hold true for F^N and $\bar{C}'N^{-2}$ replaced by $\hat{F}^N$ and $\bar{C}'N^{-1}$ respectively.

Proof. The proof of this lemma for the standard method mirrors that of Lemma 2.5.9 as (2.5.104a) and (2.4.110) still hold.

For the stabilised method we follow the steps of the proof of Lemma 2.5.9 until (2.5.93) where we instead have

$$\pm F^N[\beta_*(X_j;p) + p''(X_j)] \geqslant p''\gamma^2 - CN^{-1} + \frac{C_0C_1}{2C_3}|p|\varepsilon\gamma^2 + O(\varepsilon^3), \tag{2.5.111}$$

due to the bound (2.5.104b). In (2.5.12) we choose $p'' := \bar{C}'N^{-1}$ so that $p''\gamma^2$ dominates the term CN^{-1}. As $\dfrac{C_0C_1}{2C_3}|p|\varepsilon\gamma^2$ dominates $O(\varepsilon^3)$ we obtain the desired results. □

We now have the necessary requirements for upper and lower solutions in the entire domain and can obtain accuracy results for the discrete problem.

2.5.9 Existence and Accuracy of Discrete Upper and Lower Solutions, Pointwise Error Bounds for $\bar{\Omega}^N$.

We now show there exist ordered discrete upper and lower solutions of the form $\beta_*(X_i;\pm p)\pm p''w(X_i)$, with p'' defined in (2.5.12) for $X_i\in\bar{\Omega}^N$ and hence obtain accuracy results for the standard and stabilised methods, F^N and $\hat{F}^N$ respectively.

Theorem 2.5.5. *Let the mesh $\{r_i,l_j\}$ be the Shishkin mesh of §2.5.1. Set $C'=4C_\tau/\bar{\gamma}$. For some $\varpi\in[0,2]$ we assume $c_0\varepsilon\geqslant(C'N^{-1}\ln N)^{2+\varpi}$,*

(i) If $C_\tau>2$, then there is a discrete solution U of (2.5.13), (2.5.80), (2.5.102) such that for N sufficiently large,

$$|U(X_i)-u(X_i)|\leqslant C\begin{cases}(N^{-1}\ln N)^{2-\varpi} & \text{for } X_i\in\Omega_{(-\tau,\tau)}\\ N^{-2} & \text{for } X_i\in\bar{\Omega}^N\backslash\Omega_{(-\tau,\tau)}.\end{cases}\tag{2.5.112}$$

(ii) If $C_\tau>3$, then there is a discrete solution $\hat{U}$ of (2.5.13), (2.5.80), (2.5.102) with ε replaced by $\hat{\varepsilon}(X_i)$ such that for N sufficiently large,

$$|\hat{U}(X_i)-u(X_i)|\leqslant C\begin{cases}(N^{-1}\ln N)^{2-\varpi}+N^{-1} & \text{for } X_i\in\Omega_{(-\tau,\tau)}\\ N^{-1} & \text{for } X_i\in\bar{\Omega}^N\backslash\Omega_{(-\tau,\tau)}.\end{cases}\tag{2.5.113}$$

Proof. Firstly recalling (2.2.3) and $\hat{h}^4\leqslant c_0\varepsilon$ from Lemma 2.5.5 we have, for $\varpi\in[0,2]$,

$$c_0\varepsilon\geqslant\hat{h}^{2+\varpi}\geqslant\hat{h}^4.\tag{2.5.114}$$

We choose $\bar{p}:=\bar{C}(\varepsilon^2+\hat{h}^2+\hat{h}^4/\varepsilon)$ and we can say $\bar{p}\leqslant C\hat{h}^{2-\varpi}$ and hence by Lemma 2.5.5, Lemma 2.5.9 and Lemma 2.5.12 we have that

$$F^N[\beta_*(X_i;-\bar{p})-p''w(X_i)]\leqslant 0\leqslant F^N[\beta_*(X_i;\bar{p})+p''w(X_i)],\tag{2.5.115}$$

holds true for $X_i \in \bar{\Omega}^N$ where $0 \leqslant w(X_i) \leqslant 1$ is defined in (2.5.11) and p'' is defined in (2.5.12). By Lemma 2.4.10 with $\mu = \frac{1}{2}$ and $p''w(X_i) \geqslant 0$ we have

$$\beta(X_i, -\bar{p}) - p''w(X_i) \leqslant \beta(X_i, \bar{p}) + p''w(X_i), \tag{2.5.116}$$

for $X_i \in \bar{\Omega}^N$. By (A6) and $p''w(X_i) \geqslant 0$ we have

$$\beta(X_i, -\bar{p}) - p''w(X_i) \leqslant g(X_i) \leqslant \beta(X_i, \bar{p}) + p''w(X_i), \tag{2.5.117}$$

for $X_i \in \partial\Omega$. These properties imply $\beta(X_i, \pm\bar{p}) \pm p''w(X_i)$ are ordered discrete upper and lower solutions of the discrete problem (2.5.13), (2.5.80) and (2.5.102) with (2.5.65) and (2.5.94).

As the discrete operator F^N is a Z-field, we can invoke the theory of upper and lower solutions, Lemma 1.0.1, and we find there exists a solution $\{U_i\}$ such that

$$\beta(X_i, -\bar{p}) - p''w(X_i) \leqslant U(X_i) \leqslant \beta(X_i, \bar{p}) + p''w(X_i). \tag{2.5.118}$$

Now considering the standard and stabilised method separately, we first look at the standard method, i.e., where $p'' := \bar{C}'N^{-2}$. We note Lemma 2.4.10 for the layer region can be written as

$$|\beta(x; \pm\bar{p}) - u_{as}(x; 0)| \leqslant C_\lambda(|\bar{p}| + \hat{h}^2) + C\varepsilon|\bar{p}|. \tag{2.5.119}$$

For $X_i \in \Omega_{(-\tau,\tau)}$ combine (2.5.119) with Corollary 2.4.1 and recall (2.2.3) to get

$$|\beta(X_i; \pm\bar{p}) \pm \bar{C}'N^{-2}w(X_i) - u(X_i)| \leqslant C_\lambda(|\bar{p}| + \hat{h}^2) + C\varepsilon|\bar{p}| + CN^{-2}. \tag{2.5.120}$$

Using (2.5.118) it follows that

$$|U(X_i) - u(X_i)| \leqslant C_\lambda(|\bar{p}| + \hat{h}^2) + C\varepsilon|\bar{p}| + CN^{-2}, \tag{2.5.121}$$

and recalling the definition of $\hat{h}$ in (2.5.10) gives the first bound in (2.5.112).

For the outer region, Lemma 2.4.10 can be written as

$$|\beta(x; \pm\bar{p}) - u_{as}(x; 0)| \leqslant C_\lambda(|\bar{p}| + \hat{h}^2)e^{-(\bar{\gamma}-\lambda)\tau/\varepsilon} + C\varepsilon|\bar{p}|. \tag{2.5.122}$$

Using the same method again results in

$$|U(X_i) - u(X_i)| \leqslant C_\lambda(|\bar{p}| + \hat{h}^2)e^{-(\bar{\gamma}-\lambda)\tau/\varepsilon} + C\varepsilon|\bar{p}| + CN^{-2}, \tag{2.5.123}$$

for $X_i \in \mathring{\Omega}^N$. Recalling the choice of λ sufficiently small in (2.5.34) we get $e^{-(\bar{\gamma}-\lambda)\tau/\varepsilon} \leqslant CN^{-C'_\tau}$. In the outer region we have chosen $C'_\tau = 2$ from (2.5.53). Combining this in (2.5.123) gives

$$|U(X_i) - u(X_i)| \leqslant C(|\bar{p}| + \hat{h}^2)N^{-2} + C\varepsilon|\bar{p}| + CN^{-2}, \tag{2.5.124}$$

and hence from the choice of $\bar{p}$ and $\hat{h}$ we get the bound for $X_i \in \mathring{\Omega}^N$ in (2.5.112).

For the stabilised method,

$$|\beta(X_i; \pm\bar{p}) \pm \bar{C}'N^{-1}w(X_i) - u(X_i)| \leqslant C_\lambda(|\bar{p}| + \hat{h}^2)e^{-(\bar{\gamma}-\lambda)|\xi|} + C\varepsilon|\bar{p}| + CN^{-1}, \tag{2.5.125}$$

for $X_i \in \bar{\Omega}^N$. Following the same steps as above,

$$|U(X_i) - u(X_i)| \leqslant C(|\bar{p}| + \hat{h}^2) + CN^{-1} \quad \text{for } X_i \in \Omega_{(-\tau,\tau)}, \tag{2.5.126}$$

and $C'_\tau = 3$ from (2.5.62) we have

$$|U(X_i) - u(X_i)| \leqslant C(|\bar{p}| + \hat{h}^2)N^{-3} + C\varepsilon|\bar{p}| + CN^{-1}, \tag{2.5.127}$$

for $X_i \in \mathring{\Omega}^N$. Again noting the choice of $\bar{p}$ and $\hat{h}$ we get (2.5.113). □

As in [18] we now present a result for the case in which the relationship between N and ε is stronger than what has already been considered. We look at the case where

$$\varepsilon \leqslant CN^{-\varpi'}, \tag{2.5.128}$$

for some $\varpi' \geqslant 4 - \lambda$.

Theorem 2.5.6. *Let the mesh $\{r_i, l_j\}$ be the Shishkin mesh of §2.5.1. Fix $\lambda \in (0, 1)$. Assume that $\varepsilon \leqslant CN^{-\varpi'}$ for some $\varpi' \geqslant 4 - \lambda$ and $C > 0$, and N is sufficiently large independently of ε.*

(i) If $C_\tau > 2$, then there exists a solution U of the standard scheme (2.5.13), (2.5.80), (2.5.102) *such that*

$$|U(X_i) - u(X_i)| \leqslant CN^{-\min\{2,\varpi'-2\}} \leqslant CN^{-(2-\lambda)} \quad \textit{for } X_i \in \mathring{\Omega}^N \cup \Gamma_{\pm\tau} \tag{2.5.129}$$

(ii) If $C_\tau > 1$, then there exists a solution $\hat{U}$ of the stabilised scheme (2.5.13), (2.5.80), (2.5.102) *such that*

$$|\hat{U}(X_i) - u(X_i)| \leqslant CN^{-1} \quad \textit{for } X_i \in \mathring{\Omega}^N \cup \Gamma_{\pm\tau}. \tag{2.5.130}$$

Proof. This proof follows Theorem 5.2 [18] now considered using the discretisation of F^N in (2.5.13), (2.5.80) and (2.5.102) with (2.5.65) and (2.5.94), on the domain $\bar{\Omega}^N$.

In this theorem we simplify the upper and lower solutions. Define $\bar{p}$ as

$$\bar{p} := \bar{C}N^{-\min\{2,\varpi'-2\}}. \tag{2.5.131}$$

We base our upper and lower solutions around the function $\alpha(X_i; p; I_1)$ defined as

$$\alpha(X_i; p; I_1) := \begin{cases} \varphi_1(X_i) + p & \text{for } X_i \in I_1, \\ \varphi_2(X_i) + p & \text{for } X_i \in \bar{\Omega}^N \backslash I_1, \end{cases} \tag{2.5.132}$$

for some subdomain I_1. We claim that a discrete upper solution to problem (2.0.1) is given by

$$\hat{\alpha}(X_i) := \alpha(X_i, \bar{p}; \{X_i \in \mathring{\Omega}^{(1)} \cup \Gamma_\tau\}), \tag{2.5.133}$$

and a discrete lower solution is given by

$$\tilde{\alpha}(X_i) := \alpha(X_i, -\bar{p}; \{X_i \in \mathring{\Omega}^{(1)} \cup \Omega_{(-\tau,\tau]}\}). \tag{2.5.134}$$

These are represented in Figure 2.12.

From the definition of $\hat{\alpha}(X_i)$ and $\tilde{\alpha}(X_i)$ using (2.5.132) it can be easily seen that $\tilde{\alpha}(X_i) \leqslant \hat{\alpha}(X_i)$ for all $X_i \in \bar{\Omega}^N$. We now consider $\hat{\alpha}(X_i)$, with $\tilde{\alpha}(X_i)$ being similar, and prove that it is an upper solution, i.e., we want to show $F^N\hat{\alpha}(X_i) \geqslant 0$ for $X_i \in \Omega^N$ and $\hat{\alpha}(X_i) \geqslant g(X_i)$ for $X_i \in \partial\Omega^N$. Recalling (A6) it is clear that $\hat{\alpha}(X_i) \geqslant g(X_i)$ is true for $X_i \in \partial\Omega^N$. We calculate $F^N\hat{\alpha}(X_i) \geqslant 0$ in three regions, the layer region which has the discretisation (2.5.13), the outer region in which we have (2.5.80) and the interface between the two given by (2.5.65) and (2.5.94) combined in (2.5.102).

Recalling (A3) we have

$$b(X_i, \hat{\alpha}(X_i)) \geqslant \bar{p}\gamma^2 + O(\bar{p}^2) \geqslant \frac{1}{2}\bar{p}\gamma^2 \quad \forall X_i \in \bar{\Omega}^N, \tag{2.5.135}$$

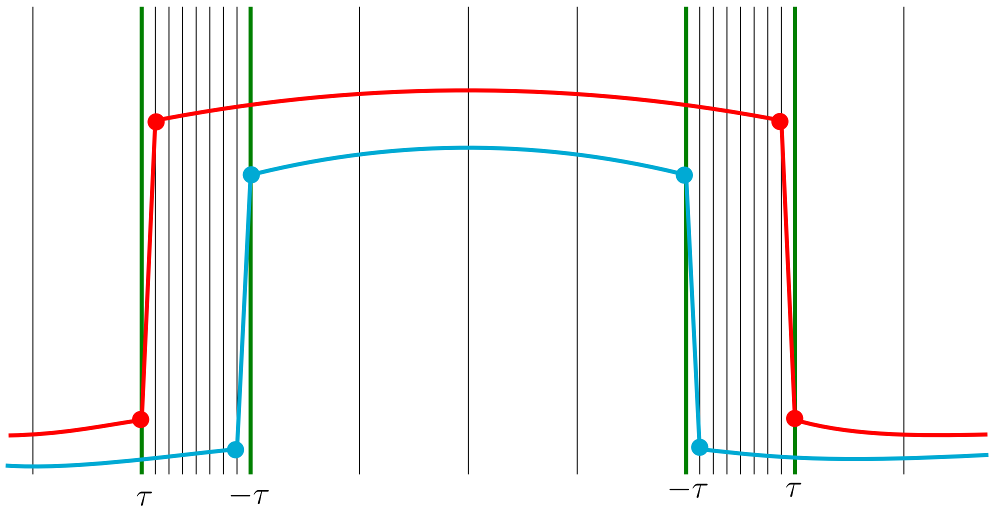


Figure 2.12: Sketch of $\hat{\alpha}(X_i)$ *and* $\tilde{\alpha}(X_i)$*, the upper and lower solutions defined using* (2.5.132)*,* (2.5.133) *and* (2.5.134)*.*

and hence $b(X_i, \hat{\alpha}(X_i)) \geqslant 0$. Note that $\varpi' \geqslant 4 - \lambda$ with $\lambda \in (0,1)$ implies $\varpi' > 3$.

For the region $x_{ij} \in \Omega_{(-\tau,\tau-h)}$, we have $\hat{\alpha}(x_{ij}) = \varphi_2(x_{ij}) + \bar{p}$. We consider $F^N\hat{\alpha}(x_{ij})$ from (2.5.13). We have

$$-\varepsilon^2\Delta^N\hat{\alpha}(x_{ij}) = -\varepsilon^2\eta_{ij}^{-1}D_r\left[\tilde{\eta}_{ij}D_r^-\varphi_2(x_{ij})\right] - \varepsilon^2\zeta_{ij}D_l\left[\tilde{\zeta}_{ij}D_l^-\varphi_2(x_{ij})\right]. \tag{2.5.136}$$

As $\varphi_2(x_{i\pm1,j}) = \varphi_2(x_{ij}) \pm h\left.\dfrac{\partial\varphi_2}{\partial x}\right|_{ij} + O(h^2)$, using (2.5.13e) and (2.5.13f), we obtain the result

$$-\varepsilon^2\Delta^N\hat{\alpha}(x_{ij}) = O(\varepsilon^2). \tag{2.5.137}$$

Combining (2.5.135) and (2.5.137) in (2.5.13) gives

$$F^N\hat{\alpha}(x_{ij}) \geqslant \frac{1}{2}\bar{p}\gamma^2 + O(\varepsilon^2) \quad \text{for } x_{ij} \in \Omega_{(-\tau,\tau-h)}. \tag{2.5.138}$$

Recalling $\varepsilon \leqslant CN^{-\varpi'}$ and using the definition of $\bar{p}$ from (2.5.131) we can

make $\bar{C}$ sufficiently large such that $\frac{1}{2}\bar{p}\gamma^2 \geqslant CN^{-2\varpi'}$ and hence $F^N\hat{\alpha}(x_{ij}) \geqslant 0$ for $x_{ij} \in \Omega_{(-\tau,\tau-h)}$.

We now consider the point $x_{N-1,j}$ i.e., where $r_{N-1} = \tau - h$, as there is a jump in the solution $\hat{\alpha}(X_i)$ here from $\varphi_2(X_i)+\bar{p}$ to $\varphi_1(X_i)+\bar{p}$. The Laplacian term is

$$\begin{aligned} -\varepsilon^2\Delta^N\hat{\alpha}(x_{N-1,j}) = & -\varepsilon^2\eta_{N-1,j}^{-1}D_r\left[\tilde{\eta}_{N-1,j}D_r^-\hat{\alpha}(x_{N-1,j})\right] \\ & -\varepsilon^2\zeta_{N-1,j}D_l\left[\tilde{\zeta}_{N-1,j}D_l^-\hat{\alpha}(x_{N-1,j})\right]. \end{aligned} \tag{2.5.139}$$

Noting the second term is $O(\varepsilon^2)$, $\eta_{i\pm1,j} = \eta_{ij} + O(h)$ and recalling the values of $\hat{\alpha}_{ij}$ at the necessary points results in

$$-\varepsilon^2\Delta^N\hat{\alpha}(x_{N-1,j}) = -\frac{\varepsilon^2}{h^2}(\varphi_1(X_{N,j})-2\varphi_2(X_{N-1,j})+\varphi_2(X_{N-2,j}))+O(\varepsilon^2(1+h^{-1})). \tag{2.5.140}$$

As $\varphi_k(X_i)$ for $k = 1, 2$ is sufficiently smooth we have

$$-\varepsilon^2\Delta^N\hat{\alpha}(x_{N-1,j}) = \frac{\varepsilon^2}{h^2}(\varphi_2(X_{N-1,j})-\varphi_1(X_{N-1,j}))+O(\varepsilon^2(1+h^{-1})), \tag{2.5.141}$$

and, from (A2), for N sufficiently large this is non-negative. Since $b(x,\hat{\alpha}) \geqslant 0$ we now have $F^N\hat{\alpha}(x_{N-1,j}) \geqslant 0$.

For $X_i \in \mathring{\Omega}^N$,

$$F^N\hat{\alpha}(X_i) := \frac{\varepsilon^2}{(1,\chi_i)}(\nabla\hat{\alpha}^I, \nabla\chi_i) + b(X_i,\hat{\alpha}), \tag{2.5.142}$$

and we want to show it is non-negative. We consider the region $\mathring{\Omega}^{(1)}$, with $\mathring{\Omega}^{(2)}$ being similar, and look for an upper bound for $\dfrac{\varepsilon^2}{(1,\chi_i)}(\nabla\hat{\alpha}^I, \nabla\chi_i)$. We apply (2.5.85) to $(\nabla\hat{\alpha}, \nabla\chi_i)$ where $\chi_i = 0$ for $X_i \in \mathring{\Omega}^{(1)}$ implies the final term

is zero and thus

$$(\nabla\hat{\alpha}^I, \nabla\chi_i) = (\nabla[\hat{\alpha}^I - \hat{\alpha}], \nabla\chi_i) - (\Delta\hat{\alpha}, \chi_i). \tag{2.5.143}$$

Recalling the interpolation error estimate, $|\nabla[\hat{\alpha}^I - \hat{\alpha}]| \leqslant N^{-1}\|\hat{\alpha}\|_{C^2(\bar{\Omega})}$, and the standard quasiuniform mesh properties, $(1, \chi_i) \geqslant CN^{-2}$ and $(1, |\nabla\chi_i|) \leqslant CN^{-1}$ from §2.5.7, we can write

$$\frac{\varepsilon^2}{(1,\chi_i)}(\nabla\hat{\alpha}^I, \nabla\chi_i) \leqslant \varepsilon^2\|\hat{\alpha}\|_{C^2(\bar{\Omega})}. \tag{2.5.144}$$

By a similar method as calculating $\|u_0\|_{C^2(\bar{\Omega})} \leqslant C$ in Lemma 2.5.6 we have $\|\hat{\alpha}\|_{C^2(\bar{\Omega})} \leqslant C$ which then, along with $\varepsilon \leqslant CN^{-\varpi'}$, gives

$$\frac{\varepsilon^2}{(1,\chi_i)}(\nabla\hat{\alpha}^I, \nabla\chi_i) \leqslant CN^{-2\varpi'}. \tag{2.5.145}$$

Recalling (2.5.135), (2.5.131) and $\varpi' > 3$,

$$CN^{-2\varpi'} \leqslant \frac{1}{2}\bar{C}\gamma^2 N^{-\min\{2,\varpi'-2\}} = \frac{1}{2}\bar{C}\gamma^2\bar{p}. \tag{2.5.146}$$

Hence,

$$\frac{\varepsilon^2}{(1,\chi_i)}(\nabla\hat{\alpha}^I, \nabla\chi_i) \leqslant b(X_i, \hat{\alpha}), \tag{2.5.147}$$

and so $F^N\hat{\alpha}(X_i) \geqslant 0$ for $X_i \in \mathring{\Omega}^{(1)}$.

For the interface Γ_τ, $F^N\hat{\alpha}(X_i)$ from (2.5.102) can be calculated with $F^N_{\Omega_{[-\tau,\tau]}}\hat{\alpha}(X_i)$ and $F^N_{\mathring{\Omega}^N}\hat{\alpha}(X_i)$ defined in (2.5.65) and (2.5.94) respectively, giving

$$F^N\hat{\alpha}(X_i) := A_1 + A_2 + b(X_N, \hat{\alpha}_N), \tag{2.5.148}$$

where A_1 is defined as

$$A_1 := \frac{h_N/2}{h_N/2 + 1/a_j}\left[\varepsilon^2 \eta_{N,j}^{-1}\tilde{\eta}_{N,j}\frac{D_r^- \hat{\alpha}_{N,j}}{h_N/2} - \varepsilon^2 \zeta_{N,j} D_l(\zeta_{N,j} D_l^- \hat{\alpha}_{N,j})\right], \quad (2.5.149)$$

and A_2 is

$$A_2 := \frac{1/a_j}{h_N/2 + 1/a_j}\left[\frac{\varepsilon^2}{(1,\chi_i)}(\nabla \hat{\alpha}^I, \nabla \chi_i)\right]. \quad (2.5.150)$$

We want to show this is non-negative by looking for an upper bound for the derivative terms and showing that $b(X_N, \hat{\alpha}_N)$ dominates this. As before we have (2.5.135) for $b(X_N, \hat{\alpha}_N)$. For A_1 we have

$$A_1 = \frac{h_N/2}{h_N/2 + 1/a_j}\left[\frac{\varepsilon^2}{h_N^2/2}(\varphi_2(X_N) - \varphi_1(X_N)) + O(\varepsilon^2(1 + h^{-1}))\right] \quad (2.5.151)$$

which can be written as

$$A_1 \leqslant C\varepsilon^2 N(h_N^{-1} + 1) + O(\varepsilon^2). \quad (2.5.152)$$

For A_2 we again use (2.5.85) giving

$$A_2 = \frac{1/a_j}{h_N/2 + 1/a_j}\frac{\varepsilon^2}{(1,\chi_i)}\left[(\nabla[\hat{\alpha}^I - \hat{\alpha}], \nabla\chi_i) - (\Delta\hat{\alpha}, \chi_i) + \oint_{\Gamma_\tau} \chi_i \frac{\partial \hat{\alpha}}{\partial n} ds\right]. \quad (2.5.153)$$

As $\left|\dfrac{\partial \hat{\alpha}}{\partial n}\right| \leqslant \|\hat{\alpha}\|_{C^2(\bar{\Omega})} \leqslant C$,

$$|A_2| \leqslant \frac{1/a_j}{h_N/2 + 1/a_j}\frac{\varepsilon^2}{(1,\chi_i)}\left[\left|(\nabla[\hat{\alpha}^I - \hat{\alpha}], \nabla\chi_i) - (\Delta\hat{\alpha}, \chi_i)\right| + C\oint_{\Gamma_\tau} \chi_i ds\right]. \quad (2.5.154)$$

Using the mesh properties and interpolation error estimates above and recalling the definition and order of a_j from (2.5.94b) and (2.5.95) we now have

$$|A_2| \leqslant C\varepsilon^2(1 + N). \quad (2.5.155)$$

Putting (2.5.152) and (2.5.155) together gives

$$|A_1 + A_2| \leqslant C\varepsilon^2 N(h_N^{-1} + 1) + C\varepsilon^2(1 + N). \tag{2.5.156}$$

Recalling $h = C\varepsilon\hat{h}$ and $\hat{h}^{-1} \leqslant CN$ results in

$$|A_1 + A_2| \leqslant C\varepsilon N^2. \tag{2.5.157}$$

Using (2.5.128) gives

$$|A_1 + A_2| \leqslant CN^{-\varpi'+2}. \tag{2.5.158}$$

As $\varpi' > 3$, then

$$CN^{-\varpi'+2} \leqslant \frac{1}{2}\bar{C}\gamma^2 N^{-\min\{2,\varpi'-2\}} = \frac{1}{2}\bar{p}\gamma^2, \tag{2.5.159}$$

and we have established that $F^N\hat{\alpha}(X_i) \geqslant 0$ for $X_i \in \Gamma_\tau$.

A similar argument holds true for $\hat{\alpha}(X_i)$ on $\Gamma_{-\tau}$ where the solution is $\varphi_2(X_i) + \bar{p}$. On this curve,

$$A_1 = \frac{h_N/2}{h_N/2 + 1/a_j}\left[\frac{\varepsilon^2}{h_N^2/2}(\varphi_2(X_N) - \varphi_2(X_N)) + O(\varepsilon^2(1 + h^{-1}))\right], \tag{2.5.160}$$

which can be simplified to

$$A_1 = O(\varepsilon^2(1 + h^{-1})). \tag{2.5.161}$$

Given A_2 from (2.5.155),

$$A_1 + A_2 \leqslant C\varepsilon^2(1 + h^{-1}) + C\varepsilon^2 N. \tag{2.5.162}$$

As before this term is dominated by $b(X_i, \hat{\alpha}_i)$ as

$$CN^{-2\varpi'+1} \leqslant \frac{1}{2}\bar{C}\gamma^2 N^{-\min\{2,\varpi'-2\}}, \tag{2.5.163}$$

and we have $F^N\hat{\alpha}(X_i) \geqslant 0$ for $X_i \in \Gamma_{-\tau}$.

As we now have the necessary requirements; $F^N\hat{\alpha}(X_i) \geqslant 0$ for $X_i \in \Omega^N$ and $\hat{\alpha}(X_i) \geqslant g(X_i)$ for $X_i \in \partial\Omega^N$, then $\hat{\alpha}(X_i)$ is an upper solution of problem (2.0.1). By a similar method we can establish that $\tilde{\alpha}(X_i)$ is a lower solution of problem (2.0.1). Hence there exists a discrete solution $U(X_i)$ such that $\tilde{\alpha}(X_i) \leqslant U(X_i) \leqslant \hat{\alpha}(X_i)$ for $X_i \in \bar{\Omega}^N$. For $X_i \in \mathring{\Omega}^N \cup \Gamma_{\pm\tau}$ with $C_\tau > 2$, $\alpha = u_0(x) + O(\varepsilon^2 + \bar{p}) = u(x) + O(N^{-\min\{2,\varpi'-2\}})$ and hence we get (2.5.129).

For the stabilised method we choose

$$\bar{p} := \bar{C}N^{-1}. \tag{2.5.164}$$

The argument for $F^N\hat{\alpha}(x_{ij}) \geqslant 0$ holds true for the region $\Omega_{(-\tau,\tau-h)}$ as $\hat{\varepsilon}(x_{ij}) = \varepsilon$ for $x_{ij} \in \Omega_{(-\tau,\tau)}$ and since $\varpi' > 3$ we have $\bar{C}N^{-1} \geqslant \bar{C}N^{-\min\{2,\varpi'-2\}}$.

For the points $x_{N-1,j}$ we have

$$-\hat{\varepsilon}^2(X_i)\Delta^N\hat{\alpha}(x_{N-1,j}) = \frac{\hat{\varepsilon}^2(X_i)}{h^2}(\varphi_2(X_{N-1,j}) - \varphi_1(X_{N-1,j})) + O((\varepsilon^2 + \hat{\varepsilon}^2)h^{-1}). \tag{2.5.165}$$

Recalling (A2) we have for sufficiently large N, $-\hat{\varepsilon}^2(X_i)\Delta^N\hat{\alpha}(x_{N-1,j}) \geqslant 0$ and hence $F^N\hat{\alpha}(x_{N-1,j}) \geqslant 0$.

For $X_i \in \mathring{\Omega}^N$ we want to show (2.5.142) is non-negative with ε replaced by $\hat{\varepsilon}(X_i) = \widehat{C}N^{-1}$ from (2.5.6). Again using the standard quasiuniform mesh properties we have the bound

$$\frac{\hat{\varepsilon}^2(X_i)}{(1,\chi_i)}(\nabla\varphi_2^I, \nabla\chi_i) \leqslant C\hat{\varepsilon}^2, \tag{2.5.166}$$

and $\dfrac{1}{2}\bar{p}\gamma^2 = \dfrac{1}{2}\bar{C}\gamma^2 N^{-1} \geqslant CN^{-2}$ and hence $F^N\hat{\alpha} \geqslant 0$ for $X_i \in \mathring{\Omega}^N$.

For $X_i \in \Gamma_\tau$ replace ε with $\hat{\varepsilon}(X_i)$ in (2.5.148), and define $\hat{A}_1$ and $\hat{A}_2$ as (2.5.149) and (2.5.150) respectively with ε replaced by $\hat{\varepsilon}(X_i)$. For $\hat{A}_1 + \hat{A}_2$ we have

$$\hat{A}_1 + \hat{A}_2 \leqslant C\hat{\varepsilon}(X_i)N^2. \tag{2.5.167}$$

We note ϕ_j survives in the calculation of (2.5.167) unlike with the standard method and so there is the extra term in $\hat{A}_1 + \hat{A}_2$, which was

$$\left(\frac{\varepsilon^2 - \hat{\varepsilon}^2}{h_N/2 + 1/a_j}\right)\phi_j = O(\hat{\varepsilon}^2(X_i)N), \tag{2.5.168}$$

however this term does not change the bound (2.5.167). Next, $\hat{A}_1 + \hat{A}_2$ is dominated by $\frac{1}{2}\bar{p}\gamma^2 = \frac{1}{2}\bar{C}\gamma^2 N^{-1} \geqslant CN^{-1}$ with $\bar{C}$ chosen sufficiently large and by recalling (2.5.6) and we have $F^N\hat{\alpha}(X_i) \geqslant 0$ for $X_i \in \Gamma_\tau$.

Finally for $X_i \in \Gamma_{-\tau}$,

$$\hat{A}_1 + \hat{A}_2 \leqslant C\varepsilon^2(1 + h^{-1}) + C\hat{\varepsilon}^2(X_i)(1 + N), \tag{2.5.169}$$

and hence

$$\hat{A}_1 + \hat{A}_2 \leqslant C\hat{\varepsilon}^2(X_i)N. \tag{2.5.170}$$

By the argument above we have $F^N\hat{\alpha}(X_i) \geqslant 0$ for $X_i \in \Gamma_{-\tau}$.

Hence $F^N\hat{\alpha}(X_i) \geqslant 0$ for $X_i \in \Omega^N$ and $\hat{\alpha}(X_i) \geqslant g(X_i)$ for $X_i \in \partial\Omega^N$ holds. By a similar method we can establish that $\tilde{\alpha}(X_i)$ is a lower solution to (2.0.1). Hence there exists a discrete solution $\hat{U}(X_i)$ such that $\tilde{\alpha}(X_i) \leqslant \hat{U}(X_i) \leqslant \hat{\alpha}(X_i)$. If $C_\tau > 1$, we can say $\alpha(x) = u(x) + O(\hat{\varepsilon}^2 + N^{-1})$ for $X_i \in \mathring{\Omega}^N \cup \Gamma_{\pm\tau}$, and hence we have proven (2.5.130). $\square$

Remark 2.5.7. Postprocessing
We also consider using postprocessing for small values of ε to gain ε-uniform accuracy in $\bar{\Omega}^N$ for all $\varepsilon \leqslant CN^{-1}$. This is done in [18] for the one-dimensional case and is extended to two dimensions here. Postprocessing is used to deal with smaller values of ε, i.e., for $\varepsilon \in (0, \bar{\varepsilon})$ with $\bar{\varepsilon} := CN^{-2} \ln^3 N$ for some positive constant N. As in the above work we will use the notation x_{ij} and $u_{ij} = u(x_{ij})$ for the mesh and solution in the layer region and X_i and $u_i = u(X_i)$ in the outer region, combining these and using X_i and $u_i = u(X_i)$ in the entire domain.

We use the notation $\bar{u}_i$ to represent the discrete solution of (2.0.1) with $\bar{\varepsilon}$ on the mesh $\bar{X}_i$ defined with the mesh transition parameter

$$\tau(\bar{\varepsilon}) := \frac{C_\tau}{\bar{\gamma}} \bar{\varepsilon} \ln N. \tag{2.5.171}$$

Note that we are redefining the notation $\bar{x}_{ij}$ from previous sections for the purposes of this proof only. The post-processed solution is defined as $\tilde{u}_{ij}$ and $\tilde{u}_i$.

For the outer region $\tilde{X}_i := \bar{X}_i$ and $\tilde{u}_i := \bar{u}_i$, and for the layer region we compress the computed solution $\bar{u}_{ij}^N$ in the r direction by writing $\tilde{x}_{ij} := x\left(\dfrac{\varepsilon \bar{r}_i}{\bar{\varepsilon}}, l_j\right)$ and $\tilde{u}_{ij} := \bar{u}_{ij}$.

We look for the bound

$$\max_i |\tilde{u}_i^N - u(\tilde{X}_i)| \leqslant \max_j |\bar{u}(\bar{X}_j) - u(\tilde{X}_j)| + \max_j |\bar{u}_j^N - \bar{u}(\bar{X}_j)|. \tag{2.5.172}$$

The requirements of Theorem 2.5.5 are met by $\bar{\varepsilon}$ with $\varpi = 0$, i.e., $c_0 \bar{\varepsilon} \geqslant (C' N^{-1} \ln N)^2$ is true, and hence

$$|\bar{u}_j^N - \bar{u}(\bar{X}_j)| \leqslant CN^{-2} \ln^2 N, \tag{2.5.173}$$

for the standard method and

$$|\bar{u}_j^N - \bar{u}(\bar{X}_j)| \leqslant CN^{-1}, \tag{2.5.174}$$

for the stabilised method. For the remaining term in (2.5.172), we have

$$|\bar{u}(\bar{X}_j) - u(\tilde{X}_j)| \leqslant |\bar{u}(\bar{X}_j) - \mathcal{U}(\bar{X}_j, \bar{\varepsilon})| + |u(\tilde{X}_j) - \mathcal{U}(\tilde{X}_j, \varepsilon)| \quad \forall\, j, \tag{2.5.175}$$

where $\mathcal{U}(X_j, \varepsilon)$ is defined in (2.4.79) and $\mathcal{U}(\bar{X}_j, \bar{\varepsilon}) = \mathcal{U}(\tilde{X}_j, \varepsilon)$.

Using Corollary 2.4.1 and Lemma 2.4.6 with $c_1 = \tau(\bar{\varepsilon})$ gives

$$|u(X_j) - U(X_j, \varepsilon)| \leqslant \tau(\bar{\varepsilon}) + C\varepsilon^2. \tag{2.5.176}$$

Hence using the definition of $\tau(\bar{\varepsilon})$ from (2.5.171) along with (2.5.175) and (2.5.176) we have

$$|\bar{u}(\bar{X}_j) - u(\tilde{X}_j)| \leqslant CN^{-2} \ln^4 N. \tag{2.5.177}$$

Combining (2.5.177), (2.5.173) and (2.5.174) in (2.5.172) results in

$$|\tilde{u}_i^N - u(\tilde{X}_i)| \leqslant CN^{-2} \ln^4 N, \quad \forall\, \tilde{X}_i \in \bar{\Omega}^N, \tag{2.5.178}$$

for the standard method and

$$|\tilde{u}_i^N - u(\tilde{X}_i)| \leqslant C \max\{N^{-2} \ln^4 N, N^{-1}\}, \quad \forall\, \tilde{X}_i \in \bar{\Omega}^N, \tag{2.5.179}$$

for the stabilised method.

2.6 Conclusions

We have obtained accuracy results for the two-dimensional elliptic interior layer problem (2.0.1). By considering an example it was clear that the stabilised method was required for this problem; the conventional method gave incorrect computed solutions. This example is considered in further detail in *Chapter 5*.

Using asymptotic analysis that had previously been carried out we obtained upper and lower solutions. In the layer region we employed the finite difference method in curvilinear coordinates while in the outer region we used lumped mass finite elements on a quasiuniform mesh. By doing this we were able to employ the theory of Z-fields. On the interface between the outer region and the layer region we employed a fictitious Neumann condition, giving a discretisation for the outer region side of the curve in the finite element method and a discretisation for the layer side of the curve in the finite difference method. These were combined to get a discretisation for the curve, eliminating the fictitious Neumann condition. Discrete upper and lower solutions were obtained and existence of an exact solution between these discrete solutions was proven.

The problem was considered on the Shishkin mesh only, using the Bakhvalov mesh would have added considerable difficulty to the problem. The truncation error of the system was found to be $O((N^{-1}\ln N)^{2-\varpi})$ and $O(N^{-2})$ for the layer region and the outer region respectively with the conventional method and $O((N^{-1}\ln N)^{2-\varpi}+N^{-1})$ and $O(N^{-1})$ for the layer region and the outer region with the stabilised method. Here $c_0\varepsilon \geqslant (C'N^{-1}\ln N)^{2+\varpi}$ with $\varpi \in [0,2]$. In the case where the relationship between ε and N was stronger than $\varepsilon \leqslant CN^{-1}$, that is $\varepsilon \leqslant CN^{-\varpi'}$ for some $\varpi' \geqslant 4-\lambda$, the truncation error was found to be $O(N^{-(2-\lambda)})$ for the standard method and $O(N^{-1})$ for the stabilised method. As these results are not ε-uniform, we performed

post-processing and found the truncation error of the standard method to be $O(N^{-2}\ln^4 N)$ and the stabilised method to be $O(\max\{N^{-2}\ln^4 N, N^{-1}\})$ with post-processing for $\varepsilon \in (0, \bar{\varepsilon})$ with $\bar{\varepsilon} := CN^{-2}\ln^3 N$ for some positive constant N. These results are consistent with the one-dimensional analysis [18].

2.7 Technical Properties of the Asymptotic Analysis

2.7.1 Order of F

This section includes technical results that have been used in §2.4. We want to find equations for the terms $v_0(x)$, $v_1(x)$ and $v_2(x)$ from the asymptotic expansion of $u(x)$. We also find the order of the system in the different regions. Here $u_{as}(x) = u_0(x)+\varepsilon^2 u_2(x)+(v_0(x)+\varepsilon v_1(x)+\varepsilon^2 v_2(x))\vartheta(x)$ where $\vartheta(x)$ is the smooth cut off function described after (2.4.29). For presentation purposes we assume $p = 0$ in the following work. For $p \neq 0$ the following is unchanged. Throughout this section we assume $x = x(r, l)$, $\bar{x}_0 = x(0, l)$ and ϑ, v_0, v_1, etc, are $\vartheta(x)$, $v_0(\xi, l; p)$, $v_1(\xi, l; p)$, etc.

The domain $\bar{\Omega}$ can be considered in three cases; the outer region $\bar{\Omega}\backslash\bar{\Omega}_{c_1}$, and two regions of the interior layer, $\frac{c_1}{2} \leqslant |r| \leqslant c_1$ and $|r| \leqslant \frac{c_1}{2}$. In the outer region $\vartheta(x) = 0$ and the $v_i(x)$ terms disappear from the asymptotic expansion. Recall $\vartheta(x)$ from (2.4.29), in the interior layer region $\vartheta(x) = 1$ when $|r| \leqslant \frac{c_1}{2}$ and goes to zero in $\frac{c_1}{2} \leqslant |r| \leqslant c_1$. Recall x can by represented by $x = \bar{x} + \varepsilon\xi n(l)$ where $\bar{x}$ is defined in (2.4.13).

We first obtain accuracy for $F(u_0 + \varepsilon^2 u_2)$ for all $x \notin \Gamma_0$ in the following lemma.

Lemma 2.7.1. *For the asymptotic expansion* $u_0(x) + \varepsilon^2 u_2(x)$ *of problem* (2.0.1) *away from* Γ_0 *we have accuracy*

$$F(u_0 + \varepsilon^2 u_2) = O(\varepsilon^4) \quad \textit{for } x \in \Omega\backslash\Gamma_0. \tag{2.7.1}$$

Proof. For the region away from the layer,

$$F(u_0+\varepsilon^2 u_2) = -\varepsilon^2\Delta(u_0+\varepsilon^2 u_2) + B(x,\varepsilon^2 u_2). \tag{2.7.2}$$

We want to prove this is $O(\varepsilon^4)$. Performing a Taylor expansion of $B(x,\varepsilon^2 u_2)$ about ε gives

$$F(u_0+\varepsilon^2 u_2) = -\varepsilon^2\Delta(u_0+\varepsilon^2 u_2)+B(x,0)+\varepsilon^2 u_2 B_s(x,0)+\frac{1}{2}\varepsilon^4 u_2^4 B_{ss}(x,\varepsilon^2 u_2\theta(x)). \tag{2.7.3}$$

combining $B(x,0)=0$, $B_{ss}(x,\varepsilon^2 u_2\theta(x)) = O(1)$ and (2.4.8) where $\Delta u_0 = u_2 B_s(x,0)$ we arrive at the desired result (2.7.1). $\square$

We have the following corollary to this lemma showing the accuracy in the region $\frac{c_1}{2} \leqslant |r|$.

Corollary 2.7.1. *For the asymptotic expansion $u_{as}(x;p)$ of problem* (2.0.1) *we have accuracy*

$$F(u_{as}(x;p)) = O(\varepsilon^4) \quad \textit{for } \frac{c_1}{2} \leqslant |r|. \tag{2.7.4}$$

Proof. In view of Lemma 2.7.1 it remains to estimate Fu_{as} for $\frac{c_1}{2} \leqslant r \leqslant c_1$. In this region $u_{as}(x;p) := u_0+\varepsilon^2 u_2+(v_0+\varepsilon v_1+\varepsilon^2 v_2)\vartheta$ where v_i with $i=0,1,2$ are exponentially small in this region and $\vartheta(x)$ takes values in $(0,1)$. For the nonlinear function,

$$\begin{aligned} B(x,\varepsilon^2 u_2+(v_0+\varepsilon v_1+\varepsilon^2 v_2)\vartheta) &= B(x,\varepsilon^2 u_2) \\ +(v_0+\varepsilon v_1+\varepsilon^2 v_2)\vartheta B_s(x,\varepsilon^2 u_2&+(v_0+\varepsilon v_1+\varepsilon^2 v_2)\vartheta\theta), \end{aligned} \tag{2.7.5}$$

for $\theta \in (0,1)$. Looking at the Laplacian of $u_{as}(x)$ we have

$$-\varepsilon^2\Delta u_{as} = -\varepsilon^2\Delta(u_0+\varepsilon^2 u_2) - \varepsilon^2\Delta((v_0+\varepsilon v_1+\varepsilon^2 u_2)\vartheta) + O(\varepsilon^4), \tag{2.7.6}$$

where the second term is exponentially small. We note the second term in (2.7.5) is exponentially small as well and combine this with (2.7.6) to get

$$Fu_{as}(x;p) = -\varepsilon^2\Delta(u_0 + \varepsilon^2 u_2) + B(x, \varepsilon^2 u_2) + O(\varepsilon^4). \tag{2.7.7}$$

Using Lemma 2.7.1 we now obtain (2.7.4). □

Next, we derive similar results for the interior layer, i.e., the region $|r| \leqslant \frac{c_1}{2}$. In this region $\vartheta(x) = 1$ and so the asymptotic expansion here is $u_{as}(x;p) := u_0(x) + \varepsilon^2 u_2(x) + v_0(\xi, l; p) + \varepsilon v_1(\xi, l; p) + \varepsilon^2 v_2(\xi, l; p)$. Also, $x = \bar{x} + \varepsilon \xi n(l)$ in these regions.

In the following lemma we use Taylor expansions to represent

$$B(x, \varepsilon^2 u_2(x) + s)\Big|_{s=0}^{s=v_0+\varepsilon v_1+\varepsilon^2 v_2}, \tag{2.7.8}$$

in a form that will be easier to differentiate with respect to ε. This will be used in Lemma 2.7.3 to calculate Fu_{as} for $|r| \leqslant \frac{c_1}{2}$.

Lemma 2.7.2. *For the nonlinear function $B(x, s)$ defined in* (2.4.26) *we have*

$$\Big[B(x, \varepsilon^2 u_2(x) + s) - B(x, s)\Big]\Big|_{s=0}^{s=v_0+\varepsilon v_1+\varepsilon^2 v_2} = \varepsilon^2 u_2(\bar{x}) B_s(\bar{x}, s)\Big|_{s=0}^{s=v_0} + O(\varepsilon^3). \tag{2.7.9}$$

Proof. Firstly,

$$\Big[B(x, \varepsilon^2 u_2(x) + s) - B(x, s)\Big]\Big|_{s=0}^{s=v_0+\varepsilon v_1+\varepsilon^2 v_2} = \varepsilon^2 u_2(x) B_s(x, s)\Big|_{s=0}^{s=v_0} + O(\varepsilon^3). \tag{2.7.10}$$

Expanding this about $x = \bar{x}$ gives

$$\begin{aligned}&\Big[B(x, \varepsilon^2 u_2(x) + s) - B(x, s)\Big]\Big|_{s=0}^{s=v_0+\varepsilon v_1+\varepsilon^2 v_2} = \\ &\left(\varepsilon^2 u_2(\bar{x}) + \varepsilon^3 \xi \left.\frac{\partial u_2}{\partial r}\right|_{x=\bar{x}+\varepsilon\xi\theta}\right) \left[B_s(\bar{x}, s) + \varepsilon \xi B_{sr}(\bar{x} + \varepsilon\xi\theta, s)\right]\Big|_{s=0}^{s=v_0} + O(\varepsilon^3),\end{aligned} \tag{2.7.11}$$

Using the mean value theorem this can be written as

$$\begin{aligned}&\Big[B(x, \varepsilon^2 u_2(x) + s) - B(x, s)\Big]\Big|_{s=0}^{s=v_0+\varepsilon v_1+\varepsilon^2 v_2} = \varepsilon^2 u_2(\bar{x}) B_s(\bar{x}, s)\Big|_{s=0}^{s=v_0} \\ &+ \varepsilon^3 u_2(\bar{x}) \xi v_0 B_{ssr}(\bar{x} + \varepsilon\xi\theta, s)\Big|_{s=\theta v_0} \\ &+ \varepsilon^3 \xi v_0 \left.\frac{\partial u_2}{\partial r}\right|_{x=\bar{x}+\varepsilon\xi\theta} \left[B_{ss}(\bar{x}, s) + \varepsilon\xi B_{ssr}(\bar{x} + \varepsilon\xi\theta, s)\right]\Big|_{s=\theta v_0} + O(\varepsilon^3),\end{aligned} \tag{2.7.12}$$

As $\xi \to \pm\infty$, v_0 decays exponentially by (2.4.50), hence $|\xi^n v_0|$ is bounded and as $B(x, s)$ and its derivatives are bounded we get the desired result.

As $B(x, s)$ and its derivatives are bounded we can write

$$\Big[B(x, \varepsilon^2 u_2(x) + s) - B(x, s)\Big]\Big|_{s=0}^{s=v_0+\varepsilon v_1+\varepsilon^2 v_2} = \varepsilon^2 u_2(\bar{x}) B_s(\bar{x}, s)\Big|_{s=0}^{s=v_0} + O\left(\varepsilon^3 \xi v_0\right). \tag{2.7.13}$$

As $\xi \to \pm\infty$, v_0 decays exponentially by (2.4.50) giving the desired result.

□

Lemma 2.7.3. *For the asymptotic expansion $u_{as}(x; p)$ of problem* (2.0.1) *we have accuracy*

$$F(u_{as}(x; p)) = O(\varepsilon^3) \quad \textit{for } |r| \leqslant \frac{c_1}{2}. \tag{2.7.14}$$

Proof. As $\vartheta(x) = 1$ in this region,

$$Fu_{as} = -\varepsilon^2\Delta(u_0+\varepsilon^2 u_2) - \varepsilon^2\Delta(v_0+\varepsilon v_1+\varepsilon^2 v_2) + B(x, \varepsilon^2 u_2 + v_0 + \varepsilon v_1 + \varepsilon^2 v_2), \tag{2.7.15}$$

and using Lemma 2.7.1 this becomes

$$Fu_{as} = -\varepsilon^2\Delta(v_0+\varepsilon v_1+\varepsilon^2 v_2) + B(x, \varepsilon^2 u_2 + s)\Big|_{s=0}^{s=v_0+\varepsilon v_1+\varepsilon^2 v_2} + O(\varepsilon^4). \tag{2.7.16}$$

We now call on Lemma 2.7.2 to find

$$\begin{aligned} Fu_{as} = &- \varepsilon^2\Delta(v_0+\varepsilon v_1+\varepsilon^2 v_2) + B(x, s)\Big|_{s=0}^{s=v_0+\varepsilon v_1+\varepsilon^2 v_2} \\ &+ \varepsilon^2 u_2(\bar{x}) B_s\Bigg|_{s=0, x=\bar{x}}^{s=v_0, x=\bar{x}} + O(\varepsilon^3). \end{aligned} \tag{2.7.17}$$

Recalling $x = \bar{x} + \varepsilon\xi n(l)$, we define $\mathcal{G}(\varepsilon)$ as

$$\mathcal{G}(\varepsilon) := B(\bar{x} + \varepsilon\xi, s)\Big|_{s=0}^{s=v_0+\varepsilon v_1+\varepsilon^2 v_2}, \tag{2.7.18}$$

and find the Taylor expansion of $\mathcal{G}(\varepsilon)$ about $\varepsilon = 0$, i.e.,

$$\mathcal{G}(\varepsilon) = \mathcal{G}(0) + \varepsilon\mathcal{G}'(0) + \frac{\varepsilon^2}{2}\mathcal{G}''(0) + \frac{\varepsilon^3}{3!}\mathcal{G}'''(\varepsilon^*), \tag{2.7.19}$$

for some $\varepsilon^* \in (0, \varepsilon)$. The necessary derivatives are

$$\mathcal{G}'(\varepsilon) = \xi\frac{\partial B}{\partial r}\Bigg|_{s=0}^{s=v_0+\varepsilon v_1+\varepsilon^2 v_2} + (v_1 + 2\varepsilon v_2)\frac{\partial B}{\partial s}\Bigg|_{s=v_0+\varepsilon v_1+\varepsilon^2 v_2}, \tag{2.7.20}$$

$$\begin{aligned}\mathcal{G}''(\varepsilon) =& \xi^2 \frac{\partial^2 B}{\partial r^2}\Bigg|_{s=0}^{s=v_0+\varepsilon v_1+\varepsilon^2 v_2} \\ &+ \left(2\xi(v_1+2\varepsilon v_2)\frac{\partial^2 B}{\partial r \partial s} + (v_1+2\varepsilon v_2)^2 \frac{\partial^2 B}{\partial s^2} + 2v_2 \frac{\partial B}{\partial s}\right)\Bigg|_{s=v_0+\varepsilon v_1+\varepsilon^2 v_2},\end{aligned} \tag{2.7.21}$$

and

$$\begin{aligned}\mathcal{G}'''(\varepsilon) = \xi^3 \frac{\partial^3 B}{\partial r^3}\Bigg|_{s=0}^{s=v_0+\varepsilon v_1+\varepsilon^2 v_2} &+ \left((v_1+2\varepsilon v_2)^3 \frac{\partial^3 B}{\partial s^3} + 3\xi^2(v_1+2\varepsilon v_2)\frac{\partial^3 B}{\partial r^2 \partial s}\right. \\ \left. +3\xi(v_1+2\varepsilon v_2)^2 \frac{\partial^3 B}{\partial r \partial s^2} + 6v_2(v_1+2\varepsilon v_2)\frac{\partial^2 B}{\partial s^2} + 6\xi v_2 \frac{\partial^2 B}{\partial s \partial r}\right)&\Bigg|_{s=v_0+\varepsilon v_1+\varepsilon^2 v_2}.\end{aligned} \tag{2.7.22}$$

If $s = 0$, $B(x, s)$ and any derivatives with respect to r are zero giving

$$\mathcal{G}(0) = B(\bar{x}, v_0), \tag{2.7.23}$$

$$\mathcal{G}'(0) = \left(\xi \frac{\partial B}{\partial r} + v_1 \frac{\partial B}{\partial s}\right)\Bigg|_{s=v_0, x=\bar{x}}, \tag{2.7.24}$$

and

$$\mathcal{G}''(0) = \left(\xi^2 \frac{\partial^2 B}{\partial r^2} + 2\xi v_1 \frac{\partial^2 B}{\partial r \partial s} + v_1^2 \frac{\partial^2 B}{\partial s^2} + 2v_2 \frac{\partial B}{\partial s}\right)\Bigg|_{s=v_0, x=\bar{x}}. \tag{2.7.25}$$

By the mean value theorem we can simplify the first term in $\mathcal{G}'''(\varepsilon)$ giving

$$\begin{aligned}\mathcal{G}'''(\varepsilon) =& \xi^3(v_0+\varepsilon v_1+\varepsilon^2 v_2)\frac{\partial^4 B}{\partial r^3 \partial s}\bigg|_{s=\theta(v_0+\varepsilon v_1+\varepsilon^2 v_2)} \\ &+\Bigg((v_1+2\varepsilon v_2)^3\frac{\partial^3 B}{\partial s^3}+3\xi^2(v_1+2\varepsilon v_2)\frac{\partial^3 B}{\partial r^2\partial s} \\ &+3\xi(v_1+2\varepsilon v_2)^2\frac{\partial^3 B}{\partial r\partial s^2}+6v_2(v_1+2\varepsilon v_2)\frac{\partial^2 B}{\partial s^2}+6\xi v_2\frac{\partial^2 B}{\partial s\partial r}\Bigg)\bigg|_{s=v_0+\varepsilon v_1+\varepsilon^2 v_2},\end{aligned} \tag{2.7.26}$$

for some $\theta \in (0,1)$. Note $|\xi^n v_i|$ is bounded for $n = 1,2,3$ and $i = 0,1,2$ as v_i decays exponentially as $\xi \to \infty$. Also note $B(x,s)$ and its derivatives are bounded, hence $\mathcal{G}'''(\varepsilon^*) = O(1)$.

Putting $\mathcal{G}(\varepsilon)$ back into Fu_{as} and rewriting Δ using (2.4.28) gives

$$\begin{aligned}Fu_{as} =& -\frac{\partial^2}{\partial\xi^2}(v_0+\varepsilon v_1+\varepsilon^2 v_2)-\varepsilon\kappa\frac{\partial}{\partial\xi}(v_0+\varepsilon v_1+\varepsilon^2 v_2) \\ &-\varepsilon^2\zeta\frac{\partial}{\partial l}\left(\zeta\frac{\partial}{\partial l}(v_0+\varepsilon v_1+\varepsilon^2 v_2)\right) \\ &+B(\bar{x},v_0)+\varepsilon\left(\xi\frac{\partial B}{\partial r}+v_1\frac{\partial B}{\partial s}\right)\bigg|_{s=v_0,x=\bar{x}} \\ &+\varepsilon^2\left(\frac{\xi^2}{2}\frac{\partial^2 B}{\partial r^2}+\xi v_1\frac{\partial^2 B}{\partial r\partial s}+\frac{v_1^2}{2}\frac{\partial^2 B}{\partial s^2}+v_2\frac{\partial B}{\partial s}\right)\bigg|_{s=v_0,x=\bar{x}} \\ &+\varepsilon^2 u_2(\bar{x})\frac{\partial B}{\partial s}\bigg|_{s=0,x=\bar{x}}^{s=v_0,x=\bar{x}}+O(\varepsilon^3).\end{aligned} \tag{2.7.27}$$

From this we can extract the equations for v_0, v_1 and v_2,

$$-\frac{\partial^2 v_0}{\partial\xi^2}+B(\bar{x},v_0)=0, \tag{2.7.28}$$

$$-\frac{\partial^2 v_1}{\partial\xi^2}+v_1\frac{\partial B}{\partial s}\bigg|_{s=v_0,x=\bar{x}}=\kappa\frac{\partial v_0}{\partial\xi}-\xi\frac{\partial B}{\partial r}\bigg|_{s=v_0,x=\bar{x}}, \tag{2.7.29}$$

$$\begin{aligned} -\frac{\partial^2 v_2}{\partial \xi^2} + v_2 \frac{\partial B}{\partial s}\bigg|_{s=v_0,x=\bar{x}} &= \kappa \frac{\partial v_1}{\partial \xi} + \zeta \frac{\partial}{\partial l}\left(\zeta \frac{\partial v_0}{\partial l}\right) \\ -\left(\frac{\xi^2}{2}\frac{\partial^2 B}{\partial r^2} + \xi v_1 \frac{\partial^2 B}{\partial r \partial s} + \frac{v_1^2}{2}\frac{\partial^2 B}{\partial s^2}\right)\bigg|_{s=v_0,x=\bar{x}} &- u_2(\bar{x})\frac{\partial B}{\partial s}\bigg|_{s=0,x=\bar{x}}^{s=v_0,x=\bar{x}}. \end{aligned} \tag{2.7.30}$$

All that remains is

$$Fu_{as} = -\varepsilon^3 \kappa \frac{\partial v_2}{\partial \xi} - \varepsilon^3 \zeta \frac{\partial}{\partial l}\left(\zeta \frac{\partial}{\partial l}(v_1 + \varepsilon v_2)\right) + O(\varepsilon^3), \tag{2.7.31}$$

and this is clearly $O(\varepsilon^3)$. □

2.7.2 Truncation Error Analysis

We present truncation error analysis for a one-dimensional Laplace operator on a uniform and a non-uniform mesh. These are combined in §2.5.3 in order to obtain results for (2.5.13).

Let $w(t)$ be some function. We consider

$$Mw := -q(t)(p(t)w'(t))', \tag{2.7.32}$$

and want to evaluate the truncation error $R_i := M^N w_i - (Mw)_i$. Assume there is a discontinuity in $w(t)$ at $t = 0$. The discrete analogue of (2.7.32) on a non uniform mesh is

$$M^N w_i^N := -q_i D\left(\tilde{p}_i D^- w_i^N\right) \quad \text{for } i = 1, ..., N-1, \tag{2.7.33a}$$

where

$$D^- w_i := (w_i - w_{i-1})/k_i, \quad D w_i := (w_{i+1} - w_i)/\hbar_i, \tag{2.7.33b}$$

$$\hbar_i := (k_i + k_{i+1})/2, \quad k_i := t_i - t_{i-1} \quad \text{and} \quad \tilde{p}_i := p_{i-\frac{1}{2}}. \tag{2.7.33c}$$

We have the following lemma for a uniform mesh.

Lemma 2.7.4. *Let $|w^{(j)}| \leqslant C$ and $|p^{(j)}| \leqslant C$ for $j = 0, ..., 6$, and $k_i = k$. The truncation error $R_i := M^N w_i - (Mw)_i$ of (2.5.15) with $t \neq 0$ is*

$$R_i = -\alpha_1 k^2 q_i \left((pw''')' + (pw')'''\right)_i + O(k^4), \tag{2.7.34}$$

with $\alpha_1 = \dfrac{1}{2^2 3!}$.

Proof. Define $v := pw'$ and $V_{i-\frac{1}{2}} := \tilde{p}_i D^- w_i$. We can write

$$R_i = q_i \left[v_i' - \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{k} - \frac{\left(V_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}\right) - \left(V_{i-\frac{1}{2}} - v_{i-\frac{1}{2}}\right)}{k} \right]. \tag{2.7.35}$$

Taking Taylor expansions of $v_{i+\frac{1}{2}}$ and $v_{i-\frac{1}{2}}$ about t_i yields

$$v_{i\pm\frac{1}{2}} = v_i \pm \frac{k}{2} v_i' + \frac{k^2}{2^2}\frac{1}{2} v_i'' \pm \frac{k^3}{2^3}\frac{1}{3!} v_i''' + \frac{k^4}{2^4}\frac{1}{4!} v_i^{(4)} + O(k^5). \tag{2.7.36}$$

Considering (2.7.35) we have

$$v_i' - \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{k} = -\alpha_1 k^2 v_i''' + O(k^4), \tag{2.7.37}$$

with $\alpha_1 := \dfrac{1}{2^2 3!}$.

Now writing $V_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}$ it in its original form we have

$$V_{i+\frac{1}{2}} - v_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} \left[\frac{w_{i+1} - w_i}{k} - w'\left(t_{i+\frac{1}{2}}\right) \right]. \tag{2.7.38}$$

A Taylor expansion on w_i and w_{i+1} about $t_{i+\frac{1}{2}}$ gives

$$\begin{aligned} w_{(i+\frac{1}{2})\pm\frac{1}{2}} =& w_{i+\frac{1}{2}} \pm \frac{k}{2} w'_{i+\frac{1}{2}} + \frac{k^2}{2^3} w''_{i+\frac{1}{2}} \pm \frac{k^3}{2^3}\frac{1}{3!} w'''_{i+\frac{1}{2}} \\ &+ \frac{k^4}{2^4}\frac{1}{4!} w^{(4)}_{i+\frac{1}{2}} \pm \frac{k^5}{2^5}\frac{1}{5!} w^{(5)}_{i+\frac{1}{2}} + O(k^6). \end{aligned} \tag{2.7.39}$$

Putting these into (2.7.38) and obtaining a similar result for the case with $t_{i-\frac{1}{2}}$ gives

$$V_{i\pm\frac{1}{2}} - v_{i\pm\frac{1}{2}} = \tilde{p}_{(i+\frac{1}{2})\pm\frac{1}{2}} \left[\alpha_1 k^2 w'''_{i\pm\frac{1}{2}} + \alpha_2 k^4 w^{(5)}_{i\pm\frac{1}{2}} \right] + O(k^5), \tag{2.7.40}$$

where $\alpha_2 := \dfrac{1}{2^4 5!}$. Recall the definition of $\tilde{p}$ in (2.5.15), i.e., $\tilde{p}_{(i+\frac{1}{2})\pm\frac{1}{2}} = p_{i\pm\frac{1}{2}}$, we now write

$$\frac{\left(V_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}\right) - \left(V_{i-\frac{1}{2}} - v_{i-\frac{1}{2}}\right)}{k} = \frac{k^2 \left(\alpha_1 p w''' + \alpha_2 k^2 p w^{(5)}\right)\Big|_{i-\frac{1}{2}}^{i+\frac{1}{2}}}{k} + O(k^4). \tag{2.7.41}$$

Simplifying this yields

$$\frac{\left(V_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}\right) - \left(V_{i-\frac{1}{2}} - v_{i-\frac{1}{2}}\right)}{k} = \alpha_1 k^2 \left((p w''')'_i + O(k^2)\right) + O(k^4), \tag{2.7.42}$$

Combining (2.7.37) and (2.7.42) in (2.7.35) and rewriting $v_i = (pw')_i$ gives the desired result. □

Now consider the truncation error for a non uniform mesh. This is used in (2.5.13) to obtain the truncation error of derivatives with respect to l.

Lemma 2.7.5. *Let $|w^{(j)}| \leqslant C$ and $|p^{(j)}| \leqslant C$ for $j = 0, ..., 3$. For a non uniform mesh the truncation error $R_i := M^N w_i - (Mw)_i$ of (2.5.15) is*

$$R_i := O(k) \tag{2.7.43}$$

where $k = \max k_i$.

Proof. We set up the problem in a similar way to the previous case. Define $v := pw'$ and $V_{i-\frac{1}{2}} := \tilde{p}_i D^- w_i$ and R_i becomes

$$R_i = q_i \left[v_i' - \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{\hbar_i} - \frac{(V_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}) - (V_{i-\frac{1}{2}} - v_{i-\frac{1}{2}})}{\hbar_i} \right]. \tag{2.7.44}$$

Taking a Taylor expansion of $v_{i+\frac{1}{2}}$ and $v_{i-\frac{1}{2}}$ about t_i gives

$$v_{i+\frac{1}{2}} = v_i + \frac{k_{i+1}}{2} v_i' + O(k^2), \quad v_{i-\frac{1}{2}} = v_i - \frac{k_i}{2} v_i' + O(k^2), \tag{2.7.45}$$

where $k = \max k_i$. Combining these gives

$$v_i' - \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{\hbar_i} = O(k). \tag{2.7.46}$$

Calculating $V_{i+\frac{1}{2}} - v_{i+\frac{1}{2}} = \tilde{p}_{i+1} \left[\frac{w_{i+1} - w_i}{k_{i+1}} - w'_{i+\frac{1}{2}} \right]$ we use (2.7.39) to $O(k^3)$ with k replaced by k_{i+1} and consider the case for $i - \frac{1}{2}$ similarly, resulting in

$$V_{i\pm\frac{1}{2}} - v_{i\pm\frac{1}{2}} = O(k^2). \tag{2.7.47}$$

Combining this information in (2.7.44) we get (2.7.43). □

Finally we consider the truncation error across the discontinuity at $t = 0$.

Lemma 2.7.6. *Let* $|w^{(j)}| \leqslant Ca$ *and* $|p^{(j)}| \leqslant C$ *for* $j = 0, ..., 3$ *and some positive constant* a*. For the case with discontinuous derivatives across* $t = 0$*, the truncation error of* (2.7.32) *on a uniform mesh is*

$$\begin{aligned} M^N w(0) - \frac{1}{2} \left\{ Mw(0^+) + Mw(0^-) \right\}_i &= \\ -q_i \frac{1}{k} \left\{ pw^{+\prime} - pw^{-\prime} \right\}_i &+ O(ka), \end{aligned} \tag{2.7.48}$$

where $w^{\pm} = w(0^{\pm})$.

Proof. As before let $v := pw'$ and define $R_i := M^N w(0) - \frac{1}{2}\left\{Mw(0^+) + Mw(0^-)\right\}_i$, i.e.,

$$R_i = q_i \left[\frac{1}{2}(v_i^{+\prime} + v_i^{-\prime}) - \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{k} - \frac{\left(V_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}\right) - \left(V_{i-\frac{1}{2}} - v_{i-\frac{1}{2}}\right)}{k} \right]. \tag{2.7.49}$$

Doing a Taylor expansion of $v_{i+\frac{1}{2}}$ and $v_{i-\frac{1}{2}}$ about the point i gives,

$$v_{i\pm\frac{1}{2}} = v_i^{\pm} \pm \frac{k}{2} v_i^{\pm\prime} + O(k^2 a), \tag{2.7.50}$$

where $v^{\pm} = v(0^{\pm})$. Putting this into (2.7.52) gives

$$\frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{k} = \frac{1}{k}(v_i^+ - v_i^-) + \frac{1}{2}(v_i^{+\prime} + v_i^{-\prime}) + O(ka). \tag{2.7.51}$$

Noting (2.7.40) we have $V_{i\pm\frac{1}{2}} - v_{i\pm\frac{1}{2}} = O(k^2 a)$. Combining this information results in

$$R_i = q_i \left[\frac{1}{2}(v_i^{+\prime} + v_i^{-\prime}) - \left\{ \frac{1}{k}(v_i^+ - v_i^-) + \frac{1}{2}(v_i^{+\prime} + v_i^{-\prime}) \right\} + O(ka) \right]. \tag{2.7.52}$$

Simplifying this and recalling $v_i := (pw')_i$ yields the desired result.

□

Chapter 3

Singularly Perturbed Nonlinear Time-Dependent Parabolic Problem with Singularly Perturbed Neumann Boundary Conditions

We consider the following nonlinear singularly perturbed time-dependent parabolic equation with singularly perturbed Neumann boundary conditions,

$$\mathcal{T}u := \varepsilon^2 \left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \right) + f(x,t,u) = 0, \tag{3.0.1a}$$

$$\text{for} \quad (x,t) \in \mathcal{D} := \{(x,t) \in [0,1] \times [0,T],\, T \in \mathbb{R}^+\}, \tag{3.0.1b}$$

$$\varepsilon \left.\frac{\partial u}{\partial x}\right|_{x=0} = g_0(t), \quad \varepsilon \left.\frac{\partial u}{\partial x}\right|_{x=1} = g_1(t) \quad \text{for } t \in [0,T] \tag{3.0.1c}$$

and

$$u(x,0) = \varphi(x) \quad \text{for } x \in [0,1], \tag{3.0.1d}$$

where $g_0(t)$, $g_1(t)$, $f(x,t,u)$ are sufficiently smooth and $0 < \varepsilon \leqslant \varepsilon_0 \ll 1$. We consider this system with a nonlinear function $f(x,t,u)$. As in *Chapter 2*, we have multiple solutions to the reduced problem, $f(x,t,u) = 0$. We again do not assume $f_u(x,t,u) > 0$ and instead make weaker local assumptions described in §3.1.

We enforce compatibility conditions at $x = 0$, $t = 0$ so that we obtain a sufficiently smooth solution and remove the existence of corner layer functions. As in the previous chapter we will obtain existence and accuracy results for (3.0.1) and present a numerical scheme for a solution to the problem. Calling on the theory of upper and lower solutions and the theory of Z-fields, accuracy results will be obtained for the computed solution.

We have the following proposition that there is a unique solution to the continuous problem.

Proposition 3.0.1. *Problem* (3.0.1) *has at most one solution.*

Proof. A similar proof can be found in [16, Proposition 2.1] for the Dirichlet problem and we make alterations to their argument for the case of Neumann boundary conditions similar to that done in [31, Proposition 3.1.1] for homogeneous boundary conditions.

Suppose there are two solutions to (3.0.1); $\tilde{u}(x,t)$ and $\bar{u}(x,t)$. Define $d(x,t) := \tilde{u}(x,t) - \bar{u}(x,t)$ where

$$\varepsilon^2 \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) d + f(x,t,\tilde{u}) - f(x,t,\bar{u}) = 0, \tag{3.0.2}$$

$$d(x,0) = 0, \quad \varepsilon \frac{\partial d}{\partial x}\bigg|_{x=0} = \varepsilon \frac{\partial d}{\partial x}\bigg|_{x=1} = 0. \tag{3.0.3}$$

We have $|\tilde{u}|$, $|\bar{u}| \leqslant K_1$ and hence,

$$f_u(x,t,u) \geqslant -K_2 \quad \text{for } x \in [0,1],\, t \in [0,T],\, u \in [-K_1, K_1], \tag{3.0.4}$$

for some positive constants K_1 and K_2. Also

$$f(x,t,\tilde{u}) - f(x,t,\bar{u}) = \int_{\bar{u}}^{\tilde{u}} f_u(x,t,\hat{s})d\hat{s}, \tag{3.0.5}$$

which using a change of variable, $\hat{s} = \bar{u} + s(\tilde{u} - \bar{u})$, becomes

$$f(x,t,\tilde{u}) - f(x,t,\bar{u}) = (\tilde{u} - \bar{u}) \int_0^1 f_u(x,t,\bar{u} + s(\tilde{u} - \bar{u}))ds. \tag{3.0.6}$$

and we define $p(x,t) := \int_0^1 f_u(x,t,\bar{u} + sd)ds$ and note $p(x,t) \geqslant -K_2$. Now using the transformation $z(x,t) := (\tilde{u} - \bar{u})e^{-K_2 t/\varepsilon^2}$ we get

$$\varepsilon^2 \left(\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} \right) + (K_2 + p(x,t))z = 0. \tag{3.0.7}$$

$$z(x,0) = 0, \quad \varepsilon \frac{\partial z}{\partial x}\bigg|_{x=0} = \varepsilon \frac{\partial z}{\partial x}\bigg|_{x=1} = 0. \tag{3.0.8}$$

By the maximum principle [28, Chapter 3], $z(x,t) = 0$ for all (x,t) as $K_2 + p(x,t) \geqslant 0$ and hence by the definition of $z(x,t)$ we have $\tilde{u}(x,t) = \bar{u}(x,t)$. □

3.1 Hypothesis for the Continuous Problem

We consider (3.0.1) under the following assumptions. Similar assumptions can be found in [2] and some resemble those of the two-dimensional interior layer problem in *Chapter 2*.

There exists $u_0(x,t)$, a sufficiently smooth solution of the reduced problem, i.e.,

$$f(x,t,u_0(x,t)) = 0, \tag{3.1.1}$$

such that

$$f_u(x,t,u_0(x,t)) > \gamma^2 > 0 \text{ for } (x,t) \in \bar{\mathcal{D}}, \tag{B1}$$

i.e., $u_0(x,t)$ is a stable solution to the reduced equation. As $f(x,t,u(x,t))$ is nonlinear $u_0(x,t)$ is not necessarily the unique solution to $f(x,t,u(x,t)) = 0$.

There exists a sufficiently smooth function $A(t)$ such that

$$\int_0^{A(t)} f(0,t,u_0(0,t)+s)ds = \frac{g_0^2(t)}{2}, \tag{B2}$$

and for the nonlinear function,

$$sf(0,t,u_0(0,t)+s) > 0 \quad \text{for } s \in (0, A(t)]'. \tag{B3}$$

This assumption can also be found in [2] in a different form.

Remark 3.1.1. We note that if (3.0.1) has the condition $f(x,t,u(x,t)) > 0$ for all $(x,t,u) \in [0,1] \times [0,T] \times \mathbb{R}$ then the problem becomes simpler as (B3) is automatically met. Conditions (B2) and (B3) are required for existence of boundary layer functions. The importance of these will be seen in Lemma 3.3.2 and a discussion on (B3) is included in Remark 3.3.1.

The initial condition is in the domain of attraction of the reduced solution, i.e.,

$$sf(x,0,u_0(x,0)+s) > 0 \quad \text{for } s \in (0, \varphi(x) - u_0(x,0)]', \tag{B4}$$

where the notation $(a,b]'$ is defined as $(a,b]$ when $a \leqslant b$ and $(b,a]$ when $b \leqslant a$. This assumption is found by considering (3.0.1a) near $t = 0$, i.e.,

$$\varepsilon^2 \frac{\partial u}{\partial t} + f(x,0,u(x,0)) \approx 0 \quad \text{for } t \approx 0. \tag{3.1.2}$$

Consider Figure 3.1 representing the domain of attraction, when $\varphi(x) < u_0(x,0)$ then we have $f(x,0,u(x,0)+s) < 0$ so that $\varphi(x)$ is in the domain of attraction of $u_0(x,0)$. As $\varphi(x) < u_0(x,0)$ then $s < 0$ and we have $sf(x,0,u_0(x,0)+s) > 0$. Alternatively when $\varphi(x) > u_0(x,0)$ then $f(x,0,u(x,0)+s) > 0$. As $\varphi(x) > u_0(x,0)$

then $s > 0$ and we again get $sf(x, 0, u_0(x, 0) + s) > 0$.

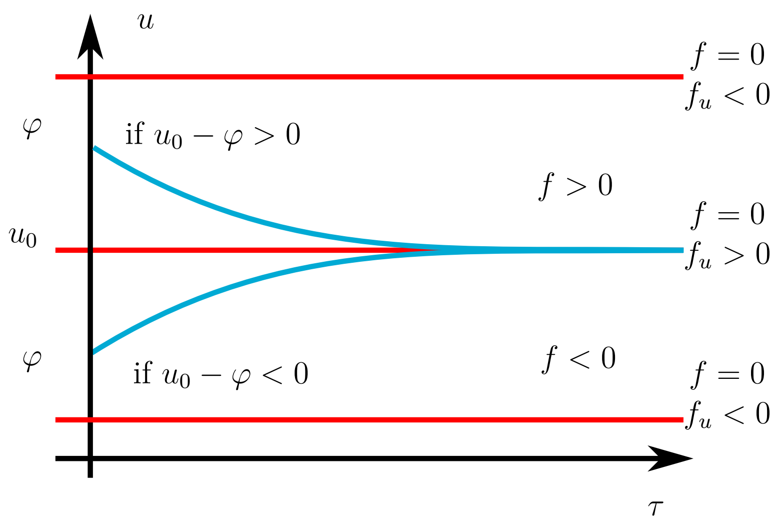


Figure 3.1: Domain of attraction of the stable solution $u_0(x, 0)$, including two unstable solutions $\bar{u}$ and $\tilde{u}$.

Under assumptions (B1)-(B4) we have a unique asymptotic solution.

Remark 3.1.2. The assumptions (B1)-(B4) are the same assumptions that appear in [2] however we present these in a different form that allows us to easily show their use in proving existence of boundary and initial layer functions in §3.3. For example see Lemma 3.3.2 for the use of (B2).

We also assume ε is small, i.e.,

$$\varepsilon \leqslant C(N^{-1} + M^{-1/2}), \tag{3.1.3}$$

where N and M are the number of space steps and time steps respectively. As in the previous chapter, the following analysis would be very different if ε were not small.

To avoid considering cases and without loss of generality we assume

$$u_0(x,0) \leqslant \varphi(x) \quad \text{for } x \in [0,1], \tag{3.1.4}$$

$$g_0(t) \leqslant 0 \quad \text{for } t \in [0,T], \tag{3.1.5}$$

and

$$\varepsilon \frac{\partial u_0}{\partial x}\bigg|_{x=1} = g_1(t) \quad \text{for } t \in [0,T]. \tag{3.1.6}$$

The final equation, (3.1.6), simplifies our presentation as there is no longer a boundary layer at $x = 1$.

3.1.1 Compatibility Condition

With an equation of type (3.0.1) there is the possibility of existence of corner layers. These corner layers can appear as the boundary condition and initial condition do not necessarily match in the corners. In this situation the corner is dealt with separately from the initial and boundary layer; both x and t are rescaled in this region, i.e., $x = \varepsilon\xi$ and $t = \varepsilon^2\tau$, and corner layer functions are considered, i.e., $q_0(\xi,\tau) + \varepsilon q_1(\xi,\tau) + O(\varepsilon^2)$. By doing this the rescaled equation remains as a parabolic partial differential equation and can be more difficult to solve. We note that these compatibility conditions are not required in [2] as time periodic solutions are considered with smooth data and hence there are no initial or corner layers.

We restrict our analysis to sufficiently smooth solutions and so introduce compatibility conditions such that the boundary and initial conditions match at the corners. At $x = 0$, $t = 0$ we require the following compatibility condition to be met; $\varepsilon \left.\dfrac{\partial \varphi}{\partial x}\right|_{x=l} = g_l(0)$ for $l = 0, 1$. Within our framework we

assume $g_l(t)$ and $\varphi(x)$ are independent of ε and hence we have

$$\left.\frac{\partial \varphi}{\partial x}\right|_{x=l} = g_l(0) = 0 \quad l = 0, 1. \tag{3.1.7}$$

In general we can have $\varphi(x, \varepsilon)$, $g_0(t, \varepsilon)$ and $f(x, t, u, \varepsilon)$ in which case (3.1.7) is not necessarily equal to zero, i.e., we can have $g_l(0) = O(\varepsilon)$ for $l = 0, 1$. We note $g_1(0) = 0$ is not required for our analysis but as we say $\varphi(x)$ and $g_l(t)$ are independent of ε we include this.

We make two further assumptions to avoid considering corner layers. Firstly we assume

$$f(l, 0, \varphi(l)) = 0 \quad l = 0, 1. \tag{3.1.8}$$

Condition (3.1.8) can be found in the Dirichlet case, i.e., [16], and is used here so that the zeroth order corner layer function is zero as described in §3.6.2. From (3.1.8) we get

$$u_0(l, 0) = \varphi(l) \quad l = 0, 1, \tag{3.1.9}$$

by the following method. Consider (B4) at $t = 0$ and, for example, $x = 0$,

$$sf(0, 0, u_0(0, 0) + s) > 0 \quad \text{ for } s \in (0, \varphi(0) - u_0(0, 0)]', \tag{3.1.10}$$

and by (3.1.4) we can say $s \geqslant 0$. Assuming $s > 0$ then,

$$f(0, 0, u_0(0, 0) + s) > 0 \quad \text{ for } s \in (0, \varphi(0) - u_0(0, 0)]', \tag{3.1.11}$$

and for a specific s i.e., $s_0 = \varphi(0) - u_0(0, 0)$, then,

$$f(0, 0, \varphi(0)) > 0. \tag{3.1.12}$$

This contradicts (3.1.8) and so $s \not> 0$, therefore $s = 0$, i.e., at $x = 0$, $t = 0$ we have $u_0(0, 0) = \varphi(0)$. A similar argument holds true for $x = 1$ giving (3.1.9).

The second assumption we make to avoid considering corner layers is

$$\left.\frac{\partial u_0}{\partial x}\right|_{x=0,t=0} = 0. \tag{3.1.13}$$

Due to (3.1.6) we do not need a version of (3.1.13) at $x = 1$.

Remark 3.1.3. For our analysis we require (3.1.13) to be met but in the case that u_0 is not readily available we can instead assume the stronger condition

$$f_x(0, 0, s) = 0, \tag{3.1.14}$$

for all s in the domain of attraction of the reduced solution. Taking the derivative of (3.1.1) we have

$$f_x(x, t, u_0(x, t)) + \frac{\partial u_0}{\partial x} f_u(x, t, u_0(x, t)) = 0. \tag{3.1.15}$$

Evaluating (3.1.15) at $x = 0$, $t = 0$ we can use (3.1.14) to see that (3.1.13) is satisfied.

Without (3.1.13) the $O(\varepsilon)$ corner layer function is not zero. Instead it is given by a homogeneous partial differential equation with constant coefficients. We refer the reader to §3.6.2 and specifically Remark 3.6.1 for a discussion of this equation.

Combining (3.1.7), (3.1.9) and (3.1.13) we have the condition

$$\left.\frac{\partial \varphi}{\partial x}\right|_{x=l} = g_l(0) = 0, \quad \varphi(l) - u_0(l, 0) = 0, \quad l = 0, 1, \tag{B5a}$$

$$\left.\frac{\partial u_0}{\partial x}\right|_{x=0,t=0} = 0. \tag{B5b}$$

In §3.6.2 we show that (B5) implies the corner layer functions $q_0(\xi, \tau)$ and $q_1(\xi, \tau)$ are zero.

3.1.2 Properties of the Auxiliary Nonlinear Function $F(x,t,u)$

We note that as in *Chapter 2* for the nonlinear function, $f(x,t,u)$, we define

$$F(x,t,s) := f(x,t,u_0+s), \tag{3.1.16}$$

and give a perturbed version, $\tilde{F}$, for p sufficiently small,

$$\tilde{F}(x,t,s;p) := f(x,t,u_0+s) - ps, \tag{3.1.17}$$

The following statements hold for $\tilde{F}(x,t,s)$:

$$\tilde{F}(x,t,0) = 0, \tag{3.1.18}$$

and

$$\left.\frac{\partial \tilde{F}}{\partial x}\right|_{s=0} = 0, \;\; \left.\frac{\partial^2 \tilde{F}}{\partial x^2}\right|_{s=0} = 0, \;\; \left.\frac{\partial \tilde{F}}{\partial t}\right|_{s=0} = 0, \tag{3.1.19}$$

and so

$$\left|\frac{\partial \tilde{F}}{\partial x}\right| \leqslant C|s|, \;\; \left|\frac{\partial^2 \tilde{F}}{\partial x^2}\right| \leqslant C|s|, \;\; \left|\frac{\partial \tilde{F}}{\partial t}\right| \leqslant C|s|. \tag{3.1.20}$$

Recalling the notation $v|_{a;b}^{a+b} = v(a+b) - v(a) - v(b)$ in §1.2.1, for the function $\tilde{F}(x,t,u)$ with $\tilde{F}(x,t,0) = F(x,t,0) = 0$, we have

$$\tilde{F}(x,t,\cdot)|_{a;b}^{a+b} = ab\tilde{F}_{ss}((a+b)\theta) = O(|ab|), \tag{3.1.21}$$

for some $\theta \in (0,1)$.

3.2 Existence and Accuracy of Discrete Solutions; Main Results

We present the main results of the chapter, that is existence and accuracy of a discrete solution to problem (3.0.1) of the standard finite difference discretisation given by (3.4.1) using the conventional method and the stabilised scheme given by (3.4.5).

Theorem 3.2.1. *For N sufficiently large, ε sufficiently small and the meshes defined in §3.4.2 and §3.4.2 there exists solutions U_{ij} and $\widehat{U}_{ij}$ to* (3.4.1) *and* (3.4.5)*. Furthermore for the bilinear interpolants of U_{ij} and $\widehat{U}_{ij}$, U^I_{ij} and $\widehat{U}^I_{ij}$, we have*

$$|U^I_{ij}(x,t) - u(x,t)| \leqslant CN^{-2}\ln^{2m} N + CM^{-1}\ln^m M, \tag{3.2.1}$$

$$|\hat{U}^I_{ij}(x,t) - u(x,t)| \leqslant CN^{-2}\ln^{2m} N + CM^{-1}\ln^m M, \tag{3.2.2}$$

for all $(x,t) \in [0,1]\times[0,T]$ where $u(x,t)$ is the unique solution of (3.0.1) *and $m = 0$ for the Bakhvalov mesh and $m = 1$ for the Shishkin mesh.*

3.3 Asymptotic Analysis

So far we know a solution to (3.0.1) lies close to the reduced solution in the majority of the domain. Near $x = 0$ there is a boundary layer and near $t = 0$ there is an initial layer. Away from these layers the asymptotic expansion of $u(x,t)$ is close to $u_0(x,t)$, which is the solution to the reduced problem, $f(x,t,u_0(x,t)) = 0$. For accuracy of the system away from $x = 0$ and $t = 0$, there is the following lemma.

Lemma 3.3.1. *For the function $u_0(x,t)$ of* (3.1.1),

$$\mathcal{T} u_0(x,t) = O(\varepsilon^2) \quad \text{for } (x,t) \in (0,1] \times (0,T]. \tag{3.3.1}$$

Proof. This proof is included in Lemma 3.6.1. □

The asymptotic expansion of $u(x,t)$ is expected to be of the form

$$u_{as}(x,t) := u_0(x,t) + v_0(\xi,t) + \varepsilon v_1(\xi,t) + w_0(x,\tau), \tag{3.3.2}$$

where

$$\xi := \frac{x}{\varepsilon}, \quad \tau := \frac{t}{\varepsilon^2}, \tag{3.3.3}$$

$v_0(\xi,t)$ and $v_1(\xi,t)$ are boundary layer functions and $w_0(x,\tau)$ is an initial layer function.

In general $u_{as}(x,t)$ also includes corner layer functions such as

$$u_{as}(x,t) := u_0(x,t)+v_0(\xi,t)+\varepsilon v_1(\xi,t)+w_0(x,\tau)+q_0(\xi,\tau)+\varepsilon q_1(\xi,\tau). \tag{3.3.4}$$

A discussion of the corner layer functions $q_0(\xi,\tau)$ and $q_1(\xi,\tau)$, solutions near $x = 0$, $t = 0$, is given in §3.6.2 where they are found to be zero by (B5).

3.3.1 The Boundary Layer: Solution Near $x = 0$

The boundary layer functions are defined in the following equations obtained during the proof of Lemma 3.3.5 and described in §3.6.1. For x close to 0 rescaling the system (3.0.1) and using $\xi := \dfrac{x}{\varepsilon}$ gives

$$-\frac{\partial^2 v_0}{\partial \xi^2} + F(0,t,v_0) = 0, \tag{3.3.5a}$$

$$\left.\frac{\partial v_0}{\partial \xi}\right|_{\xi=0} = g_0(t), \quad v_0(\infty,t) = 0. \tag{3.3.5b}$$

For the perturbed boundary layer function, $\tilde{v}_0(\xi, t; p)$, we have

$$-\frac{\partial^2 \tilde{v}_0}{\partial \xi^2} + F(0, t, \tilde{v}_0) - p\tilde{v}_0 = 0, \tag{3.3.6a}$$

$$\left.\frac{\partial \tilde{v}_0}{\partial \xi}\right|_{\xi=0} = g_0(t), \quad \tilde{v}_0(\infty, t; p) = 0. \tag{3.3.6b}$$

Note $\tilde{v}_0(\xi, t; 0) = v_0(\xi, t)$.

Defining the operator

$$\mathcal{L}_\xi[v(\cdot)] := -\frac{\partial^2 v}{\partial \xi^2} + v\left.\frac{\partial F}{\partial s}\right|_{s=v_0, x=0}, \tag{3.3.7}$$

the function $v_1(\xi, t)$ is given by

$$\mathcal{L}_\xi[v_1] = -\xi\left.\frac{\partial F}{\partial x}\right|_{s=v_0, x=0}, \tag{3.3.8a}$$

$$\left.\frac{\partial v_1}{\partial \xi}\right|_{\xi=0} = -\left.\frac{\partial u_0}{\partial x}\right|_{x=0}, \quad v_1(\infty, t) = 0. \tag{3.3.8b}$$

Let

$$\gamma_L^2 := \min_{t\in[0,T]} f_u(0, t, u_0(0, t)) > \gamma^2, \tag{3.3.9}$$

$p_0 \in (0, \gamma_L^2)$ and

$$\hat{\chi}(\xi, t) := \begin{cases} \dfrac{1}{g_0(t)}\dfrac{\partial v_0}{\partial \xi}, & v_0(0, t) > 0, \\ e^{-\gamma_L \xi}, & v_0(0, t) = 0. \end{cases} \tag{3.3.10}$$

This is well defined as when $\left.\dfrac{\partial v_0}{\partial \xi}\right|_{\xi=0} = 0$ we have $v_0(0, t) = 0$.

We now consider the auxiliary problem;

$$\frac{\partial^2 \nu}{\partial \xi^2} = \phi(\nu), \quad \nu'(0) = \hat{g} \leqslant 0, \quad \nu(\infty) = 0. \tag{3.3.11}$$

For this problem, define

$$\hat{\chi}(\xi) := \begin{cases} \hat{g}^{-1}\nu'(\xi), & \nu(0) > 0, \\ e^{-\sqrt{\phi'(0)}\xi}, & \nu(0) = 0. \end{cases} \tag{3.3.12}$$

For $\nu(0) > 0$ and $\nu(0) = 0$ the derivative of (3.3.11) with respect to ξ exists, and $\hat{\chi}(\xi)$ satisfies

$$\hat{\chi}''(\xi) = \phi'(\nu)\hat{\chi}(\xi), \tag{3.3.13a}$$

$$\hat{\chi}(0) = 1, \quad \hat{\chi}(\infty) = 0. \tag{3.3.13b}$$

Again we note that when $\nu'(0) = 0$ then $\nu(0) = 0$, i.e., (3.3.12) is well defined.

Lemma 3.3.2. *(i) Suppose $\phi(s)$ is a sufficiently smooth function satisfying*

$$\phi(0) = 0, \quad \phi'(0) \geqslant \bar{\gamma}^2 > 0, \tag{3.3.14}$$

and

$$\exists \bar{A} : \int_0^{\bar{A}} \phi(s)ds = \frac{\hat{g}^2}{2} \quad \textit{and} \quad s\phi(s) > 0 \quad \forall s \in \left(0, \bar{A}\right]'. \tag{3.3.15}$$

Then there exists a solution $\nu(\xi)$ to (3.3.11) with $0 \leqslant \nu(\xi) \leqslant \bar{A}$, and for any arbitrarily small but fixed $\delta \in (0, \phi'(0))$ there is a positive constant $\bar{C}_\delta$ such that

$$|\nu^{(k)}(\xi)| \leqslant \bar{C}_\delta e^{-(\bar{\gamma}-\delta)\xi} \quad \textit{for } k = 0, ..., 4, \tag{3.3.16}$$

and $\chi(\xi) := \nu'(\xi)$ satisfying

$$C'\nu(\xi) \leqslant |\chi(\xi)| \leqslant C''\nu(\xi). \tag{3.3.17}$$

(ii) There exists a solution $\bar{\nu}(\xi)$ to the second auxiliary problem

$$-\frac{\partial^2 \bar{\nu}}{\partial \xi^2} + \bar{\nu}\phi'(\nu) = \Psi(\xi), \quad \bar{\nu}'(0) = \bar{g}, \quad \bar{\nu}(\infty) = 0, \tag{3.3.18}$$

where $\nu(\xi)$ is the solution from part (i) and $|\Psi(\xi)| \leqslant C^*(1+\xi^m)\hat{\chi}(\xi)$ for some $m > 0$ and $\hat{\chi}(\xi)$ defined in (3.3.12)*. The solution $\bar{\nu}(\xi)$ satisfies*

$$|\bar{\nu}(\xi)| \leqslant C\hat{\chi}(\xi)\{|\bar{g}| + C^*(1+\xi^{m+1})\}. \tag{3.3.19}$$

Furthermore if $\bar{g} \leqslant 0$ and $\Psi(\xi) \geqslant 0$ then $\bar{\nu}(\xi) \geqslant 0$ for all $\xi \geqslant 0$.

Proof. (i)
To prove existence of $\nu(\xi)$ and to obtain the bound (3.3.16) we consider [6, Lemma 2.1] and [17, Lemma 2.1] which give results for the case of Dirichlet boundary conditions. Here we make alterations for the case of Neumann boundary conditions. From (3.3.11) we have the system

$$\nu'(\xi) = V, \tag{3.3.20a}$$

$$V'(\xi) = \phi(\nu), \tag{3.3.20b}$$

with boundary conditions

$$\nu'(0) = V(0) = \hat{g}, \quad \nu(\infty) = V(\infty) = 0. \tag{3.3.20c}$$

As $\phi(0) = 0$ the fixed points of the system include $(0, 0)$. We note that as $\phi(\nu)$ may have more than one solution there is the possibility of other fixed points to the system. The Jacobian for the system is

$$\begin{pmatrix} 0 & 1 \\ \phi'(0) & 0 \end{pmatrix}. \tag{3.3.21}$$

From this the eigenvalues are $\pm\sqrt{\phi'(0)}$ which are real and of opposite sign, as $\phi'(0) > 0$, hence the fixed point $(0,0)$ is a saddle point.

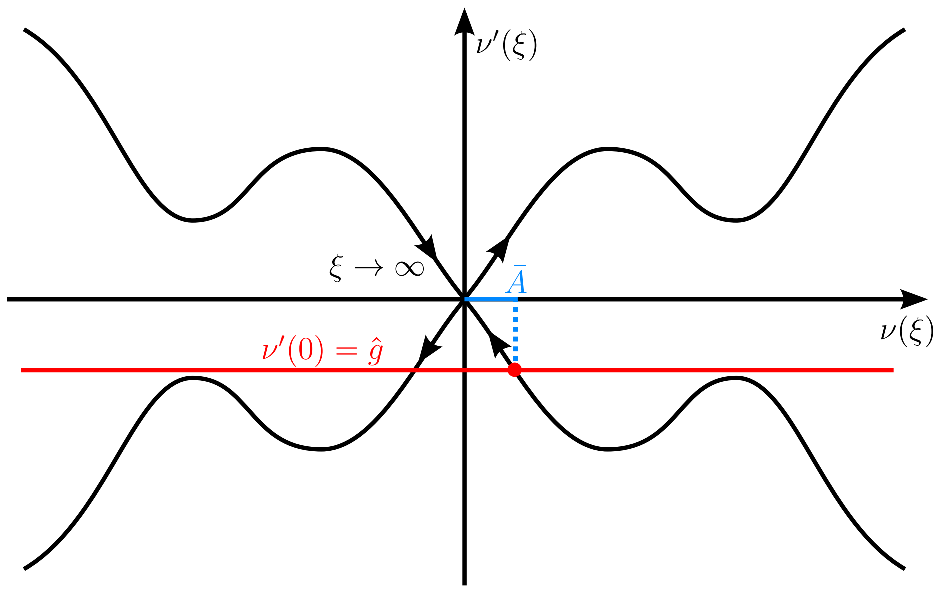


Figure 3.2: Phase plane of (3.3.20) *including the Neumann boundary condition* $\nu'(0) = V(0) = \hat{g}$.

Figure 3.2 shows the phase plane of (3.3.20). We are looking for a trajectory of this system that intersects the line $\nu'(0) = V(0) = \hat{g}$ and goes to 0 as $\xi \to \infty$. The upper half plane can be ruled out as the boundary condition is negative because of (3.1.5). For $\nu(\xi) < 0$, $\nu'(\xi) < 0$ the trajectory goes to infinity so the required trajectory can only lie in the fourth quadrant.

Solving the system $V'(\xi) = \phi(\nu)$, gives

$$V(\xi) = \pm\sqrt{2\int_0^{\nu} \phi(s)ds + C}. \tag{3.3.22}$$

Calculating (3.3.22) at $\xi = \infty$ and recalling $\nu(\infty) = 0$ and $V(\infty) = 0$ gives

$C = 0$. Also, since $V(\xi) \leqslant 0$ then

$$V(\xi) = -\sqrt{2\int_0^{\nu} \phi(s)ds}. \tag{3.3.23}$$

We now require the trajectory $V(\xi)$ to exist from $\xi = 0$ to $\xi = \infty$. In other words (3.3.23) must exist for all $v \in (0, \bar{A}]'$ where $\bar{A}$ is defined in (3.3.15) and $(\bar{A}, \hat{g})$ is the point of intersection of $\nu'(0) = \hat{g}$ and the trajectory we are looking for. Recalling (B3) the integral in (3.3.23) is positive as when $\phi(s) > 0$ then $\nu > 0$ and when $\phi(s) < 0$ then $\nu < 0$ and $V(\xi)$ exists from $\xi = 0$ to $\xi = \infty$. Therefore there exists a separatrix that intersects the line $\nu'(0) = \hat{g}$ and goes to 0, i.e., there exists a solution $\nu(\xi)$ to (3.3.20).

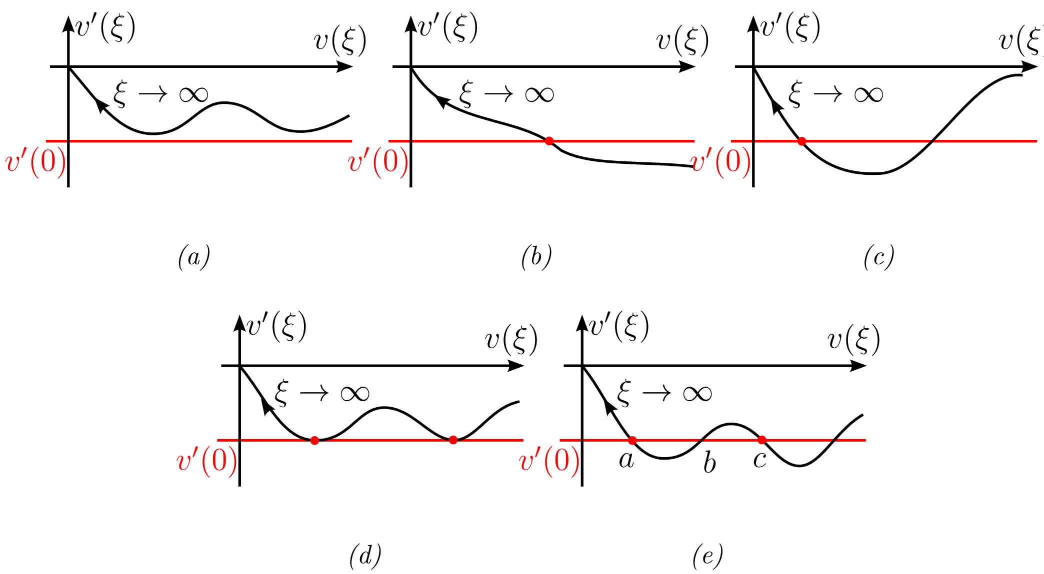


Figure 3.3: Possible phase planes for $(\nu(\xi), \nu'(\xi))$.

Figure 3.3 gives other possible phase planes for the system (3.3.11) not described by Figure 3.2. Considering Figure 3.3a, (3.3.15) is not met and so there is no solution to (3.3.11). Figure 3.3d also does not meet (B3) and gives no solution. Figure 3.3b, Figure 3.3c and Figure 3.3e are phase

planes that satisfy our conditions and give solutions. We note a phase plane like that described by Figure 3.3b or Figure 3.3c will have a unique solution while Figure 3.3e has multiple solutions. In the case of Figure 3.3e we refer the reader to Remark 3.3.3 to get uniqueness of the time-dependent original problem.

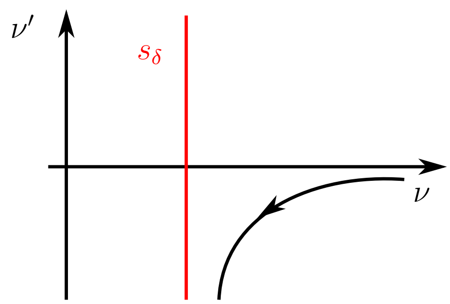


Figure 3.4: The phase plane $(\nu(\xi), \nu'(\xi))$ with the condition $\nu(\xi) > s_\delta$.

Next recalling (3.3.14) for any $\delta \in (0, \phi'(0))$ there exists $s_0 > 0$ such that

$$(\bar{\gamma} - \delta)^2 s \leqslant \phi(s) \quad \text{for } 0 \leqslant s \leqslant s_0. \tag{3.3.24}$$

If $\nu(\xi) > s_0$ for all ξ then $V(\xi) \leqslant -C$ and so $\nu(\xi) \to -\infty$ as $\xi \to \infty$ which contradicts (3.3.20c) and is represented in Figure 3.4. Therefore there exists $\xi_0 > 0$ such that $\nu(\xi_0) \leqslant s_0$. As $\nu(\xi)$ is a decreasing function, for all $\xi \leqslant \xi_0$ we have $\nu(\xi) \leqslant s_0$. Next fix ξ_0 with $0 \leqslant \nu(\xi_0) \leqslant s_0$. By (3.3.23) and (3.3.24) we have, for $\nu(\xi) \in (0, s_0]$,

$$V \leqslant -\sqrt{2\int_0^\nu (\bar{\gamma} - \delta)^2 s ds}, \tag{3.3.25}$$

which can be rewritten as

$$\frac{1}{\nu}\partial\nu \leqslant -(\bar{\gamma} - \delta)\partial\xi \quad \text{for } \xi_0 \leqslant \xi \leqslant \infty. \tag{3.3.26}$$

Integrating (3.3.26) from ξ_0 to ξ gives

$$\nu(\xi) \leqslant \nu(\xi_0)e^{(\bar{\gamma}-\delta)\xi_0}e^{-(\bar{\gamma}-\delta)\xi} \quad \text{for } \xi_0 \leqslant \xi \leqslant \infty. \tag{3.3.27}$$

Choosing $\bar{C}_\delta \geqslant \nu(\xi_0)e^{(\bar{\gamma}-\delta)\xi_0}$ gives

$$\nu(\xi) \leqslant \bar{C}_\delta e^{-(\bar{\gamma}-\delta)\xi} \quad \text{for } \xi_0 \leqslant \xi \leqslant \infty. \tag{3.3.28}$$

For $0 \leqslant \xi \leqslant \xi_0$ we recall $\nu(\xi)$ is a decreasing function and so $\nu(\xi_0) \leqslant \nu(\xi) \leqslant \nu(0)$. Therefore,

$$\nu(\xi) \leqslant \bar{C}_\delta e^{-(\bar{\gamma}-\delta)\xi_0} \tag{3.3.29}$$

with $\bar{C}_\delta := \nu(0)e^{(\bar{\gamma}-\delta)\xi_0}$. Hence as $\xi \leqslant \xi_0$ we have

$$\nu(\xi) \leqslant \bar{C}_\delta e^{-(\bar{\gamma}-\delta)\xi} \quad \text{for } 0 \leqslant \xi \leqslant \xi_0. \tag{3.3.30}$$

Therefore, we have (3.3.16) for $k = 0$ with

$$\bar{C}_\delta := \max\left\{\nu(\xi_0), \nu(0)\right\} e^{(\bar{\gamma}-\delta)\xi_0}. \tag{3.3.31}$$

To obtain a bound for $\nu'(\xi)$ we rewrite (3.3.25) as

$$\nu'(\xi) = -\sqrt{2\int_0^\nu s\phi'(0)ds} = -\nu(\xi)\sqrt{\phi'(0)}, \tag{3.3.32}$$

and using the bound obtained for $\nu(\xi)$ we obtain (3.3.16) for $k = 1$. Taking the absolute value of (3.3.32) and noting that we can find two positive constants C' and C'' such that $C' \leqslant \sqrt{\phi'(0)} \leqslant C''$ we obtain (3.3.17).

For $\nu''(\xi)$ we use (3.3.11) to get

$$|\nu''(\xi)| \leqslant \phi'(\nu(\xi)\theta)\nu(\xi), \tag{3.3.33}$$

for some $\theta \in (0,1)$. As $\nu(\xi)$ is bounded by (3.3.16) with $k=0$ and $\phi(s)$ is sufficiently smooth we get (3.3.16) with $k=2$. For $\nu'''(\xi)$ and $\nu^{(4)}(\xi)$ we take derivatives of (3.3.11), i.e.,

$$\nu'''(\xi) = \nu'(\xi)\phi'(\nu), \quad \nu^{(4)}(\xi) = (\nu'(\xi))^2\phi''(\nu) + \nu''(\xi)\phi'(\nu). \tag{3.3.34}$$

As $\phi(s)$ is sufficiently smooth and calling on (3.3.16) with $k=1,2$ we have (3.3.16) for $k=3,4$.

(ii)

Next we consider the second auxiliary function $\bar{\nu}(\xi)$ described by (3.3.18). Multiplying (3.3.18) by $\hat{\chi}(\xi)$, gives

$$-\hat{\chi}(\xi)\frac{\partial^2\bar{\nu}}{\partial\xi^2} + \bar{\nu}(\xi)\hat{\chi}(\xi)\phi'(\nu) = \hat{\chi}(\xi)\Psi(\xi), \tag{3.3.35}$$

which, using (3.3.13), we can now write (3.3.35) as

$$-\frac{\partial}{\partial\xi}\left(\hat{\chi}^2(\xi)\frac{\partial}{\partial\xi}\left(\frac{\bar{\nu}(\xi)}{\hat{\chi}(\xi)}\right)\right) = \hat{\chi}(\xi)\Psi(\xi). \tag{3.3.36}$$

Taking the integral from ξ to ∞ of (3.3.36) results in

$$\hat{\chi}^2(\xi)\frac{\partial}{\partial\xi}\left(\frac{\bar{\nu}(\xi)}{\hat{\chi}(\xi)}\right) - \hat{\chi}^2(\infty)\left.\frac{\partial}{\partial\xi}\left(\frac{\bar{\nu}(\xi)}{\hat{\chi}(\xi)}\right)\right|_{\xi=\infty} = \int_{\xi}^{\infty}\hat{\chi}(\tilde{\eta})\Psi(\tilde{\eta})d\tilde{\eta}. \tag{3.3.37}$$

The second term in (3.3.37) can be simplified to $\lim_{\tilde{\eta}\to\infty}\left(\bar{\nu}'(\tilde{\eta})\hat{\chi}(\tilde{\eta}) - \bar{\nu}(\tilde{\eta})\hat{\chi}'(\tilde{\eta})\right)$. We note that as $\bar{\nu}(\infty) = 0$ then $\bar{\nu}'(\infty) = 0$. Refer to Remark 3.3.2 for an explanation. As $\hat{\chi}'(\xi)$ is bounded by (3.3.16) and $\hat{\chi}(\infty) = 0$ we have $\lim_{\tilde{\eta}\to\infty}\left(\bar{\nu}'(\tilde{\eta})\hat{\chi}(\tilde{\eta}) - \bar{\nu}(\tilde{\eta})\hat{\chi}'(\tilde{\eta})\right) = 0$ and so the second term is zero.

Next dividing (3.3.37) by $\hat{\chi}^2(\xi)$, integrating from 0 to ξ and rearranging

terms yields

$$\bar{\nu}(\xi) = \hat{\chi}(\xi) \int_0^{\xi} \hat{\chi}^{-2} \int_{\eta}^{\infty} \hat{\chi}(\tilde{\eta})\Psi(\tilde{\eta})d\tilde{\eta}d\eta + C\hat{\chi}(\xi). \tag{3.3.38}$$

Exploiting the Neumann boundary condition, we differentiate (3.3.38) with respect to ξ, recalling the boundary condition (3.3.13b), and we find the constant of integration to be

$$C = \frac{\bar{g}}{\hat{\chi}'(0)} - \frac{1}{\hat{\chi}'(0)} \int_0^{\infty} \hat{\chi}(\tilde{\eta})\Psi(\tilde{\eta})d\tilde{\eta} - \lim_{a \to 0^+} \int_0^{a} \int_0^{\infty} \hat{\chi}(\tilde{\eta})\Psi(\tilde{\eta})d\tilde{\eta}d\eta. \tag{3.3.39}$$

As $\hat{\chi}(\xi)$ and $\Psi(\xi)$ are sufficiently smooth the final term in (3.3.39) is zero. Putting (3.3.39) into (3.3.38), the solution $\bar{\nu}(\xi)$ is now

$$\bar{\nu}(\xi) = \hat{\chi} \int_0^{\xi} \hat{\chi}^{-2}(\eta) \int_{\eta}^{\infty} \hat{\chi}(\tilde{\eta})\Psi(\tilde{\eta})d\tilde{\eta}d\eta + \hat{\chi}(\xi) \left[\frac{\bar{g}}{\hat{\chi}'(0)} - \frac{1}{\hat{\chi}'(0)} \int_0^{\infty} \hat{\chi}(\tilde{\eta})\Psi(\tilde{\eta})d\tilde{\eta} \right]. \tag{3.3.40}$$

Recalling (B3) we have $\nu(0) \neq 0$ implies $\hat{\chi}'(0) \neq 0$ and by the definition of $\hat{\chi}(\xi)$ for $\nu(0) = 0$ we have $\hat{\chi}'(0) = \sqrt{\phi'(0)} \geqslant \bar{\gamma} > 0$. Therefore $\bar{\nu}(\xi)$ is well defined and there exists a solution $\bar{\nu}(\xi)$ to (3.3.18).

We pause to consider the term $\int_{\eta}^{\infty} \hat{\chi}(\tilde{\eta})\Psi(\tilde{\eta})d\tilde{\eta}$ in (3.3.40), firstly considering $\nu(0) > 0$. By (3.3.17), there exists $\bar{C}'' > 0$ sufficiently large such that $\hat{\chi} \leqslant \bar{C}''\nu$ as $|\hat{g}| \leqslant C$. Multiplying both sides by $\hat{\chi}$, recalling (3.3.12) and noting $\hat{g}^{-1} \leqslant -C$, we can again make $\bar{C}''$ sufficiently large such that

$$\hat{\chi}^2 d\tilde{\eta} \leqslant -\bar{C}''\nu d\nu, \tag{3.3.41}$$

is also satisfied. Equation (3.3.41) can be rewritten as $\hat{\chi}^2 d\tilde{\eta} \leqslant -(\bar{C}''/2)d\nu^2$. Multiplying $|\Psi(\xi)| \leqslant C^*(1 + \xi^m)|\hat{\chi}(\xi)|$ by $\hat{\chi}(\tilde{\eta})d\tilde{\eta}$ and using the above inequalities gives $|\hat{\chi}(\tilde{\eta})\Psi(\tilde{\eta})d\tilde{\eta}| \leqslant -(\bar{C}''C^*/2)(1 + \tilde{\eta}^m)d\nu^2$. Taking the integral

of both sides gives

$$\left|\int_{\eta}^{\infty} \Psi(\tilde{\eta})\hat{\chi}(\tilde{\eta})d\tilde{\eta}\right| \leqslant -\frac{\bar{C}''C^{*}}{2}\int_{\eta}^{\infty}(1+\tilde{\eta}^{m})d\nu^{2}, \tag{3.3.42}$$

and integrating by parts yields

$$\begin{aligned}\left|\int_{\eta}^{\infty} \Psi(\tilde{\eta})\chi(\tilde{\eta})d\tilde{\eta}\right| \leqslant & \frac{\bar{C}''C^{*}}{2}\nu^{2}(\eta)(1+\eta^{m}) - \frac{\bar{C}''C^{*}}{2}\lim_{\tilde{\eta}\to\infty}\nu^{2}(\tilde{\eta})(1+\tilde{\eta}^{m}) \\ & + \frac{m\bar{C}''C^{*}}{2}\int_{\eta}^{\infty}\tilde{\eta}^{m-1}\nu^{2}(\tilde{\eta})d\tilde{\eta}.\end{aligned} \tag{3.3.43}$$

We integrating by parts a further $m-1$ times, each time noting $\nu^{2}(\tilde{\eta})d\tilde{\eta} \leqslant -Cd\nu^{2}$, and get

$$\left|\int_{\eta}^{\infty} \Psi(\tilde{\eta})\chi(\tilde{\eta})d\tilde{\eta}\right| \leqslant CC^{*}\nu^{2}(\eta)(1+\eta^{m}). \tag{3.3.44}$$

Recalling (3.3.16), the second and third term in (3.3.44) are bounded and negative and $C\nu \leqslant \hat{\chi}(\xi)$ for $\nu(0) > 0$ giving

$$\left|\int_{\eta}^{\infty} \Psi(\tilde{\eta})\hat{\chi}(\tilde{\eta})d\tilde{\eta}\right| \leqslant CC^{*}\hat{\chi}^{2}(\eta)(1+\eta^{m}). \tag{3.3.45}$$

Secondly we consider the case with $\nu(0) = 0$, we again look to calculate $\int_{\eta}^{\infty} \Psi(\tilde{\eta})\hat{\chi}(\tilde{\eta})d\tilde{\eta}$ now with $\hat{\chi}(\xi) = e^{-\sqrt{\phi'(0)}\xi}$, and have

$$\left|\int_{\eta}^{\infty} \Psi(\tilde{\eta})\hat{\chi}(\tilde{\eta})d\tilde{\eta}\right| \leqslant \int_{\eta}^{\infty} C^{*}(1+\tilde{\eta}^{m})e^{-2\sqrt{\phi'(0)}\tilde{\eta}}d\tilde{\eta}. \tag{3.3.46}$$

Calculating the integral gives

$$\begin{aligned}\left|\int_{\eta}^{\infty}\Psi(\tilde{\eta})\hat{\chi}(\tilde{\eta})d\tilde{\eta}\right| \leqslant &-\lim_{\tilde{\eta}\to\infty} 2C^*\sqrt{\phi'(0)}(1+\tilde{\eta}^m)e^{-2\sqrt{\phi'(0)}\tilde{\eta}} \\ &+ 2C^*\sqrt{\phi'(0)}(1+\eta^m)e^{-2\sqrt{\phi'(0)}\eta} \\ &- mC^*\int_{\tilde{\eta}}^{\infty}\tilde{\eta}^{m-1}e^{-2\sqrt{\phi'(0)}\tilde{\eta}}d\tilde{\eta}.\end{aligned} \tag{3.3.47}$$

As the first term and the final term are negative we obtain (3.3.45) for $\hat{\chi}(\xi)$ with $\nu(0) = 0$.

Putting (3.3.45) into (3.3.40) yields

$$|\bar{\nu}(\xi)| \leqslant C\hat{\chi}(\xi)\int_0^{\xi} C^*(1+\eta^m)d\eta + \left|\frac{\bar{g}}{\hat{\chi}'(0)}\hat{\chi}(\xi)\right|. \tag{3.3.48}$$

Calculating the integral and rearranging terms gives

$$|\bar{\nu}(\xi)| \leqslant C\hat{\chi}(\xi)\{C^*(1+\xi^{m+1}) + |\bar{g}|\}, \tag{3.3.49}$$

i.e., we have the desired bound (3.3.19) for $\bar{\nu}(\xi)$.

Finally we consider (3.3.40) with $\Psi(\xi) \geqslant 0$ and $\bar{g} \leqslant 0$. The first term of (3.3.40) is positive. By (B3) we have that $\phi(\nu) > 0$, that is $\nu''(0) > 0$, as $\nu(0) > 0$ so the final term is positive. Again by (B3), $\dfrac{\hat{\chi}(\xi)}{\hat{\chi}'(0)} < 0$ and so if $\bar{g} \leqslant 0$ then $\dfrac{\bar{g}\hat{\chi}(\xi)}{\hat{\chi}'(0)} \geqslant 0$ for all ξ. Hence we have $\bar{\nu}(\xi) \geqslant 0$ for all ξ if the conditions $\Psi(\xi) \geqslant 0$ and $\bar{g} \leqslant 0$ are met. □

Remark 3.3.1. Considering (3.3.40) we see that without the condition $\nu''(0) \neq 0$ for $\nu(0) \neq 0$, i.e., that $\hat{\chi}'(0) \neq 0$, from (B3) the solution $\bar{\nu}(\xi)$ will not exist as it involves $1/\hat{\chi}'(0)$. Furthermore without $\nu''(0) > 0 (< 0)$ for $\nu(0) > 0 (< 0)$ in (B3) we will not have the result $\bar{\nu}(\xi) \geqslant 0$ for $\bar{g} \leqslant 0$ and $\Psi(\xi) \geqslant 0$ as we require $\hat{\chi}'(0) < 0$ and so will not be able to obtain $\dfrac{\partial \tilde{v}_0}{\partial p} \geqslant 0$ which is used to give ordered upper and lower solutions in §3.3.3.

Remark 3.3.2. We want to show $\bar{\nu}(\infty) = 0$ implies $\bar{\nu}'(\infty) = 0$. Considering (3.3.18) we have, for $\xi > \bar{\xi}$ with $\bar{\xi}$ sufficiently large,

$$\phi'(\nu) > \bar{\gamma}^2 + \nu\phi''(\nu\theta) \geqslant 0, \tag{3.3.50}$$

for some $\theta \in (0,1)$ and so we can use the maximum principle, as the second term is sufficiently small, to find

$$|\bar{\nu}(\xi)| \leqslant Ce^{-\bar{\gamma}\xi} \quad \forall\xi. \tag{3.3.51}$$

Next calculating $\bar{\nu}'(\xi)$ from (3.3.18) we have

$$\bar{\nu}'(\xi) = \int_{\xi}^{\infty} \phi'(\nu)\bar{\nu}(\eta) - \Psi(\eta)d\eta. \tag{3.3.52}$$

As $|\Psi(\xi)| \leqslant (1+\xi^m)|\hat{\chi}(\xi)|$ using (3.3.16), $\phi'(\nu)$ is sufficiently smooth and by (3.3.51) then we can obtain

$$|\bar{\nu}'(\xi)| \leqslant Ce^{-\bar{\gamma}\xi}, \tag{3.3.53}$$

which gives the result $\bar{\nu}'(\infty) = 0$.

Remark 3.3.3. As for $t = 0$ we have $A(0) = 0$ and we are considering this problem with continuous time the intersection of the trajectory and the Neumann boundary condition will not be able to switch from the point close to zero to a different intersection point. A switch of this kind would mean (B3) is broken in the transition. From this we can rule out points b and c in Figure 3.3(e) and we have a unique solution to (3.3.5).

We now apply Lemma 3.3.2 to problems $\tilde{v}_0(\xi, t; p)$, $v_0(\xi, t)$ and $v_1(\xi, t)$.

Lemma 3.3.3. *There exists functions $v_0(\xi,t)$, $\tilde{v}_0(\xi,t;p)$ and $v_1(\xi,t)$ which satisfy* (3.3.5), (3.3.6) *and* (3.3.8) *with the properties*

$$0 \leqslant \tilde{v}_0(\xi,t;p) \leqslant A(t), \quad \frac{\partial \tilde{v}_0}{\partial \xi} \leqslant 0, \quad \tilde{v}_0(\xi,t;p) + \varepsilon|v_1(\xi,t)| \leqslant Ct, \quad \frac{\partial \tilde{v}_0}{\partial p} \geqslant 0, \tag{3.3.54}$$

for $\xi, t \geqslant 0$. Furthermore for $\delta \in (0, \gamma_L - \sqrt{p_0})$, there exists a positive constant $\bar{C}_\delta$ such that

$$\left|\frac{\partial^k \tilde{v}_0}{\partial \xi^k}\right| + \left|\frac{\partial^k v_1}{\partial \xi^k}\right| + \left|\frac{\partial^l \tilde{v}_0}{\partial t^l}\right| + \left|\frac{\partial^l v_1}{\partial t^l}\right| + \left|\frac{\partial \tilde{v}_0}{\partial p}\right| \leqslant \bar{C}_\delta e^{-\left(\gamma_L - \sqrt{p_0} - \delta\right)\xi}, \tag{3.3.55}$$

for $k = 0, ..., 4$ and $l = 0, 1, 2$.

Proof. We apply Lemma 3.3.2 to problems (3.3.5), (3.3.6) and (3.3.8).

We consider $\tilde{v}_0(\xi,t;p)$ noting $\tilde{v}_0(\xi,t;0) = v_0(\xi,t)$. To consider $\tilde{v}_0(\xi,t;p)$ we first note (3.3.6) is of the form of (3.3.11). From (3.3.6) we have $\phi(s) = F(0,t,s) - ps$ where $\phi(0) = 0$ and $\phi'(s) = F_s(0,t,\tilde{v}_0) - p$. We require (B1) and (B2) to be satisfied for the perturbed problem, i.e., we need to show $\tilde{F}_s(0,t,0) > 0$ and

$$\int_0^v \tilde{F}(0,t,s)ds > 0 \quad v \in (0, \tilde{v}_0(0,t)]'. \tag{3.3.56}$$

Firstly recalling (3.1.17), then $\tilde{F}_s(0,t,0) = F_s(0,t,0) - ps \geqslant \gamma_L^2 - p_0 > 0$ as $p_0 \in (0.\gamma_L^2)$.

To show (3.3.56) is met we extend the argument in [17, Lemma 2.2] to two dimensions. We also note the two-dimensional analysis can be found in [31, Lemma 2.4.5]. We first consider $s \in (0, s_0)$ for some $s_0 \leqslant \tilde{v}_0(0,t)$. Taking a Taylor expansion of $\tilde{F}(0,t,s)$ and noting $\tilde{F}(0,t,0) = 0$, C_6 can be found such that $|F_{ss}(0,t,\hat{s})| \leqslant C_6$ and so

$$F(0,t,s) - ps \geqslant s[F_s(0,t,0) - p_0] - C_6 s^2. \tag{3.3.57}$$

Suppose $p_0 \leqslant \min_{t\in[0,T]} F_s(0,t,0)/2$ hence for $|p| \leqslant p_0$ we have

$$F(0,t,s) - ps \geqslant \frac{sF_s(0,t,0)}{2} - C_6 s^2. \tag{3.3.58}$$

Choosing $s_0 := F_s(0,t,0)/(2C_6)$ and as $0 < s < s_0$ we have $F(0,t,s) - ps > 0$ and so (3.3.56) holds for $s \in (0, s_0)$.

Next we assume $s_0 < \tilde{v}_0(0,t)$ and consider $s \in [s_0, \tilde{v}_0(0,t)]$. Define

$$m := \min_{t\in[0,T]} \left(\min_{\tilde{v}_0(\xi,t)\in[s_0,\tilde{v}_0(0,t)]} \int_0^{\tilde{v}_0} F(0,t,s)ds \right). \tag{3.3.59}$$

We find $m > 0$ by (B2). We choose $p_0 = \min_{t\in[0,T]} \min \left\{ \frac{m}{\tilde{v}_0^2(0,t)}, \frac{F_s(0,t,0)}{2} \right\}$, as $s \in [s_0, \tilde{v}_0(\xi,t)]$ then m is bounded away from 0 and so p_0 is bounded away from 0. We can calculate

$$\int_0^{\tilde{v}_0} psds = \frac{p\tilde{v}_0^2(\xi,t)}{2}. \tag{3.3.60}$$

For $|p| \leqslant p_0$, (3.3.60) becomes

$$\int_0^{\tilde{v}_0} psds \leqslant \frac{m}{2} < \int_0^{\tilde{v}_0} F(0,t,s)ds, \tag{3.3.61}$$

given the definition of m in (3.3.59) and we have (3.3.56) for $s \in [s_0, \tilde{v}_0(0,t)]$. Hence there exists $p_0 \in (0, F_s(0,t,0))$ such that (3.3.56) is met and the above phase plane analysis holds for $\tilde{v}_0(\xi,t;p)$.

We can now apply Lemma 3.3.2 to (3.3.6) and find there exists a solution $\tilde{v}_0(\xi,t;p)$ such that $0 \leqslant \tilde{v}_0(\xi,t;p) \leqslant A(t)$ and $\chi(\xi,t) \leqslant 0$. We note $0 \leqslant p_0 \leqslant \gamma_L^2$ gives the inequality $\sqrt{\gamma_L^2 - p_0} \geqslant \gamma_L - \sqrt{p_0}$ we have the bounds for $\frac{\partial^k \tilde{v}_0}{\partial \xi^k}$ with $k = 0, ..., 4$ in (3.3.55).

We consider the equation for $\frac{\partial \tilde{v}_0}{\partial p}$, taking the derivative with respect to p

of (3.3.6) gives

$$-\frac{\partial^2}{\partial\xi^2}\frac{\partial\tilde{v}_0}{\partial p}+\frac{\partial\tilde{v}_0}{\partial p}\left(F_s(0,t,\tilde{v}_0)-p\right)=\tilde{v}_0,\qquad \frac{\partial}{\partial\xi}\frac{\partial\tilde{v}_0}{\partial p}\bigg|_{\xi=0}=0,\qquad \frac{\partial\tilde{v}_0}{\partial p}\bigg|_{\xi=\infty}=0. \tag{3.3.62}$$

We can see (3.3.62) is of the form (3.3.18) hence $\frac{\partial\tilde{v}_0}{\partial p}$ exists and $\left|\frac{\partial\tilde{v}_0}{\partial p}\right|\leqslant C(1+\xi^m)|\hat{\chi}|$. As we have established a bound for $\hat{\chi}(\xi,t)$ we can now obtain (3.3.55) for $\frac{\partial\tilde{v}_0}{\partial p}$. Next we have $\Psi(\xi,t)=\tilde{v}_0(\xi,t)\geqslant 0$ and $\frac{\partial}{\partial\xi}\frac{\partial\tilde{v}_0}{\partial p}\Big|_{\xi=0}=0$ so using Lemma 3.3.2(ii) we have $\frac{\partial\tilde{v}_0}{\partial p}\geqslant 0$ for all $\xi,t\geqslant 0$.

For bounds for $v_1(\xi,t)$ we use (3.3.8) noting it is of the form (3.3.18) with its right hand side satisfying $|\Psi(\xi,t)|\leqslant C(1+\xi^m)|\hat{\chi}(\xi,t)|$ and $\phi(s)=F(0,t,s)$ satisfying (3.3.14). Hence by Lemma 3.3.2(ii) the solution $v_1(\xi,t)$ exists and satisfies $|v_1(\xi,t)|\leqslant C(1+\xi^k)|\chi(\xi,t)|$. Using the bound for $\hat{\chi}(\xi,t)$ we again obtain the bounds for $v_1(\xi,t)$ with respect to ξ in (3.3.55).

For the remaining bounds we differentiate equations (3.3.5a), (3.3.6a) l times with respect to t and (3.3.8a) k times with respect to ξ and l times with respect to t. The results are equations of type (3.3.18) with $\phi(s)$ satisfying (3.3.14) and $|\Psi(\xi,t)|\leqslant C(1+\xi^m)|\hat{\chi}(\xi,t)|$ hence we obtain the necessary bounds. These equations are included in §3.6.3 for the reader.

To show $|v_0(\xi,t)|\leqslant Ct$ we recall condition (B5) and see that it implies $\frac{\partial\tilde{v}_0}{\partial\xi}\Big|_{\xi=0,t=0}=0$. Now considering (3.3.6) we can see for $t=0$, $\tilde{v}_0(\xi,0;p)=0$ is a solution. Hence we can take a Taylor expansion about $t=0$ and as $\frac{\partial\tilde{v}_0}{\partial t}$ is bounded by (3.3.55) we have $|\tilde{v}_0(\xi,t;p)|\leqslant Ct$. Finally to show $|v_1(\xi,t)|\leqslant Ct$ we consider (3.3.8) at $t=0$. By (B5b) we have that the boundary condition is zero and as we have shown $v_0(\xi,0)=0$ the right hand side of (3.3.8a) is also zero. Therefore $v_1(\xi,0)=0$ is a solution and as we have the necessary bounds we can write $|v_1(\xi,t)|\leqslant Ct$ giving (3.3.54). $\square$

Lemma 3.3.4. *For* $\tilde{v}_0(\xi, t; p)$ *and* $v_0(\xi, t)$ *defined by* (3.3.6) *and* (3.3.5) *we have*

$$\varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right](\tilde{v}_0 - v_0) = -\, F(\varepsilon\xi, t, \cdot)\big|_{v_0+\varepsilon v_1}^{\tilde{v}_0+\varepsilon v_1} + p v_0 + O(\varepsilon^2 + p^2). \quad (3.3.63)$$

Proof. To obtain (3.3.63) we begin with

$$\varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right](\tilde{v}_0 - v_0) = -\frac{\partial^2}{\partial \xi^2}(\tilde{v}_0 - v_0) + O(\varepsilon^2). \quad (3.3.64)$$

Using (3.3.5a) and (3.3.6a) yields

$$-\frac{\partial^2}{\partial \xi^2}(\tilde{v}_0 - v_0) = -\, F(0, t, \cdot)\big|_{v_0}^{\tilde{v}_0} + p\tilde{v}_0. \quad (3.3.65)$$

Now as

$$\begin{aligned} F(\varepsilon\xi, t, \cdot)\big|_{v_0+\varepsilon v_1}^{\tilde{v}_0+\varepsilon v_1} &= F(0, t, \cdot)\big|_{v_0}^{\tilde{v}_0} + \varepsilon v_1\, F_s(0, t, \cdot)\big|_{v_0+\varepsilon\theta v_1}^{\tilde{v}_0+\varepsilon\theta v_1} \\ &\quad + \varepsilon\xi\, F_x(\theta x, t, \cdot)\big|_{v_0+\varepsilon\theta v_1}^{\tilde{v}_0+\varepsilon\theta v_1}, \end{aligned} \quad (3.3.66)$$

for some $\theta \in (0, 1)$, then

$$\begin{aligned} -\frac{\partial^2}{\partial \xi^2}(\tilde{v}_0 - v_0) &= -\, F(\varepsilon\xi, t, \cdot)\big|_{v_0+\varepsilon v_1}^{\tilde{v}_0+\varepsilon v_1} + \varepsilon v_1\, F_s(0, t, \cdot)\big|_{v_0+\varepsilon\theta v_1}^{\tilde{v}_0+\varepsilon\theta v_1} \\ &\quad + \varepsilon\xi\, F_x(\theta x, t, \cdot)\big|_{v_0+\varepsilon\theta v_1}^{\tilde{v}_0+\varepsilon\theta v_1} + p\tilde{v}_0. \end{aligned} \quad (3.3.67)$$

As $F_s(0, t, \cdot)\big|_{v_0+\varepsilon\theta v_1}^{\tilde{v}_0+\varepsilon\theta v_1} = (\tilde{v}_0 - v_0)F_{ss}(0, t, \theta v_1) + O(\varepsilon)$, $F_x(\theta x, t, \cdot)\big|_{v_0+\varepsilon\theta v_1}^{\tilde{v}_0+\varepsilon\theta v_1} = (\tilde{v}_0 - v_0)F_{sx}(0, t, \theta v_1) + O(\varepsilon)$ and $(1 + \xi)|\tilde{v}_0 - v_0| = O(p)$ we have

$$-\frac{\partial^2}{\partial \xi^2}(\tilde{v}_0 - v_0) = -\, F(\varepsilon\xi, t, \cdot)\big|_{v_0+\varepsilon v_1}^{\tilde{v}_0+\varepsilon v_1} + p\tilde{v}_0 + O(\varepsilon p). \quad (3.3.68)$$

Using (3.3.64) and as $O(\varepsilon p) = O(p^2 + \varepsilon^2)$ we obtain (3.3.63). □

Lemma 3.3.5. *For the asymptotic expansion* $u_0(x,t) + v_0(\xi,t) + \varepsilon v_1(\xi,t)$ *of problem* (3.0.1) *near* $x = 0$ *we have accuracy*

$$\mathcal{T}[u_0(x,t) + v_0(\xi,t) + \varepsilon v_1(\xi,t)] = O(\varepsilon^2). \tag{3.3.69}$$

Proof. This proof is included in Lemma 3.6.2 of the technical section. □

3.3.2 The Initial Layer: Solution Near $t = 0$

The equation for the initial layer function $w_0(x,\tau)$ is obtained during the proof of Lemma 3.6.3. For t close to 0 we rescale (3.0.1) using $\tau := t/\varepsilon^2$. The equation for $w_0(x,\tau)$ is

$$\frac{\partial w_0}{\partial \tau} + F(x,0,w_0) = 0, \tag{3.3.70a}$$

$$w_0(x,0) = \varphi(x) - u_0(x,0) \quad \text{and} \quad w_0(x,\infty) = 0. \tag{3.3.70b}$$

The perturbed version of $w_0(x,\tau)$, $\tilde{w}_0(x,\tau;p)$ where $\tilde{w}_0(x,\tau;0) = w_0(x,\tau)$, is given by

$$\frac{\partial \tilde{w}_0}{\partial \tau} + F(x,0,\tilde{w}_0) - p\tilde{w}_0 = 0, \tag{3.3.71a}$$

$$\tilde{w}_0(x,0;p) = \varphi(x) - u_0(x,0), \tag{3.3.71b}$$

and

$$\tilde{w}_0(x,\infty;p) = 0. \tag{3.3.71c}$$

Let

$$\gamma_T^2 := \min_{x\in[0,1]} f_u(x,0,u_0(x,0)) > \gamma^2, \tag{3.3.72}$$

and $p_0 \in (0,\gamma_T^2)$.

We now consider the auxiliary initial value problem

$$\frac{\partial \omega}{\partial \tau} = -\phi(\omega) \quad \text{for } \tau > 0, \quad \omega(0) = \omega_0 \geqslant 0, \quad \omega(\infty) = 0. \tag{3.3.73}$$

For this problem we have

$$\hat{\omega}(\tau) := \begin{cases} \omega/\omega_0, & \omega_0 > 0, \\ e^{-\phi'(0)\tau}, & \omega_0 = 0. \end{cases} \tag{3.3.74}$$

Lemma 3.3.6. *(i) Let a sufficiently smooth function $\phi(s)$ satisfy*

$$\phi(0) = 0, \quad \phi'(0) > 0, \quad \phi(s) > 0 \quad \forall s \in (0, \omega_0]. \tag{3.3.75}$$

Then there exists a solution to (3.3.73) *with* $0 \leqslant \omega(\tau) \leqslant \omega_0$*, and for arbitrarily small but fixed* $\delta \in (0, \phi'(0))$ *there is a constant* $\bar{C}_\delta > 0$ *such that*

$$|\omega| + |\omega'| + |\omega''| \leqslant \omega_0 \bar{C}_\delta e^{-(\phi'(0)-\delta)\tau} \quad \textit{for } \tau \geqslant 0. \tag{3.3.76}$$

(ii) Let $\hat{\omega}$ be defined by (3.3.74)*. For the second auxiliary problem*

$$\frac{\partial \bar{\omega}}{\partial \tau} + \bar{\omega}\phi'(\omega) = \Psi(\tau), \quad \bar{\omega}(0) = \bar{\omega}_0, \quad \bar{\omega}(\infty) = 0, \tag{3.3.77}$$

where $|\Psi(\tau)| \leqslant C^*(1 + \tau^m)\hat{\omega}(\tau)$*,* $\omega(\tau)$ *is the solution from part (i) and* C^* *is a sufficiently large positive constant, there exists a solution* $\bar{\omega}(\tau)$ *such that*

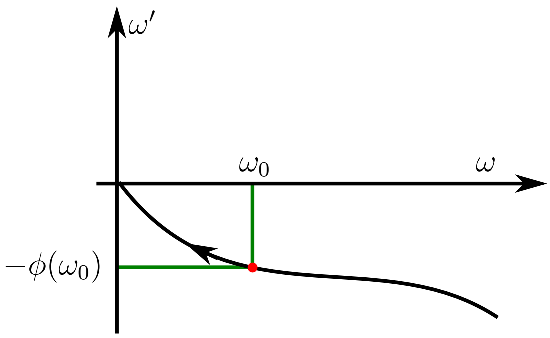
$$|\bar{\omega}(\tau)| \leqslant C(\bar{\omega}_0 + C^*(1 + \tau^{m+1})\hat{\omega}(\tau)). \tag{3.3.78}$$

Also if $\bar{\omega}_0 \geqslant 0$ *and* $\Psi(\tau) \geqslant 0$ *then* $\bar{\omega}(\tau) \geqslant 0$ *for all* $\tau \geqslant 0$*.*

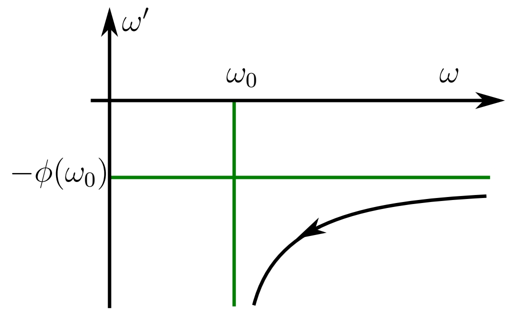
Proof. The proof of this lemma can be found in [16, Lemma 4.2]. For completeness we include the proof here.

Proof of (i)

If $\omega_0 = 0$ then $\omega(\tau) = 0$ and so the above statements of (i) follow.



(a) The phase plane for (3.3.73).



(b) The phase plane for (3.3.73) *with the condition that there exists* $\tau_\delta > 0$ *such that* $\omega(\tau) > s_\delta$ *for all* τ.

Figure 3.5: Phase planes for (3.3.73).

For $\omega_0 > 0$ we carry out phase plane analysis of $\omega' = -\phi(\omega)$. For the existence of a solution there must exist a trajectory leaving $\omega(\tau) = \omega_0$ at $\tau = 0$ and entering $\omega(\tau) = 0$ as $\tau \to \infty$. We consider the (ω, ω') plane, this is represented in Figure 3.5a. Using (3.3.75) we have $\omega' = -\phi(\omega_0) < 0$ and so the trajectory is in the fourth quadrant. From this we see $0 \leqslant \omega(\tau) \leqslant \omega_0$ and $\omega(\tau)$ is a decreasing function. As $\tau \to \infty$,

$$(\omega_0, -\phi(\omega_0)) \to (\omega(\infty), -\phi(\omega(\infty))) = (0,0). \tag{3.3.79}$$

Therefore there exists a trajectory leaving $\omega(0) = \omega_0$ and entering $\omega(\tau) = 0$ as $\tau \to \infty$ and hence the solution $\omega(\tau)$ exists.

There are now two cases to consider. Firstly that there exists $\tau_\delta > 0$ such that $\omega(\tau) > s_\delta$ for all τ, which implies $\omega'(\tau) \leqslant -C$ for all τ. The trajectory is bounded away from zero and must remain $\omega(\tau) > s_\delta > 0$ and $\omega'(\tau) \leqslant -C$ and so $\omega(\infty) = -\infty$, i.e., there is a contradiction. This is represented in Figure 3.5b.

The second case is there exists $\tau_\delta > 0$ such that $\omega(\tau_\delta) = s_\delta$. In this case

we can say for any $\delta \in (0, \phi'(0))$ there exists $s_\delta \in (0, \omega_0)$ such that

$$\left|\frac{\phi(s)}{s} - \phi'(0)\right| \leqslant \delta \quad \forall s \in [0, s_\delta], \tag{3.3.80}$$

i.e.,

$$[\phi'(0) - \delta]s \leqslant \phi(s) \leqslant [\phi'(0) + \delta]s \quad \forall s \in [0, s_\delta]. \tag{3.3.81}$$

Recalling (3.3.73) we now have, for all $\tau \geqslant \tau_\delta$,

$$[\phi'(0) - \delta]\omega(\tau) \leqslant -\omega'(\tau) \leqslant [\phi'(0) + \delta]\omega(\tau) \tag{3.3.82}$$

Solving (3.3.82) yields

$$e^{-(\phi'(0)+\delta)\tau} \leqslant \omega(\tau)/\omega(\tau_\delta) \leqslant e^{-(\phi'(0)-\delta)\tau} \quad \text{ for } \tau \geqslant \tau_\delta. \tag{3.3.83}$$

As $\omega(\tau)$ is positive and decreasing we have $\omega(\tau_\delta) < \omega_0$, hence

$$\omega(\tau) \leqslant \omega_0 e^{-(\phi'(0)-\delta)\tau}, \tag{3.3.84}$$

i.e., we have (3.3.76) for $\omega(\tau)$. As $-[\phi'(0) + \delta]\omega \leqslant [\phi'(0) - \delta]\omega$ we can write

$$|\omega'| \leqslant [\phi'(0) + \delta]\omega \leqslant [\phi'(0) + \delta]\omega_0 e^{-(\phi'(0)-\delta)\tau}. \tag{3.3.85}$$

For some positive constant $\bar{C}_\delta$, we now have the bound for $\omega'(\tau)$ in (3.3.76). Taking the derivative of (3.3.77) with respect to τ we have $\omega'' = -\omega'\phi'(\omega)$ yielding (3.3.76) for $\omega''(\tau)$ as $\phi(\omega)$ is sufficiently smooth.

Proof of (ii)

For the second auxiliary equation we first look at the homogeneous normalised version of (3.3.77);

$$\frac{\partial \theta}{\partial \tau} + \theta(\tau)\phi'(\omega) = 0, \quad \theta(0) = 1, \quad \theta(\infty) = 0. \tag{3.3.86}$$

This has positive solution, $\theta(\tau) \geqslant 0$ such that $\theta(0) = 1$.

For $\omega_0 > 0$, we have $\omega' < 0$, and $[\phi'(0) - \delta]\omega \leqslant -\omega' \leqslant [\phi'(0) + \delta]\omega$ which implies $C^{-1} \leqslant |\omega'|/\omega \leqslant C$. We choose $\theta := \omega'/\omega'(0) > 0$, such that

$$C^{-1} \leqslant \theta/\hat{\omega} \leqslant C. \tag{3.3.87}$$

Now to get the unique solution, multiply (3.3.77) by $\theta(\tau)$ and as $\theta'(\tau) = -\theta(\tau)\phi'(\omega)$, we can write

$$\bar{\omega}'(\tau)\theta(\tau) - \theta'(\tau)\bar{\omega}(\tau) = \theta(\tau)\Psi(\tau). \tag{3.3.88}$$

Using the quotient rule, (3.3.88) transforms into

$$\frac{\partial}{\partial \tau}\left(\frac{\bar{\omega}(\tau)}{\theta(\tau)}\right) = \frac{\Psi(\tau)}{\theta(\tau)}. \tag{3.3.89}$$

Integrating both sides and dividing by $\theta(\tau)$ gives

$$\bar{\omega}(\tau) = \bar{\omega}_0\theta(\tau) + \theta(\tau)\int_0^\tau \frac{\Psi(\tilde{\tau})}{\theta(\tilde{\tau})}d\tilde{\tau}. \tag{3.3.90}$$

From (3.3.87) we have $\hat{\omega} \leqslant C\theta$, and as $|\Psi(\tau)| \leqslant C^*(1 + \tau^m)\hat{\omega}(\tau)$ we have $|\Psi(\tau)| \leqslant CC^*(1 + \tau^m)\theta(\tau)$ and can write

$$|\bar{\omega}(\tau)| \leqslant |\bar{\omega}_0|\theta(\tau) + C\theta(\tau)\int_0^\tau C^*(1 + \tilde{\tau}^m)d\tilde{\tau}, \tag{3.3.91}$$

and again by (3.3.87) we get (3.3.78).

Next for $\omega_0 = 0$ we have $\omega(\tau) = 0$ and $\phi'(s) = \phi'(0) > 0$ so we choose $\theta(\tau) := e^{-\phi'(0)\tau} = \hat{\omega}(\tau)$. Here we solved $\theta'(\tau) + \theta(\tau)\phi'(0) = 0$ and as $\phi'(0) > 0$ and $\theta(0) = 1$ the solution is $\theta(\tau) = e^{-\phi'(0)\tau}$. The above argument holds and we again obtain (3.3.78).

Finally if $\bar{\omega}_0 \geqslant 0$, and $\Psi(\tau) \geqslant 0$ then as $\theta(\tau) > 0$ this gives $\bar{\omega}(\tau) \geqslant 0$ by

(3.3.90). ☐

Lemma 3.3.7. *There exists $p_0 \in (0, \gamma_T^2)$ such that for all $|p| \leqslant p_0$ problem (3.3.70) has a solution $\tilde{w}_0(x,\tau;p)$ with*

$$0 \leqslant \tilde{w}_0(x,\tau;p) \leqslant \varphi(x) - u_0(x,0), \quad \frac{\partial \tilde{w}_0}{\partial p} \geqslant 0, \quad \left|\frac{\partial \tilde{w}_0}{\partial x}\right| \leqslant Cx, \tag{3.3.92}$$

for $\tau \geqslant 0$ and $x \in [0,1]$.

Furthermore, for $\delta \in (0, \gamma_T^2 - p_0)$, there exists a positive constant $\bar{C}_\delta$ such that

$$\left|\frac{\partial^l \tilde{w}_0}{\partial \tau^l}\right| + \left|\frac{\partial^k \tilde{w}_0}{\partial x^k}\right| + \left|\frac{\partial \tilde{w}_0}{\partial p}\right| \leqslant \bar{C}_\delta e^{-(\gamma_T^2 - p_0 - \delta)\tau}, \tag{3.3.93}$$

for $k = 0, ..., 4$, $l = 0, 1, 2$, $\tau \geqslant 0$ and $x \in [0,1]$.

Proof. We consider $\tilde{w}_0(x,\tau;p)$ noting $\tilde{w}_0(x,\tau;0) = w_0(x,\tau)$. To obtain similar results for $\tilde{w}_0(x,\tau;p)$ we first need a version of (B4) for the perturbed problem, i.e., we require $s\tilde{F}(x,0,s) = s(F(x,0,s) - ps) > 0$ for $s \in (0, \varphi(x) - u_0(x,0)]'$. By (3.1.17),

$$s(F(x,0,s) - ps) = s(F(x,0,0) + sF_s(x,0,0) + s^2 F_{ss}(x,0,s\theta) - ps), \tag{3.3.94}$$

for some $\theta \in (0,1)$. Choosing a sufficiently large positive constant C_7 and recalling (3.3.72) gives

$$s(F(x,0,s) - ps) \geqslant s^2(\gamma_T^2 - C_7 s - p_0). \tag{3.3.95}$$

We now consider two cases; $0 < s < \dfrac{\gamma_T^2}{2C_7}$ and $s \geqslant \dfrac{\gamma_T^2}{2C_7}$. In the first case, (3.3.95) becomes

$$s(F(x,0,s) - ps) > s^2\left(\frac{\gamma_T^2}{2} - p_0\right). \tag{3.3.96}$$

By choosing $p_0 \leqslant \dfrac{\gamma_T^2}{2}$ we get $s(F(x,0,s) - ps) > 0$ for $0 < s < \dfrac{\gamma_T^2}{2C_7}$.

In the second case, $s_{\max} := \varphi(x) - u_0(x,0) > \dfrac{\gamma_T^2}{2C_7}$ and so

$$ps \leqslant p_0(\varphi(x) - u_0(x,0)), \tag{3.3.97}$$

and by choosing $p_0 = \min_{\gamma_T^2/(2C_7) \leqslant s \leqslant s_{\max}, x \in [0,1]} f(x,0,u_0(x,0)+s)/s$ we have $s\tilde{F}(x,0,s) > 0$ for $s \geqslant \dfrac{\gamma_T^2}{2C_7}$. Combining these results in

$$p_0 := \min\left\{ \min_{\gamma_T^2/(2C_7) \leqslant s \leqslant s_{\max}, x \in [0,1]} \frac{f(x,0,u_0(x,0)+s)}{s}, \frac{\gamma_T^2}{2C_7} \right\}, \tag{3.3.98}$$

and thus

$$ps \leqslant \min_{x \in [0,1]} f(x,0,u_0(x,0)+s) < f(x,0,u_0(x,0)+s). \tag{3.3.99}$$

This is the desired result for all s, i.e., we have obtained a perturbed version of (B4).

For $\tilde{w}_0(x,\tau;p)$ we have (3.3.71) with $\phi(s) = \tilde{F}(x,0,s)$ which can be written as $\phi(s) = F(x,0,s) - ps$ and by (B3) with p_0 sufficiently small then $\phi(s) > 0$. We have $\phi'(s) = F_s(x,0,s) - p$ and so $\phi'(0) = F_s(x,0,0) - p \geqslant \gamma_T^2 - p_0 > 0$, i.e., $\phi'(0) > 0$ is satisfied. We also note $\phi(0) = \tilde{F}(x,0,0) = 0$. Hence the requirements of Lemma 3.3.6 are met and the solution $\tilde{w}_0(x,\tau;p)$ exists and as $\tilde{w}_0(x,0;p) = \varphi(x) - u_0(x,0) \geqslant 0$,

$$0 \leqslant \tilde{w}_0(x,\tau;p) \leqslant \varphi(x) - u_0(x,0). \tag{3.3.100}$$

We note $\varphi(x) - u_0(x,0) \leqslant C$ as $\varphi(x)$ and $u_0(x,0)$ are sufficiently smooth, and get, by (3.3.76),

$$\left| \frac{\partial^l \tilde{w}_0}{\partial \tau^l} \right| \leqslant \bar{C}_\delta e^{-(\gamma_T^2 - p_0 - \delta)\tau}, \tag{3.3.101}$$

for $l = 0, 1, 2$, $\tau \geqslant 0$ and $x \in [0, 1]$.

For $\dfrac{\partial \tilde{w}_0}{\partial p}$ we can take the derivative with respect to p of (3.3.71) giving

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \tilde{w}_0}{\partial p}\right) + \frac{\partial \tilde{w}_0}{\partial p}\left(F_s(x, 0, \tilde{w}_0) - p\right) = \tilde{w}_0. \tag{3.3.102}$$

This equation is of type (3.3.77), with $\Psi(x, \tau) = \tilde{w}_0$ and $\phi'(s) = F_s(x, 0, \tilde{w}_0) - p$, hence we find the assumptions of Lemma 3.3.6 again hold and so $\dfrac{\partial \tilde{w}_0}{\partial p}$ exists with $\left|\dfrac{\partial \tilde{w}_0}{\partial p}\right| \leqslant C(1 + \tau^{m+1})\tilde{w}_0$, i.e., by the bound for $\tilde{w}_0$ in (3.3.93) we have $\left|\dfrac{\partial \tilde{w}_0}{\partial p}\right| \leqslant \bar{C}_\delta e^{-(\gamma_T^2 - p_0 - \delta)\tau}$. Furthermore as $\bar{\omega}_0 = \left.\dfrac{\partial \tilde{w}_0}{\partial p}\right|_{t=0} = 0$ and $\Psi(x, \tau) \geqslant 0$ then Lemma 3.3.6 gives $\dfrac{\partial \tilde{w}_0}{\partial p} \geqslant 0$.

We now take the k^{th} derivative of equation (3.3.71) with respect to x for $k = 1, ..., 4$. This results in equations of type (3.3.77). We take the first derivative as an example and note the remaining equations are included in §3.6.3. The equation for $\dfrac{\partial \tilde{w}_0}{\partial x}$ is

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \tilde{w}_0}{\partial x}\right) + \frac{\partial \tilde{w}_0}{\partial x}\left(F_s(x, 0, \tilde{w}_0) - p\right) = -F_x(x, 0, \tilde{w}_0). \tag{3.3.103a}$$

$$\left.\frac{\partial \tilde{w}_0}{\partial x}\right|_{\tau=0} = \frac{\partial \varphi}{\partial x} - \left.\frac{\partial u_0}{\partial x}\right|_{t=0}, \quad \left.\frac{\partial \tilde{w}_0}{\partial x}\right|_{\tau=\infty} = 0. \tag{3.3.103b}$$

Here we have $|\Psi(x, \tau)| = |F_x(x, 0, \tilde{w}_0)| \leqslant C|\tilde{w}_0|$ i.e., $|\Psi(x, \tau)| \leqslant C(1 + \tau^m)\tilde{w}_0$ is satisfied. Noting $\bar{\omega}_0 = \dfrac{\partial \varphi}{\partial x} - \left.\dfrac{\partial u_0}{\partial x}\right|_{t=0}$ by (3.3.78) we have

$$\left|\frac{\partial \tilde{w}_0}{\partial x}\right| \leqslant C\left(\frac{\partial \varphi}{\partial x} - \left.\frac{\partial u_0}{\partial x}\right|_{t=0} + 1 + \tau^{m+1}\right)\tilde{w}_0. \tag{3.3.104}$$

Recalling (3.3.93) for $\tilde{w}_0$ and as $\left.\dfrac{\partial \varphi}{\partial x} - \dfrac{\partial u_0}{\partial x}\right|_{t=0} \leqslant C$ then

$$\left|\frac{\partial \tilde{w}_0}{\partial x}\right| \leqslant \bar{C}_\delta e^{-(\gamma_T^2 - p_0 - \delta)\tau}. \tag{3.3.105}$$

To establish $\left|\dfrac{\partial \tilde{w}_0}{\partial x}\right| \leqslant Cx$ we look at (3.3.103) with $x = 0$, i.e., the equation for $\left.\dfrac{\partial \tilde{w}_0}{\partial x}\right|_{x=0}$. Using (B5) we see that the boundary conditions are zero. By (B5a) the right hand side of (3.3.103a) is zero hence $\left.\dfrac{\partial \tilde{w}_0}{\partial x}\right|_{x=0} = 0$ is a solution and as we have established bounds for the derivatives of $\tilde{w}_0$ we have $\left|\dfrac{\partial \tilde{w}_0}{\partial x}\right| \leqslant Cx$. □

We now establish a similar result to Lemma 3.3.1 for the functions $\tilde{w}_0(x, \tau; p)$ and $w_0(x, \tau)$.

Lemma 3.3.8. *For* $\tilde{w}_0(x, \tau; p)$ *and* $w_0(x, \tau)$ *defined by* (3.3.71) *and* (3.3.70) *we have*

$$\varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right](\tilde{w}_0 - w_0) = -\left. F(x, \varepsilon^2\tau, \cdot)\right|_{w_0}^{\tilde{w}_0} + p w_0 + O(\varepsilon^2 + p^2). \tag{3.3.106}$$

Proof. We consider the derivative terms in (3.3.106), i.e.,

$$\varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right](\tilde{w}_0 - w_0) = \frac{\partial}{\partial \tau}(\tilde{w}_0 - w_0) + O(\varepsilon^2). \tag{3.3.107}$$

Now using (3.3.5a) and (3.3.6a), $\dfrac{\partial}{\partial \tau}(\tilde{w}_0 - w_0)$ becomes

$$\frac{\partial}{\partial \tau}(\tilde{w}_0 - w_0) = -\left. F(t, 0, \cdot)\right|_{w_0}^{\tilde{w}_0} + p\tilde{w}_0, \tag{3.3.108}$$

and as

$$F(\varepsilon\xi, t, \cdot)\big|_{w_0}^{\tilde{w}_0} = F(x, 0, \cdot)\big|_{w_0}^{\tilde{w}_0} + \varepsilon^2\tau\, F_t(x, \theta t, \cdot)\big|_{w_0}^{\tilde{w}_0}, \tag{3.3.109}$$

for some $\theta \in (0,1)$, we get

$$\frac{\partial}{\partial\tau}(\tilde{w}_0 - w_0) = -\, F(x, \varepsilon^2\tau, \cdot)\Big|_{w_0}^{\tilde{w}_0} + \varepsilon^2\tau\, F_t(x, \hat{t}, \cdot)\Big|_{w_0}^{\tilde{w}_0} + p\tilde{w}_0. \tag{3.3.110}$$

As $F_t(x, 0, \cdot)|_{w_0}^{\tilde{w}_0} = (\tilde{w}_0 - w_0)F_{st}(x, 0, \theta(\tilde{w}_0 - w_0))$ and we have $(1+\tau)|\tilde{w}_0 - w_0| = O(p)$ then

$$\frac{\partial}{\partial\tau}(\tilde{w}_0 - w_0) = -\, F(x, \varepsilon^2\tau, \cdot)\Big|_{w_0}^{\tilde{w}_0} + p\tilde{w}_0 + O(\varepsilon p) \tag{3.3.111}$$

and finally putting this into (3.3.107) we obtain (3.3.106). □

Lemma 3.3.9. *For the asymptotic expansion $u_0(x,t) + w_0(x,\tau)$ of problem* (3.0.1) *near $t = 0$ we have accuracy*

$$\mathcal{T}[u_0(x,t) + w_0(x,\tau)] = O(\varepsilon^2). \tag{3.3.112}$$

Proof. This proof is included in Lemma 3.6.3 of the technical section. □

Lemma 3.3.10. *Near $x = 0$ and $t = 0$ the asymptotic expansion $u_{as}(x,t) := u_0(x,t) + v_0(\xi,t) + \varepsilon v_1(\xi,t) + w_0(x,\tau)$ of problem* (3.0.1) *has accuracy*

$$\mathcal{T}u_{as}(x,t) = O(\varepsilon^2). \tag{3.3.113}$$

Furthermore the boundary conditions satisfy

$$\varepsilon \left.\frac{\partial u_{as}}{\partial x}\right|_{x=0} = g_0(t), \quad \varepsilon \left.\frac{\partial u_{as}}{\partial x}\right|_{x=1} = g_1(t) + O(\varepsilon^2), \tag{3.3.114}$$

and the initial condition satisfies

$$u_{as}(x,0) = \varphi(x). \tag{3.3.115}$$

Proof. This proof is included in Lemma 3.6.4 in the technical section. □

3.3.3 Upper and Lower Solutions

We create upper and lower solutions for the system by taking the asymptotic expansion (3.3.2) and perturbing it, to get

$$\begin{aligned}\beta(x,t;p) := {} & u_0(x,t) + \tilde{v}_0(\xi,t;p) + \varepsilon v_1(\xi,t) \\ & + \tilde{w}_0(x,\tau;p) + C_0 p(1 + e^{-c_0 x/\varepsilon} + e^{-c_0(1-x)/\varepsilon}),\end{aligned} \tag{3.3.116}$$

for some small positive p, some sufficiently large constant $C_0 > 0$ and some sufficiently small constant $c_0 > 0$. The exponential terms are included so that at $x = 0$ and $x = 1$ the boundary conditions will meet the requirement (3.3.120c) and (3.3.120d). We can rewrite (3.3.116) as

$$\beta(x,t;p) := u_{as}(x,t) + V(\xi,t;p) + W(x,\tau;p) + C_0 p\rho(x), \tag{3.3.117}$$

where $V := \tilde{v}_0(\xi,t;p) - v_0(\xi,t)$, $W := \tilde{w}_0(x,\tau;p) - w_0(x,\tau)$ and $\rho(x) := 1 + e^{-c_0 x/\varepsilon} + e^{-c_0(1-x)/\varepsilon}$. We note

$$(1+\xi)|V| + \left|\frac{\partial V}{\partial \xi}\right| \leqslant Cp, \tag{3.3.118}$$

and

$$(1+\tau)|W| + \left|\frac{\partial W}{\partial x}\right| \leqslant Cp. \tag{3.3.119}$$

These are found by linearising, for example, $\tilde{v}_0(\xi, t; p)$ and cancelling terms to get $(1+\xi)|V| \leqslant C(1+\xi)p\dfrac{\partial \tilde{v}_0}{\partial p}$. From Lemma 3.3.3 we have $\dfrac{\partial \tilde{v}_0}{\partial p} \leqslant \bar{C}_\delta e^{-\left(\gamma_L - \sqrt{p_0} - \delta\right)\xi}$, resulting in (3.3.118). A similar argument holds for (3.3.119).

We now need to prove that $\beta(x, t; \pm p)$ are in fact upper and lower solutions to the system by showing that $\beta(x, t; \pm p)$ satisfies

$$\beta(x, t; -p) \leqslant \beta(x, t; p), \tag{3.3.120a}$$

$$\mathcal{T}\beta(x, t; -p) \leqslant 0 \leqslant \mathcal{T}\beta(x, t; p), \tag{3.3.120b}$$

$$\varepsilon \left.\frac{\partial \beta(x, t; -p)}{\partial x}\right|_{x=0} \geqslant g_0(t) \geqslant \varepsilon \left.\frac{\partial \beta(x, t; p)}{\partial x}\right|_{x=0}, \tag{3.3.120c}$$

$$\varepsilon \left.\frac{\partial \beta(x, t; -p)}{\partial x}\right|_{x=1} \leqslant g_1(t) \leqslant \varepsilon \left.\frac{\partial \beta(x, t; p)}{\partial x}\right|_{x=1}, \tag{3.3.120d}$$

and

$$\beta(x, 0; -p) \leqslant \varphi(x) \leqslant \beta(x, 0; p). \tag{3.3.120e}$$

Lemma 3.3.11. *For $\beta(x, t; p)$ of (3.3.116) we have*

$$\beta(x, t; p) = u_{as}(x, t) + O(p) \quad \textit{for } (x, t) \in [0, 1] \times [0, T]. \tag{3.3.121}$$

Furthermore for $p \geqslant 0$,

$$\beta(x, t; -p) \leqslant u_{as}(x, t) - C_0 p \leqslant u_{as}(x, t) + C_0 p \leqslant \beta(x, t; p), \tag{3.3.122}$$

for $(x, t) \in [0, 1] \times [0, T]$.

Proof. Considering $\beta(x, t; \pm p)$ in the form (3.3.117), using (3.3.118), (3.3.119)

and as $1 \leqslant \rho \leqslant 3$ we have (3.3.121).

Again as $\dfrac{\partial \tilde{v}_0}{\partial p} \geqslant 0$ and $\dfrac{\partial \tilde{w}_0}{\partial p} \geqslant 0$ both $\tilde{v}_0(\xi, t; p)$ and $\tilde{w}_0(x, \tau; p)$ are increasing with respect to p therefore $\tilde{v}_0(\xi, t; p) - v_0(\xi, t) \geqslant 0$ and $\tilde{w}_0(x, \tau; p) - w_0(x, \tau) \geqslant 0$. Combining this with $\rho(x) \geqslant 1$ we obtain $u_{as}(x, t) + C_0 p \leqslant \beta(x, t; p)$. Similarly we can find $\beta(x, t; -p) \leqslant u_{as}(x, t) - C_0 p\rho$. As $u_{as}(x, t) - C_0 p\rho \leqslant u_{as}(x, t) + C_0 p\rho$ we have established (3.3.122). □

Lemma 3.3.12. *For* $\beta(x, t; p)$ *defined in* (3.3.116) *we have*

$$\begin{aligned} \mathcal{T}\beta(x, t; p) = & C_0 p\rho F_s(x, t, 0) + p(v_0 + w_0)[1 + C_0\rho\lambda] \\ & - c_0^2 C_0 p(e^{-c_0 x/\varepsilon} + e^{-c_0(1-x)/\varepsilon}) + O(\varepsilon^2 + p^2), \end{aligned} \tag{3.3.123}$$

with $\lambda = \lambda(x, t) := F_{ss}(x, t, (v_0 + w_0)\theta)$ *and some* $\theta \in (0, 1)$.

Proof. As we have established the order of $\mathcal{T}u_{as}$ in (3.3.113) we consider $\mathcal{T}\beta - \mathcal{T}u_{as}$, i.e.,

$$\mathcal{T}\beta - \mathcal{T}u_{as} = \varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right](V + W + C_0 p\rho) + f(x, t, \cdot)|_{u_{as}}^{\beta}. \tag{3.3.124}$$

Calculating $\varepsilon^2 \left[\dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial x^2}\right] C_0 p\rho$ and using (3.3.63) and (3.3.106) we have

$$\begin{aligned} \varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right](V + W + C_0 p\rho) = & - F(\varepsilon\xi, t, \cdot)|_{v_0+\varepsilon v_1}^{\tilde{v}_0+\varepsilon v_1} - F(x, \varepsilon^2\tau, \cdot)\Big|_{w_0}^{\tilde{w}_0} \\ & + p(v_0 + w_0) - c_0^2 C_0 p(e^{-c_0 x/\varepsilon} + e^{-c_0(1-x)/\varepsilon}) + O(\varepsilon^2 + p^2). \end{aligned} \tag{3.3.125}$$

Using a Taylor expansion we find

$$\begin{aligned} \varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right](V + W + C_0 p\rho) = & -V F_s(\varepsilon\xi, t, v_0 + \varepsilon v_1) - W F_s(x, \varepsilon^2\tau, w_0) \\ & + p(v_0 + w_0) - c_0^2 C_0 p(e^{-c_0 x/\varepsilon} + e^{-c_0(1-x)/\varepsilon}) + O(\varepsilon^2 + p^2). \end{aligned} \tag{3.3.126}$$

From (3.3.124) we have

$$f(x,t,\cdot)|_{u_{as}}^{\beta} = f(x,t,\cdot)|_{u_{as}+V+W}^{\beta} + f(x,t,\cdot)|_{u_{as}}^{u_{as}+V+W}. \tag{3.3.127}$$

The second term in (3.3.127) can be written as

$$f(x,t,\cdot)|_{u_{as}}^{u_{as}+V+W} = [V+W]F_s(x,t,v_0+\varepsilon v_1+w_0) + O(p^2) \tag{3.3.128}$$

and taking Taylor expansions of $F_s(x,t,s)$ we obtain

$$\begin{aligned} f(x,t,\cdot)|_{u_{as}}^{u_{as}+V+W} &= VF_s(x,t,v_0+\varepsilon v_1) + w_0VF_{ss}(x,t,v_0+\varepsilon v_1+w_0\theta) \\ &+WF_s(x,t,w_0) + (v_0+\varepsilon v_1)WF_{ss}(x,t,w_0+(v_0+\varepsilon v_1)\theta) + O(p^2). \end{aligned} \tag{3.3.129}$$

for some $\theta \in (0,1)$. Recalling (3.3.54) and (3.3.119) we have $(v_0+\varepsilon v_1)WF_{ss}(x,t,w_0+(v_0+\varepsilon v_1)\theta) = O(\varepsilon^2\tau W) = O(\varepsilon^2 p)$. Recalling (B5a) we can say $|w_0(x,\tau)| \leqslant Cx$ giving $|w_0V| \leqslant Cxp \leqslant C\varepsilon p$ and so we have $w_0VF_{ss}(x,t,v_0+\varepsilon v_1+w_0\theta) = O(\varepsilon p)$. Now (3.3.129) becomes

$$f(x,t,\cdot)|_{u_{as}}^{u_{as}+V+W} = VF_s(x,t,v_0+\varepsilon v_1) + WF_s(x,t,w_0) + O(\varepsilon^2+p^2). \tag{3.3.130}$$

The first term in (3.3.127) can be written as

$$f(x,t,\cdot)|_{u_{as}+V+W}^{\beta} = C_0 p\rho F_s(x,t,v_0+w_0) + O(\varepsilon^2+p^2), \tag{3.3.131}$$

which can be simplified to be

$$f(x,t,\cdot)|_{u_{as}+V+W}^{\beta} = C_0 p\rho[F_s(x,t,0) + (v_0+w_0)\lambda] + O(\varepsilon^2+p^2), \tag{3.3.132}$$

where $\lambda = \lambda(x,t) := F_{ss}(x,t,(v_0+w_0)\theta)$ for some $\theta \in (0,1)$. Putting

(3.3.126), (3.3.130) and (3.3.132) into (3.3.124) gives

$$\begin{aligned}\mathcal{T}\beta - \mathcal{T}u_{as} = & p(v_0 + w_0) - c_0^2 C_0 p(e^{-c_0 x/\varepsilon} + e^{-c_0(1-x)/\varepsilon}) \\ & + C_0 p\rho[F_s(x,t,0) + (v_0 + w_0)\lambda] + O(\varepsilon^2 + p^2).\end{aligned} \tag{3.3.133}$$

Finally including (3.3.113) we obtain (3.3.123). □

Corollary 3.3.1. *There exists positive constants C_0 and C_1 such that for all $0 \leqslant p \leqslant p_0$ we have*

$$\mathcal{T}\beta(x,t;p) \geqslant C_0 p\gamma^2 - C_1(p^2 + \varepsilon^2), \tag{3.3.134}$$

$$\mathcal{T}\beta(x,t;-p) \leqslant -C_0 p\gamma^2 + C_1(p^2 + \varepsilon^2). \tag{3.3.135}$$

Proof. Taking $p > 0$ as an example, Lemma 3.3.12 with (B1) gives

$$\begin{aligned}\mathcal{T}\beta(x,t;p) \geqslant & C_0 p\rho\gamma^2 + p(v_0 + w_0)[1 + C_0\rho\lambda] \\ & - c_0^2 C_0 p(e^{-c_0 x/\varepsilon} + e^{-c_0(1-x)/\varepsilon}) - C_1(\varepsilon^2 + p^2).\end{aligned} \tag{3.3.136}$$

We note $v_0 + w_0 \geqslant 0$ by (3.3.54) and (3.3.92) and $e^{-c_0 x/\varepsilon} + e^{-c_0(1-x)/\varepsilon} \leqslant 2$. As $1 \leqslant \rho(x) \leqslant 3$ and $|\lambda| \leqslant C$ we can choose C_0 such that $1 + C_0\rho\lambda \geqslant 0$ and (3.3.136) becomes

$$\mathcal{T}\beta(x,t;p) \geqslant C_0 p\gamma^2 - 2c_0^2 C_0 p - C_1(\varepsilon^2 + p^2). \tag{3.3.137}$$

Choosing c_0 sufficiently small such that $\gamma^2/2 \geqslant 2c_0^2$ we get (3.3.134). A similar argument holds true for $\mathcal{T}\beta(x,t;-p)$. □

Theorem 3.3.4. *For $\beta(x,t;p)$ defined in* (3.3.116)*,* (3.3.120) *holds and there exists sufficiently small $\varepsilon_0 > 0$ such that for all $\varepsilon \leqslant \varepsilon_0$ there exists a unique solution $u(x,t)$ to problem* (3.0.1) *with*

$$\beta(x,t;-\bar{p}) \leqslant u(x,t) \leqslant \beta(x,t;\bar{p}). \tag{3.3.138}$$

Furthermore we have

$$|u(x,t) - u_{as}(x,t)| \leqslant C\varepsilon^2 \quad \textit{for } (x,t) \in [0,1] \times [0,T]. \tag{3.3.139}$$

Proof. Define $\bar{p} := C_2\varepsilon^2$, with $C_2 \geqslant 2C_1/(C_0\gamma^2)$ so that $C_0\bar{p}\gamma^2 \geqslant 2C_1\varepsilon^2$. For $\varepsilon \leqslant 1/C_2$, we can say $\bar{p} \leqslant \varepsilon$ and hence $C_1(\varepsilon^2 + \bar{p}^2) \leqslant 2C_1\varepsilon^2$. Now Corollary 3.3.1 gives

$$\mathcal{T}\beta(x,t;\bar{p}) \geqslant 0. \tag{3.3.140}$$

Similarly we can find $\mathcal{T}\beta(x,t;-\bar{p}) \leqslant 0$.

Considering the boundary condition at $x = 0$ for $p = \bar{p}$, recalling (3.3.114), we have

$$-\varepsilon \left.\frac{\partial\beta}{\partial x}\right|_{x=0,p=\bar{p}} = -g_0(t) - \left.\frac{\partial V}{\partial\xi}\right|_{\xi=0,p=\bar{p}} - \varepsilon \left.\frac{\partial W}{\partial x}\right|_{x=0,p=\bar{p}} + c_0C_0\bar{p}[1 - e^{-c_0/\varepsilon}]. \tag{3.3.141}$$

Recalling (3.3.5b) and (3.3.6b) we have $\left.\frac{\partial V}{\partial\xi}\right|_{\xi=0,p=\bar{p}} = 0$ and by (3.3.92) we find $\left.\frac{\partial W}{\partial x}\right|_{x=0,p=\bar{p}} = 0$. As $(1 - e^{-c_0/\varepsilon}) \geqslant 1/2$, (3.3.141) becomes

$$-\varepsilon \left.\frac{\partial\beta}{\partial x}\right|_{x=0,p=\bar{p}} \geqslant -g_0(t) + \frac{c_0C_0\bar{p}}{2} \geqslant -g_0(t). \tag{3.3.142}$$

A similar argument holds for $p = -\bar{p}$ and we obtain (3.3.120c).

A similar method is used for $x = 1$. Considering β again in the form (3.3.117) we have

$$\varepsilon \left.\frac{\partial \beta}{\partial x}\right|_{x=1,p=\bar{p}} = \varepsilon \left.\frac{\partial u_{as}}{\partial x}\right|_{x=1} + \varepsilon \left.\frac{\partial V}{\partial x}\right|_{x=1,p=\bar{p}} + \varepsilon \left.\frac{\partial W}{\partial x}\right|_{x=1,p=\bar{p}} - c_0 C_0 \bar{p}[e^{-c_0/\varepsilon} - 1]. \tag{3.3.143}$$

Recalling (3.3.114), (3.3.118), (3.3.119) and $e^{-c_0/\varepsilon} - 1 \leqslant -1/2$, we choose $C_8 > 0$ such that $\varepsilon \left.\frac{\partial u_{as}}{\partial x}\right|_{x=1} \geqslant g_1(t) - C_8\varepsilon^2$ and $C_9 > 0$ such that $\varepsilon \left.\frac{\partial V}{\partial x}\right|_{x=1,p=\bar{p}} + \varepsilon \left.\frac{\partial W}{\partial x}\right|_{x=1,p=\bar{p}} \geqslant -C_9\varepsilon\bar{p}$. Now (3.3.143) becomes

$$\varepsilon \left.\frac{\partial \beta}{\partial x}\right|_{x=1,p=\bar{p}} \geqslant g_1(t) - C_8\varepsilon^2 - C_9\varepsilon\bar{p} + \frac{c_0 C_0 \bar{p}}{2}. \tag{3.3.144}$$

By choosing C_0 sufficiently large such that we have $C_8\varepsilon^2 - C_9\varepsilon\bar{p} \leqslant \frac{c_0 C_0 \bar{p}}{4}$ and hence

$$\varepsilon \left.\frac{\partial \beta}{\partial x}\right|_{x=1,p=\bar{p}} \geqslant g_1(t) + \frac{c_0 C_0 \bar{p}}{4} \geqslant g_1(t). \tag{3.3.145}$$

Again a similar argument holds for $p = -\bar{p}$ and we get (3.3.120d).

For the initial condition, by (3.3.115) and $\rho(x) \geqslant 0$, we have

$$\beta(x, 0; -\bar{p}) = \varphi(x) - C_0\bar{p}\rho \leqslant \varphi(x) \leqslant \varphi(x) + C_0\bar{p}\rho = \beta(x, 0; \bar{p}). \tag{3.3.146}$$

Hence the requirements (3.3.120) are met and so $\beta(x, t; \bar{p})$ and $\beta(x, t; -\bar{p})$ are ordered upper and lower solutions to problem (3.0.1). By Lemma 3.3.11 we have

$$\beta(x, t; \bar{p}) = u_{as}(x, t) + O(\bar{p}) = u_{as}(x, t) + O(\varepsilon^2). \tag{3.3.147}$$

As $\beta(x, t; \pm\bar{p})$ are ordered upper and lower solutions we recall Lemma 1.0.1

and have that the solution of (3.0.1) satisfies

$$\beta(x,t;-\bar{p}) \leqslant u(x,t) \leqslant \beta(x,t;\bar{p}), \tag{3.3.148}$$

and so by (3.3.147) we have (3.3.139). $\square$

3.4 Discrete Space: Analysis of the Numerical Method

To solve (3.0.1) we will use Newton's method with a finite difference discretisation considered on two layer adapted meshes, the Bakhvalov mesh and the Shishkin mesh. The discrete system is

$$\mathcal{T}^h U_{ij} := \varepsilon^2 \left[D_t^- - \delta_x^2\right] U_{ij} + f(x_i, t_j, U_{ij}) = 0, \tag{3.4.1a}$$

for $i = 1, ..., N$, and $j = 1, ..., M$,

$$\mathcal{T}^h U_{0j} = \varepsilon^2 \left[D_t^- U_{0j} - \frac{2}{h_1} D_x^- U_{1j}\right] + \frac{2\varepsilon}{h_1} g_0(t_j) + f(x_0, t_j, U_{0j}) = 0, \tag{3.4.1b}$$

$$\mathcal{T}^h U_{Nj} = \varepsilon^2 \left[D_t^- U_{Nj} + \frac{2}{h_N} D_x^- U_{Nj}\right] - \frac{2\varepsilon}{h_N} g_1(t_j) + f(x_N, t_j, U_{Nj}) = 0, \tag{3.4.1c}$$

for $j = 1, ..., M$ and initial condition

$$U_{i,0} = \varphi(x_i) \quad \text{for } i = 0, ..., N. \tag{3.4.1d}$$

The following notation is used

$$D_t^- U_{ij} := \frac{U_{ij} - U_{i,j-1}}{k_j}, \quad \delta_x^2 U_{ij} := \frac{2}{h_i + h_{i+1}} \left(D_x^- U_{i+1,j} - D_x^- U_{ij}\right), \tag{3.4.1e}$$

$$D_x U_{ij} := \frac{U_{i+1,j} - U_{i-1,j}}{2h_i}, \quad D_x^- U_{ij} := \frac{U_{ij} - U_{i-1,j}}{h_i}, \tag{3.4.1f}$$

$h_i = x_i - x_{i-1}$, $k_j = t_j - t_{j-1}$ and $0 = x_0 < x_1 < ... < x_{N-1} < x_N = 1$, $0 = t_0 < t_1 < ... < t_{M-1} < t_M = T$.

Here we obtained (3.4.1b) by calculating (3.4.1a) at $i = 0$, i.e.,

$$\mathcal{T}^h U_{0j} = \varepsilon^2 \left[\frac{U_{0j} - U_{0,j-1}}{k_j} - \frac{2}{h_0 + h_1} \left(\frac{U_{1,j} - U_{0j}}{h_1} - \frac{U_{0j} - U_{-1,j}}{h_0} \right) \right] + f(x_i, t_j, U_{0j}), \tag{3.4.2}$$

where the point $x_{-1} = -h_1$ is a fictitious point with $h_0 := h_1$. To eliminate $U_{-1,j}$ we discretise (3.0.1c) using central differencing to obtain

$$\varepsilon D_x U_{0,j} = \varepsilon \frac{U_{1,j} - U_{-1,j}}{2h_0} = g_0(t_j) \quad \text{for } j = 1, ..., M. \tag{3.4.3}$$

Eliminating $U_{-1,j}$ by combining (3.4.2) and (3.4.3) gives (3.4.1b). The discretisation (3.4.1c) is found using a similar method with

$$\varepsilon D_x U_{N,j} = \varepsilon \frac{U_{N+1,j} - U_{N-1,j}}{2h_N} = g_1(t_j) \quad \text{for } j = 1, ..., M, \tag{3.4.4}$$

where $x_{N+1} = 1 + h_N$ is a fictitious point with $h_{N+1} := h_N$.

3.4.1 The Stabilised Method

As we may obtain incorrect computed solutions as discussed in §..., we again employ the stabilisation method described in §2.5.2. For problems of type (3.0.1) the stabilisation method is, for some constant $\widehat{C} \geqslant 0$ and $j = 1, \ldots, M$,

$$\widehat{\mathcal{T}}^h \hat{U}_{ij} := \hat{\varepsilon}^2(t_j) D_t^- \hat{U}_{ij} - \varepsilon^2 \delta_x^2 \hat{U}_{ij} + f(x_i, t_j, \hat{U}_{ij}) = 0, \tag{3.4.5a}$$

$$\widehat{\mathcal{G}}^h \hat{U}_{0j} := \frac{h_1}{2\varepsilon} \hat{\varepsilon}^2(t_j) D_t^- \hat{U}_{0j} - \varepsilon D_x^- \hat{U}_{1,j} + g_0(t_j) + \frac{h_1}{2\varepsilon} f(x_0, t_j, \hat{U}_{0j}) = 0, \tag{3.4.5b}$$

$$\widehat{\mathcal{G}}^h \hat{U}_{Nj} := \frac{h_N}{2\varepsilon} \hat{\varepsilon}^2(t_j) D_t^- \hat{U}_{Nj} + \varepsilon D_x^- \hat{U}_{N,j} - g_1(t_j) + \frac{h_N}{2\varepsilon} f(x_N, t_j, \hat{U}_{Nj}) = 0, \tag{3.4.5c}$$

and

$$\hat{U}_{i,0} = \varphi(x_i), \quad i = 0, ...N, \tag{3.4.5d}$$

where

$$\hat{\varepsilon}^2(t_j) := \max\{\varepsilon^2, \widehat{C}k_j\}, \tag{3.4.6}$$

where $\widehat{C}$ is chosen using Proposition 3.4.1. This new parameter $\hat{\varepsilon}^2(t_j)$ is picked up if $\varepsilon^2 < \widehat{C}k_j$. On a non uniform mesh the resulting system will pick up $\hat{\varepsilon}$ on the coarse section of the mesh and will remain unchanged on the fine section. On a uniform mesh this parameter is picked up throughout the domain. Note $\hat{\varepsilon}^2(t_j)$ only depends on the time step and not the space step.

As the stabilised method is a generalisation of the standard method we consider $\hat{U}_{ij}$ in the following work.

Proposition 3.4.1. *Let $\hat{U}_{ij}$ be a solution of* (3.4.5) *and let $\hat{\varepsilon}_j^2 > C^*k_j$ for some $C^* \geqslant 0$. If $f_u(x,t,u) \geqslant -C^*$ for all x, t, u then $\hat{U}_{ij}$ is a unique computed solution of* (3.4.5)*. Furthermore if $f_u(x,t,u) \geqslant -C^*$ and $K_1 \leqslant \hat{U}_{ij} \leqslant K_2$ for all x, t and $u \in [K_1, K_2]$ then $\hat{U}_{ij}$ is a unique computed solution between K_1 and K_2.*

Proof. The proof of the case with Dirichlet boundary conditions can be found in [16, Proposition 2.2] and we modify this for the case of Neumann boundary conditions. This proof can be found in [31, Proposition 3.5.1] and is included here for completeness.

Suppose there exists two solutions to the discrete problem (3.4.5), $\tilde{U}_{ij}$ and $\bar{U}_{ij}$. Define $V_{ij} := \tilde{U}_{ij} - \bar{U}_{ij}$ where

$$\hat{\varepsilon}^2 D_t^- V_{ij} - \varepsilon^2 \delta_x^2 V_{ij} + f(x_i, t_j, \tilde{U}_{ij}) - f(x_i, t_j, \bar{U}_{ij}) = 0, \tag{3.4.7}$$

with

$$V_{i,0} = 0, \quad D_x V_{0,j} = D_x V_{N,j} = 0. \tag{3.4.8}$$

Similar to $p(x,t)$ in Proposition 3.0.1, we define $p_{ij} := \int_0^1 f_u(x_i, t_j, \hat{U}_{ij} - s[\bar{U}_{ij} -$

$\hat{U}_{ij}])ds \geqslant -C^*$ and $Z_{ij} := V_{ij}\Pi_{l=1}^{j}(1 + k_l\mu_l\hat{\varepsilon}_l^{-2})^{-1}$ and hence we have

$$\frac{\hat{\varepsilon}^2}{1 + k_j\mu_j\hat{\varepsilon}_j^{-2}} D_t^- Z_{ij} - \varepsilon^2\delta_x^2 Z_{ij} + \left(\frac{\mu_j}{1 + k_j\mu_j\hat{\varepsilon}_j^{-2}} + p_{ij}\right) Z_{ij} = 0. \tag{3.4.9}$$

Using (3.4.1b) the boundary condition equation is

$$\frac{\hat{\varepsilon}_j^2}{1 + k_j\mu_j\hat{\varepsilon}_j^{-2}} \left(\frac{Z_{0,j} - Z_{0,j-1}}{k_j}\right) - \varepsilon^2 \frac{Z_{1,j} - Z_{0,j}}{h_1^2/2} + \left(\frac{\mu_j}{1 + k_j\mu_j\hat{\varepsilon}_j^{-2}} + p_{0,j}\right) Z_{0,j} = 0, \tag{3.4.10}$$

and

$$\begin{aligned} &\frac{\hat{\varepsilon}_j^2}{1 + k_j\mu_j\hat{\varepsilon}_j^{-2}} \left(\frac{Z_{N,j} - Z_{N,j-1}}{k_j}\right) - \varepsilon^2 \frac{Z_{N,j} - Z_{N-1,j}}{h_N^2/2} \\ &\quad + \left(\frac{\mu_j}{1 + k_j\mu_j\hat{\varepsilon}_j^{-2}} + p_{N,j}\right) Z_{N,j} = 0. \end{aligned} \tag{3.4.11}$$

For some constant $\tilde{C}^* \geqslant C^*$ we have $\hat{\varepsilon}_j^2 \geqslant \tilde{C}^* k_j$ and hence we can choose μ_j sufficiently large such that $\dfrac{\mu_j}{1 + \mu_j/\tilde{C}^*}$ is sufficiently close to $\tilde{C}^*$ and exceeds C^* hence the coefficient of Z_{ij} in the final terms of (3.4.9), (3.4.10) and (3.4.11) are positive, i.e., $\dfrac{\mu_j}{1 + k_j\mu_j\hat{\varepsilon}_j^{-2}} + p_{ij} \geqslant 0$. Therefore we get a matrix with negative off diagonal entries and positive, dominating diagonal entries, that is we have an M-matrix discretisation. Applying the discrete maximum principle we get $Z_{ij} = 0$ and so $\bar{U}_{ij} = \hat{U}_{ij}$ for all i and j. □

3.4.2 Layer Adapted Meshes

We use the layer adapted meshes described in §1.0.3 and §1.0.3 and give the full description for those meshes on problem (3.4.1) in the following two sections.

The Bakhvalov Mesh [16]

Recall the Bakhvalov mesh discussed in §1.0.3. The mesh points are (x_i, t_j) with $x_i := x(i/N)$ and $t_j := t(j/M)$ where the mesh generating functions are given by

$$x(\zeta) = \begin{cases} \dfrac{2\varepsilon}{\gamma} \ln\left(\dfrac{1}{1-4\zeta}\right), & \zeta \in [0, \theta_x], \\ \dfrac{1}{2} - d_x\left(\dfrac{1}{2} - \zeta\right), & \zeta \in (\theta_x, 1/2], \\ 1 - x(1-\zeta), & \zeta \in (1/2, 1], \end{cases} \quad t(\eta) = \begin{cases} \dfrac{\varepsilon^2}{\gamma^2} \ln\left(\dfrac{1}{1-2\eta}\right), & \eta \in [0, \theta_t], \\ T - d_t(1-\eta), & \eta \in (\theta_t, 1], \end{cases} \tag{3.4.12}$$

with

$$\theta_x := 1/4 - C_3\varepsilon, \quad \theta_t := 1/2 - C_4\varepsilon^2, \tag{3.4.13}$$

for some positive constants C_3 and C_4, and d_x and d_t are chosen such that $x(\zeta)$ and $t(\eta)$ are continuous. Recall the Bakhvalov mesh is a graded mesh with a fine mesh in the layer regions and graded to a coarse mesh elsewhere.

The mesh generating functions $x(\zeta)$ and $t(\eta)$ in (3.4.12) are only valid for $\varepsilon \leqslant \dfrac{1}{8}\min\{\gamma, 2C_3^{-1}\}$ and $\varepsilon^2 \leqslant \dfrac{1}{2}\min\{\gamma^2 T, C_4^{-1}\}$ respectively. When these are not met the mesh becomes uniform with $x(\zeta) = \zeta$ and $t(\eta) = \eta/T$.

The Shishkin Mesh [16][32]

The mesh transition point in the x direction, σ_x, and in the t direction, σ_t, are defined as

$$\sigma_x := \min\left\{\frac{2\varepsilon}{\gamma}\ln N, \frac{1}{4}\right\}, \quad \sigma_t := \min\left\{\frac{\varepsilon^2}{\gamma^2}\ln M, \frac{T}{2}\right\}. \tag{3.4.14}$$

For the piecewise uniform mesh $\{x_i\}_{i=1}^{N+1}$ we divide the region into three intervals, $[0, \sigma_x]$, $[\sigma_x, 1-\sigma_x]$ and $[1-\sigma_x, 1]$ and divide these into $N/4$, $N/2$ and $N/4$ equidistant subintervals respectively. The mesh $\{t_j\}_{j=1}^{M+1}$ is divided into two

intervals, $[0, \sigma_t]$ and $[\sigma_t, T]$ and these are then divided into $M/2$ equidistant subintervals.

This results in a mesh that is fine in the layer regions, near $x = 0$, $x = 1$ and $t = 0$ i.e., on $[0, \sigma_x]$, $[1 - \sigma_x, 1]$ and $[0, \sigma_t]$, and coarse elsewhere, i.e., on $[\sigma_x, 1 - \sigma_x]$ and $[\sigma_t, T]$.

3.4.3 Truncation Error and Accuracy Results

We recall (3.0.1c) and define

$$\mathcal{G}u(0,t) := -\varepsilon \left.\frac{\partial u}{\partial x}\right|_{x=0} + g_0(t) = 0, \quad x = 0,\, t \in [0,T], \tag{3.4.15a}$$

$$\mathcal{G}u(1,t) := \varepsilon \left.\frac{\partial u}{\partial x}\right|_{x=1} - g_1(t) = 0, \quad x = 1,\, t \in [0,T]. \tag{3.4.15b}$$

Lemma 3.4.1. *Let $\beta(x_i, t_j; p)$ be defined by* (3.3.116) *and let the mesh be either that of §3.4.2 or §3.4.2. For all $|p| \leqslant p_0$ the truncation error of the system is, for $i = 1, \dots, N-1$ and $j = 1, \dots, M$,*

$$|\widehat{\mathcal{T}}^h \beta(x_i, t_j; p) - \mathcal{T}\beta(x_i, t_j; p)| \leqslant C(N^{-2} \ln^{2m} N + M^{-1} \ln^m M), \tag{3.4.16}$$

$$|\widehat{\mathcal{G}}^h \beta(x_0, t_j; p) - \mathcal{G}\beta(x_0, t_j; p)| \leqslant \frac{h_1}{2\varepsilon}\mathcal{T}\beta(x_0, t_j; p) + C(N^{-2} \ln^{2m} N + M^{-1} \ln^m M), \tag{3.4.17}$$

and

$$\begin{aligned} |\widehat{\mathcal{G}}^h \beta(x_N, t_j; p) - \mathcal{G}\beta(x_N, t_j; p)| \leqslant & \frac{h_N}{2\varepsilon}\mathcal{T}\beta(x_N, t_j; p) \\ & + C(N^{-2} \ln^{2m} N + M^{-1} \ln^m M), \end{aligned} \tag{3.4.18}$$

where $m = 0$ for the Bakhvalov mesh and $m = 1$ for the Shishkin mesh.

Proof. For points in the interior of the domain this proof follows [16, Lemma 5.1] and [17, Lemma 3.3] and is included here for completeness. For $x = 0$ and $x = 1$ the proof is similar to that found in [31, Lemma 3.5.2] where homogeneous boundary conditions are considered.

We first consider points inside the domain, i.e., for $i = 1, \ldots, N-1$ and $j = 1, \ldots, M$. We aim to estimate

$$\begin{aligned} |\widehat{\mathcal{T}}^h\beta(x_i,t_j;p) - \mathcal{T}\beta(x_i,t_j;p)| = &- \varepsilon^2\left[\delta_x^2 - \frac{\partial^2}{\partial x^2}\right]\beta_{ij} \\ &+ \varepsilon^2\left[D_t^- - \frac{\partial}{\partial t}\right]\beta_{ij} \\ &+ (\hat{\varepsilon}^2(t_j) - \varepsilon^2)D_t^-\beta_{ij}. \end{aligned} \tag{3.4.19}$$

Recalling β defined in (3.3.116) we have

$$-\varepsilon^2\left[\delta_x^2 - \frac{\partial^2}{\partial x^2}\right](u_0 + \tilde{w}_0 + C_0 p\rho) = O(\varepsilon^2), \tag{3.4.20}$$

and

$$\varepsilon^2\left[D_t^- - \frac{\partial}{\partial t}\right](u_0 + \tilde{v}_0 + \varepsilon v_1 + C_0 p\rho) = O(\varepsilon^2). \tag{3.4.21}$$

Define

$$\begin{gathered} R_1 := -\varepsilon^2\left[\delta_x^2 - \frac{\partial^2}{\partial x^2}\right](\tilde{v}_0 + \varepsilon v_1), \quad R_2 := \varepsilon^2\left[D_t^- - \frac{\partial}{\partial t}\right]\tilde{w}_0, \\ \text{and } R_3 := (\hat{\varepsilon}^2(t_j) - \varepsilon^2)D_t^-\tilde{w}_0. \end{gathered} \tag{3.4.22}$$

and note

$$|\widehat{\mathcal{T}}^h\beta(x_i,t_j;p) - \mathcal{T}\beta(x_i,t_j;p)| = |R_1| + |R_2| + |R_3| + O(\varepsilon^2) \tag{3.4.23}$$

We can consider R_2 and R_3 for $i = 0, \ldots, N$ and will use these results while calculating the truncation error for the Neumann boundary conditions.

Truncation error R_1 in space.

We first consider R_1 and define $v(\xi,t;p) := \tilde{v}_0(\xi,t;p) + \varepsilon v_1(\xi,t)$. Taking a Taylor expansion of $v_{i\pm1,j}$ about i we have

$$\begin{aligned} v_{i\pm1,j} =& v_{ij} \pm h_{(i+1/2)\pm1/2}\frac{\partial v_{ij}}{\partial x} + \frac{(h_{(i+1/2)\pm1/2})^2}{2}\frac{\partial^2 v_{ij}}{\partial x^2} \\ &\pm \frac{(h_{(i+1/2)\pm1/2})^3}{3!}\frac{\partial^3 v_{ij}}{\partial x^3} + \frac{(h_{(i+1/2)\pm1/2})^4}{4!}\left.\frac{\partial^4 v}{\partial x^4}\right|_{x_i\pm\theta x_{i\pm1},t_j}. \end{aligned} \tag{3.4.24}$$

Recalling (3.4.1e), $\delta_x^2 v_{ij}$ is

$$\delta_x^2 v_{ij} = \frac{\partial^2 v_{ij}}{\partial x^2} + \frac{h_{i+1}-h_i}{3}\frac{\partial^3 v_{ij}}{\partial x^3} + \frac{h_{i+1}^2 - 2h_ih_{i+1} + h_i^2}{12}\left.\frac{\partial^4 v}{\partial x^4}\right|_{x_i\pm\theta x_{i\pm1},t_j}, \tag{3.4.25}$$

and hence the bound for R_1 in (3.4.22) can be written as

$$|R_1| \leqslant C\varepsilon^2 \min\left\{|h_{i+1}-h_i|\left|\frac{\partial^3 v_{ij}}{\partial x^3}\right| + (h_{i+1}+h_i)^2\left|\left.\frac{\partial^4 v}{\partial x^4}\right|_{x_i\pm\theta x_{i\pm1},t_j}\right|, \left|\frac{\partial^2 v_{ij}}{\partial x^2}\right|\right\}. \tag{3.4.26}$$

We note $\gamma_L - \sqrt{p_0} > \gamma$ and choose δ sufficiently small such that $\gamma_L - \sqrt{p_0} - \delta > \gamma$, hence

$$e^{-(\gamma_L - \sqrt{p_0} - \delta)x/\varepsilon} \leqslant e^{-\gamma x/\varepsilon}. \tag{3.4.27}$$

As $\dfrac{\partial^k}{\partial \xi^k} = \varepsilon^{-k}\dfrac{\partial^k}{\partial x^k}$ from (3.3.55) we find

$$\max_{x\in[x_{i-1},x_{i+1}]}\left|\frac{\partial^k v}{\partial x^k}\right| \leqslant \varepsilon^{-k}e^{-\gamma x_{i-1}/\varepsilon} \qquad \text{for } k = 0,\ldots,4. \tag{3.4.28}$$

Equation (3.4.26) becomes

$$|R_1| \leqslant C\min\left\{\frac{|h_{i+1}-h_i|}{\varepsilon} + \frac{(h_{i+1}+h_i)^2}{\varepsilon^2}, 1\right\}e^{-\gamma x_{i-1}/\varepsilon}. \tag{3.4.29}$$

We now consider the Bakhvalov mesh and the Shishkin mesh separately.

(a) The Bakhvalov mesh

For the Bakhvalov mesh of §3.4.2 we consider two cases; $iN^{-1} \geqslant \theta_x - C_6N^{-1}$ and $iN^{-1} < \theta_x - C_6N^{-1} = 1/4 - C_3\varepsilon - C_6N^{-1}$.

Case A: $iN^{-1} \geqslant \theta_x - C_6N^{-1} = 1/4 - C_3\varepsilon - C_6N^{-1}$ with $C_6 \geqslant 2$. Firstly we have $h_{i+1} \pm h_i = O(N^{-1})$ and so recalling (3.1.3) we have $\frac{|h_{i+1} - h_i|}{\varepsilon} + \frac{(h_{i+1} + h_i)^2}{\varepsilon^2} \leqslant C$. For x_{i-1} we recall (3.4.12) and have

$$x_{i-1} = x\left(\frac{i-1}{N}\right) \geqslant -\frac{2\varepsilon}{\gamma} \ln\left(1 - 4\left(\frac{i-1}{N}\right)\right), \tag{3.4.30}$$

which can be written as

$$x_{i-1} = x\left(\frac{i-1}{N}\right) \geqslant -\frac{2\varepsilon}{\gamma} \ln\left(4C_3\varepsilon + 4(C_6+1)N^{-1}\right), \tag{3.4.31}$$

Now $e^{-\gamma x_{i-1}/\varepsilon} \leqslant e^{2\ln\left(4C_3\varepsilon+4(C_6+1)N^{-1}\right)} \leqslant C(\varepsilon + N^{-1})^2$ and by (3.4.29) we get $|R_1| \leqslant CN^{-2}$ for $iN^{-1} \geqslant \theta_x - C_6N^{-1}$.

Case B: $iN^{-1} < \theta_x - C_6N^{-1} = 1/4 - C_3\varepsilon - C_6N^{-1}$ with $C_6 \geqslant 2$. We can calculate $h_{i+1} + h_i$ and $|h_{i+1} - h_i|$ using Taylor expansions and the mean value theorem as

$$h_{i+1} + h_i = x_{i+1} - x_{i-1} = N^{-1}x'(\tilde{\zeta}) \leqslant CN^{-1}x'(\zeta_{i+1}), \tag{3.4.32}$$

and

$$|h_{i+1} - h_i| = |\zeta_{i+1}x'(\bar{\zeta}) - \zeta_{i-1}x'(\bar{\bar{\zeta}})| \leqslant CN^{-2}|(\bar{\zeta} - \bar{\bar{\zeta}})x''(\tilde{\zeta})| \leqslant CN^{-2}x''(\zeta_{i+1}), \tag{3.4.33}$$

where $\zeta_{i-1} < \tilde{\zeta} < \zeta_{i+1}$, $\zeta_{i-1} < \bar{\bar{\zeta}} < \zeta_i < \bar{\zeta} < \zeta_{i+1}$, and $x'(\zeta) = \frac{8\varepsilon}{\gamma}(1 - 4\zeta)^{-1}$

and $x''(\zeta) = \dfrac{32\varepsilon}{\gamma}(1-4\zeta)^{-2}$. Using (3.4.29) becomes

$$|R_1| \leqslant CN^{-2}(1-4\zeta_{i+1})^{-2}e^{-\gamma x_{i-1}/\varepsilon}. \tag{3.4.34}$$

Considering $e^{-\gamma x_{i-1}/\varepsilon}$ we have $x_{i-1} = -\dfrac{2\varepsilon}{\gamma}\ln(1-4\zeta_{i-1})$. Thus,

$$e^{-\gamma x_{i-1}/\varepsilon} = (1-4\zeta_{i-1})^2\,. \tag{3.4.35}$$

Noting $\zeta_i = iN^{-1}$, (3.4.34) is now

$$|R_1| \leqslant CN^{-2}\left(\frac{1-4(i-1)N^{-1}}{1-4(i+1)N^{-1}}\right)^2. \tag{3.4.36}$$

Finally as

$$\frac{1-4(i-1)N^{-1}}{1-4(i+1)N^{-1}} = 1 + \frac{8N^{-1}}{1-4(i+1)N^{-1}}, \tag{3.4.37}$$

$iN^{-1}-1/4 < -C_3\varepsilon - C_6N^{-1} < -C_6N^{-1}$ and so $1-4(i+1)N^{-1} > 4(C_6+1)N^{-1}$ and so

$$\frac{1-4(i-1)N^{-1}}{1-4(i+1)N^{-1}} < 1 + \frac{8N^{-1}}{4(C_6+1)N^{-1}} \leqslant C \tag{3.4.38}$$

and so (3.4.36) gives $|R_1| \leqslant CN^{-2}$ for $iN^{-1} < \theta_x - C_6N^{-1}$ and hence

$$|R_1| \leqslant CN^{-2}, \tag{3.4.39}$$

for the Bakhvalov mesh for $i = 1, \ldots, N-1$.

(b) The Shishkin mesh

For the Shishkin mesh of §3.4.2,

$$h_i = \begin{cases} 4\sigma_x N^{-1}, & i \leqslant N/4, \\ 2(1-2\sigma_x)N^{-1}, & i > N/4 \end{cases} \tag{3.4.40}$$

If $\sigma_x = 1/4$ then $1/4 \leqslant 2\gamma^{-1}\varepsilon\ln N$ and so $\ln^{-1}N \leqslant \varepsilon$ and $\varepsilon \leqslant N^{-1}$ is broken,

therefore we consider $\sigma_x = 2\gamma^{-1}\varepsilon \ln N$.

For $i \leqslant N/4 - 1$ we have $h_i = 8\gamma^{-1}\varepsilon N^{-1} \ln N$ and $h_{i+1} - h_i = 0$ giving

$$\frac{(h_{i+1} + h_i)^2}{\varepsilon^2} \leqslant CN^{-2} \ln^2 N. \tag{3.4.41}$$

As $e^{-\gamma x_{i-1}/\varepsilon} \leqslant 1$ from (3.4.29) we get $|R_1| \leqslant CN^{-2} \ln^2 N$ for $i \leqslant N/4 - 1$.

For $i \geqslant N/4 + 1$ we have $h_i = 2(1 - 4\gamma^{-1}\varepsilon \ln N)N^{-1} \leqslant 2N^{-1}$ and again $h_{i+1} - h_i = 0$ giving

$$\frac{|h_{i+1} - h_i|}{\varepsilon} + \frac{(h_{i+1} + h_i)^2}{\varepsilon^2} \leqslant C. \tag{3.4.42}$$

Considering $e^{-\gamma x_{i-1}/\varepsilon}$ we have $x_{i-1} \geqslant 4N^{-1}\sigma_x(i-1)$ and as $i - 1 \geqslant N/4$ and recalling (3.4.13) we get $x_{i-1} \geqslant 2\gamma^{-1}\varepsilon \ln N$. Thus $e^{-\gamma x_{i-1}/\varepsilon} \leqslant N^{-2}$ which gives $|R_1| \leqslant CN^{-2}$.

Finally considering $i = N/4$ we note $h_{i+1} \neq h_i$ as $h_i = 4\sigma_x N^{-1}$ and $h_{i+1} = 2(1 - 2\sigma_x)N^{-1}$. Therefore $h_{i+1} - h_i = (2 - 8\sigma_x)N^{-1}$ and $h_{i+1} + h_i = 2N^{-1}$ which gives

$$\frac{(h_{i+1} + h_i)^2}{\varepsilon^2} = 4N^{-2}\varepsilon^{-2}, \quad \frac{|h_{i+1} - h_i|}{\varepsilon} = 2N^{-1}\varepsilon^{-1} + 8\sigma_x N^{-1}\varepsilon^{-1}. \tag{3.4.43}$$

By (3.1.3) and (3.4.13), (3.4.43) becomes

$$\frac{(h_{i+1} + h_i)^2}{\varepsilon^2} \leqslant C, \quad \frac{|h_{i+1} - h_i|}{\varepsilon} \leqslant C. \tag{3.4.44}$$

As $x_i \geqslant \sigma_x$ we get $e^{-\gamma x_{i-1}/\varepsilon} \leqslant N^{-2}$ and so by (3.4.29) we have $|R_1| \leqslant CN^{-2}$. Hence on the Shishkin mesh,

$$|R_1| \leqslant CN^{-2} \ln^2 N \quad \text{for } i = 1, \ldots, N - 1. \tag{3.4.45}$$

Truncation Error R_2 and R_3 in time

We now consider R_2 defined in (3.4.22). To simplify notation in this section we define $w(x,\tau;p) := \tilde{w}_0(x,\tau;p)$. Taking Taylor expansions about ij we have

$$w_{i,j-1} = w_{ij} - k_j \frac{\partial w_{ij}}{\partial t} + \frac{k_j^2}{2} \left.\frac{\partial^2 w}{\partial t^2}\right|_{x_i, t_j - \theta t_{j-1}} \tag{3.4.46}$$

and so $D_t^- w_{ij} = \dfrac{\partial w_{ij}}{\partial t} + \dfrac{k_j}{2} \left.\dfrac{\partial^2 w}{\partial t^2}\right|_{x_i, t_j - \theta t_{j-1}}$ i.e., we have

$$|R_2| \leqslant C\varepsilon^2 \min\left\{ k_j \left| \left.\frac{\partial^2 w}{\partial t^2}\right|_{x_i, t_j - \theta t_{j-1}} \right|, \left| \frac{\partial w_{ij}}{\partial t} \right| \right\}. \tag{3.4.47}$$

Note that $\gamma_T^2 - p_0 > \gamma^2$ and also choose δ sufficiently small such that $\gamma_T^2 - p_0 - \delta > \gamma^2$, hence

$$e^{-(\gamma_T^2 - p_0 - \delta)t/\varepsilon} \leqslant e^{-\gamma^2 t/\varepsilon}. \tag{3.4.48}$$

As $\dfrac{\partial^l}{\partial t^l} = \varepsilon^{-2l} \dfrac{\partial^l}{\partial \tau^l}$, we can write (3.3.93) in the form

$$\max_{t \in [t_{j-1}, t_{j+1}]} \left| \frac{\partial^l w}{\partial t^l} \right| \leqslant \varepsilon^{-2l} e^{-\gamma^2 t_{j-1}/\varepsilon^2} \quad \text{for } l = 0, 1, 2, \tag{3.4.49}$$

and so (3.4.47) becomes

$$|R_2| \leqslant C \min\{k_j \varepsilon^{-2}, 1\} e^{-\gamma^2 t_{j-1}/\varepsilon^2}. \tag{3.4.50}$$

For R_3 defined in (3.4.22) we have

$$|R_3| \leqslant C \frac{(\hat{\varepsilon}^2(t_j) - \varepsilon^2)}{k_j} w_{ij} \leqslant C \frac{(\hat{\varepsilon}^2(t_j) - \varepsilon^2)}{k_j} e^{-\gamma^2 t_{j-1}/\varepsilon}. \tag{3.4.51}$$

Using the definition of $\hat{\varepsilon}_j$ in (3.4.6), (3.4.51) becomes

$$|R_3| \leqslant C \max\{0, \widehat{C} - \varepsilon^2/k_j\} e^{-\gamma^2 t_{j-1}/\varepsilon^2}. \tag{3.4.52}$$

The analysis for R_2 and R_3 hold true for all i, i.e., $i = 0, \dots, N$. We again consider each mesh separately.

(a) The Bakhvalov Mesh

We consider two cases; Case A: $jM^{-1} \leqslant \theta_t - C_5 M^{-1}$ and Case B: $jM^{-1} > \theta_t - C_5 M^{-1}$.

Case A: $jM^{-1} \leqslant \theta_t - C_5 M^{-1}$ with $C_5 \geqslant 1$. We have

$$k_j \leqslant M^{-1} t'(jM^{-1}) = M^{-1}\varepsilon^2/\gamma^2 (1/2 - jM^{-1})^{-1}, \tag{3.4.53}$$

and $e^{-\gamma^2 t_{j-1}/\varepsilon^2} = 1 - 2(j-1)M^{-1}$. Combining this in (3.4.50) gives

$$|R_2| \leqslant C \min\{M^{-1}(1 - 2jM^{-1})^{-1}, 1\} \left(1 - 2(j-1)M^{-1}\right) \tag{3.4.54}$$

which can be written as

$$|R_2| \leqslant CM^{-1} \left(\frac{1 - 2(j-1)M^{-1}}{1 - 2jM^{-1}} \right) = CM^{-1} \left(1 + \frac{M^{-1}}{1 - 2jM^{-1}} \right). \tag{3.4.55}$$

As we have $jM^{-1} \leqslant \theta_t - C_5 M^{-1}$ and by recalling (3.4.13) we can write $1/2 - jM^{-1} \geqslant C_5 M^{-1}$ and so obtain

$$|R_2| \leqslant CM^{-1} \quad \text{for } jM^{-1} \leqslant \theta_t - C_5 M^{-1}. \tag{3.4.56}$$

Considering R_3 defined in (3.4.52) which again by (3.4.13) we have $\theta_t - jM^{-1} \geqslant C_5 M^{-1}$ giving $1/2 - jM^{-1} \geqslant C_5 M^{-1}$ and so recalling (3.4.53) yields

$$k_j \leqslant \frac{\varepsilon^2}{C_5 \gamma^2}. \tag{3.4.57}$$

Using (3.4.57) with (3.4.52) results in

$$\widehat{C} - \frac{\varepsilon^2}{k_j} \leqslant \widehat{C} - \gamma^2 C_5. \tag{3.4.58}$$

By choosing C_5 sufficiently large we have $\widehat{C} - \dfrac{\varepsilon^2}{k_j} \leqslant 0$ and so $R_3 = 0$.

Case B: $jM^{-1} > \theta_t - C_5 M^{-1}$. Recalling θ_t from (3.4.13) we have $j/M > 1/2 - C_4\varepsilon^2 - C_5 M^{-1}$ and

$$t_{j-1} = -\varepsilon^2/\gamma^2 \ln(1 - 2(j-1)M^{-1}) \geqslant -\varepsilon^2/\gamma^2 \ln(2M^{-1}(1 + C_5) + 2C_4\varepsilon^2) \tag{3.4.59}$$

and so

$$e^{-\gamma^2 t_{j-1}/\varepsilon^2} \leqslant e^{\ln(2M^{-1}(1+C_5)+2C_4\varepsilon^2)} \leqslant C(M^{-1} + \varepsilon^2). \tag{3.4.60}$$

Recalling R_2 from (3.4.50) we have

$$|R_2| \leqslant C \min\{k_j/\varepsilon^2, 1\}(M^{-1} + \varepsilon^2), \tag{3.4.61}$$

and as $k_j \leqslant M^{-1}$ and by (3.1.3) we obtain

$$|R_2| \leqslant CM^{-1} \quad \text{for } j = 1, \ldots, M. \tag{3.4.62}$$

For $|R_3|$ we have $\widehat{C} - \varepsilon^2 k_j^{-1} > 0$ implies $\varepsilon^2 \leqslant CM^{-1}$ hence (3.4.52) is

$$|R_3| \leqslant C(\widehat{C} - \varepsilon^2 k_j^{-1}) e^{-\gamma^2 t_{j-1}/\varepsilon^2} \leqslant C(M^{-1} + \varepsilon^2) \leqslant CM^{-1}. \tag{3.4.63}$$

Hence for the Bakhvalov mesh we have

$$|R_2| \leqslant CM^{-1}, \quad |R_3| \leqslant CM^{-1} \quad \text{for } i = 0, \ldots, N,\ j = 0, \ldots, M. \tag{3.4.64}$$

(b) The Shishkin Mesh

For the Shishkin mesh we first note if $\sigma_t = T/2$ then $T/2 \leqslant \dfrac{\varepsilon^2}{\gamma^2}\ln M$ which contradicts (3.1.3) and so $\sigma_t = \dfrac{\varepsilon^2}{\gamma^2}\ln M$.

For $j \leqslant M/2$ we have $k_j = 2\sigma_t M^{-1}$, i.e., $k_j = C\varepsilon^2 M^{-1}\ln M$, and so $k_j/\varepsilon^2 \leqslant CM^{-1}\ln M$, and as $e^{-\gamma^2 t_{j-1}/\varepsilon^2} \leqslant 1$ then

$$|R_2| \leqslant CM^{-1}\ln M. \tag{3.4.65}$$

As $\widehat{C} - \dfrac{\varepsilon^2}{k_j} \leqslant \widehat{C} - 1/2\gamma^2 M^{-1}\ln M \leqslant 2\gamma - 2M^{-1}\ln M \leqslant C\dfrac{k_j}{\varepsilon^2}$ we also have

$$|R_3| \leqslant CM^{-1}\ln M. \tag{3.4.66}$$

For $j > M/2$, we have $e^{-\gamma^2 t_{j-1}/\varepsilon^2} \leqslant e^{-\gamma^2\sigma_t/\varepsilon^2} = M^{-1}$, as $T - \dfrac{\varepsilon^2}{\gamma^2}\ln M \geqslant T/2$ and recalling $\varepsilon^2 \leqslant CM^{-1}$ we get

$$\widehat{C}k_j - \varepsilon^2 = \widehat{C}[2T - 2\frac{\varepsilon^2}{\gamma^2}\ln M]M^{-1} - \varepsilon^2 \geqslant \widehat{C}TM^{-1} - \varepsilon^2 \geqslant 0 \tag{3.4.67}$$

and so we have

$$|R_2| \leqslant CM^{-1}, \quad |R_3| \leqslant CM^{-1}. \tag{3.4.68}$$

Hence for the Shishkin mesh we have

$$|R_2| \leqslant CM^{-1}\ln M, \quad |R_3| \leqslant CM^{-1}\ln M \qquad \text{for } i = 0,\dots,N,\ j = 0,\dots,M. \tag{3.4.69}$$

Truncation Error for the Neumann Boundary Conditions

Finally for the points x_0 and x_N we consider (3.4.5b), (3.4.5c) and (3.4.15) and take x_0 as an example. We look to find $|\widehat{\mathcal{G}}^h\beta(x_0,t_j;p) - \mathcal{G}\beta(x_0,t_j;p)|$.

Recalling (3.4.15) and (3.4.5) we define R_4 as

$$\begin{aligned} R_4 := |\widehat{\mathcal{G}}^h\beta_{0j} - \mathcal{G}\beta_{0j}| =& \frac{h_1}{2\varepsilon}(\hat{\varepsilon}^2(t_j) - \varepsilon^2)D_t^-\beta_{0j} + \frac{h_1\varepsilon}{2}D_t^-\beta_{0j} - \varepsilon D_x^-\beta_{1,j} \\ &+ \frac{h_1}{2\varepsilon}f(x_0,t_j,\beta_{0j}) + \varepsilon\left.\frac{\partial\beta}{\partial x}\right|_{x=0}. \end{aligned} \tag{3.4.70}$$

From (3.4.70) we consider $-\varepsilon D_x^-\beta_{1,j} + \varepsilon\left.\frac{\partial\beta}{\partial x}\right|_{x=0}$ and take the Taylor expansion of β_{1j} to get

$$-\varepsilon D_x^-\beta_{1,j} + \varepsilon\left.\frac{\partial\beta}{\partial x}\right|_{x=0} = -\frac{\varepsilon h_1}{2}\left.\frac{\partial^2\beta}{\partial x^2}\right|_{x=0} - \frac{\varepsilon h_1^2}{3!}\left.\frac{\partial^3\beta}{\partial x^3}\right|_{x=x_0+h_1\theta} \tag{3.4.71}$$

Looking at the final term in (3.4.71), we recall (3.3.55), note $x_0 = 0$ and find

$$\frac{\varepsilon h_1^2}{3!}\left.\frac{\partial^3\beta}{\partial x^3}\right|_{x=x_0+h_1\theta} = \frac{h_1^2}{3!\varepsilon^2}\left.\frac{\partial^3\beta}{\partial\xi^3}\right|_{\xi=x_0/\varepsilon+h_1\theta/\varepsilon} = O\left(\frac{h_1^2}{\varepsilon^2}\right). \tag{3.4.72}$$

For the Shishkin mesh of §3.4.2 we have $h_1 = \dfrac{8\varepsilon\ln N}{\gamma N}$, which gives

$$\frac{h_1^2}{\varepsilon^2} = O(N^{-2}\ln^{2m} N). \tag{3.4.73}$$

For the Bakhvalov mesh of §3.4.2 we can write $h_1 = N^{-1}x'(0+\theta)$ for some $\theta \in (0,1)$. Recalling $x'(\theta/N) = \dfrac{8\varepsilon}{\gamma}(1-4\theta/N)^{-1}$. For N sufficiently large we can say $|(1-4\theta/N)^{-1}| \leqslant C$ and so $|x'(\theta/N)| \leqslant C\varepsilon$ and $h_1 \leqslant C\varepsilon N^{-1}$, i.e., $h_1/\varepsilon \leqslant CN^{-1}$. Combining these results we have

$$\frac{h_1^2}{\varepsilon^2} = O(N^{-2}\ln^{2m} N), \tag{3.4.74}$$

with $m = 1$ for the Shishkin mesh and $m = 0$ for the Bakhvalov mesh.

Combining (3.4.72) and (3.4.74), (3.4.71) is

$$-\varepsilon D_x^- \beta_{1,j} + \varepsilon \left.\frac{\partial \beta}{\partial x}\right|_{x=0} = -\frac{\varepsilon h_1}{2} \left.\frac{\partial^2 \beta}{\partial x^2}\right|_{x=0} + O(N^{-2} \ln^{2m} N). \tag{3.4.75}$$

Considering the term $\frac{h_1 \hat{\varepsilon}^2(t_j)}{2\varepsilon} D_t^- \beta_{0j}$ from (3.4.70) we have

$$\frac{h_1 \hat{\varepsilon}^2(t_j)}{2\varepsilon} D_t^- \beta_{0j} = \frac{h_1}{2\varepsilon}\left[\varepsilon^2 D_t^- \beta_{0j} + (\hat{\varepsilon}^2(t_j) - \varepsilon^2) D_t^- \beta_{0j}\right] \tag{3.4.76}$$

Recalling (3.4.64) and (3.4.69) we have

$$(\hat{\varepsilon}^2(t_j) - \varepsilon^2) D_t^- \beta_{0j} = O(M^{-1} \ln^m M), \tag{3.4.77}$$

while taking a Taylor expansion of $D_t^- \beta_{0j}$ about t_j, and noting (3.1.3), we get

$$\frac{h_1 \varepsilon}{2} D_t^- \beta_{0j} = \frac{h_1 \varepsilon}{2} \frac{\partial \beta_{0j}}{\partial t} + O\left(M^{-1} \ln^m M\right). \tag{3.4.78}$$

Combining (3.4.75), (3.4.77) and (3.4.78),

$$\begin{aligned} R_4 =& \frac{h_1 \varepsilon}{2} \frac{\partial \beta_{0j}}{\partial t} - \frac{\varepsilon h_1}{2} \left.\frac{\partial^2 \beta}{\partial x^2}\right|_{x=0} + \frac{h_1}{2\varepsilon} f(x_0, t_j, \beta_{0j}) \\ &+ O(N^{-2} \ln^{2m} N + M^{-1} \ln^m M), \end{aligned} \tag{3.4.79}$$

i.e.,

$$R_4 = \frac{h_1}{2\varepsilon} \mathcal{T} \beta_{0j} + O(N^{-2} \ln^{2m} N + M^{-1} \ln^m M). \tag{3.4.80}$$

A similar method can be used to obtain (3.4.18). □

Lemma 3.4.2. *The discretisation* (3.4.5) *is a Z-field.*

Proof. We take the solution matrix $\hat{U}_{ij}$ with $i = 0, \ldots, N$ and $j = 0, \ldots, M$ and rewrite it as a vector time level by time level, i.e.,

$$\vec{\hat{U}} = \begin{pmatrix} \hat{U}_{00} \\ \hat{U}_{10} \\ \vdots \\ \hat{U}_{N0} \\ \hat{U}_{10} \\ \vdots \\ \hat{U}_{1N} \\ \vdots \\ \hat{U}_{NM} \end{pmatrix} \tag{3.4.81}$$

$$\hat{U}_{i0} = \varphi(x_i) \quad i = 0, \ldots, N. \tag{3.4.82}$$

The matrix representation of our system is

$$\hat{\mathcal{T}}^N \vec{\hat{U}} = A\vec{\hat{U}} + \vec{F}(\vec{X}, \vec{T}, \vec{\hat{U}}) = \mathbf{0}, \tag{3.4.83}$$

where A is defined as

$$A := \begin{pmatrix} I_{N+1} & & & & \\ C_1 & B_1 & & & \\ & C_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & C_M & B_M \end{pmatrix} \tag{3.4.84}$$

where I_{N+1} is the $(N+1) \times (N+1)$ identity matrix and C_j and B_j are defined

as

$$B_j := \begin{pmatrix} \frac{\varepsilon}{h_1} + \frac{h_1\hat{\varepsilon}^2(t_j)}{2\varepsilon k_j} & -\frac{\varepsilon}{h_1} & & & \\ -\frac{\varepsilon^2}{h_1\hbar_1} & \frac{\hat{\varepsilon}^2(t_j)}{k_j} + \varepsilon^2 a_1 & -\frac{\varepsilon^2}{h_2\hbar_1} & & \\ & & \ddots & & \\ & & -\frac{\varepsilon^2}{h_{N-1}\hbar_{N-1}} & \frac{\hat{\varepsilon}^2(t_j)}{k_j} + \varepsilon^2 a_{N-1} & -\frac{\varepsilon^2}{h_N\hbar_{N-1}} \\ & & & -\frac{\varepsilon}{h_N} & \frac{\varepsilon}{h_N} + \frac{h_N\hat{\varepsilon}^2(t_j)}{2\varepsilon k_j} \end{pmatrix}, \tag{3.4.85}$$

where $a_i := \frac{1}{h_i\hbar_i} + \frac{1}{\hbar_i h_{i+1}}$,

$$C_j := \hat{\varepsilon}^2(t_j) \begin{pmatrix} -\frac{h_1}{2\varepsilon k_j} & & & & & 0 \\ & -\frac{1}{k_j} & & & & \\ & & -\frac{1}{k_j} & & & \\ & & & \ddots & & \\ & & & & -\frac{1}{k_j} & \\ & & & & & -\frac{h_N}{2\varepsilon k_j} \end{pmatrix} \tag{3.4.86}$$

and F defined as

$$F := \begin{pmatrix} -\varphi(x_0) \\ \vdots \\ -\varphi(x_N) \\ \frac{h_1}{2\varepsilon} f(x_0, t_1, \hat{U}_{01}) + g_0(t_1) \\ f(x_1, t_1, \hat{U}_{11}) \\ \vdots \\ f(x_{N-1}, t_1, \hat{U}_{N-1,1}) \\ \frac{h_N}{2\varepsilon} f(x_N, t_1, \hat{U}_{N,1}) - g_1(t_1) \\ \frac{h_1}{2\varepsilon} f(x_0, t_2, \hat{U}_{02}) + g_0(t_2) \\ f(x_1, t_2, \hat{U}_{12}) \\ \vdots \\ f(x_{N-1}, t_M, \hat{U}_{N-1,M}) \\ \frac{h_N}{2\varepsilon} f(x_N, t_M, \hat{U}_{N,M}) - g_1(t_M) \end{pmatrix} \tag{3.4.87}$$

The Jacobian of the system can be written as $J = A + D$ where D is defined as

$$D := \begin{pmatrix} \mathbf{0} & & & & & \\ & D_1 & & & & \\ & & \ddots & & & \\ & & & D_j & & \\ & & & & \ddots & \\ & & & & & D_M \end{pmatrix}, \tag{3.4.88}$$

with

$$D_j := \begin{pmatrix} \frac{h_1}{2\varepsilon} f_u(x_0, t_j, \hat{U}_{0j}) & & & & \\ & f_u(x_1, t_j, \hat{U}_{1j}) & & & \\ & & \ddots & & \\ & & & f_u(x_{N-1}, t_j, \hat{U}_{N-1j}) & \\ & & & & \frac{h_N}{2\varepsilon} f_u(x_N, t_j, \hat{U}_{Nj}) \end{pmatrix} \tag{3.4.89}$$

Now considering the off-diagonal entries in the matrix J we have that all off-diagonal entries in D are zero. In B and C all off-diagonal entries are non-positive. Hence the operator is a Z-field as all off-diagonal entries of the Jacobian are non-positive.

□

The following theorem gives existence and accuracy of the unique discrete solution.

Theorem 3.4.1. *With* $\beta(x_i, t_j; \pm p)$ *defined in* (3.3.116) *and the meshes defined in §3.4.2 and §3.4.2 there exists solutions* U_{ij} *and* $\widehat{U}_{ij}$ *such that*

$$\beta(x_i, t_j; -p) \leqslant U_{ij} \leqslant \beta(x_i, t_j; p), \tag{3.4.90}$$

$$\beta(x_i, t_j; -p) \leqslant \hat{U}_{ij} \leqslant \beta(x_i, t_j; p). \tag{3.4.91}$$

Furthermore for the bilinear interpolants of U_{ij} *and* $\widehat{U}_{ij}$, U^I_{ij} *and* $\widehat{U}^I_{ij}$, *we have*

$$|U^I_{ij}(x,t) - u(x,t)| \leqslant CN^{-2}\ln^{2m} N + CM^{-1}\ln^m M, \tag{3.4.92}$$

$$|\hat{U}^I_{ij}(x,t) - u(x,t)| \leqslant CN^{-2}\ln^{2m} N + CM^{-1}\ln^m M, \tag{3.4.93}$$

for all $(x,t) \in [0,1] \times [0,T]$ *where* $u(x,t)$ *is the unique solution of* (3.0.1) *and* $m = 0$ *for the Bakhvalov mesh and* $m = 1$ *for the Shishkin mesh.*

Proof. We call on the theory of Z-fields from §1.0.2 and note as

$$\widehat{\mathcal{T}}^N U = \hat{\varepsilon}_j^2 D_t^- U_{ij} - \varepsilon^2 \delta_x^2 U_{ij} + f(x,t,U)_{ij}, \tag{3.4.94}$$

where the mapping

$$\mathcal{T}^N : \mathbb{R}^{(N+1)+(M+1)} \to \mathbb{R}^{(N+1)+(M+1)}, \tag{3.4.95}$$

is a Z-field.

By Lemma 3.4.1, for $C_7 > 0$ sufficiently large, we have

$$|\widehat{\mathcal{T}}^h \beta(x_i, t_j; \bar{p}) - \mathcal{T}\beta(x_i, t_j; \bar{p})| \leqslant C_7(N^{-2} \ln^{2m} N + M^{-1} \ln^m M). \tag{3.4.96}$$

Choose $\bar{p} := C_6(N^{-2} \ln^{2m} N + M^{-1} \ln^m M)$ with C_6 sufficiently large such that $C_7(N^{-2} \ln^{2m} N + M^{-1} \ln^m M) \leqslant C_0 \bar{p}\gamma^2/3$ i.e., $C_7 \leqslant C_0 C_6 \gamma^2/3$, i.e.,

$$|\widehat{\mathcal{T}}^N \beta(x_i, t_j; \bar{p}) - \mathcal{T}\beta(x_i, t_j; \bar{p})| \leqslant C_0 \bar{p}\gamma^2/3. \tag{3.4.97}$$

Recalling (3.1.3) we can see $C_1\varepsilon^2 \leqslant C_1' \bar{p}^2$ and as $\bar{p}$ becomes sufficiently small $C_1'\bar{p}^2 \leqslant C_0\bar{p}\gamma^2/3$. Again as $\bar{p}$ becomes sufficiently small we have $C_1\bar{p}^2 \leqslant C_0\bar{p}\gamma^2/3$. Now (3.3.134) becomes, for $\bar{p} \geqslant 0$,

$$\mathcal{T}\beta(x, t; \bar{p}) \geqslant C_0\bar{p}\gamma^2 - 2C_0\bar{p}\gamma^2/3. \tag{3.4.98}$$

Combining (3.4.97) and (3.4.98) gives

$$\widehat{\mathcal{T}}^h \beta(x_i, t_j; \bar{p}) \geqslant 0. \tag{3.4.99}$$

Similarly for $-\bar{p}$ we find

$$\widehat{\mathcal{T}}^h \beta(x_i, t_j; -\bar{p}) \leqslant 0. \tag{3.4.100}$$

Choosing C_7 sufficiently large we recall (3.4.17) and have

$$\widehat{\mathcal{G}}^h\beta(x_0,t_j;\bar{p})-\mathcal{G}\beta(x_0,t_j;\bar{p}) \geqslant \frac{h_1}{2\varepsilon}\mathcal{T}\beta(x_0,t_j;\bar{p})-C_7(N^{-2}\ln^{2m}N+M^{-1}\ln^m M) \tag{3.4.101}$$

By (3.3.134) we have $\mathcal{T}\beta(x_0,t_j;\bar{p}) \geqslant 0$. We recall (3.3.142) and (3.4.101) becomes

$$\widehat{\mathcal{G}}^h\beta(x_0,t_j;\bar{p}) \geqslant \frac{c_0C_0\bar{p}}{2} - C_7(N^{-2}\ln^{2m}N + M^{-1}\ln^m M). \tag{3.4.102}$$

By choosing C_6 sufficiently large such that $C_7 \leqslant \dfrac{c_0C_0C_6}{4}$ and so (3.4.103) is

$$\widehat{\mathcal{G}}^h\beta(x_0,t_j;\bar{p}) \geqslant -\frac{c_0C_0\bar{p}}{4} + \frac{c_0C_0\bar{p}}{2} \geqslant 0. \tag{3.4.103}$$

Hence we have established a discrete analogue of (3.3.120c) and (3.3.120d) is found by a similar method.

As $\beta(x_i,t_j;-\bar{p}) \leqslant \beta(x_i,t_j;\bar{p})$, we have established that the boundary conditions satisfy discrete versions of (3.3.120c) and (3.3.120d), and as (3.3.146) holds giving a discrete version of (3.3.120e) we have $\beta(x_i,t_j,\bar{p})$ and $\beta(x_i,t_j,-\bar{p})$ are discrete upper and lower solutions of (3.0.1).

Next by (3.3.121) we have $\beta(x_i,t_j,\pm\bar{p}) = u_{as}(x_i,t_j)+O(\bar{p})$. By Lemma 1.0.1 a solution, $\widehat{U}_{i,j}$, exists with $\beta(x_i,t_j,-\bar{p}) \leqslant \hat{U}_{ij} \leqslant \beta(x_i,t_j,\bar{p})$. Again by (3.3.121) we have $\hat{U}_{ij} = u_{as}(x_i,t_j) + O(\bar{p})$ and $\hat{U}^I_{ij} = u^I_{as}(x_i,t_j) + O(\bar{p})$ where u^I_{as} is the bilinear interpolant of u_{as}.

As $|u^I_{as} - u_{as}| \leqslant C(N^{-2} + M^{-1})\|D^2u_{as}\|_{L^\infty(\mathcal{D})}$ with

$$D^2u := \left(\sum_{i,j=1}\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2\right)^{1/2}, \tag{3.4.104}$$

and by (3.3.2) with the bounds (3.3.55) and (3.3.93) we have $\|D^2u_{as}\|_{L^\infty(\mathcal{D})} \leqslant$

C. By the definition of $\bar{p}$ above we can say $u_{as}^I = u_{as} + O(\bar{p})$ hence $\hat{U}_{ij}^I = u_{as}(x_i, t_j) + O(\bar{p})$. Finally with (3.3.139) and (3.1.3) we have $u_{as}(x,t) = u(x,t) + O(\varepsilon^2) = u(x,t) + O(\bar{p})$ which gives the desired result. $\square$

3.5 Conclusions

We considered a singularly perturbed time-dependent reaction-diffusion equation with singularly perturbed Neumann boundary conditions. Certain assumptions were made to give a sufficiently smooth asymptotic solution and to avoid existence of corner layers. The stabilised method was used replacing ε^2 in front of the time derivative with a new parameter, $\hat{\varepsilon}^2(t_j)$ defined by $\hat{\varepsilon}(x_{ij}) := \max\{\varepsilon^2, \widehat{C}k_j\}$ where k_j is the time step. By creating upper and lower solutions the truncation error of the discrete system was estimated. The problem was considered on two meshes; the Shishkin mesh on which the truncation error was found to be $O(N^{-2}\ln^2 N + M^{-1}\ln M)$ and the Bakhvalov mesh on which the truncation error was found to be $O(N^{-2} + M^{-1})$, where N and M are the number of space steps and time steps respectively. Existence of a unique computed solution was proven and by showing the discretisation was a Z-field we proved existence of the computed solution and obtained accuracy results for this solution.

3.6 Technical Properties of the Asymptotic Analysis

3.6.1 Order of $\mathcal{T}u_{as}(x,t)$

We now consider the order of our system in four regions, the outer region, the boundary layer region, the initial layer region and the corner layer region.

Lemma 3.6.1. *For the asymptotic expansion $u_0(x)$ of problem (3.0.1) away from $x = 0$ and $t = 0$ we have accuracy*

$$\mathcal{T}u_0(x,t) = O(\varepsilon^2) \quad \text{for } x \in (0,1] \times (0,T]. \tag{3.6.1}$$

Proof. We want to find the order of

$$\mathcal{T}u_0(x,t) := \varepsilon^2 \frac{\partial u_0}{\partial t} - \varepsilon^2 \frac{\partial u_0}{\partial x^2} + F(x,t,0). \tag{3.6.2}$$

As $F(x,t,0) = 0$,

$$\mathcal{T}u_0(x,t) = \varepsilon^2 \frac{\partial u_0}{\partial t} - \varepsilon^2 \frac{\partial u_0}{\partial x^2} = O(\varepsilon^2), \tag{3.6.3}$$

giving the desired result. □

Lemma 3.6.2. *For the asymptotic expansion $u_0(x,t) + v_0(\xi,t) + \varepsilon v_1(\xi,t)$ of problem (3.0.1) with $\xi = x/\varepsilon$, near $x = 0$, we have accuracy*

$$\mathcal{T}\left[u_0(x,t) + v_0(\xi,t) + \varepsilon v_1(\xi,t)\right] = O(\varepsilon^2). \tag{3.6.4}$$

Proof. By Lemma 3.6.1 we can write

$$\begin{aligned}\mathcal{T}\left[u_0(x,t)+v_0(\xi,t)+\varepsilon v_1(\xi,t)\right] = &\left(\varepsilon^2\frac{\partial}{\partial t}-\frac{\partial^2}{\partial \xi^2}\right)(v_0+\varepsilon v_1)\\ &+ F(\varepsilon\xi,t,s)\big|_{s=0}^{s=v_0+\varepsilon v_1+\varepsilon^2 v_2}+O(\varepsilon^4).\end{aligned} \tag{3.6.5}$$

We now define $\mathcal{G}(\varepsilon) := F(\varepsilon\xi,t,s)\Big|_{s=0}^{s=v_0+\varepsilon v_1}$ and take a Taylor expansion about $\varepsilon = 0$, giving

$$\mathcal{G}(\varepsilon)=\mathcal{G}(0)+\varepsilon\mathcal{G}'(0)+\frac{\varepsilon^2}{2}\mathcal{G}''(\varepsilon^*), \tag{3.6.6}$$

where $0<\varepsilon^*<\varepsilon$. Using $\mathcal{G}'(\varepsilon)$ and $\mathcal{G}''(\varepsilon)$ from Lemma 2.7.3, by replacing $\bar{x}$ with 0 and r with x, then

$$\mathcal{G}(0)=F(0,t,v_0), \tag{3.6.7}$$

and

$$\mathcal{G}'(0)=\left(\xi\frac{\partial F}{\partial x}+v_1\frac{\partial F}{\partial s}\right)\Bigg|_{s=v_0,x=0}. \tag{3.6.8}$$

We note $\mathcal{G}''(\varepsilon^*)$ is

$$\mathcal{G}''(\varepsilon^*)=\left(\xi^2\frac{\partial^2 F}{\partial x^2}+2\xi v_1\frac{\partial^2 F}{\partial x\partial s}+v_1^2\frac{\partial^2 F}{\partial s^2}\right)\Bigg|_{s=0,x=\varepsilon^*\xi}^{s=v_0+\varepsilon^* v_1}, \tag{3.6.9}$$

which can be rewritten using the mean value theorem as

$$\mathcal{G}''(\varepsilon^*)=(v_0+\varepsilon^* v_1)\xi^2\frac{\partial^3 F}{\partial x^2\partial s}\Bigg|_{s=(v_0+\varepsilon^* v_1)\theta}+\left(2\xi v_1\frac{\partial^2 F}{\partial x\partial s}+v_1^2\frac{\partial^2 F}{\partial s^2}\right)\Bigg|_{s=v_0+\varepsilon^* v_1}, \tag{3.6.10}$$

for some $\theta\in(0,1)$.

As v_i decays exponentially as $\xi\to\infty$ by (3.3.55) then $|\xi^n v_i|$ is bounded for $n=1,2$ and $i=0,1$ and as $F(x,t,u)$ and its derivatives are bounded, we

get $\mathcal{G}''(\varepsilon^*) = O(1)$. Now for $G(\varepsilon)$ from (3.6.6) we have

$$\mathcal{G}(\varepsilon) = F(0, t, v_0) + \varepsilon \left(\xi \frac{\partial F}{\partial x} + v_1 \frac{\partial F}{\partial s} \right) \bigg|_{s=v_0, x=0} + O(\varepsilon^2). \tag{3.6.11}$$

Putting (3.6.11) back into (3.6.5) gives

$$\begin{aligned} \mathcal{T}\left[u_0(x,t) + v_0(\xi,t) + \varepsilon v_1(\xi,t)\right] &= \left(\varepsilon^2 \frac{\partial}{\partial t} - \frac{\partial^2}{\partial \xi^2} \right) (v_0 + \varepsilon v_1) + F(0,t,v_0) \\ &\quad + \varepsilon \left(\xi \frac{\partial F}{\partial r} + v_1 \frac{\partial F}{\partial s} \right) \bigg|_{s=v_0, x=0} + O(\varepsilon^2). \end{aligned} \tag{3.6.12}$$

Extracting the equations for $v_0(\xi, t)$ and $v_1(\xi, t)$ gives

$$-\frac{\partial^2 v_0}{\partial \xi^2} + F(0, t, v_0) = 0, \tag{3.6.13}$$

and

$$-\frac{\partial^2 v_1}{\partial \xi^2} + v_1 \frac{\partial F}{\partial s} \bigg|_{s=v_0, x=0} = -\xi \frac{\partial F}{\partial r} \bigg|_{s=v_0, x=0}. \tag{3.6.14}$$

We are now left with

$$\mathcal{T}\left[u_0(x,t) + v_0(\xi,t) + \varepsilon v_1(\xi,t)\right] = \varepsilon^2 \frac{\partial}{\partial t}(v_0 + \varepsilon v_1) + O(\varepsilon^2), \tag{3.6.15}$$

and as these remaining terms are $O(\varepsilon^2)$ we obtain the desired result. □

Lemma 3.6.3. *For the asymptotic expansion $u_0(x,t) + w_0(x,\tau)$ of problem (3.0.1) with $\tau = t/\varepsilon^2$, near $t = 0$ we have accuracy*

$$\mathcal{T}\left[u_0(x,t) + w_0(x,\tau)\right] = O(\varepsilon^2). \tag{3.6.16}$$

Proof. As in the boundary layer case we start by using Lemma 3.6.1 giving

$$\mathcal{T}\left[u_0(x,t) + w_0(x,\tau)\right] = \left(\frac{\partial}{\partial\tau} - \varepsilon^2\frac{\partial^2}{\partial x^2}\right)w_0 + F(x,\varepsilon^2\tau,s)\Big|_{s=0}^{s=w_0} + O(\varepsilon^2). \tag{3.6.17}$$

We define $\mathcal{H}(\varepsilon)$ as

$$\mathcal{H}(\varepsilon) := F(x,\varepsilon^2\tau,s)\Big|_{s=0}^{s=w_0}, \tag{3.6.18}$$

and take a Taylor expansion of this about $\varepsilon = 0$, giving

$$\mathcal{H}(\varepsilon) = \mathcal{H}(0) + \varepsilon\mathcal{H}'(0) + \frac{\varepsilon^2}{2}\mathcal{H}''(\varepsilon^*), \tag{3.6.19}$$

with $0 < \varepsilon^* < \varepsilon$. Taking the necessary derivatives of (3.6.19) yields

$$\mathcal{H}'(\varepsilon) = 2\varepsilon\tau\frac{\partial F}{\partial t}\Big|_{s=0}^{s=w_0}, \tag{3.6.20}$$

and

$$\mathcal{H}''(\varepsilon) = (2\varepsilon\tau)^2\frac{\partial^2 F}{\partial t^2}\Big|_{s=0}^{s=w_0} + 2\tau\frac{\partial F}{\partial t}\Big|_{s=0}^{s=w_0}. \tag{3.6.21}$$

Evaluating $\mathcal{H}(0)$ and $\mathcal{H}'(0)$ we have

$$\mathcal{H}(0) = F(x,0,s)\Big|_{s=0}^{s=w_0}, \tag{3.6.22}$$

and

$$\mathcal{H}'(0) = 0. \tag{3.6.23}$$

For $\mathcal{H}''(\varepsilon)$ we can use the mean value theorem and get

$$\mathcal{H}''(\varepsilon) = (2\varepsilon\tau)^2 w_0\frac{\partial^3 F}{\partial t^2\partial s}\Big|_{s=w_0\theta} + 2\tau w_0\frac{\partial^2 F}{\partial t\partial s}\Big|_{s=w_0\theta}, \tag{3.6.24}$$

for some $\theta \in (0,1)$. For the initial layer function, similar to the boundary layer function, we have $|\tau^n w_0|$ is bounded for $n = 1, 2$ as w_0 decays exponentially as

$\tau \to \infty$. As $F(x,t,s)$ and its derivatives are bounded we have $\mathcal{H}''(\varepsilon^*) = O(1)$. Noting $\left.\dfrac{\partial^m F}{\partial t^m}\right|_{s=0} = 0$, (3.6.19) is now

$$\mathcal{H}(\varepsilon) = F(x,0,w_0) + O(\varepsilon^2), \tag{3.6.25}$$

and from (3.6.17) we obtain

$$\mathcal{T}\left[u_0(x,t) + w_0(x,\tau)\right] = \left(\frac{\partial}{\partial \tau} - \varepsilon^2 \frac{\partial^2}{\partial x^2}\right) w_0 + F(x,0,w_0) + O(\varepsilon^2). \tag{3.6.26}$$

Extracting the equation for the initial layer function $w_0(x,\tau)$ yields

$$\frac{\partial w_0}{\partial \tau} + F(x,0,w_0) = 0 \tag{3.6.27}$$

The remaining terms are

$$\mathcal{T}\left[u_0(x,t) + w_0(x,\tau)\right] = -\varepsilon^2 \frac{\partial^2 w_0}{\partial x^2} + O(\varepsilon^2), \tag{3.6.28}$$

which are $O(\varepsilon^2)$, i.e., we have the desired result. $\square$

3.6.2 Corner Layers; near $x = 0$, $t = 0$

We consider the existence of corner layer functions in the region where both x and t are small, i.e., we are looking at the asymptotic expansion

$$\begin{aligned} u_{as}(\varepsilon\xi, \varepsilon^2\tau) :=& \, u_0(\varepsilon\xi, \varepsilon^2\tau) + v_0(\xi, \varepsilon^2\tau) + \varepsilon v_1(\xi, \varepsilon^2\tau) \\ &+ w_0(\varepsilon\xi, \tau) + q_0(\xi,\tau) + \varepsilon q_1(\xi,\tau), \end{aligned} \tag{3.6.29}$$

where $q_0(\xi,\tau)$ and $q_1(\xi,\tau)$ are corner layer functions with

$$\lim_{\xi\to\infty} q_0(\xi,\tau) = 0, \quad \lim_{\tau\to\infty} q_0(\xi,\tau) = 0, \tag{3.6.30}$$

and

$$\lim_{\xi\to\infty} q_1(\xi,\tau) = 0, \quad \lim_{\tau\to\infty} q_1(\xi,\tau) = 0. \tag{3.6.31}$$

In this section we will show that by the condition (B5) we have $q_0(\xi,\tau)$ and $q_1(\xi,\tau)$ are equal to zero.

Taking the zero order terms we have, near $x = 0$ and $t = 0$, the equation

$$\frac{\partial}{\partial\tau}(w_0+q_0)-\frac{\partial^2}{\partial\xi^2}(v_0+q_0)+F(0,0,v_0(\xi,0)+w_0(0,\tau)+q_0(\xi,\tau)) = 0. \tag{3.6.32}$$

By using (3.3.5a) and (3.3.70a) we get

$$\frac{\partial q_0}{\partial\tau}-\frac{\partial^2 q_0}{\partial\xi^2}+F(0,0,v_0(\xi,0)+w_0(0,\tau)+q_0)-F(0,0,w_0(0,\tau))-F(0,0,v_0(\xi,0)) = 0. \tag{3.6.33}$$

For the boundary condition, using (3.3.5b) and (3.3.8b), we have

$$\varepsilon\left.\frac{\partial w_0}{\partial x}\right|_{x=0} + \left.\frac{\partial q_0}{\partial x}\right|_{x=0} + \varepsilon\left.\frac{\partial q_1}{\partial x}\right|_{x=0} = 0, \tag{3.6.34}$$

and equating terms of the same order gives

$$\left.\frac{\partial q_0}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial q_1}{\partial x}\right|_{x=0} = -\left.\frac{\partial w_0}{\partial x}\right|_{x=0}. \tag{3.6.35}$$

Using (3.3.70a) the initial condition becomes

$$v_0(\xi,0)+\varepsilon v_1(\xi,0)+q_0(\xi,0)+\varepsilon q_1(\xi,0) = 0, \tag{3.6.36}$$

and again equating terms of the same order gives

$$q_0(\xi,0) = -v_0(\xi,0), \quad q_1(\xi,0) = -v_1(\xi,0). \tag{3.6.37}$$

As $F(0,0,0) = 0$ and by (3.3.5) and (B5a) we have $v_0(\xi,0) = 0$, the leading

order corner layer function is now defined by

$$\frac{\partial q_0}{\partial \tau} - \frac{\partial^2 q_0}{\partial \xi^2} + F(0,0,w_0(0,\tau)+q_0) - F(0,0,w_0(0,\tau)) = 0, \tag{3.6.38a}$$

with

$$\left.\frac{\partial q_0}{\partial \xi}\right|_{\xi=0} = 0, \quad q_0(\xi,0)=0, \quad \lim_{\xi\to\infty} q_0(\xi,\tau)=0, \quad \lim_{\tau\to\infty} q_0(\xi,\tau)=0. \tag{3.6.38b}$$

From (3.6.38) it can be seen that $q_0(\xi,\tau)=0$ is a solution to the system.

To obtain a similar equation for $q_1(\xi,\tau)$ we consider $u_{as}(\varepsilon\xi,\varepsilon^2\tau)$ from (3.6.29) with $q_0(\xi,\tau)=0$ and the problem becomes

$$\begin{aligned}\left(\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \xi^2}\right)&\left[u_0(\varepsilon\xi,\varepsilon^2\tau)+v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)\right]\\ &+F(\varepsilon\xi,\varepsilon^2\tau,v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau))=0,\end{aligned} \tag{3.6.39}$$

which using (3.6.4) and (3.6.16) gives

$$\frac{\partial q_1}{\partial \tau} - \frac{\partial^2 q_1}{\partial \xi^2} + F(\varepsilon\xi,\varepsilon^2\tau,\cdot)\Big|_{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau);w_0(\varepsilon\xi,\tau)}^{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)} = O(\varepsilon^2), \tag{3.6.40}$$

where the notation $F(\cdot)|_{a;b}^{a+b} = F(a+b)-F(a)-F(b)$ is used. Define

$$G(\varepsilon) := F(\varepsilon\xi,\varepsilon^2\tau,\cdot)\Big|_{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau);w_0(\varepsilon\xi,\tau)}^{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)}, \tag{3.6.41}$$

and taking a Taylor expansion of this about $\varepsilon=0$ we have

$$G(\varepsilon) = G(0) + \varepsilon G'(0) + \varepsilon^2 G''(\varepsilon^*), \tag{3.6.42}$$

for $0 < \varepsilon^* < \varepsilon$. Calculating $G'(\varepsilon)$ we have

$$\begin{aligned} G'(\varepsilon) &= \xi F_x(\varepsilon\xi, \varepsilon^2\tau, \cdot)|_{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau);w_0(\varepsilon\xi,\tau)}^{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)} \\ &+2\varepsilon\tau F_t(\varepsilon\xi, \varepsilon^2\tau, \varepsilon\xi, \varepsilon^2\tau, \cdot)|_{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau);w_0(\varepsilon\xi,\tau)}^{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)} \\ &+[2\varepsilon\tau v_{0t} + 2\varepsilon^2\tau v_{1t} + v_1 + \xi w_{0x} + q_1]F_s(\varepsilon\xi, \varepsilon^2\tau, g) \\ &-[2\varepsilon\tau v_{0t} + 2\varepsilon^2\tau v_{1t} + v_1]F_s(\varepsilon\xi, \varepsilon^2\tau, v_0(\xi, \varepsilon^2\tau) + \varepsilon v_1(\xi, \varepsilon^2\tau)) \\ &-\xi w_{0x}F_s(\varepsilon\xi, \varepsilon^2\tau, w_0(\varepsilon\xi, \tau)) \end{aligned} \tag{3.6.43}$$

defining $g := v_0(\xi, \varepsilon^2\tau)+\varepsilon v_1(\xi, \varepsilon^2\tau)+w_0(\varepsilon\xi, \tau)+\varepsilon q_1(\xi, \tau)$ temporarily for presentation purposes. By a further calculation it can be seen that $G''(\varepsilon^*) = O(1)$. Evaluating $G(0)$ and $G'(0)$ we have

$$G(0) = F(0, 0, \cdot)|_{v_0(\xi,0);w_0(0,\tau)}^{v_0(\xi,0)+w_0(0,\tau)}, \tag{3.6.44}$$

and

$$\begin{aligned} G'(0) &= \xi F_x(0, 0, \cdot)|_{v_0(\xi,0);w_0(0,\tau)}^{v_0(\xi,0)+w_0(0,\tau)} \\ &+[v_1 + \xi w_{0x} + q_1]F_s(0, 0, v_0(\xi, 0) + w_0(0, \tau)) \\ &-v_1F_s(0, 0, v_0(\xi, 0)) - \xi w_{0x}F_s(0, 0, w_0(0, \tau)). \end{aligned} \tag{3.6.45}$$

Now $G(\varepsilon)$ from (3.6.42) is

$$\begin{aligned} G(\varepsilon) &= F(0, 0, \cdot)|_{v_0(\xi,0);w_0(0,\tau)}^{v_0(\xi,0)+w_0(0,\tau)} \\ &+\varepsilon \Big[\xi F_x(0, 0, \cdot)|_{v_0(\xi,0);w_0(0,\tau)}^{v_0(\xi,0)+w_0(0,\tau)} + (v_1 + \xi w_{0x} + q_1)F_s(0, 0, v_0(\xi, 0) + w_0(0, \tau)) \\ &-v_1F_s(0, 0, v_0(\xi, 0)) - \xi w_{0x}F_s(0, 0, w_0(0, \tau))\Big] + O(\varepsilon^2). \end{aligned} \tag{3.6.46}$$

We note $F(0, 0, \cdot)|_{v_0(\xi,0);w_0(0,\tau)}^{v_0(\xi,0)+w_0(0,\tau)} = 0$ as $v_0(\xi, 0) = 0$.

Extracting the $O(\varepsilon)$ terms the equation for $q_1(\xi,\tau)$ is

$$\begin{aligned}
&\frac{\partial q_1}{\partial \tau} - \frac{\partial^2 q_1}{\partial \xi^2} + q_1 F_s(0,0,v_0(\xi,0)+w_0(0,\tau)) \\
&= -[v_1 + \xi w_{0x}]F_s(0,0,v_0(\xi,0)+w_0(0,\tau)) + v_1 F_s(0,0,v_0(\xi,0)) \\
&\quad + \xi w_{0x} F_s(0,0,w_0(0,\tau)) - \xi F_x(0,0,\cdot)|_{v_0(\xi,0);w_0(0,\tau)}^{v_0(\xi,0)+w_0(0,\tau)}.
\end{aligned} \tag{3.6.47}$$

As $v_0(\xi,0)=0$ and $v_1(\xi,0)=0$, (3.6.47) simplifies to

$$\frac{\partial q_1}{\partial \tau} - \frac{\partial^2 q_1}{\partial \xi^2} + q_1 F_s(0,0,w_0(0,\tau)) = 0 \tag{3.6.48a}$$

From (3.6.35) and (3.6.37) the boundary and initial conditions are

$$\begin{aligned}
&q_1(\xi,0)=0, \quad \left.\frac{\partial q_1}{\partial \xi}\right|_{\xi=0} = -\left.\frac{\partial w_0}{\partial x}\right|_{x=0}, \\
&\lim_{\xi\to\infty} q_1(\xi,\tau)=0, \quad \lim_{\tau\to\infty} q_1(\xi,\tau)=0.
\end{aligned} \tag{3.6.48b}$$

Recalling (3.3.92) we have $\left.\frac{\partial w_0}{\partial x}\right|_{x=0} = 0$ which implies $q_1(\xi,\tau)=0$ is a solution of problem (3.6.48). Hence the corner layer functions $q_0(\xi,\tau)$ and $q_1(\xi,\tau)$ are both zero and we do not need to consider the corner layer separately.

We now consider the order of the system near $x=0$ and $t=0$ with the asymptotic expansion $u_{as}(x,t) := u_0(x,t)+v_0(\xi,t)+\varepsilon v_1(\xi,t)+w_0(x,\tau)$ as we have found $q_0(\xi,\tau)=q_1(\xi,\tau)=0$.

Lemma 3.6.4. *For the asymptotic expansion $u_{as}(x,t) := u_0(x,t)+v_0(\xi,t)+\varepsilon v_1(\xi,t)+w_0(x,\tau)$ near $x=0$, $t=0$ we have accuracy*

$$\mathcal{T} u_{as}(x,t) = O(\varepsilon^2). \tag{3.6.49}$$

$$\varepsilon \left.\frac{\partial u_{as}}{\partial x}\right|_{x=0} = g_0(t), \quad \varepsilon \left.\frac{\partial u_{as}}{\partial x}\right|_{x=1} = g_1(t) + O(\varepsilon^2). \tag{3.6.50}$$

$$u_{as}(x,0) = \varphi(x). \tag{3.6.51}$$

Proof. We want to calculate $\mathcal{T}[u_0+v_0+\varepsilon v_1+w_0]$. Using Lemma 3.6.1, Lemma 3.6.2 and Lemma 3.6.3 we have

$$\mathcal{T}u_{as}(x,t) = F(x,t,\cdot)|_{v_0+\varepsilon v_1;w_0}^{v_0+\varepsilon v_1+w_0} + O(\varepsilon^2). \tag{3.6.52}$$

Using (3.1.21), (3.1.16) and (3.1.1), (3.6.52) can be simplified to

$$F(x,t,\cdot)|_{v_0+\varepsilon v_1;w_0}^{v_0+\varepsilon v_1+w_0} = w_0(v_0+\varepsilon v_1)F_{ss}(x,t,(v_0+\varepsilon v_1+w_0)\theta), \tag{3.6.53}$$

for some $\theta \in (0,1)$. As $F(x,t,u)$ is a sufficiently smooth bounded function and recalling (3.3.54) and (3.3.93), we have

$$\mathcal{T}u_{as}(x,t) = O(te^{-(\gamma_T^2-p_0-\delta)\tau}) + O(\varepsilon^2), \tag{3.6.54}$$

Finally noting $te^{-(\gamma_T^2-p_0-\delta)\tau} = \varepsilon^2\tau e^{-(\gamma_T^2-p_0-\delta)\tau}$ and as $\tau \to \infty$, $e^{-(\gamma_T^2-p_0-\delta)\tau}$ is exponentially decaying and so $\varepsilon^2\tau e^{-(\gamma_T^2-p_0-\delta)\tau}$ is dominated by $C\varepsilon^2$ and we get the desired result.

For the boundary condition at $x=0$, we recall (3.3.5b) and (3.3.8b) and have

$$\varepsilon\,\frac{\partial u_{as}}{\partial x}\bigg|_{x=0} = g_0(t) + \varepsilon\,\frac{\partial w_0}{\partial x}\bigg|_{x=0}. \tag{3.6.55}$$

By (3.3.92) the final term is zero and we obtain (3.6.50) for $x=0$.

For the boundary condition at $x=1$ we recall (3.1.6) and the bounds (3.3.55) giving $\left|\dfrac{\partial(v_0+\varepsilon v_1)}{\partial\xi}\bigg|_{\xi=1/\varepsilon}\right| \leqslant \bar{C}_\delta e^{-(\gamma_L-\sqrt{p_0}-\delta)/\varepsilon} \leqslant C\varepsilon^2$ and so

$$\varepsilon\,\frac{\partial u_{as}}{\partial x}\bigg|_{x=1} = g_1(t) + \varepsilon\,\frac{\partial w_0}{\partial x}\bigg|_{x=1} + O(\varepsilon^2). \tag{3.6.56}$$

Considering (3.3.70) at $x = 1$ and using (3.1.7) we have $\left.\dfrac{\partial w_0}{\partial x}\right|_{x=1} = 0$ and so we obtain (3.6.50) for $x = 1$. By (3.3.54) and (3.3.70b) we get (3.6.51). $\square$

Remark 3.6.1. We make a remark on the corner layer solution without the assumption (B5b). In this case the analysis for $q_0(\xi, \tau)$ remains the same and gives $q_0(\xi, \tau) = 0$ however the analysis for $q_1(\xi, \tau)$ differs. With $v_0(\xi, 0) = w_0(0, \tau) = 0$, we recall equations for $v_1(\xi, t)$ and $w_0(x, \tau)$ and define $\gamma_0^2 := F_s(0, 0, 0)$ giving the equation for $q_1(\xi, \tau)$ as

$$\frac{\partial q_1}{\partial \tau} - \frac{\partial^2 q_1}{\partial \xi^2} + \gamma_0^2 q_1 = 0, \tag{3.6.57a}$$

$$\begin{gathered} q_1(\xi, 0) = -v_1(\xi, 0), \quad \left.\frac{\partial q_1}{\partial \xi}\right|_{\xi=0} = \left.\frac{\partial \varphi}{\partial x}\right|_{x=0} - \left.\frac{\partial u_0}{\partial x}\right|_{x=0}, \\ \lim_{\xi \to \infty} q_1(\xi, \tau) = 0, \quad \lim_{\tau \to \infty} q_1(\xi, \tau) = 0. \end{gathered} \tag{3.6.57b}$$

At the corner the derivative of the initial condition should equal the boundary condition, i.e.,

$$\left.\frac{\partial v_1}{\partial \xi}\right|_{\xi=0, t=0} = \left.\frac{\partial \varphi}{\partial x}\right|_{x=0} - \left.\frac{\partial u_0}{\partial x}\right|_{x=0}, \tag{3.6.58}$$

which by (B5a) and (3.3.8a) is satisfied. We note that (3.6.57) is a partial differential equation with constant coefficients.

If $q_1(\xi, \tau)$ and its derivatives are bounded then the truncation error will remain the same so we leave finding $q_1(\xi, \tau)$ as future work. We continue with the result $q_1(\xi, \tau) = 0$ and the assumption (B5b).

3.6.3 Equations for Derivatives of $\tilde{v}_0(\xi, t; p)$, $v_1(\xi, t)$, $\tilde{w}_0(x, \tau; p)$

In this section we give equations for boundary layer functions, initial layer functions and necessary derivatives of each.

Boundary Layer Functions

For the boundary layer function $\tilde{v}_0$ we have the following equations for first derivatives with respect to p and k^{th} derivatives with respect to t for $k = 0, ..., 2$.

We recall the notation $\mathcal{L}_\xi[v(\cdot)]$ defined in (3.3.7) and introduce an extension to this notation for the perturbed case:

$$\tilde{\mathcal{L}}_\xi[v(\cdot)] := -\frac{\partial^2 v}{\partial \xi^2} + v \left(\frac{\partial F}{\partial s}\bigg|_{s=v_0, x=0} - p \right). \tag{3.6.59}$$

We have the following systems for $\tilde{v}_0(\xi, l; p)$ and $\chi(\xi, l; p)$,

$$-\frac{\partial^2 \tilde{v}_0}{\partial \xi^2} + F(0, t, \tilde{v}_0) - p\tilde{v}_0 = 0, \tag{3.6.60a}$$

$$\frac{\partial \tilde{v}_0}{\partial \xi}\bigg|_{\xi=0} = g_0(t), \quad \tilde{v}_0(\infty, t; p) = 0. \tag{3.6.60b}$$

$$\frac{\partial^2 \chi}{\partial \xi^2} + \chi(F_s(0, t, v_0) - p) = 0, \tag{3.6.61a}$$

$$\frac{\partial \chi}{\partial \xi}\bigg|_{\xi=0} = F(0, t, \tilde{v}_0(0, t)) - p\tilde{v}_0(0, t), \quad \chi(\infty, t) = 0. \tag{3.6.61b}$$

We note (3.6.61) can be given by a Dirichlet boundary condition, $\chi(0, t) = g_0(t)$, rather than the Neumann boundary condition above. We consider the problem with the Neumann boundary condition so that Lemma 3.3.2 can be

applied to all problems.

$$\tilde{\mathcal{L}}_\xi\left[\frac{\partial \tilde{v}_0}{\partial p}\right] = \tilde{v}_0, \quad \frac{\partial}{\partial \xi}\frac{\partial \tilde{v}_0}{\partial p}\bigg|_{\xi=0} = 0, \quad \frac{\partial \tilde{v}_0}{\partial p}\bigg|_{\xi=\infty} = 0. \tag{3.6.62a}$$

$$\tilde{\mathcal{L}}_\xi\left[\frac{\partial \tilde{v}_0}{\partial t}\right] = -F_t(0,t,\tilde{v}_0), \quad \frac{\partial}{\partial \xi}\frac{\partial \tilde{v}_0}{\partial t}\bigg|_{\xi=0} = \frac{\partial g_0}{\partial t}, \quad \frac{\partial \tilde{v}_0}{\partial t}\bigg|_{\xi=\infty} = 0. \tag{3.6.63a}$$

$$\tilde{\mathcal{L}}_\xi\left[\frac{\partial^2 \tilde{v}_0}{\partial t^2}\right] = -F_{tt}(0,t,\tilde{v}_0) - 2\frac{\partial \tilde{v}_0}{\partial t}F_{ts}(0,t,\tilde{v}_0) - \left(\frac{\partial \tilde{v}_0}{\partial t}\right)^2 F_{ss}(0,t,\tilde{v}_0), \tag{3.6.64a}$$

$$\frac{\partial}{\partial \xi}\frac{\partial^2 \tilde{v}_0}{\partial t^2}\bigg|_{\xi=0} = \frac{\partial^2 g_0}{\partial t^2}, \quad \frac{\partial^2 \tilde{v}_0}{\partial t^2}\bigg|_{\xi=\infty} = 0. \tag{3.6.64b}$$

For the boundary layer function v_1 we have the following equations for derivatives with respect to p and k^{th} derivatives with respect to t for $k = 0, ..., 4$.

$$\mathcal{L}_\xi[v_1] = -\xi F_x(0,t,v_0), \tag{3.6.65a}$$

$$\frac{\partial v_1}{\partial \xi}\bigg|_{\xi=0} = -\left.\frac{\partial u_0}{\partial x}\right|_{x=0}, \quad v_1(\infty,t) = 0. \tag{3.6.65b}$$

$$\mathcal{L}_\xi\left[\frac{\partial v_1}{\partial \xi}\right] = -\xi\frac{\partial v_0}{\partial \xi}F_{xs}(0,t,v_0) - v_1\frac{\partial v_0}{\partial \xi}F_{ss}(0,t,v_0) - F_x(0,t,v_0), \tag{3.6.66a}$$

$$\frac{\partial^2 v_1}{\partial \xi^2}\bigg|_{\xi=0} = v_1(0,t)F_s(0,t,v_0(0,t)), \quad \frac{\partial v_1}{\partial \xi}\bigg|_{\xi=\infty} = 0. \tag{3.6.66b}$$

Note $\left.\dfrac{\partial v_1}{\partial \xi}\right|_{\xi=\infty} = 0$ is found by recalling Remark 3.3.2.

$$\begin{aligned}\mathcal{L}_\xi\left[\frac{\partial^2 v_1}{\partial \xi^2}\right] &= -\xi\frac{\partial^2 v_0}{\partial \xi^2}F_{xs}(0,t,v_0) - v_1\frac{\partial^2 v_0}{\partial \xi^2}F_{ss}(0,t,v_0)\\ &-\xi\left(\frac{\partial v_0}{\partial \xi}\right)^2 F_{xss}(0,t,v_0) - v_1\left(\frac{\partial v_0}{\partial \xi}\right)^2 F_{sss}(0,t,v_0)\\ &-2\frac{\partial v_1}{\partial \xi}\frac{\partial v_0}{\partial \xi}F_{ss}(0,t,v_0) - 2\frac{\partial v_0}{\partial \xi}F_{xs}(0,t,v_0),\end{aligned} \tag{3.6.67a}$$

$$\left.\frac{\partial^3 v_1}{\partial \xi^3}\right|_{\xi=0} = \left.\frac{\partial u_0}{\partial x}\right|_{x=0} F_s(0,t,v_0(0,t)) - v_1(0,t)g_0(t)F_{ss}(0,t,v_0(0,t)) - F_x(0,t,v_0(0,t)), \tag{3.6.67b}$$

$$\left.\frac{\partial^2 v_1}{\partial \xi^2}\right|_{\xi=\infty} = 0. \tag{3.6.67c}$$

$$\begin{aligned}\mathcal{L}_\xi\left[\frac{\partial^3 v_1}{\partial \xi^3}\right] &= -3\left(\frac{\partial v_0}{\partial \xi}\right)^2 F_{xss}(0,t,v_0) - 3\frac{\partial^2 v_0}{\partial \xi^2}F_{xs}(0,t,v_0)\\ &-\xi\left(\frac{\partial v_0}{\partial \xi}\right)^3 F_{xsss}(0,t,v_0) - 3\xi\frac{\partial v_0}{\partial \xi}\frac{\partial^2 v_0}{\partial \xi^2}F_{xss}(0,t,v_0)\\ &-\xi\frac{\partial^3 v_0}{\partial \xi^3}F_{xs}(0,t,v_0) - 3\frac{\partial^2 v_1}{\partial \xi^2}\frac{\partial v_0}{\partial \xi}F_{ss}(0,t,v_0)\\ &-3\frac{\partial v_1}{\partial \xi}\left(\frac{\partial v_0}{\partial \xi}\right)^2 F_{sss}(0,t,v_0) - 3\frac{\partial v_1}{\partial \xi}\frac{\partial^2 v_0}{\partial \xi^2}F_{ss}(0,t,v_0)\\ &-v_1\left(\frac{\partial v_0}{\partial \xi}\right)^3 F_{ssss}(0,t,v_0) - 3v_1\frac{\partial v_0}{\partial \xi}\frac{\partial^2 v_0}{\partial \xi^2}F_{sss}(0,t,v_0)\\ &-v_1\frac{\partial^3 v_0}{\partial \xi^3}F_{ss}(0,t,v_0),\end{aligned} \tag{3.6.68a}$$

$$\left.\frac{\partial^4 v_1}{\partial \xi^4}\right|_{\xi=0} = -\left.\frac{\partial^2 v_1}{\partial \xi^2}\right|_{\xi=0} F_s(0,t,v_0(0,t))$$
$$-v_1(0,t)\left.\frac{\partial^2 v_0}{\partial \xi^2}\right|_{\xi=0} F_{ss}(0,t,v_0(0,t)) - v_1(0,t)g_0^2(t)F_{sss}(0,t,v_0(0,t)) \quad (3.6.68b)$$
$$+2\left.\frac{\partial u_0}{\partial x}\right|_{x=0} g_0(t)F_{ss}(0,t,v_0(0,t)) - 2g_0(t)F_{xs}(0,t,v_0(0,t)),$$

$$\left.\frac{\partial^3 v_1}{\partial \xi^3}\right|_{\xi=\infty} = 0. \quad (3.6.68c)$$

$$\mathcal{L}_\xi\left[\frac{\partial^4 v_1}{\partial \xi^4}\right] = -6\xi\left(\frac{\partial v_0}{\partial \xi}\right)^2\frac{\partial^2 v_0}{\partial \xi^2}F_{xsss}(0,t,v_0) - \xi\left(\frac{\partial v_0}{\partial \xi}\right)^4 F_{xssss}(0,t,v_0)$$
$$-3\xi\left(\frac{\partial^2 v_0}{\partial \xi^2}\right)^2 F_{xss}(0,t,v_0) - 4\xi\frac{\partial v_0}{\partial \xi}\frac{\partial^3 v_0}{\partial \xi^3}F_{xss}(0,t,v_0)$$
$$-6\frac{\partial^2 v_1}{\partial \xi^2}\left(\frac{\partial v_0}{\partial \xi}\right)^2 F_{sss}(0,t,v_0) - 12\frac{\partial v_1}{\partial \xi}\frac{\partial v_0}{\partial \xi}\frac{\partial^2 v_0}{\partial \xi^2}F_{sss}(0,t,v_0)$$
$$-4\frac{\partial v_1}{\partial \xi}\left(\frac{\partial v_0}{\partial \xi}\right)^3 F_{ssss}(0,t,v_0) - 6v_1\left(\frac{\partial v_0}{\partial \xi}\right)^2\frac{\partial^2 v_0}{\partial \xi^2}F_{ssss}(0,t,v_0)$$
$$-v_1\left(\frac{\partial v_0}{\partial \xi}\right)^4 F_{sssss}(0,t,v_0) - 3v_1\left(\frac{\partial^2 v_0}{\partial \xi^2}\right)^2 F_{sss}(0,t,v_0)$$
$$-4v_1\frac{\partial v_0}{\partial \xi}\frac{\partial^3 v_0}{\partial \xi^3}F_{sss}(0,t,v_0) - 4\frac{\partial^3 v_0}{\partial \xi^3}F_{xs}(0,t,v_0)$$
$$-\xi\frac{\partial^4 v_0}{\partial \xi^4}F_{xs}(0,t,v_0) - 4\left(\frac{\partial v_0}{\partial \xi}\right)^3 F_{xsss}(0,t,v_0)$$
$$-12\frac{\partial v_0}{\partial \xi}\frac{\partial^2 v_0}{\partial \xi^2}F_{xss}(0,t,v_0) - v_1\frac{\partial^4 v_0}{\partial \xi^4}F_{ss}(0,t,v_0)$$
$$-4\frac{\partial v_1}{\partial \xi}\frac{\partial^3 v_0}{\partial \xi^3}F_{ss}(0,t,v_0) - 6\frac{\partial^2 v_1}{\partial \xi^2}\frac{\partial^2 v_0}{\partial \xi^2}F_{ss}(0,t,v_0)$$
$$-4\frac{\partial v_0}{\partial \xi}\frac{\partial^3 v_1}{\partial \xi^3}F_{ss}(0,t,v_0),$$
$$(3.6.69a)$$

$$\begin{aligned}\left.\frac{\partial^5 v_1}{\partial \xi^5}\right|_{\xi=0} =& -\left.\frac{\partial^3 v_1}{\partial \xi^3}\right|_{\xi=0} F_s(0,t,v_0(0,t)) - 3g_0^2(t)F_{xss}(0,t,v_0(0,t)) \\ &-3\left.\frac{\partial^2 v_0}{\partial \xi^2}\right|_{\xi=0} F_{xs}(0,t,v_0(0,t)) - 3\left.\frac{\partial^2 v_1}{\partial \xi^2}\right|_{\xi=0} g_0(t)F_{ss}(0,t,v_0(0,t)) \\ &+3\left.\frac{\partial u_0}{\partial x}\right|_{x=0} g_0^2(t)F_{sss}(0,t,v_0(0,t)) + 3\left.\frac{\partial u_0}{\partial x}\right|_{x=0}\left.\frac{\partial^2 v_0}{\partial \xi^2}\right|_{\xi=0} F_{ss}(0,t,v_0(0,t)) \\ &-v_1(0,t)g_0^3(t)F_{ssss}(0,t,v_0(0,t)) - 3v_1(0,t)g_0(t)\left.\frac{\partial^2 v_0}{\partial \xi^2}\right|_{\xi=0} F_{sss}(0,t,v_0(0,t)) \\ &-v_1(0,t)\left.\frac{\partial^3 v_0}{\partial \xi^3}\right|_{\xi=0} F_{ss}(0,t,v_0(0,t)),\end{aligned} \tag{3.6.69b}$$

$$\left.\frac{\partial^4 v_1}{\partial \xi^4}\right|_{\xi=\infty} = 0. \tag{3.6.69c}$$

$$\begin{aligned}\mathcal{L}_\xi\left[\frac{\partial v_1}{\partial t}\right] =& -\xi\left(F_{xt}(0,t,v_0) + \frac{\partial v_0}{\partial t}F_{xs}(0,t,v_0)\right) \\ &-v_1\left(F_{ts}(0,t,v_0) + \frac{\partial v_0}{\partial t}F_{ss}(0,t,v_0)\right),\end{aligned} \tag{3.6.70a}$$

$$\left.\frac{\partial}{\partial \xi}\left(\frac{\partial v_1}{\partial t}\right)\right|_{\xi=0} = -\left.\left(\frac{\partial}{\partial x}\frac{\partial u_0}{\partial t}\right)\right|_{x=0}, \quad \left.\frac{\partial v_1}{\partial t}\right|_{\xi=\infty} = 0. \tag{3.6.70b}$$

$$\begin{aligned}\mathcal{L}_\xi\left[\frac{\partial^2 v_1}{\partial t^2}\right] =& -\xi\left(F_{xtt}(0,t,v_0) + \frac{\partial v_0}{\partial t}F_{xts}(0,t,v_0)\right) \\ -\xi\frac{\partial v_0}{\partial t}&\left(F_{xts}(0,t,v_0) + \frac{\partial v_0}{\partial t}F_{xss}(0,t,v_0)\right) - 2\frac{\partial v_1}{\partial t}\left(F_{ts}(0,t,v_0) + \frac{\partial v_0}{\partial t}F_{ss}(0,t,v_0)\right) \\ -v_1&\left(F_{tts}(0,t,v_0) + \frac{\partial v_0}{\partial t}F_{tss}(0,t,v_0)\right) - v_1\frac{\partial v_0}{\partial t}\left(F_{tss}(0,t,v_0) + \frac{\partial v_0}{\partial t}F_{sss}(0,t,v_0)\right) \\ &-\xi\frac{\partial^2 v_0}{\partial t^2}F_{xs}(0,t,v_0) - v_1\frac{\partial^2 v_0}{\partial t^2}F_{ss}(0,t,v_0),\end{aligned} \tag{3.6.71a}$$

$$\left.\frac{\partial}{\partial \xi}\left(\frac{\partial^2 v_1}{\partial t^2}\right)\right|_{\xi=0} = -\left.\frac{\partial}{\partial x}\left(\frac{\partial^2 u_0}{\partial t^2}\right)\right|_{x=0}, \quad \left.\frac{\partial^2 v_1}{\partial t^2}\right|_{\xi=\infty} = 0. \tag{3.6.71b}$$

Initial Layer Functions

For the initial layer function $\tilde{w}_0$ we have the following equations for derivatives with respect to p and k^{th} derivatives with respect to x for $k = 0, ..., 4$.

We define $\mathcal{L}_\tau$ to be, Unsure how useful this is but keeping for the moment.

$$\mathcal{L}_\tau[\omega(\cdot)] := \frac{\partial \omega}{\partial \tau} + \omega \left.\frac{\partial F}{\partial s}\right|_{s=\omega_0, t=0}. \tag{3.6.72}$$

$$\tilde{\mathcal{L}_\tau}[\omega(\cdot)] := \frac{\partial \omega}{\partial \tau} + \omega \left(\left.\frac{\partial F}{\partial s}\right|_{s=\tilde{\omega}_0, t=0} - p \right). \tag{3.6.73}$$

We define a perturbed version of this, $\tilde{\mathcal{L}_\tau}[\omega(\cdot)]$,

$$\tilde{\mathcal{L}_\tau}[\omega(\cdot)] := \frac{\partial \omega}{\partial \tau} + \omega \left(\left.\frac{\partial F}{\partial s}\right|_{s=\omega_0, t=0} - p \right). \tag{3.6.74}$$

$$\tilde{\mathcal{L}_\tau}\left[\frac{\partial \tilde{w}_0}{\partial p}\right] = \tilde{w}_0. \quad \left.\frac{\partial \tilde{w}_0}{\partial p}\right|_{\tau=0} = 0, \quad \left.\frac{\partial \tilde{w}_0}{\partial p}\right|_{\tau=\infty} = 0. \tag{3.6.75}$$

$$\tilde{\mathcal{L}_\tau}\left[\frac{\partial \tilde{w}_0}{\partial x}\right] = \frac{\partial}{\partial \tau}\left(\frac{\partial \tilde{w}_0}{\partial x}\right) + \frac{\partial \tilde{w}_0}{\partial x}\left(F_s(x, 0, \tilde{w}_0) - p\right) = -F_x(x, 0, \tilde{w}_0). \tag{3.6.76a}$$

$$\left.\frac{\partial \tilde{w}_0}{\partial x}\right|_{\tau=0} = \frac{\partial \varphi}{\partial x} - \left.\frac{\partial u_0}{\partial x}\right|_{t=0}, \quad \left.\frac{\partial \tilde{w}_0}{\partial x}\right|_{\tau=\infty} = 0. \tag{3.6.76b}$$

$$\tilde{\mathcal{L}_\tau}\left[\frac{\partial^2 \tilde{w}_0}{\partial x^2}\right] = -F_{xx}(x, 0, \tilde{w}_0) - 2\frac{\partial \tilde{w}_0}{\partial x}F_{xs}(x, 0, \tilde{w}_0) - \left(\frac{\partial \tilde{w}_0}{\partial x}\right)^2 F_{ss}(x, 0, \tilde{w}_0), \tag{3.6.77a}$$

$$\left.\frac{\partial^2 \tilde{w}_0}{\partial x^2}\right|_{\tau=0} = \frac{\partial^2 \varphi}{\partial x^2} - \left.\frac{\partial^2 u_0}{\partial x^2}\right|_{t=0}, \quad \left.\frac{\partial^2 \tilde{w}_0}{\partial x^2}\right|_{\tau=\infty} = 0. \tag{3.6.77b}$$

$$\begin{aligned}\tilde{\mathcal{L}}_\tau\left[\frac{\partial^3\tilde{w}_0}{\partial x^3}\right] =& - F_{xxx}(x,0,\tilde{w}_0) - 3\frac{\partial \tilde{w}_0}{\partial x}F_{xxs}(x,0,\tilde{w}_0) - 3\left(\frac{\partial \tilde{w}_0}{\partial x}\right)^2 F_{xss}(x,0,\tilde{w}_0)\\ &- \left(\frac{\partial \tilde{w}_0}{\partial x}\right)^3 F_{sss}(x,0,\tilde{w}_0) - 3\frac{\partial^2 \tilde{w}_0}{\partial x^2}F_{xs}(x,0,\tilde{w}_0)\\ &- \left(\left(\frac{\partial \tilde{w}_0}{\partial x}\right)^2 + 2\frac{\partial \tilde{w}_0}{\partial x}\frac{\partial^2 \tilde{w}_0}{\partial x^2}\right) F_{ss}(x,0,\tilde{w}_0),\end{aligned} \tag{3.6.78a}$$

$$\left.\frac{\partial^3\tilde{w}_0}{\partial x^3}\right|_{\tau=0} = \frac{\partial^3\varphi}{\partial x^3} - \left.\frac{\partial^3 u_0}{\partial x^3}\right|_{\tau=0}, \qquad \left.\frac{\partial^3\tilde{w}_0}{\partial x^3}\right|_{\tau=\infty} = 0. \tag{3.6.78b}$$

$$\begin{aligned}\tilde{\mathcal{L}}_\tau\left[\frac{\partial^4\tilde{w}_0}{\partial x^4}\right] =& - F_{xxxx}(x,0,\tilde{w}_0) - 4\frac{\partial \tilde{w}_0}{\partial x}F_{xxxs}(x,0,\tilde{w}_0) - 6\left(\frac{\partial \tilde{w}_0}{\partial x}\right)^2 F_{xxss}(x,0,\tilde{w}_0)\\ &- 4\left(\frac{\partial \tilde{w}_0}{\partial x}\right)^3 F_{xsss}(x,0,\tilde{w}_0) - \left(\frac{\partial \tilde{w}_0}{\partial x}\right)^4 F_{ssss}(x,0,\tilde{w}_0)\\ &- 6\frac{\partial^2 \tilde{w}_0}{\partial x^2}F_{xxs}(x,0,\tilde{w}_0) - 12\frac{\partial \tilde{w}_0}{\partial x}\frac{\partial^2 \tilde{w}_0}{\partial x^2}F_{xss}(x,0,\tilde{w}_0)\\ &- 6\left(\frac{\partial \tilde{w}_0}{\partial x}\right)^2 \frac{\partial^2 \tilde{w}_0}{\partial x^2}F_{sss}(x,0,\tilde{w}_0)\\ &- \left(3\left(\frac{\partial^2 \tilde{w}_0}{\partial x^2}\right)^2 + 4\frac{\partial \tilde{w}_0}{\partial x}\frac{\partial^3 \tilde{w}_0}{\partial x^3}\right) F_{ss}(x,0,\tilde{w}_0) - 4\frac{\partial^3 \tilde{w}_0}{\partial x^3}F_{xs}(x,0,\tilde{w}_0),\end{aligned} \tag{3.6.79a}$$

$$\left.\frac{\partial^4\tilde{w}_0}{\partial x^4}\right|_{\tau=0} = \frac{\partial^4\varphi}{\partial x^4} - \left.\frac{\partial^4 u_0}{\partial x^4}\right|_{\tau=0}, \qquad \left.\frac{\partial^4\tilde{w}_0}{\partial x^4}\right|_{\tau=\infty} = 0. \tag{3.6.79b}$$

Chapter 4

Singularly Perturbed Nonlinear Elliptic Problem with Singularly Perturbed Neumann Boundary Conditions

We consider the following nonlinear singularly perturbed elliptic equation;

$$\mathcal{F}u := -\varepsilon^2 \Delta u + b(x,u) = 0 \quad \text{for} \quad x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \tag{4.0.1a}$$

with singularly perturbed Neumann boundary condition,

$$\varepsilon \frac{\partial u}{\partial n}\bigg|_{x \in \partial\Omega} = g(x) \quad \text{for } x \in \partial\Omega, \tag{4.0.1b}$$

where $b(x,u)$ and $g(x)$ are sufficiently smooth functions, ε is a small parameter such that $0 < \varepsilon \leqslant \varepsilon_0 \ll 1$ and $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. The two-dimensional domain is represented by Ω with boundary $\partial\Omega$ which is a smooth closed curve and n is the outward normal to this curve.

To obtain existence and accuracy for this problem we will employ methods from the previous chapters. We consider problems with boundary layer

solutions and use curvilinear coordinates as in *Chapter 2*. In this case, the curvilinear coordinates are used in a narrow strip along the boundary of the domain. We consider this problem with many of the assumptions made in *Chapter 3*, these will be discussed in §4.1. Compared with *Chapter 3* there are no corner layers or initial layers considered here. We will obtain upper and lower solutions and employ the theory of Z-fields to obtain existence and accuracy of a solution.

In contrast to *Chapter 2*, there will now be an error on the boundary coming from the approximation for the Neumann boundary condition, because of this truncation error analysis will be carried out on the boundary. We add a term to the upper and lower solutions that differs from that in [14] and to previous chapters, see the exponential function in the final term in (4.3.28). This is to dominate terms coming from the truncation error and so that $\beta(x_i, t_j; \pm p)$ satisfies the necessary requirements of upper and lower solutions on the boundary of the domain, see Theorem 4.4.1 in particular (4.4.83) onwards.

We note that the two-dimensional steady state problem considered here can be generalised to a time-dependent two-dimensional problem by combining this work with the previous chapter. We do not do this here as it will complicate the analysis. In this case the compatibility condition, (B5), would be extended to two dimensions.

4.1 Hypotheses of the Problem

Setting $\varepsilon = 0$ we assume the reduced problem, $b(x,u) = 0$, has a sufficiently smooth solution $u_0(x)$, i.e.,

$$b(x, u_0(x)) = 0. \tag{4.1.1}$$

As $b(x,u)$ is nonlinear there may be multiple solutions to the reduced problem. We now make a number of hypotheses for (4.0.1), similar to those in §2.2 and §3.1.

The reduced solution satisfies

$$b_u(x, u_0(x)) > \gamma^2 > 0 \quad \text{for } x = (x_1, x_2) \in \bar{\Omega}. \tag{C1}$$

There exists a sufficiently smooth function $A(x)$ with $x \in \partial\Omega$ such that

$$\int_0^{A(x)} f(x, u_0(x) + s)ds = \frac{g^2(x)}{2}, \tag{C2}$$

This assumption is a version of (B2). The similarity is to be expected as the phase plane analysis of Lemma 3.3.2 remains true in the case of (4.0.1).

We have a version of (B3) for the two-dimensional problem (4.0.1), i.e.,

$$sf(x, u_0(x) + s) > 0 \quad \text{for } s \in (0, A(x)]', \, x \in \partial\Omega. \tag{C3}$$

This assumption has been discussed in §3.1.

For presentation purposes we assume, without loss of generality,

$$g(x) \leqslant 0 \quad \text{for } x \in \partial\Omega, \tag{C4}$$

and again assume

$$\varepsilon \leqslant CN^{-1}. \tag{4.1.2}$$

4.2 Existence and Accuracy of Discrete Solutions; Main Results

This section gives the main result of the chapter; that is existence and accuracy of a discrete solution to problem (4.0.1) for the discretisation (4.4.4), (4.4.67) and (4.4.74) with (4.4.55) and (4.4.70) that will be given in §4.4.

Theorem 4.2.1. *For the Bakhvalov and Shishkin meshes described in §4.4.1, N sufficiently large and ε sufficiently small, there exists a solution U_i of the discrete problem with*

$$|U(X_i) - u(X_i)| \leqslant CN^{-2} \ln^{2m} N, \tag{4.2.1}$$

for all mesh nodes $X_i \in \bar{\Omega}$ where $u(X_i)$ is the exact solution of (4.0.1) *and $m = 0$ for the Bakhvalov mesh and $m = 1$ for the Shishkin mesh.*

4.3 Analysis of the Method

4.3.1 Curvilinear Coordinates

As the reduced solution $u_0(x)$ does not necessarily meet the boundary condition (4.0.1b) we look for solutions that exhibit a boundary layer near $x \in \partial\Omega$. Therefore we use the curvilinear coordinate system described in §2.4.1 with the alteration that $r = 0$ representing the curve $\partial\Omega$, i.e., we use the curvilinear coordinate system in a narrow region inside the boundary $\partial\Omega$. We have curvilinear local coordinates (r, l) in a narrow strip along the boundary $\partial\Omega$. These are defined by

$$x_1 = q_1(l) + rn_1(l), \quad x_2 = q_2(l) + rn_2(l), \tag{4.3.1}$$

where r is the distance between a point and the curve $\partial\Omega$ along the outward unit normal vector to $\partial\Omega$, $n = n(l)$, at the point on the curve represented by $(q_1(l), q_2(l))$. We note that choosing the outward normal results in $r \leqslant 0$ throughout the layer region. However, this choice keeps the notation similar to that in *Chapter 2*. As $\partial\Omega$ is a smooth closed curve in a narrow region inside this the coordinates $x(r,l)$ are well defined, i.e., the mapping (x_1, x_2) and $x(r,l)$ correspond one-to-one and is invertible. If $\partial\Omega$ is not a smooth closed curve then $x(r,l)$ is not well defined and the following work does not hold true. Recall Lemma 2.4.1 where the Laplace operator is given as

$$\Delta u := \eta^{-1}\frac{\partial}{\partial r}\left(\eta\frac{\partial u}{\partial r}\right) + \zeta\frac{\partial}{\partial l}\left(\zeta\frac{\partial u}{\partial l}\right), \tag{4.3.2}$$

where

$$\eta(r,l) := 1 + \kappa r, \quad \zeta(r,l) := (\tau\eta)^{-1}, \tag{4.3.3}$$

and

$$\tau = \tau(l) = \sqrt{q_1'^2 + q_2'^2}, \quad \kappa = \kappa(l) = \frac{q_1' q_2'' - q_2' q_1''}{\tau^3}. \tag{4.3.4}$$

As r points in the same direction as n we have

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r}. \tag{4.3.5}$$

4.3.2 Regions

The region we are considering is $\Omega \in \mathbb{R}^2$ with smooth closed boundary $\partial\Omega$. Using the notation

$$\Omega_{(a,b)} := \{x(r,l) \in \bar{\Omega} : a < r < b\}, \tag{4.3.6}$$

of *Chapter 2* we define the layer region as $\Omega_{[-c_1,0)}$ and the closed layer region as $\Omega_{[-c_1,0]}$ where c_1 is a sufficiently small positive constant. The outer region

is given by $\mathring{\Omega}$ and we note $\mathring{\Omega} := \bar{\Omega}\backslash\Omega_{[-c_1,0]}$. These regions are represented in Figure 4.1.

We adopt the notation $\Gamma_a := \{x(r,l) : r = a\}$ from *Chapter 2* and call the interface between the layer region and outer region Γ_{-c_1}. The boundaries of $\Omega_{[-c_1,0]}$ are $\partial\Omega$ and Γ_{-c_1}. Points on $\partial\Omega$ are defined as $\bar{x} = x(0,l)$, i.e., $\bar{x} \in \partial\Omega$ and we note $\Gamma_0 = \partial\Omega$ with the notation given here.

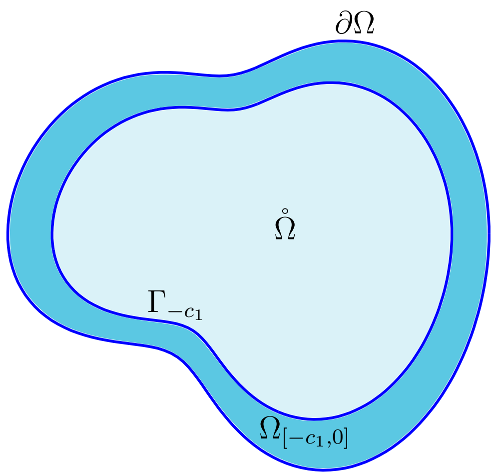


Figure 4.1: Regions of Ω*.*

4.3.3 Asymptotic Expansion

We define the stretched variable ξ as

$$r = \varepsilon\xi. \tag{4.3.7}$$

With this rescaling (4.3.2) becomes

$$\varepsilon^2\Delta u := \varepsilon\kappa\frac{\partial u}{\partial \xi} + \frac{\partial^2 u}{\partial \xi^2} + \varepsilon^2\zeta\frac{\partial \zeta}{\partial l}\frac{\partial u}{\partial l} + \varepsilon^2\zeta^2\frac{\partial^2 u}{\partial l^2}, \tag{4.3.8}$$

and by (4.3.5) and (4.3.7) the normal derivative becomes

$$\frac{\partial u}{\partial n} = \frac{1}{\varepsilon}\frac{\partial u}{\partial \xi}. \tag{4.3.9}$$

The asymptotic expansion of the solution to order ε^2 is

$$u_{as}(x) := u_0(x) + [v_0(\xi, l) + \varepsilon v_1(\xi, l)]\vartheta(x), \tag{4.3.10}$$

where $\vartheta(x) \in [0,1]$ is a smooth positive cut off function described after (2.4.29), i.e., $\vartheta(x) = 1$ for $|r| \leqslant \frac{c_1}{2}$ and $\vartheta(x)$ vanishes for $r \in \bar{\Omega}\backslash\Omega_{[-c_1,0]}$. Although $v_i(\xi, l)$, $i = 0, 1$ are negligible outside the layer region they are not necessarily zero and as $x(r, l)$ is only well defined within the layer region this cut off function eliminates $v_i(\xi, l)$ outside the layer.

We define

$$B(x, s) := b(x, u_0(x) + s). \tag{4.3.11}$$

We note $B(x, 0) = 0$ and, as in the previous chapters, we have

$$\left.\frac{\partial^m B}{\partial r^m}\right|_{s=0} = 0, \quad m = 0, 1, 2 \tag{4.3.12}$$

and so

$$\left|\frac{\partial^m B}{\partial r^m}\right| \leqslant C|s|, \quad m = 0, 1, 2. \tag{4.3.13}$$

For the function $v_0(\xi, l)$ we have the following equation;

$$-\frac{\partial^2 v_0}{\partial \xi^2} + B(\bar{x}, v_0) = 0, \tag{4.3.14a}$$

$$\left.\frac{\partial v_0}{\partial \xi}\right|_{\xi=0} = g(\bar{x}), \quad v_0(-\infty, l) = 0. \tag{4.3.14b}$$

The perturbed version of $v_0(\xi, l)$ is given by the function $\tilde{v}_0(\xi, l; p)$ with

$\tilde{v}_0(\xi, l; 0) = v_0(\xi, l)$ and is described by

$$-\frac{\partial^2 \tilde{v}_0}{\partial \xi^2} + B(\bar{x}, \tilde{v}_0) - p\tilde{v}_0 = 0, \tag{4.3.15a}$$

$$\left.\frac{\partial \tilde{v}_0}{\partial \xi}\right|_{\xi=0} = g(\bar{x}). \quad \tilde{v}_0(\infty, l; p) = 0, \tag{4.3.15b}$$

As in *Chapter 2* and *Chapter 3* we define

$$\mathcal{L}_\xi[v(\cdot)] := -\frac{\partial^2 v}{\partial \xi^2} + v \left.\frac{\partial B}{\partial s}\right|_{s=v_0, x=\bar{x}}, \tag{4.3.16}$$

and so $v_1(\xi, l)$ is defined by

$$\mathcal{L}_\xi[v_1] = \kappa \frac{\partial v_0}{\partial \xi} - \xi \left.\frac{\partial B}{\partial r}\right|_{x=\bar{x}, s=v_0}, \tag{4.3.17a}$$

$$\left.\frac{\partial v_1}{\partial \xi}\right|_{\xi=0} = -\left.\frac{\partial u_0}{\partial r}\right|_{x=\bar{x}}, \quad v_1(-\infty, l) = 0. \tag{4.3.17b}$$

We also define $\tilde{\mathcal{L}}_\xi[v(\cdot)]$ for later use as

$$\tilde{\mathcal{L}}_\xi[v(\cdot)] := \mathcal{L}_\xi[v(\cdot)] - pv(\cdot). \tag{4.3.18}$$

Lemma 4.3.1. *For the asymptotic expansion $u_{as}(x)$ from* (4.3.10) *we have*

$$\mathcal{F}u_{as}(x) = O(\varepsilon^2) \quad \textit{for } x \in \bar{\Omega}, \tag{4.3.19}$$

$$\varepsilon \left.\frac{\partial u_{as}}{\partial n}\right|_{x=\bar{x}} = g(\bar{x}). \tag{4.3.20}$$

Proof. To obtain $\mathcal{F}u_{as}$ in the necessary regions we call on Lemma 2.7.1, Corol-

lary 2.7.1 and Lemma 2.7.3 of *Chapter 2* with $u_{as}(x) = u_0(x) + v_0(\xi,l) + \varepsilon v_1(\xi,l)$ rather than $u_{as}(x) = u_0(x) + \varepsilon^2 u_2(x) + v_0(\xi,l) + \varepsilon v_1(\xi,l) + \varepsilon^2 v_2(\xi,l)$ and obtain

$$\mathcal{F}u_0(x) = O(\varepsilon^2) \quad \text{for } x \in \Omega\backslash\partial\Omega, \tag{4.3.21}$$

$$\mathcal{F}u_{as}(x) = O(\varepsilon^2) \quad \text{for } -c_1 \leqslant r \leqslant -\frac{c_1}{2}, \tag{4.3.22}$$

$$\mathcal{F}u_{as}(x) = O(\varepsilon^2) \quad \text{for } -\frac{c_1}{2} \leqslant r \leqslant 0, \tag{4.3.23}$$

giving (4.3.19).

We find (4.3.20) by noting (4.3.5), taking derivatives of (4.3.10) with respect to r and recalling (4.3.15b) and (4.3.17b).

□

We define

$$\gamma_L^2 := \min_{l\in[0,L]} b_u(\bar{x}, u_0(\bar{x})) > \gamma^2, \tag{4.3.24}$$

with $p_0 \in (0, \gamma_L^2)$ and γ defined in (C1).

Lemma 4.3.2. *There exists functions $v_0(\xi,l)$, $\tilde{v}_0(\xi,l;p)$ and $v_1(\xi,l)$ satisfying (4.3.14), (4.3.15) and (4.3.17) respectively with the properties*

$$0 \leqslant \tilde{v}_0(\xi,l;p) \leqslant A(l), \quad \chi(\xi,l) := \frac{\partial \tilde{v}_0}{\partial \xi} \leqslant 0, \quad \frac{\partial \tilde{v}_0}{\partial p} \geqslant 0, \tag{4.3.25}$$

for $\xi \leqslant 0$ and $l \geqslant 0$. Furthermore for $\delta \in (0, \gamma_L - \sqrt{p_0})$, there exists $\bar{C}_\delta > 0$ such that

$$\left|\frac{\partial^{k+m}\tilde{v}_0}{\partial\xi^k\partial l^m}\right| + \left|\frac{\partial^{k+m}v_1}{\partial\xi^k\partial l^m}\right| + \left|\frac{\partial\tilde{v}_0}{\partial p}\right| + \left|\frac{\partial^2\tilde{v}_0}{\partial p\partial\xi}\right| \leqslant \bar{C}_\delta e^{-(\gamma_L-\sqrt{p_0}-\delta)|\xi|}, \tag{4.3.26}$$

for $k = 0,1,2$, $m = 0,1,2$ and $k+m \leqslant 3$.

Proof. By (C1), (C2), (C3), and since $|\Psi(\xi,l)| \leqslant C(1+|\xi|^k)|\chi(\xi,l)|$ holds for $\Psi(\xi,l)$ equal to the right hand sides of equations for $\tilde{v}_0(\xi,l)$ and $v_1(\xi,l)$

we can apply Lemma 3.3.3 and get the results for derivatives with respect to r, l and p. Similarly taking mixed derivatives of (4.3.15a) and (4.3.17a) and applying Lemma 3.3.3 gives the bounds for mixed derivatives.

To obtain a bound for $\dfrac{\partial^2 \tilde{v}_0}{\partial p \partial \xi}$ we have

$$\tilde{\mathcal{L}}_\xi \left[\frac{\partial^2 \tilde{v}_0}{\partial p \partial \xi} \right] = \frac{\partial \tilde{v}_0}{\partial \xi} - \frac{\partial \tilde{v}_0}{\partial \xi} \frac{\partial \tilde{v}_0}{\partial p} B_{ss}(\bar{x}, \tilde{v}_0), \tag{4.3.27a}$$

$$\frac{\partial}{\partial \xi} \frac{\partial^2 \tilde{v}_0}{\partial p \partial \xi}\bigg|_{\xi=0} = -\tilde{v}_0(0,l;p) + \frac{\partial \tilde{v}_0}{\partial p}\bigg|_{\xi=0} \left(B_s(0,t,\tilde{v}_0(0,l;p)) - p \right), \quad \frac{\partial^2 \tilde{v}_0}{\partial p \partial \xi}\bigg|_{\xi=\infty} = 0 \tag{4.3.27b}$$

and as the requirements of Lemma 3.3.2(ii) of *Chapter 3* are met, applying the lemma we have the bound for $\dfrac{\partial^2 \tilde{v}_0}{\partial p \partial \xi}$ in (4.3.26). $\square$

4.3.4 Upper and Lower Solutions

Define $\beta(x;p)$ as

$$\beta(x;p) := u_0(x) + [\tilde{v}_0(\xi,l;p) + \varepsilon v_1(\xi,l)]\vartheta(x) + C_0 p(1 + e^{-c_0|r|/\varepsilon}\vartheta(x)), \tag{4.3.28}$$

where C_0 is a sufficiently large positive constant, c_0 is a sufficiently small positive constant and $e^{-c_0|r|/\varepsilon}$ is introduced such that $\beta(x;\pm p)$ will satisfy (4.3.31c). The addition of $C_0 p$ ensures $(\operatorname{sgn} p)\mathcal{F}\beta(x;p) \geqslant 0$. We note $\beta(x;p)$ of (4.3.28) can be written as

$$\beta(x;p) = u_{as}(x) + V(\xi,l;p) + C_0 p \rho(r) \tag{4.3.29}$$

where $\rho(r) := 1 + e^{-c_0|r|/\varepsilon}\vartheta(x)$ and

$$V(\xi,l;p) := [\tilde{v}_0(\xi,l;p) - v_0(\xi,l)]\vartheta(x). \tag{4.3.30}$$

We claim $\beta(x;\pm p)$ are upper and lower solutions to (4.0.1) i.e., they satisfy

$$\beta(x;-p) \leqslant \beta(x;p), \tag{4.3.31a}$$

$$\mathcal{F}\beta(x;-p) \leqslant 0 \leqslant \mathcal{F}\beta(x;p), \tag{4.3.31b}$$

$$\varepsilon \left.\frac{\partial \beta(x;-p)}{\partial n}\right|_{x=\bar{x}} \leqslant g(\bar{x}) \leqslant \varepsilon \left.\frac{\partial \beta(x;p)}{\partial n}\right|_{x=\bar{x}}. \tag{4.3.31c}$$

Corollary 4.3.1. *Let $V(\xi,l;p)$ be defined by* (4.3.30). *For $|p| \leqslant p_0$ we have*

$$|V| + \left|\frac{\partial V}{\partial \xi}\right| \leqslant C_V|p|, \tag{4.3.32}$$

for some sufficiently large positive constant C_V.

Proof. Recalling (4.3.30), it is clear that $|V| \leqslant |\tilde{v}_0 - v_0|$ as $0 \leqslant \vartheta \leqslant 1$. Now taking a Taylor expansion of the perturbed terms of $|V| + |V_\xi|$ about $p = 0$ and cancelling terms gives

$$|V| + \left|\frac{\partial V}{\partial \xi}\right| \leqslant \left| p \left.\frac{\partial \tilde{v}_0}{\partial p}\right|_{p=\tilde{p}}\right| + \left| p \left.\frac{\partial^2 \tilde{v}_0}{\partial \xi \partial p}\right|_{p=\tilde{p}}\right|, \tag{4.3.33}$$

with $\tilde{p} \in (0,p)$. By Lemma 4.3.2, $\dfrac{\partial^2 \tilde{v}_0}{\partial \xi \partial p}$ and $\dfrac{\partial \tilde{v}_0}{\partial p}$ are bounded and exponentially decaying as $\xi \to -\infty$ hence we can find C_V sufficiently large such that (4.3.32) holds. $\square$

Corollary 4.3.2. *Let $\beta(x;p)$ be defined in* (4.3.28). *For $|\bar{p}| \leqslant p_0$ we have*

$$\beta(x;\bar{p}) = u_{as}(x) + O(\bar{p}), \tag{4.3.34}$$

and for $\bar{p} \geqslant 0$,

$$\beta(x;-\bar{p}) \leqslant u_{as}(x,t) - C_0\bar{p} \leqslant u_{as}(x,t) + C_0\bar{p} \leqslant \beta(x;\bar{p}), \tag{4.3.35}$$

$$\left.\frac{\partial \beta}{\partial r}\right|_{r=0,p=-\bar{p}} \leqslant g(\bar{x}) \leqslant \left.\frac{\partial \beta}{\partial r}\right|_{r=0,p=\bar{p}}. \tag{4.3.36}$$

Proof. This proof follows that of Lemma 3.3.11.

Considering $\beta(x;\bar{p})$ in the form of (4.3.29) we recall (4.3.32) and get (4.3.34).

As $\dfrac{\partial \tilde{v}_0}{\partial p} \geqslant 0$ by (4.3.25), then $\tilde{v}_0(\xi,l;-\bar{p}) - v_0(\xi,l) \leqslant 0$, and as $1 \leqslant \rho(r) \leqslant 2$ and $0 \leqslant \vartheta(x) \leqslant 1$ we have

$$\beta(x;-\bar{p}) \leqslant u_{as}(x) - C_0\bar{p}. \tag{4.3.37}$$

Next $u_{as}(x) - C_0\bar{p} \leqslant u_{as}(x) + C_0\bar{p}$ as $\bar{p} \geqslant 0$. Finally, as $\tilde{v}_0(\xi,l;\bar{p}) - v_0(\xi,l) \geqslant 0$ we also have

$$u_{as}(x) + C_0\bar{p} \leqslant \beta(x;\bar{p}), \tag{4.3.38}$$

hence (4.3.35) is satisfied.

For the boundary conditions we recall (4.3.20), (4.3.15b) and (4.3.14b) and get

$$\left.\frac{\partial \beta}{\partial r}\right|_{r=0,p=\bar{p}} = g(\bar{x}) + C_0\bar{p}\varepsilon \left.\frac{\partial \rho}{\partial r}\right|_{r=0}. \tag{4.3.39}$$

Calculating $\dfrac{\partial \rho}{\partial r}$ we note $\dfrac{\partial |r|}{\partial r} = -1$ for $r \in \bar{\Omega}$ and have

$$\varepsilon \left.\frac{\partial \rho}{\partial r}\right|_{r=0} = c_0. \tag{4.3.40}$$

Now, noting C_0, c_0 and $\bar{p}$ are positive, (4.3.39) becomes

$$\left.\frac{\partial \beta}{\partial r}\right|_{r=0, p=\bar{p}} = g(\bar{x}) + C_0 c_0 \bar{p} \geqslant g(\bar{x}). \tag{4.3.41}$$

Similarly,

$$\left.\frac{\partial \beta}{\partial r}\right|_{r=0, p=-\bar{p}} = g(\bar{x}) - C_0 c_0 \bar{p} \leqslant g(\bar{x}), \tag{4.3.42}$$

and hence (4.3.36) holds. □

Lemma 4.3.3. *Let $\beta(x;\bar{p})$ be defined by (4.3.28). For $0 < \bar{p} \leqslant p_0$ we have*

$$\mathcal{F}\beta(x;\bar{p}) = C_0\bar{p}\rho B_s(\bar{x},0) + \bar{p}v_0(1+C_0\rho\lambda) - c_0^2 C_0 \bar{p} e^{-c_0|r|/\varepsilon} + O(\varepsilon^2+\bar{p}^2), \tag{4.3.43}$$

where $\lambda := B_{ss}(\bar{x}, v_0\theta)$ and $\theta(x) \in (0,1)$.

Proof. As we have established (4.3.19) we now consider

$$\mathcal{F}\beta - \mathcal{F}u_{as} = -\varepsilon^2\Delta(V + C_0\bar{p}\rho) + b(x,s)\big|_{s=u_{as}}^{s=\beta}. \tag{4.3.44}$$

Recalling (4.3.2) and (4.3.32) we have

$$-\varepsilon^2\Delta V = -\frac{\partial^2 V}{\partial \xi^2} + O(\varepsilon^2 + \bar{p}^2). \tag{4.3.45}$$

From (4.3.14a) and (4.3.15a) we can write

$$\frac{\partial^2 V}{\partial \xi^2} = B(\bar{x}, \tilde{v}_0) - \bar{p}\tilde{v}_0 - B(\bar{x}, v_0), \tag{4.3.46}$$

and taking a Taylor expansion (4.3.46) becomes

$$\frac{\partial^2 V}{\partial \xi^2} = V B_s(\bar{x}, v_0) - \bar{p} v_0 + O(\bar{p}^2). \tag{4.3.47}$$

Next, by (4.3.44),

$$-C_0 \bar{p} \varepsilon^2 \Delta \rho = -C_0 \bar{p}(\varepsilon \kappa c_0 - c_0^2) e^{-c_0|r|/\varepsilon}, \tag{4.3.48}$$

which can be written as

$$-C_0 \bar{p} \varepsilon^2 \Delta \rho = c_0^2 C_0 \bar{p} e^{-c_0|r|/\varepsilon} + O(\varepsilon^2 + \bar{p}^2). \tag{4.3.49}$$

Next, the last term in (4.3.44) can be written as

$$b(x,s)|_{s=u_{as}}^{s=\beta} = b(x,s)_{s=u_{as}+V}^{s=\beta} + b(x,s)_{s=u_{as}}^{s=u_{as}+V}. \tag{4.3.50}$$

The first term in (4.3.50) can be written as

$$b(x,s)_{s=u_{as}+V}^{s=\beta} = C_0 \bar{p} \rho B_s(x, v_0 + \varepsilon v_1 + V) + O(\bar{p}^2), \tag{4.3.51}$$

and a Taylor expansion of (4.3.51) about $\bar{x}$ gives

$$b(x,s)_{s=u_{as}+V}^{s=\beta} = C_0 \bar{p} \rho B_s(\bar{x}, v_0 + \varepsilon v_1 + V) + O(\varepsilon^2 + \bar{p}^2). \tag{4.3.52}$$

Taking another Taylor expansion and noting (4.3.32) we obtain

$$b(x,s)_{s=u_{as}+V}^{s=\beta} = C_0 \bar{p} \rho B_s(\bar{x}, 0) + C_0 \bar{p} \rho v_0 \lambda + O(\varepsilon^2 + \bar{p}^2), \tag{4.3.53}$$

where $\lambda(x) := B_{ss}(\bar{x}, v_0 \theta)$. For the second term of (4.3.50) we have

$$b(x,s)_{s=u_{as}}^{s=u_{as}+V} = V B_s(x, v_0 + \varepsilon v_1) + O(\bar{p}^2) = V B_s(\bar{x}, v_0) + O(\bar{p}^2 + \varepsilon^2). \tag{4.3.54}$$

Hence,

$$b(x,s)|_{s=u_{as}}^{s=\beta} = C_0\bar{p}\rho B_s(\bar{x},0) + C_0\bar{p}\rho v_0\lambda + VB_s(\bar{x},v_0) + O(\varepsilon^2+\bar{p}^2). \quad (4.3.55)$$

Combining (4.3.47), (4.3.49) and (4.3.55) and recalling Lemma 4.3.1 we get (4.3.43). □

Corollary 4.3.3. *Let* $\beta(x;p)$ *be defined by* (4.3.28)*. There exists sufficiently large positive constants* C_0 *and* C_1 *such that for* $0 < \bar{p} \leqslant p_0$ *we have*

$$\mathcal{F}\beta(x;\bar{p}) \geqslant \frac{C_0\bar{p}\gamma^2}{2} - C_1(\varepsilon^2+\bar{p}^2), \quad (4.3.56)$$

and

$$\mathcal{F}\beta(x;-\bar{p}) \leqslant -\frac{C_0\bar{p}\gamma^2}{2} + C_1(\varepsilon^2+\bar{p}^2). \quad (4.3.57)$$

Proof. Considering the case with $p=\bar{p}$ we recall (4.3.43) and have

$$\mathcal{F}\beta(x;\bar{p}) = C_0\bar{p}\rho B_s(\bar{x},0) + \bar{p}v_0(1+C_0\rho\lambda) - c_0^2C_0\bar{p}e^{-c_0|r|/\varepsilon} + O(\varepsilon^2+\bar{p}^2). \quad (4.3.58)$$

Choose C_0 sufficiently large such that $0 < C_0 \leqslant |\lambda(x)|^{-1}$. As $1 \leqslant \rho \leqslant 2$ and recalling (C1) and (4.3.25) we find

$$\mathcal{F}\beta(x;\bar{p}) \geqslant C_0\bar{p}\gamma^2 - c_0^2C_0\bar{p} - C_1(\varepsilon^2+\bar{p}^2). \quad (4.3.59)$$

Choosing c_0 sufficiently small such that

$$c_0^2 \leqslant \frac{\gamma^2}{2}, \quad (4.3.60)$$

we get (4.3.56). The second bound (4.3.57) is obtained by a similar method. □

From Corollary 4.3.3 if $\bar{p} > 0$ and $\dfrac{C_0\bar{p}\gamma^2}{2}$ dominates $C_1(\varepsilon^2 + \bar{p}^2)$ then $\mathcal{F}\beta(x;-\bar{p}) \leqslant 0 \leqslant \mathcal{F}\beta(x;\bar{p})$ and as $\beta(x;-\bar{p}) \leqslant \beta(x;\bar{p})$ from Corollary 4.3.2, $\beta(x;-p)$ and $\beta(x;\bar{p})$ are ordered upper and lower solutions to (4.0.1). This is now shown in Corollary 4.3.4.

Corollary 4.3.4. *Let $\beta(x;p)$ be defined by* (4.3.28)*. For $0 < \bar{p} \leqslant \bar{p}$ we have*

$$\beta(x;-\bar{p}) \leqslant u(x) \leqslant \beta(x;\bar{p}), \tag{4.3.61}$$

and

$$|u(x) - u_{as}(x)| \leqslant C\varepsilon^2. \tag{4.3.62}$$

Proof. We consider terms in (4.3.56) and choose $\bar{p} := C_2\varepsilon^2$ where $C_2 \geqslant \dfrac{2C_1}{C_0\gamma^2}$, i.e.,

$$C_0\gamma^2\bar{p} \geqslant 2C_1\varepsilon^2. \tag{4.3.63}$$

Now considering $C_1\bar{p}^2$ we choose C_2 sufficiently large such that $\varepsilon \leqslant 1/C_2$. Hence, $\bar{p} = C_2\varepsilon^2 \leqslant \varepsilon$ and so $C_1\bar{p}^2 \leqslant C_1\varepsilon^2$ implying

$$-C_1(\varepsilon^2 + \bar{p}^2) \geqslant -2C_1\varepsilon^2. \tag{4.3.64}$$

Combining (4.3.63) and (4.3.64) in (4.3.56) we find

$$\mathcal{F}\beta(x;\bar{p}) \geqslant 2C_1\varepsilon^2 - 2C_1\varepsilon^2 = 0. \tag{4.3.65}$$

Similarly we obtain

$$\mathcal{F}\beta(x;-\bar{p}) \leqslant 0. \tag{4.3.66}$$

Hence the requirements (4.3.31) are met, i.e., we have (4.3.35), (4.3.36) and $\mathcal{F}\beta(x;-\bar{p}) \leqslant 0 \leqslant \mathcal{F}\beta(x;\bar{p})$, and so $\beta(x;\pm\bar{p})$ are ordered upper and lower solutions to (4.0.1) satisfying (4.3.61). Recalling (4.3.34) and $\bar{p} = O(\varepsilon^2)$ we

have

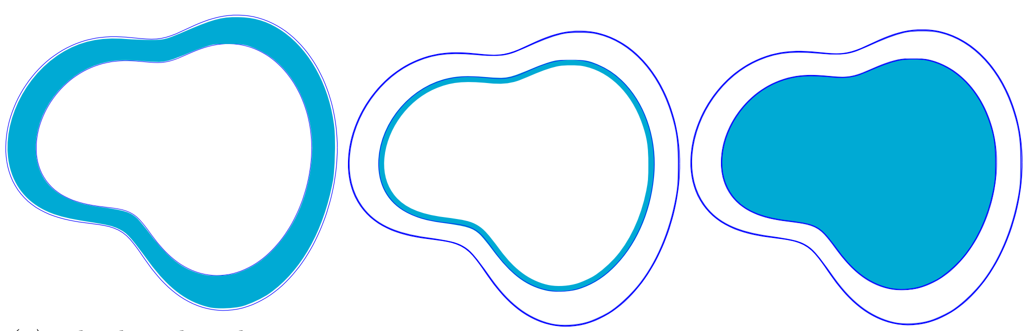
$$\beta(x;\bar{p}) = u_{as}(x) + O(\varepsilon^2), \tag{4.3.67}$$

and by (4.3.61) we have shown (4.3.62). □

4.4 Discrete Space: Analysis of the Numerical Method

In this section we find a discretisation for problem (4.0.1). This discretisation is similar to that in *Chapter 2*. We give a finite difference discretisation for the layer region and a finite element discretisation for the outer region. We consider the interface between the layer region and outer region from both sides of the interface curve and combine the results to obtain results for the interface curve itself. We again call on Z-fields as described in §1.0.2 to obtain existence and accuracy of the discrete solution U_{ij}.

As in *Chapter 2* we define τ as a small positive parameter such that $\tau \leqslant c_1$. The curve $\Gamma_{-\tau} := \{x(r,l) \in \bar{\Omega}^N : r = -\tau\}$ is a smooth closed curve in Ω^N that does not intersect $\partial\Omega^N$. We discretise the layer region $\Omega_{(-c_1,0]}$ by $\Omega_{(-\tau,0]} := \{x(r,l) : -\tau < r \leqslant 0\}$. Let $\mathring{\Omega}$ be defined as the interior of $\Gamma_{-\tau}$. These regions are represented in Figure 4.2.



(a) The boundary layer region $\Omega_{(-\tau,0]}$. *(b) The curve $\Gamma_{-\tau}$.* *(c) The outer region $\mathring{\Omega}$.*

Figure 4.2: Discrete Regions of Ω^N.

We use finite differences in curvilinear coordinates in $\Omega_{(-\tau,0]}$ and employ a tensor product mesh. Again as in *Chapter 2* for the outer region $\mathring{\Omega}^N$ we use piecewise linear finite element on a quasiuniform Delaunay triangulation.

As in [14], we are using curvilinear coordinates in a narrow region along the boundary and so have not replaced the boundary of the domain $\partial\Omega$ by a polygonal boundary. If a polygonal boundary was used a significant portion of the boundary layer would be lost.

We note that this section differs from §2.5 by the location of the layer region. In this chapter we have a boundary layer region. The curve Γ_0 in *Chapter 2* is similar to $\partial\Omega$ in this chapter. We use negative indices here so that the notation matches that of *Chapter 2* for $r < 0$.

4.4.1 Layer Adapted Meshes

As in the previous chapters we consider layer adapted meshes for this problem. We consider the Shishkin mesh and the Bakhvalov mesh of §1.0.3. We rescale our mesh in the r direction as this is the direction of rapid change of our solution. As the asymptotic expansion in this chapter is simpler than that of *Chapter 2*, the Bakhvalov mesh can be used without much further complication.

We define the mesh transition point τ for both the Shishkin and Bakhvalov meshes as

$$\tau := 2\gamma^{-1}\varepsilon \ln N, \tag{4.4.1}$$

where γ is defined in (C1). Define h as the maximum side length of any triangle, i.e., $h := \max_i h_i$. We recall the description of the tensor product mesh and the fictitious Neumann boundary condition in § 2.5.1. These hold for this problem with $r < 0$.

We use the Shishkin mesh as described in *Chapter 2* §2.5.1 with the alteration that the boundary layer is located around $\partial\Omega$ rather than Γ_0 and that r is always negative. As in *Chapter 2*, we have a uniform mesh $\{r_i\}_{i=0}^{N}$ on

$[-\tau, 0]$, with $r_i - r_{i-1} = \tau/N = 2\gamma^{-1}\varepsilon N^{-1}\ln N$. For the Shishkin mesh,

$$\hat{h} := \varepsilon^{-1}\min_j h_j = CN^{-1}\ln N. \tag{4.4.2}$$

We use the Bakhvalov mesh as described in *Chapter 3* § 3.4.2 i.e., the mesh is defined as

$$r_i := r([1-\varepsilon]i/N),\ i = 0..., N, \quad r(\zeta) := -2\gamma^{-1}\varepsilon\ln(1-\zeta) \ \text{ for } \zeta \in [0, 1-\varepsilon]. \tag{4.4.3}$$

As was the case in the continuous space, in the discrete space we may write $x_{ij} = (r_i, l_j)$ and $x_{ij} = (\xi_i, l_j)$ for $x_{ij} = x(r_i, l_j)$ and $x_{ij} = x(\xi_i, l_j)$ respectively.

4.4.2 Discretisation of the Boundary Layer Region $\Omega_{(-\tau,0]}$

We use Newton's method with finite differences to solve (4.0.1) in the boundary layer region in which $-\tau < r \leqslant 0$. This is represented in Figure 4.2(a). The discrete system $\mathcal{F}^N U_{ij}$ for $i = -1, ..., -N+1$ and $j = 0, ..., N_l - 1$, is

$$\mathcal{F}^N U_{ij} := -\varepsilon^2\eta_{ij}^{-1}D_r\left[\tilde{\eta}_{ij}D_r^- U_{ij}\right] - \varepsilon^2\zeta_{ij}D_l\left[\tilde{\zeta}_{ij}D_l^- U_{ij}\right] + b\left(x_{ij}, U_{ij}\right) = 0, \tag{4.4.4a}$$

and for $i = 0$, $j = 0, ..., N_l - 1$ we define the discrete system $\mathcal{G}^N$ as

$$\begin{aligned}\mathcal{G}^N U_{0j} :=& \frac{2\varepsilon^2}{h_0}D_r^- U_{0j} - \frac{2\varepsilon}{h_0}\left(1 + \frac{h_0\kappa}{2}\right)g_{0j} \\ &- \varepsilon^2\zeta_{0j}D_l\left[\tilde{\zeta}_{0j}D_l^- U_{0j}\right] + b\left(x_{0j}, U_{0j}\right) = 0\end{aligned} \tag{4.4.4b}$$

$$U_{i,N_l} = U_{i,0}, \quad U_{i,-1} = U_{i,N_l-1}, \tag{4.4.4c}$$

where ζ and η are described in (4.3.3) and

$$x_{ij} := x(r_i, l_j), \quad \eta_{ij} := \eta(r_i, l_j), \quad \zeta_{ij} := \zeta(r_i, l_j), \tag{4.4.4d}$$

$$\tilde{\eta}_{ij} := \eta(r_{i-1/2}, l_j), \quad \tilde{\zeta}_{ij} := \zeta(r_{i-1/2}, l_j), \tag{4.4.4e}$$

$$D_r^- v_{ij} := \frac{v_{ij} - v_{i-1,j}}{r_i - r_{i-1}}, \quad D_r v_{ij} := \frac{v_{i+1,j} - v_{ij}}{(r_{i+1} - r_{i-1})/2}, \tag{4.4.4f}$$

$$D_l^- v_{ij} := \frac{v_{ij} - v_{ij-1}}{l_j - l_{j-1}}, \quad D_l v_{ij} := \frac{v_{ij+1} - v_{ij}}{(l_{j+1} - l_{j-1})/2}, \tag{4.4.4g}$$

$$h_i := r_i - r_{i-1}, \quad g_j := g(\bar{x}) = g(0, l). \tag{4.4.4h}$$

The discretisation along the boundary given by (4.4.4b) was found by taking a central difference discretisation of the Neumann boundary condition (4.0.1b), i.e.,

$$\varepsilon \frac{U_{1,j} - U_{-1,j}}{2h_0} = g(0, l) = g_j, \tag{4.4.5}$$

where $U_{1,j}$ is the solution at the ghost point r_1 located at a distance of h_0 outside the domain, i.e., $h_1 := r_1 - r_0$ where $h_1 = h_0$. Then (4.4.5) is combined with (4.4.4a) to eliminate the ghost point $U_{1,j}$ and give

$$\begin{aligned} \mathcal{G}^N U_{0j} :=& \frac{\varepsilon^2}{h_0} \eta_{0j}^{-1} (\tilde{\eta}_{1j} + \tilde{\eta}_{0j}) D_r^- U_{0j} - \eta_{0j}^{-1} \tilde{\eta}_{1j} \frac{2\varepsilon}{h_0} g_j \\ & - \varepsilon^2 \zeta_{0j} D_l \left[\tilde{\zeta}_{0j} D_l^- U_{0j} \right] + b\left(x_{0j}, U_{0j}\right) = 0. \end{aligned} \tag{4.4.6}$$

This is the same method used in *Chapter 3* for the boundary condition of the time-dependent problem and in *Chapter 2* on the interface for the fictitious Neumann boundary condition. Taking Taylor expansions of $\tilde{\eta}_{1j}$ and $\tilde{\eta}_{0j}$ and noting $\eta' = \kappa$, then

$$\tilde{\eta}_{1j} + \tilde{\eta}_{0j} = 2\eta_{0j}, \quad \tilde{\eta}_{1j} = \eta_{0j} + \frac{h_0 \kappa}{2}. \tag{4.4.7}$$

We also note $\eta_{0j} = 1$ as $r_0 = 0$ and recall the definition of η in (4.3.3). Hence we can write (4.4.6) as (4.4.4b).

We introduce the function $\mathcal{G}u(\bar{x})$, similar to (3.4.15) in *Chapter 3*, as

$$\mathcal{G}u(\bar{x}) := \varepsilon \left.\frac{\partial u}{\partial n}\right|_{x=\bar{x}} - g(\bar{x}) = 0 \tag{4.4.8}$$

In order to calculate the truncation error of the system on the Bakhvalov mesh we first present an auxiliary result for a non uniform mesh to simplify presentation. This is similar to Lemma 2.5.2 however we require the result to a higher degree than in *Chapter 2*.

Let $w(t)$ be some function. Consider

$$Mw := -q(t)(p(t)w'(t))', \tag{4.4.9}$$

and, for the case with a non uniform mesh, take the discrete analogue of (4.4.9) to be

$$M^N w_i := -q_i D\left(\tilde{p}_i D^- w_i\right) \qquad \text{for } i = 1, ..., N-1, \tag{4.4.10a}$$

where

$$D^- w_i := (w_i - w_{i-1})/k_i, \quad D w_i := (w_{i+1} - w_i)/\hbar_i, \tag{4.4.10b}$$

$$\hbar_i := (k_i + k_{i+1})/2, \quad k_i := t_i - t_{i-1} \quad \text{and} \quad \tilde{p}_i := p_{i-\frac{1}{2}}. \tag{4.4.10c}$$

We want to evaluate the truncation error $R_i := M^N w_i - (Mw)_i$.

Lemma 4.4.1. *Let $|w^{(j)}| \leqslant C$ and $|p^{(j)}| \leqslant C$ for $j = 0, ..., 3$. For a non uniform mesh the truncation error $R_i := M^N w_i - (Mw)_i$ of (4.4.10) is*

$$R_i := q_i \left[-\frac{k_{i+1} - k_i}{2}(pw')''_i - \frac{k_{i+1}^2 - k_i k_{i+1} + k_i^2}{2^2 3!}(pw')'''_i - \frac{(p_i \hbar_i^2 w'''_i)'}{2^2 3!} + O(k^3) \right] \tag{4.4.11}$$

where $k = \max k_i$.

Proof. Define $v := pw'$ and $V_{i-\frac{1}{2}} := \tilde{p}_i D^- w_i$. For a non-uniform mesh,

$$R_i := q_i \Bigg[\underbrace{v_i' - \frac{v_{i+1/2} - v_{i-1/2}}{\hbar_i}}_{A} - \underbrace{\frac{(V_{i+1/2} - v_{i+1/2}) - (V_{i-1/2} - v_{i-1/2})}{\hbar_i}}_{B} \Bigg]. \tag{4.4.12}$$

Taking a Taylor expansion of A and cancelling terms we have

$$A = -\frac{k_{i+1} - k_i}{2} v_i'' - \frac{k_{i+1}^2 - k_i k_{i+1} + k_i^2}{2^2 3!} v_i''' + O(k^3). \tag{4.4.13}$$

Recalling that $v := pw'$, (4.4.13) becomes

$$A = -\frac{k_{i+1} - k_i}{2} (pw')_i'' - \frac{k_{i+1}^2 - k_i k_{i+1} + k_i^2}{2^2 3!} (pw')_i''' + O(k^3). \tag{4.4.14}$$

Similarly,

$$B = \frac{p_{i+1/2} k_{i+1}^2 w_{i+1/2}''' - p_{i-1/2} k_i^2 w_{i-1/2}'''}{2^2 3! \hbar_i} + O(k^4), \tag{4.4.15}$$

where $k := \max k_i$. Using the mean value theorem, and noting $k_{i+1/2} = \hbar_i$, this can be written as

$$B = \frac{(p_i \hbar_i^2 w_i''')'}{2^2 3!} + O(k^4), \tag{4.4.16}$$

giving (4.4.11). □

Lemma 4.4.2. *Let $\beta(x;p)$ be defined by (4.3.28), and the mesh $\{r_i\}_{i=0}^{-N}$ be the Shishkin mesh or Bakhvalov mesh of §4.4.1. Then for all $|p| \leqslant p_0$ we have*

$$|\mathcal{F}^N \beta(x_{ij}, p) - \mathcal{F}\beta(x_{ij}, p)| \leqslant CN^{-2} \ln^{2m} N \quad x_{ij} \in \Omega_{(-\tau,0)}, \tag{4.4.17}$$

$$|\mathcal{G}^N \beta(x_{0j}, p) - \mathcal{G}\beta(x_{0j}, p)| \leqslant \frac{h_0}{2\varepsilon} \mathcal{F}\beta(x_{0j}, p) + CN^{-2} \ln^{2m} N \quad x_{ij} \in \partial\Omega^N, \tag{4.4.18}$$

where $m = 0$ for the Bakhvalov mesh and $m = 1$ for the Shishkin mesh.

Proof. We first want to consider

$$\mathcal{F}^N \beta_{ij} - \mathcal{F}\beta(x_{ij}) \quad \text{for } x_{ij} \in \Omega_{(-\tau,0)}. \tag{4.4.19}$$

For derivatives with respect to l, we use Lemma 2.5.2 of *Chapter 2* with $w = \beta$, $p = \zeta$, $q = \zeta$, $t = l$ and $k_i = l_i - l_{i-1}$, i.e., $k_i \leqslant CN^{-1} \leqslant C\hat{h}$ and find

$$-\varepsilon^2 \zeta_{ij} D_l \left[\tilde{\zeta}_{ij} D_l^- \beta_{ij}\right] + \varepsilon^2 \zeta(\zeta\beta_l)_l = O(\varepsilon^2 N^{-1}). \tag{4.4.20}$$

Using (4.4.20) and noting $N^{-1} \leqslant C\hat{h}$ we have

$$\mathcal{F}^N \beta_{ij} - \mathcal{F}\beta(x_{ij}) = -\varepsilon^2 \eta_{ij}^{-1} D_r \left[\tilde{\eta}_{ij} D_r^- U_{ij}\right] + \varepsilon^2 \eta^{-1}(\eta\beta_r)_r + O(\varepsilon^2 \hat{h}). \tag{4.4.21}$$

From this we rescale r and define

$$R_{ij} := -\eta_{ij}^{-1} D_\xi \left[\tilde{\eta}_{ij} D_\xi^- \beta_{ij}\right] + \eta^{-1}(\eta\beta_\xi)_\xi \Big|_{x_{ij}} \tag{4.4.22}$$

We now estimate R_{ij} on the Shishkin mesh for points in the interior of $\Omega_{[-\tau,0]}$, i.e., in $\Omega_{(-\tau,0)}$. In this region the mesh is uniform and so we can call on Lemma 2.5.1 of *Chapter 2* with $w = \beta$, $p = \eta$, $q = \eta^{-1}$, $t = \xi$ and $k_i = \xi_i - \xi_{i-1}$, i.e., $k = \hat{h}$ and have,

$$R_{ij} = -\frac{1}{2^2 3!} \hat{h}^2 \eta_{ij}^{-1} \left\{ \frac{\partial}{\partial \xi}\left(\tilde{\eta}\frac{\partial^3 \beta}{\partial \xi^3}\right) + \frac{\partial^3}{\partial \xi^3}\left(\tilde{\eta}\frac{\partial \beta}{\partial \xi}\right) \right\}\Bigg|_{x_{ij}} + O(\hat{h}^4). \tag{4.4.23}$$

As $\left|\dfrac{\partial^k \beta}{\partial \xi^k}\right| \leqslant C$, $\left|\dfrac{\partial^k \tilde{\eta}}{\partial \xi^k}\right| \leqslant C$ and $\left|\dfrac{\partial^k \tilde{\eta}^{-1}}{\partial \xi^k}\right| \leqslant C$ then

$$|R_{ij}| \leqslant C\hat{h}^2. \tag{4.4.24}$$

Recalling the definition of $\hat{h}$ in (4.4.2) the truncation error of the system in

the layer region on a Shishkin mesh is

$$|R_{ij}| \leqslant CN^{-2}\ln^2 N. \tag{4.4.25}$$

We now look to find a similar result for the Bakhvalov mesh. Applying Lemma 4.4.1 with $w = \beta$, $p = \eta$, $q = \eta^{-1}$, $t = \xi$ and $k_i = \xi_i - \xi_{i-1}$, i.e., $k_i = h_i/\varepsilon$ we get

$$R_{ij} := \eta_{ij}^{-1}\Bigg[\underbrace{-\frac{h_{i+1}-h_i}{2\varepsilon}\frac{\partial^2}{\partial\xi^2}\left(\eta\frac{\partial\beta}{\partial\xi}\right)_{ij}}_{A_1} \underbrace{-\frac{h_{i+1}^2-h_ih_{i+1}+h_i^2}{2^2 3!\varepsilon^2}\frac{\partial^3}{\partial\xi^3}\left(\eta\frac{\partial\beta}{\partial\xi}\right)_{ij}}_{A_2} \underbrace{-\frac{1}{2^2 3!\varepsilon^2}\frac{\partial}{\partial\xi}\left(\hbar_i^2\eta_{ij}\frac{\partial^3\beta}{\partial\xi^3}_{ij}\right)}_{A_3} + O\left(\frac{h^3}{\varepsilon^3}\right)\Bigg], \tag{4.4.26}$$

where $h := \max_i h_i$.

Considering the Bakhvalov mesh of §4.4.1 we first consider the region with $iN^{-1} < \tau - C_6N^{-1}$ with $C_6 \geqslant 2$. Using Lemma 3.4.1 of *Chapter 3* we have that the following results hold true in the two-dimensional case, with the alterations that r is rescaled using (4.3.7) and the Bakhvalov mesh is defined (4.4.3),

$$|h_{i+1}+h_i| \leqslant C\varepsilon N^{-1}\xi'_{i+1}, \quad |h_{i+1}-h_i| \leqslant C\varepsilon N^{-2}\xi''_{i+1}, \tag{4.4.27}$$

and

$$\xi'(\zeta) = 2\gamma^{-1}(1-\zeta)^{-1}, \quad \xi''(\zeta) = 2\gamma^{-1}(1-\zeta)^{-2}. \tag{4.4.28}$$

Considering $e^{-\gamma|\xi_{i-1}|}$ we have $\xi_{i-1} = -2\gamma^{-1}\ln(1-\zeta_{i-1})$ and so

$$e^{-\gamma|\xi_{i-1}|} = (1-\zeta_{i-1})^2. \tag{4.4.29}$$

and so

$$\left|\frac{\partial^k \beta}{\partial \xi^k}\right| \leqslant C\left(1-\zeta_{i-1}\right)^2. \tag{4.4.30}$$

We consider the three terms of (4.4.26) separately and we first consider A_1. Using (4.4.27) and (4.4.28) we can write

$$|A_1| = \left|\frac{h_{i+1}-h_i}{2\varepsilon}\frac{\partial^2}{\partial \xi^2}\left(\eta\frac{\partial \beta}{\partial \xi}\right)\right| \leqslant CN^{-2}\xi''_{i+1}e^{-\gamma|\xi_{i-1}|}, \tag{4.4.31}$$

i.e.,

$$|A_1| \leqslant CN^{-2}\left(\frac{1-(i+1)N^{-1}}{1-(i-1)N^{-1}}\right)^2, \tag{4.4.32}$$

and by recalling $iN^{-1} < \tau - C_6N^{-1}$ with $C_6 \geqslant 2$ we have $(i+1)N^{-1} < \tau$ and $(i-1)N^{-1} < \tau$ giving

$$|A_1| \leqslant CN^{-2}. \tag{4.4.33}$$

To calculate A_2 we first consider $(h_{i+1}^2 - h_ih_{i+1} + h_i^2)/\varepsilon$. As $h_{i+1}/\varepsilon = (\zeta_{i+1}-\zeta_i)\xi'(\bar{\zeta})$ and $h_i/\varepsilon = (\zeta_i - \zeta_{i-1})\xi'(\bar{\bar{\zeta}})$ where $\zeta_{i-1} \leqslant \bar{\bar{\zeta}} \leqslant \zeta_i \leqslant \bar{\zeta} \leqslant \zeta_{i+1}$. Note $\zeta_{i+1}-\zeta_i = \zeta_i - \zeta_{i-1} = N^{-1}$ we can write

$$\frac{h_{i+1}^2 - h_ih_{i+1} + h_i^2}{\varepsilon^2} = N^{-2}(\xi'(\bar{\zeta}))^2 - N^{-2}\xi'(\bar{\zeta})\xi'(\bar{\bar{\zeta}}) + N^{-2}(\xi'(\bar{\bar{\zeta}}))^2, \tag{4.4.34}$$

and as $\xi'(\zeta_i) \leqslant \xi'(\zeta_{i+1})$, (4.4.34) becomes

$$\left|\frac{h_{i+1}^2 - h_ih_{i+1} + h_i^2}{\varepsilon^2}\right| \leqslant CN^{-2}(\xi'(\zeta_{i+1}))^2. \tag{4.4.35}$$

For A_2 we can use (4.4.35) to get

$$|A_2| = \left|\frac{h_{i+1}^2 - h_ih_{i+1} + h_i^2}{2^2 3!\varepsilon^2}\frac{\partial^3}{\partial \xi^3}\left(\eta\frac{\partial \beta}{\partial \xi}\right)_{ij}\right| \leqslant CN^{-2}(\xi'(\zeta_{i+1}))^2 e^{-\gamma|\xi_{i-1}|}, \tag{4.4.36}$$

and using (4.4.28) and (4.4.29) we have

$$|A_2| \leqslant CN^{-2}\left(\frac{1-(i+1)N^{-1}}{1-(i-1)N^{-1}}\right)^2 \leqslant CN^{-2}. \tag{4.4.37}$$

Finally for A_3, the quotient rule gives

$$A_3 = \frac{\hbar_i^2}{2^2 3! \varepsilon^2}\frac{\partial}{\partial \xi}\left(\eta \frac{\partial^3 \beta}{\partial \xi^3}\right)\bigg|_{ij} + \frac{2\hbar_i \hbar_i' \eta_{ij}}{2^2 3! \varepsilon^2}\frac{\partial^3 \beta}{\partial \xi^3}\bigg|_{ij} + O\left(\frac{h^4}{\varepsilon^4}\right) \tag{4.4.38}$$

Now considering the first term we have

$$\left|\frac{\hbar_i^2}{2^2 3! \varepsilon^2}\frac{\partial}{\partial \xi}\left(\eta \frac{\partial^3 \beta}{\partial \xi^3}\right)\right| \leqslant CN^{-2}(\xi_{i+1}')^2 e^{-\gamma|\xi_{i-1}|} \tag{4.4.39}$$

and again using (4.4.28) and (4.4.29) gives

$$\left|\frac{\hbar_i^2}{2^2 3! \varepsilon^2}\frac{\partial}{\partial \xi}\left(\eta \frac{\partial^3 \beta}{\partial \xi^3}\right)\right| \leqslant CN^{-2}\left(\frac{1-(i+1)N^{-1}}{1-(i-1)N^{-1}}\right)^2 \leqslant CN^{-2}. \tag{4.4.40}$$

For the second term in A_3 we have

$$\left|\frac{2\hbar_i \hbar_i' \eta_{ij}}{2^2 3! \varepsilon^2}\frac{\partial^3 \beta}{\partial \xi^3}_{ij}\right| \leqslant CN^{-2}\frac{(1-(i-1)N^{-1})^2}{(1-(i+1)N^{-1})^3} \leqslant \frac{CN^{-2}}{(1-(i+1)N^{-1})}. \tag{4.4.41}$$

Since $1-(i+1)N^{-1} > (C_6-1)N^{-1}$ as $iN^{-1} < \tau - C_6 N^{-1}$ and so

$$\left|\frac{2\hbar_i \hbar_i' \eta_{ij}}{2^2 3! \varepsilon^2}\frac{\partial^3 \beta}{\partial \xi^3}_{ij}\right| \leqslant \frac{CN^{-2}}{(C_6-1)N^{-1}} \leqslant CN^{-1}, \tag{4.4.42}$$

and by (4.1.2) and (4.4.38),

$$|A_3| \leqslant C\left(N^{-2} + \frac{h^4}{\varepsilon^4}\right). \tag{4.4.43}$$

Noting that

$$\varepsilon^{-1}h_i = \xi_i - \xi_{i-1} \leqslant N^{-1}\xi'(i/N), \tag{4.4.44}$$

and recalling (4.4.28), for negative i, we can write $\xi'(i/N) \leqslant C$. Hence, recalling (4.1.2) we have

$$\varepsilon^{-1}h = O(N^{-1}). \tag{4.4.45}$$

Putting the results for A_1, A_2 and A_3, (4.4.33), (4.4.37) and (4.4.43), into (4.4.26) and noting (4.4.45) and $|\eta^{-1}| \leqslant C$ we get

$$|R_{ij}| \leqslant CN^{-2}, \tag{4.4.46}$$

for $x_{ij} \in \Omega_{(-\tau,0)}$ on the Bakhvalov mesh.

For the boundary condition we multiply (4.4.4b) by $\dfrac{h_0}{2\varepsilon}$ and have

$$\begin{aligned}\mathcal{G}^N\beta_{0j} - G\beta_{0j} =& \frac{h_0}{2\varepsilon}\left[\frac{2\varepsilon^2}{h_0}D_r^-\beta_{0j} - \frac{2\varepsilon}{h_0}\left(1+\frac{h_0\kappa}{2}\right)g_{0j}\right.\\ &\left. - \varepsilon^2\zeta_{0j}D_l\left[\tilde{\zeta}_{0j}D_l^-\beta_{0j}\right] + b\left(x_{0j},\beta_{0j}\right)\right]\\ &- \varepsilon\left.\frac{\partial\beta}{\partial r}\right|_{x_{0j}} + g(x_{0j}).\end{aligned} \tag{4.4.47}$$

Cancelling terms gives

$$\begin{aligned}\mathcal{G}^N\beta_{0j} - G\beta_{0j} =& \varepsilon D_r^-\beta_{0j} - \frac{h_0\kappa}{2}g(x_{0j}) - \varepsilon\left.\frac{\partial\beta}{\partial r}\right|_{x_{0j}}\\ &+ \frac{h_0}{2\varepsilon}\left[- \varepsilon^2\zeta_{0j}D_l\left[\tilde{\zeta}_{0j}D_l^-\beta_{0j}\right] + b\left(x_{0j},\beta_{0j}\right)\right].\end{aligned} \tag{4.4.48}$$

Lemma 2.5.2 of *Chapter 2* gives

$$\varepsilon^2\zeta_{0j}D_l\left[\tilde{\zeta}_{0j}D_l^-\beta_{0j}\right] = \varepsilon^2[\zeta(\zeta\beta_l)_l]_{0j} + O(\varepsilon^2N^{-1}). \tag{4.4.49}$$

Taking a Taylor expansion of $D_r^-\beta_{0j}$ and noting $\left|\dfrac{\partial^k\beta}{\partial r^k}\right| \leqslant C\varepsilon^{-k}$ yields

$$D_r^-\beta_{0j} = \left.\frac{\partial\beta}{\partial r}\right|_{0j} - \frac{h_0}{2}\left.\frac{\partial^2\beta}{\partial r^2}\right|_{0j} + O(\varepsilon^{-3}h_0^2), \tag{4.4.50}$$

and so (4.4.48) can be written as

$$\begin{aligned}\mathcal{G}^N\beta_{0j} - G\beta_{0j} =& \varepsilon\left.\frac{\partial\beta}{\partial r}\right|_{x_{0j}} - \varepsilon\frac{h_0}{2}\left.\frac{\partial^2\beta}{\partial r^2}\right|_{x_{0j}} - \frac{h_0\kappa}{2}g(x_{0j}) - \varepsilon\left.\frac{\partial\beta}{\partial r}\right|_{x_{0j}}\\ &+ \frac{h_0}{2\varepsilon}\left[-\varepsilon^2\,\zeta(\zeta\beta_l)_l\big|_{x_{0j}} + b\,(x_{0j},\beta_{0j})\right] + O(h_0\varepsilon N^{-1} + \varepsilon^{-2}h_0^2).\end{aligned} \tag{4.4.51}$$

We note that for the Bakhvalov mesh $h_0 \leqslant C\varepsilon N^{-1}$ and for the Shishkin mesh $h_0 \leqslant C\varepsilon N^{-1}\ln N$. Now recalling (4.4.8) we can write

$$\begin{aligned}\mathcal{G}^N\beta_{0j} - G\beta_{0j} =& \frac{h_0}{2\varepsilon}\left[-\varepsilon^2\left.\frac{\partial^2\beta}{\partial r^2}\right|_{x_{0j}} - \varepsilon^2\kappa\left.\frac{\partial\beta}{\partial r}\right|_{x_{0j}} - \varepsilon^2\,\zeta(\zeta\beta_l)_l\big|_{x_{0j}} + b\,(x_{0j},\beta_{0j})\right]\\ &+ O(N^{-2}\ln^{2m}N),\end{aligned} \tag{4.4.52}$$

with $m = 0$ for the Bakhvalov mesh and $m = 1$ for the Shishkin mesh, and by (4.0.1a) and (4.3.2) we have shown (4.4.18). □

4.4.3 Discretisation of the Interface-Boundary of the Boundary Layer Region $\Gamma_{-\tau} \cap \Omega_{[-\tau,0]}$

We now consider the interface between the inner and outer region, that is, $\Gamma_{-\tau} \cap \Omega_{[-\tau,0]}$. This interface is located along the curve $r = -\tau$, i.e., at a distance of τ inside the boundary of the domain. This is represented in Figure 4.2(b). We have a fictitious Neumann boundary condition on the

interface between the layer region and the outer region. This condition is given as

$$\frac{\partial u}{\partial n} = \phi(x) \quad \text{for } x \in \Gamma_{-\tau} \cap \partial\mathring{\Omega}^N, \tag{4.4.53}$$

$$\frac{\partial u}{\partial r} = \phi(x) \quad \text{for } x \in \Gamma_{-\tau} \cap \Omega_{[-\tau,0]}, \tag{4.4.54}$$

where $\Gamma_{-\tau} \cap \partial\mathring{\Omega}^N$ and $\Gamma_{-\tau} \cap \Omega_{[-\tau,0]}$ are again used to distinguish the sides of the curve $\Gamma_{-\tau}$.

As we are away from the boundary of the domain much of the following work resembles §2.5.4, §2.5.5 and §2.5.7 of *Chapter 2*. However we note the form of $\beta(x;p)$ in the current chapter is simpler than that in *Chapter 2* and contains an exponential term not found in (2.4.84).

The finite difference discretisation along the curve $r = -\tau$ is

$$\mathcal{F}^N_{\Omega_{[-\tau,0]}} U_{-N,j} := -\varepsilon^2 \delta_r^2 U_{-N,j} - \varepsilon^2 \zeta_{-N,j} D_l[\tilde{\zeta}_{-N,j} D_l^- U_{-N,j}] + b(x_{-N,j}, U_{-N,j}) = 0, \tag{4.4.55a}$$

$$U_{-N,N_l} = U_{-N,0}, \quad U_{-N,-1} = U_{-N,N_l-1}, \tag{4.4.55b}$$

where

$$\delta_r^2 U_{-Nj} := -\frac{2}{h_{-N}}\phi_j + \eta_{-N}^{-1}\kappa\phi_j + \frac{2}{h_{-N}} D_r^+ U_{-Nj}, \tag{4.4.55c}$$

$$D_r^+ v_{ij} := \frac{v_{i+1,j} - v_{ij}}{r_{i+1} - r_i}, \quad h_{-N} := r_{-N} - r_{-N-1}, \quad \phi_j := \phi(x_{-N,j}), \tag{4.4.55d}$$

D_l and D_l^- are given in (4.4.4g), κ is given in (4.3.4) and r_{-N-1} is a fictitious point with $h_{-N} := r_{-N} - r_{-N-1}$.

The discretisation, (4.4.55), is found by using a central difference approximation for the Neumann boundary condition, i.e.,

$$D_r U_{-N,j} = \frac{U_{-N+1,j} - U_{-N-1,j}}{2h_{-N}} = \phi_j, \tag{4.4.56}$$

where $U_{-N-1,j}$ is the solution at the ghost point r_{-N-1} located outside the

domain with $h_{-N} := r_{-N} - r_{-N-1}$ and $h_{-N} = h_{-N+1}$. We combine (4.4.56) with (4.4.4) and eliminate the term U_{-N-1} giving

$$\delta_r^2 U_{-Nj} := \eta_{-Nj}^{-1} \frac{-2\tilde{\eta}_{-Nj}\phi_j + (\tilde{\eta}_{-N+1,j} + \tilde{\eta}_{-Nj}) D_r^+ U_{-Nj}}{h_{-N}}. \tag{4.4.57}$$

Note that

$$\eta_{-Nj}^{-1}\tilde{\eta}_{-Nj} = \eta_{-Nj}^{-1}\left(\eta_{-Nj} - \frac{h_{-N}}{2}\kappa\right) = 1 - \eta_{-N}^{-1}\frac{h_{-N}\kappa}{2}, \tag{4.4.58}$$

and $\tilde{\eta}_{-N+1,j} + \tilde{\eta}_{-Nj} = 2\eta_{-Nj}$ and combining this with (4.4.57) gives (4.4.55c).

Lemma 4.4.3. *Let $\tilde{\Omega} \supset \bar{\Omega} \backslash \Omega_{[-\tau+h_{-N},0]}$ be the interior of the curve $r = -\tau + h_{-N}$ where τ is chosen in* (4.4.1)*. For $\beta(x;p)$ defined in* (4.3.28) *we have*

$$\|\beta\|_{C^2(\tilde{\Omega})} \leqslant C\left(1 + \varepsilon^{-2}N^{-2}\right). \tag{4.4.59}$$

Proof. Recall Lemma 4.3.2 and note mixed derivatives can be calculated by again obtaining equations of type

$$\mathcal{L}_\xi[v(\xi,l)] = \psi(\xi,l) \quad \text{with} \quad |\psi(\xi,l)| \leqslant C(1 + |\xi|^k)|\chi(\xi,l)|, \tag{4.4.60}$$

and applying Lemma 3.3.2. To calculate $\|u_0 + \tilde{v}_0 + \varepsilon v_1\|_{C^2(\tilde{\Omega})}$ we can use Lemma 2.5.6 and find,

$$\|u_0 + \tilde{v}_0 + \varepsilon v_1\|_{C^2(\tilde{\Omega})} \leqslant C(1 + \varepsilon^{-2} e^{-\gamma|r|/\varepsilon}). \tag{4.4.61}$$

For the Bakhvalov and Shishkin meshes we can write $e^{-\gamma|r|/\varepsilon} = e^{-\gamma\tau/\varepsilon} = N^{-2}$ and so (4.4.61) becomes

$$\|u_0 + \tilde{v}_0 + \varepsilon v_1\|_{C^2(\tilde{\Omega})} \leqslant C(1 + \varepsilon^{-2}N^{-2}). \tag{4.4.62}$$

To calculate $C_0\bar{p}\|\rho(r)\|_{C^2(\tilde{\Omega})}$ we take derivatives with respect to r, and get

$$C_0\bar{p}\|\rho(r)\|_{C^2(\tilde{\Omega})} \leqslant C\bar{p}\varepsilon^{-2}e^{-c_0|r_i|/\varepsilon} \leqslant C\bar{p}\varepsilon^{-2}. \tag{4.4.63}$$

Recalling $\bar{p} = C_2N^{-2}$ we have

$$\|C_0\bar{p}\rho(r)\|_{C^2(\tilde{\Omega})} \leqslant C. \tag{4.4.64}$$

Hence combining (4.4.62) and (4.4.64) yields (4.4.59). □

Lemma 4.4.4. *Let $\beta(x;p)$ be defined by* (4.3.28)*, and the mesh $\{r_i\}_{i=0}^{-N}$ be the Shishkin mesh or Bakhvalov mesh of §4.4.1. Then for all $|p| \leqslant p_0$ at all interface-boundary nodes $x_{-Nj} \in \Gamma_{-\tau} \cap \Omega_{[-\tau,0)}$ we have*

$$\mathcal{F}^N_{\Omega_{[-\tau,0]}}\beta(x_{-Nj},\bar{p}) - \mathcal{F}\beta(x_{-Nj},\bar{p}) = \frac{2\varepsilon^2}{h_{-N}}\left(-\frac{\partial\beta}{\partial r} + \phi\right)\bigg|_{x_{-Nj}} + O(N^{-2}). \tag{4.4.65}$$

Proof. As we are away from the boundary we call on the proof of Lemma 2.5.7, and have

$$\mathcal{F}^N_{\Omega_{[-\tau,0]}}\beta_{-Nj} - \mathcal{F}\beta(x_{-Nj}) = \frac{2\varepsilon^2}{h_{-N}}\left(-\frac{\partial\beta}{\partial r} + \phi\right)\bigg|_{x_{-Nj}} - \varepsilon^2\eta^{-1}\kappa\frac{\partial\beta}{\partial r} + O(\varepsilon^2\|\beta\|_{C^2(\tilde{\Omega})}). \tag{4.4.66}$$

As $|\eta^{-1}| \leqslant C$ and $\kappa \leqslant C$ then $\varepsilon^2\eta^{-1}\kappa\dfrac{\partial\beta}{\partial r} = O(\varepsilon^2\|\beta\|_{C^2(\tilde{\Omega})})$ and using (4.4.59) gives (4.4.65). □

4.4.4 Discretisation of the Outer Region $\mathring{\Omega}^N$

For the outer region $\mathring{\Omega}^N := \bar{\Omega}^N \backslash \Omega_{(-\tau,0)}$, represented in Figure 4.2c, we apply the standard finite element method used in §2.5.7 of *Chapter 2* and get the

same equation as (2.5.80), i.e.,

$$F^N U_i := \frac{\varepsilon^2}{(1,\chi_i)}(\nabla U, \nabla \chi_i) + b(X_i, U_i) = 0, \quad \forall\, \chi_i \in \mathring{\Omega}^N. \tag{4.4.67}$$

Lemma 4.4.5. *Let $\beta(x;p)$ be defined by* (4.3.28) *and $\beta^I \in S^N$ be its piecewise linear interpolant such that $\beta^I(X_i) = \beta(X_i)$ at all mesh nodes $X_i \in \mathring{\Omega}^N$. Furthermore let τ be defined by* (4.4.1)*. Then for all $|p| \leqslant p_0$ we have*

$$|\mathcal{F}^N \beta_i^I - \mathcal{F}\beta(X_i)| \leqslant CN^{-2} \quad \forall X_i \in \mathring{\Omega}^N. \tag{4.4.68}$$

Proof. We recall Lemma 2.5.8 and note that as we are away from the boundary (2.5.86) is true for $\mathcal{F}^N\beta_i - \mathcal{F}\beta(X_i)$, i.e., we have

$$|\mathcal{F}^N \beta_i - \mathcal{F}\beta(X_i)| \leqslant C\varepsilon^2 \|\beta\|_{C^2(\tilde{\Omega})}. \tag{4.4.69}$$

Calling on (4.4.59) we show (4.4.68). □

4.4.5 Discretisation in the Interface of the Outer Region $\Gamma_{-\tau} \cap \partial\mathring{\Omega}^N$

We now consider the interface of the outer region, $\Gamma_{-\tau} \cap \partial\mathring{\Omega}^N$. This is represented in Figure 4.2b. Again as we are away from the boundary of the domain this section follows §2.5.8 and so we have the system

$$\mathcal{F}^N_{\mathring{\Omega}} U_j := \frac{\varepsilon^2}{(1,\chi_j)}(\nabla U, \nabla \chi_j) + b(X_j, U_j) = \varepsilon^2 a_j \phi_j \quad \forall X_j \in \Gamma^N_{-\tau} \cap \partial\mathring{\Omega}^N, \tag{4.4.70a}$$

where

$$a_j = \frac{1}{(1,\chi_j)} \oint_{\Gamma^N_{-\tau}} \chi_j ds, \quad \phi_j := \phi(X_j). \tag{4.4.70b}$$

Note

$$a_j = O(N). \tag{4.4.71}$$

Lemma 4.4.6. *Under the conditions of Lemma 4.4.5 for all* $|p| \leqslant p_0$ *we have*

$$\mathcal{F}^N_{\mathring{\Omega}} \beta^I_j - \mathcal{F}\beta(X_j) = a_j \varepsilon^2 \left(\frac{\partial \beta}{\partial r} - \phi \right) \bigg|_{X_j} + O(N^{-2}). \tag{4.4.72}$$

Proof. Calling on Lemma 2.5.10 and we have

$$\mathcal{F}^N_{\mathring{\Omega}} \beta^I_j - \mathcal{F}\beta(X_j) = a_j \varepsilon^2 \left(\frac{\partial \beta}{\partial r} - \phi \right) \bigg|_{X_j} + O(\varepsilon^2 \|\beta\|_{C^2(\bar{\Omega})}). \tag{4.4.73}$$

By (4.4.59) we get (4.4.72). □

To get the discretisation on the interface we combine the discretisation $\mathcal{F}^N_{\Omega_{[-\tau,0]}}$ and $\mathcal{F}^N_{\mathring{\Omega}}$ using

$$\mathcal{F}^N U_j := \frac{(h_{-N}/2)\mathcal{F}^N_{\Omega_{[-\tau,0]}} U_j + (1/a_j)\mathcal{F}^N_{\mathring{\Omega}} U_j}{h_{-N}/2 + 1/a_j} \quad \forall X_j \in \Gamma_{-\tau}. \tag{4.4.74}$$

Lemma 4.4.7. *Under the conditions of Lemma 4.4.4 for* $\mathcal{F}^N$ *of* (4.4.74) *we have*

$$|\mathcal{F}^N \beta_j - \mathcal{F}\beta(X_j)| \leqslant CN^{-2}, \tag{4.4.75}$$

for $X_j \in \Gamma_{-\tau}$.

Proof. Following Lemma 2.5.11 we get the desired results. □

4.4.6 Existence and Accuracy

We now prove existence and ε-uniform accuracy for the discrete solution.

Theorem 4.4.1. *Let $\beta(X_i, \bar{p})$ be defined by (4.3.28). For all $0 < |\bar{p}| \leqslant p_0$, $\beta(X_i; -\bar{p})$ and $\beta(X_i; \bar{p})$ are discrete upper and lower solutions to the discrete problem $\mathcal{F}^N$ and there exists solution U_i of the discrete problem with*

$$|U(X_i) - u(X_i)| \leqslant CN^{-2} \ln^{2m} N, \tag{4.4.76}$$

for all mesh nodes $X_i \in \bar{\Omega}$ where $u(X_i)$ is the exact solution of (4.0.1) and $m = 0$ for the Bakhvalov mesh and $m = 1$ for the Shishkin mesh.

Proof. Recalling (4.4.17), (4.4.68) and (4.4.7) we choose C_4 sufficiently large such that

$$\mathcal{F}^N\beta(X_i, \bar{p}) - \mathcal{F}\beta(X_i, \bar{p}) \geqslant -C_4 N^{-2} \ln^{2m} N \quad \forall X_i \in \bar{\Omega}^N \backslash \partial\Omega^N. \tag{4.4.77}$$

With (4.3.56), we can write (4.4.77) as

$$\mathcal{F}^N\beta(X_i, \bar{p}) \geqslant \frac{C_0 \bar{p}\gamma^2}{2} - C_1(\varepsilon^2 + \bar{p}^2) - C_4 N^{-2} \ln^{2m} N. \tag{4.4.78}$$

Let $\bar{p} := C_3 N^{-2} \ln^{2m} N$. We choose N sufficiently large such that

$$\bar{p} \leqslant \frac{C_0\gamma^2}{6C_1}, \quad i.e., \quad C_1\bar{p}^2 \leqslant \frac{C_0\gamma^2\bar{p}}{6}, \tag{4.4.79}$$

and

$$C_1\varepsilon^2 \leqslant \frac{C_0\gamma^2\bar{p}}{6}. \tag{4.4.80}$$

By choosing C_3 sufficiently large we can say

$$C_4 \leqslant \frac{C_3 C_0 \gamma^2}{6}. \tag{4.4.81}$$

Combining (4.4.79), (4.4.80) and (4.4.81) we write (4.4.78) as

$$\mathcal{F}^N\beta(X_i,\bar{p}) \geqslant \frac{C_0\bar{p}\gamma^2}{2} - \frac{C_0\bar{p}\gamma^2}{2} = 0. \tag{4.4.82}$$

Similarly we can find $\mathcal{F}^N\beta(X_i,-\bar{p}) \leqslant 0$.

Choosing C_5 sufficiently large and recalling (4.4.18), for points on the boundary, we can say

$$\mathcal{G}^N\beta(X_i,\bar{p}) \geqslant \frac{h_0}{2\varepsilon}\mathcal{F}\beta(X_i,\bar{p}) - C_5N^{-2}\ln^{2m}N + \mathcal{G}\beta(X_i,\bar{p}). \tag{4.4.83}$$

Recalling (4.3.65) and (4.3.41) we can write (4.4.83) as

$$\mathcal{G}^N\beta(X_i,\bar{p}) \geqslant C_0c_0\bar{p} - C_5N^{-2}\ln^{2m}N. \tag{4.4.84}$$

By choosing C_3 sufficiently large such that $C_3C_0c_0 \geqslant C_5$, then

$$\mathcal{G}^N\beta(X_i,\bar{p}) \geqslant 0. \tag{4.4.85}$$

Similarly,

$$\mathcal{G}^N\beta(X_i,-\bar{p}) \leqslant 0. \tag{4.4.86}$$

Now by the theory of upper and lower solutions, $\beta(X_i,-\bar{p})$ and $\beta(X_i,\bar{p})$ are discrete upper and lower solutions. As the discretisation $\mathcal{F}^N$ is a Z-field by Lemma (1.0.1) there exists a discrete solution $U(X_i)$ to (4.4.4), (4.4.67) and (4.4.74) such that

$$\beta(X_i,-\bar{p}) \leqslant U(X_i) \leqslant \beta(X_i,\bar{p}). \tag{4.4.87}$$

By (4.3.34) we can say $U(X_i) = u_{as}(X_i) + O(\bar{p})$ and combining this with (4.3.62) we have

$$|U(X_i) - u(X_i)| \leqslant C(\bar{p} + \varepsilon^2), \tag{4.4.88}$$

and recalling (4.1.2) and the definition of $\bar{p}$ we get (4.4.76). □

4.5 Conclusions

We considered a two-dimensional singularly perturbed reaction-diffusion equation with singularly perturbed Neumann boundary condition and made certain assumptions on the system in order to ensure existence of a boundary layer solution. We introduced curvilinear coordinates in a narrow region near the boundary of the domain and rescaled the system in this region. By creating an asymptotic expansion and perturbing it we obtained upper and lower solutions.

As the asymptotic expansion of the problem was less complex than that in *Chapter 2* we were able to calculate the truncation error on the Bakhvalov mesh as well as the Shishkin mesh. We used the finite element method to discretise the outer region away from the boundary layer and the finite difference method inside the boundary layer region. A fictitious Neumann condition was used on the interface curve with one side of the curve being discretised by the finite element method and the other by the finite difference method. These discretisations were then combined to eliminate the fictitious Neumann boundary condition. In using both the finite difference method and finite element method we were able to refine the mesh in the layer region and still have a Delaunay triangulation in the outer region, i.e., we had an M-matrix discretisation of the system and so could use Z-fields.

The truncation error was estimated using discrete upper and lower solutions and was found to be bounded by CN^{-2} for the Bakhvalov mesh and $CN^{-2}\ln^2 N$ for the Shishkin mesh where C is a positive constant and N is the number of mesh nodes. Existence of the computed solution was shown by using the property that the discretisation was a Z-field.

Chapter 5

Numerical Results

This chapter consists of numerical results for examples of problems from *Chapter 2* and *Chapter 3*. These are used to illustrate the theoretical results of each chapter.

5.1 Numerical Results for *Chapter 2*

We present results for a two-dimensional reaction-diffusion test problem exhibiting interior layer solutions to illustrate the need for the stabilised method in this case.

5.1.1 Test Problem for *Chapter 2*

We examine (2.0.1) with the following nonlinear function,

$$b(x, u) = (u - \varphi_b)u(u + 1), \tag{5.1.1a}$$

where

$$\varphi_b(x) = 1.5 - \frac{\rho_b}{\rho_b + 1}, \tag{5.1.1b}$$

and

$$\rho_b(x) = \left(\frac{x_1}{0.5}\right)^2 + \left(\frac{x_2}{0.4}\right)^2, \tag{5.1.1c}$$

and boundary condition

$$g(x) = -1 \quad x \in \partial\Omega. \tag{5.1.2}$$

The domain Ω is given by $x_1 = \varphi(l) := R\sin\theta$ and $x_2 = \psi(l) := 1.5R\cos\theta$ with $l \in [0, 2\pi]$ and

$$R = R(l) := 0.4 + 1/2\cos^2 l \quad \theta = \theta(l) := l + e^{l/2-5/2}\sin(l/2)\sin l. \tag{5.1.3}$$

This domain is represented by the outer curve (blue) in Figure 5.1. Recall this is the problem we considered in §2.1 to show the difficulties involved in such a problem.

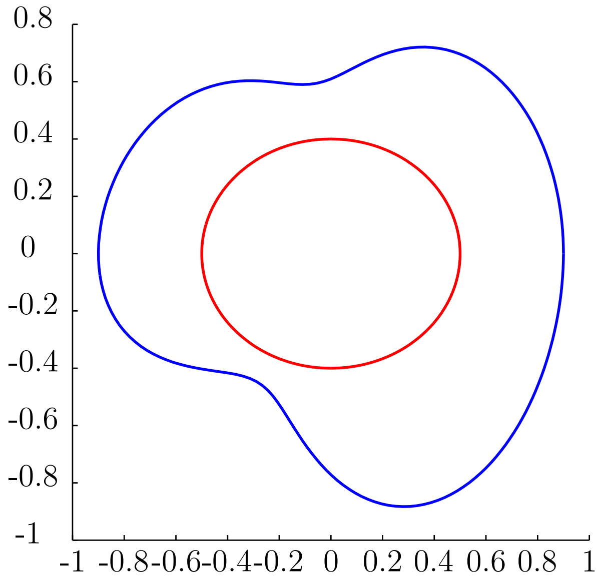


Figure 5.1: The boundary of the domain Ω (the outer curve in blue) and the transition curve Γ_0 (the inner curve in red).

We now check that assumptions (A1)-(A6) hold true for (2.0.1) with (5.1.1) and (5.1.2).

Assumptions (A1) *and* (A2)*:*

The reduced problem $b(x, u(x)) = 0$ has three solutions,

$$\varphi_0(x) = 0, \quad \varphi_1(x) = -1 \quad \text{and} \quad \varphi_2(x) = \varphi_b(x). \tag{5.1.4}$$

These solutions are ordered, $-1 < 0 < \varphi_b(x)$ for all $x \in \bar{\Omega}$ and there is no other solution between -1 and $\varphi_b(x)$. Assumptions (A1) and (A2) are satisfied.

Assumptions (A3) *and* (A4)*:*

The derivative of $b(x, u)$ with respect to u is

$$b_u(x, u(x)) = (u - \varphi_b)u + (u - \varphi_b)(u + 1) + u(u + 1). \tag{5.1.5}$$

We note $\varphi_b(x) > 0$ for all $x \in \bar{\Omega}$. For each $\varphi_i(x)$ in (5.1.4) we have,

$$b_u(x, \varphi_1(x)) = 1 + \varphi_b(x) > 0, \tag{5.1.6a}$$

$$b_u(x, \varphi_0(x)) = -\varphi_b(x) < 0, \tag{5.1.6b}$$

and

$$b_u(x, \varphi_2(x)) = \varphi_b(x)(1 + \varphi_b(x)) > 0. \tag{5.1.6c}$$

Hence (A3) and (A4) are satisfied.

Assumption (A5)*:*

For the nonlinear function (5.1.1), the integral $\mathcal{I}(x)$ from (2.2.2) is

$$\mathcal{I}(x) = -\frac{1}{4}{\varphi_1}^4 + \frac{1}{4}{\varphi_2}^4 + \frac{1}{3}\left(-\varphi_2 + 1\right)\left(-{\varphi_1}^3 + {\varphi_2}^3\right) - \frac{1}{2}\varphi_2\left(-{\varphi_1}^2 + {\varphi_2}^2\right), \tag{5.1.7}$$

and using the values of φ_0, φ_1 and φ_2 in (5.1.4) we get

$$\mathcal{I}(x) = \frac{1}{4}({\varphi_b}^4 - 1) + \frac{1}{3}\left(-\varphi_b + 1\right)\left(1 + {\varphi_b}^3\right) - \frac{1}{2}\varphi_b\left(-1 + {\varphi_b}^2\right). \qquad (5.1.8)$$

Solving (A5a), that is $\mathcal{I}(x) = 0$, gives the solution $\varphi_b(x) = 1$ and solving this for (x_1, x_2) we have

$$16x_1^2 + 25x_2^2 = 4. \qquad (5.1.9)$$

This can be written in a similar form to $\partial\Omega$, that is the curve Γ_0 is given by $x_1 = 1/2\cos(l)$ and $x_2 = 2/5\sin(l)$ for $l \in [0, 2\pi]$. This defines the curve Γ_0 which is the inner (red) curve in Figure 5.1.

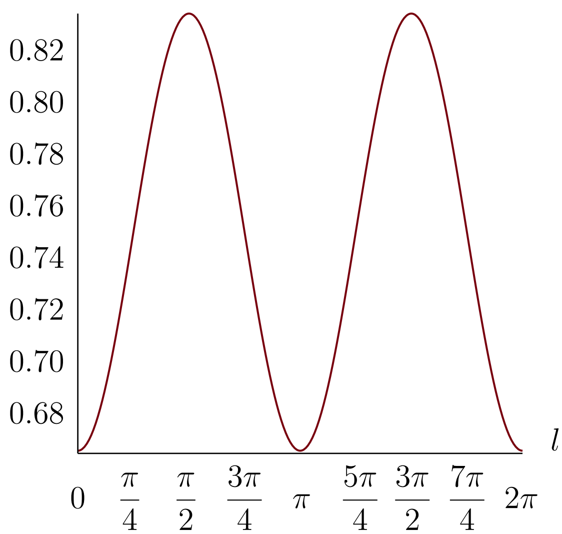


Figure 5.2: Plot of $\partial_n\mathcal{I}(x)$ for $x \in \Gamma_0$ with $x_1 = 1/2\cos(l)$ and $x_2 = 2/5\sin(l)$ for $l \in [0, 2\pi]$.

Taking the outward normal of $\mathcal{I}(x)$ from (5.1.8), we obtain $\delta_n\mathcal{I}(x)$ which is represented in Figure 5.2 and as this is bounded away from zero, (A5b) is satisfied.

Assumption (A6)*:*
The condition (A6) is met as

$$\varphi_1(x) = -1 = g(x), \quad \Delta\varphi_1(x) = \Delta(-1) = 0, \tag{5.1.10}$$

for all $x \in \partial\Omega$.

We now consider the possibility of a boundary layer solution to (2.0.1) with (5.1.1) and (5.1.2). If a boundary layer exists then it must have $u_0(x) = \varphi_b(x)$. Considering the other stable reduced solution, $u_0(x) = -1$, this cannot have a boundary layer solution as $u_0(x) = g(x)$ for all $x \in \bar{\Omega}$. For existence of a boundary layer solution the following condition must be met

$$\int_{u_0(x)}^{v} b(x,s)ds > 0 \quad \forall v \in (u_0(x), g(x)]', x \in \partial\Omega \tag{5.1.11}$$

As $u_0(x) = \varphi_b(x)$, $0.6 \leqslant \varphi_b(x) \leqslant 1.5$ and $g(x) = -1$ we consider the case $v = -0.5$, that is

$$\int_{-0.5}^{\varphi_b(x)} b(x,s)ds \quad x \in \partial\Omega. \tag{5.1.12}$$

We represent the solution in Figure 5.3. This solution is not strictly positive for $v = -0.5$ for all values of $x \in \partial\Omega$ and hence is not strictly positive for all $v \in [-1, u_0(x))$. Hence (5.1.11) is broken and a boundary layer solution to (2.0.1) with (5.1.1) does not exist.

5.1.2 Implementation of the Finite Element Method

We present results for (2.0.1) with (5.1.1) using the finite element method on a quasiuniform Delaunay triangulation as described in §2.5.7. We describe the implementation of this here. A mesh is created using the MATLAB function *initmesh* with a maximum edge size of $1/N$ where $N+1$ is the number of mesh points.

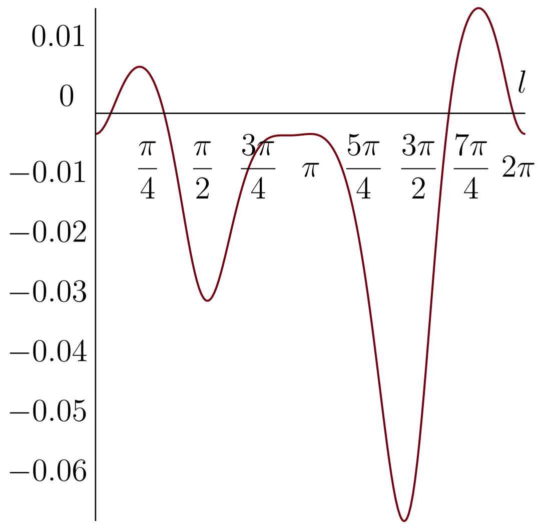


Figure 5.3: Solution of (5.1.12) *with* $v = -0.5$ *for* $x \in \partial\Omega$ *defined in* (5.1.3) *and* $l \in [0, 2\pi]$.

For the local triangle (x_1, y_1), (x_2, y_2) and (x_3, y_3) represented in Figure 5.4a, define

$$a_1 = y_2 - y_3, \quad a_2 = y_3 - y_1, \quad a_3 = y_1 - y_2, \tag{5.1.13}$$

$$b_1 = x_3 - x_2, \quad b_2 = x_1 - x_3, \quad b_3 = x_2 - x_1, \tag{5.1.14}$$

$$c_1 = x_2 y_3 - x_3 y_2, \quad c_2 = x_3 y_1 - x_1 y_3, \quad c_3 = x_1 y_2 - x_2 y_1. \tag{5.1.15}$$

The local element stiffness matrix K_{local} is defined as

$$K_{local} = \frac{1}{4S} \begin{pmatrix} a_1^2 + b_1^2 & a_1 a_2 + b_1 b_2 & a_1 a_3 + b_1 b_3 \\ a_1 a_2 + b_1 b_2 & a_2^2 + b_2^2 & a_2 a_3 + b_2 b_3 \\ a_1 a_3 + b_1 b_3 & a_2 a_3 + b_2 b_3 & a_3^2 + b_3^2 \end{pmatrix}, \tag{5.1.16}$$

and the local mass matrix, M_{local}, is created using lumped mass and is given

as

$$M_{local} = \frac{S}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.1.17}$$

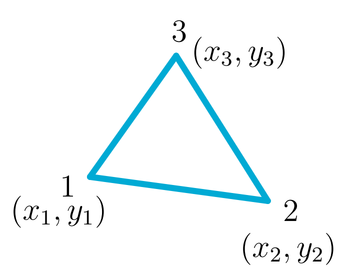
where

$$S := \frac{1}{4}\det\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \tag{5.1.18}$$

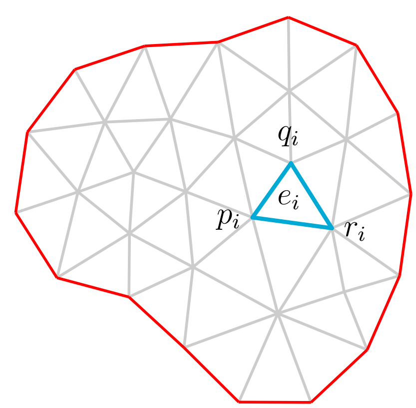
Let the global index matrix be defined as

$$L = \begin{pmatrix} \dots & p_i & \dots & \\ \dots & q_i & \dots & \\ \dots & r_i & \dots & \end{pmatrix} \tag{5.1.19}$$

where p_i, q_i and r_i are global numberings of mesh nodes that correspond to the local numberings 1, 2 and 3 of triangle e_i. This is represented in Figure 5.4b on a coarse mesh for presentation purposes.



(a) Local triangle numbering.



(b) Global triangle numbering with a coarse mesh for presentation purposes.

Figure 5.4: Relationship between local and global triangle numbering.

The global stiffness matrix is written as K_{global} and is of size $(N+1) \times (N+1)$ and the global mass matrix is given by M_{global} and is of size $(N+1) \times (N+1)$. These are defined by an iterative process. First define

$$K_{global}(1:N+1, 1:N+1) = 0, \quad M_{global}(1:N+1, 1:N+1) = 0 \quad (5.1.20)$$

The iterative process is then:
For $i = 1 : N+1$,

$$K_{global}(L(:,i), L(:,i)) = K_{global}(L(:,i), L(:,i)) + K^{i}_{local}, \quad (5.1.21)$$

$$M_{global}(L(:,i), L(:,i)) = M_{global}(L(:,i), L(:,i)) + M^{i}_{local}, \quad (5.1.22)$$

where K^i_{local} and M^i_{local} are the local stiffness and mass matrices for triangle i described in (5.1.16) and (5.1.17). The solution $U^{[k]}$ is defined as

$$U^{[k]} = [U^{[k]}_1, \ldots, U^{[k]}_{N+1}]^T. \quad (5.1.23)$$

We aim to solve the system

$$F^N U := \varepsilon^2 KU + Mb(U), \quad (5.1.24)$$

$$U_i = g(X_i) \quad \text{for } X_i \in \partial\Omega^N \quad (5.1.25)$$

for U. To do this we use Newton's method, i.e., we aim to solve

$$\left(\varepsilon^2 K_{global} + M_{global} b_u(V^{[k]})\right) W^{[k+1]} = -\left(\varepsilon^2 K_{global} V^{[k]} + M_{global} b(V^{[k]})\right), \quad (5.1.26)$$

for $W^{[k]}$ with $V^{[k+1]} := W^{[k+1]} + V^{[k]}$ where k is the iteration number. The

system (5.1.26) can be written as

$$\left(\varepsilon^2 M_{global}^{-1} K_{global} + b_u(V^{[k]})\right) W^{[k]} = -\left(\varepsilon^2 M_{global}^{-1} K_{global} V^{[k]} + b(V^{[k]})\right). \tag{5.1.27}$$

The nodes on $\partial\Omega$ and the nodes in the interior are separated. We denote the nodes on $\partial\Omega$ as d_i and the interior nodes as n_i. The matrices K_{global} and M_{global} are both now separated into two matrices;

$$K_{int} = K_{global}(n_i, n_i), \quad K_{Dir} = K_{global}(n_i, d_i), \tag{5.1.28}$$

$$M_{int} = M_{global}(n_i, n_i) \quad \text{and} \quad M_{Dir} = M_{global}(n_i, d_i). \tag{5.1.29}$$

The system (5.1.27) becomes

$$\begin{aligned}\left(\varepsilon^2 M_{int}^{-1} K_{int} + b_u(V_{int}^{[k]})\right) W_{int}^{[k]} = &-\left(\varepsilon^2 M_{int}^{-1} K_{int} V_{int}^{[k]} + b(V_{int}^{[k]})\right) \\ &- (\varepsilon^2 M_{Dir}^{-1} K_{Dir} V_{Dir}^{[k]} + b(V_{Dir}^{[k]})).\end{aligned} \tag{5.1.30}$$

We note $b(V_{Dir}^{[k]}) = b(g(x)) = 0$ and so the final term vanishes. The Jacobian is defined as $J(V_{int}^{[k]}) := \varepsilon^2 M_{int}^{-1} K_{int} + b_u(V_{int}^{[k]})$ and (5.1.24) becomes $F^N(V^{[k]}) := \varepsilon^2 M_{int}^{-1} K_{int} V_{int}^{[k]} + b(V_{int}^{[k]}) + \varepsilon^2 M_{Dir}^{-1} K_{Dir} V_{Dir}^{[k]}$, i.e., we solve

$$W^{[k+1]} = -J(V^{[k]}) \backslash F^N(V^{[k]}), \tag{5.1.31}$$

where "\" is the MATLAB backslash command.

Using (5.1.31), $W^{[k+1]}$ is calculated and the following are defined

$$V^{[k+1]}(n_i, 1) := W_{int}^{[k]} - V_{int}^{[k]} \tag{5.1.32a}$$

and

$$V^{[k+1]}(d_i, 1) := g(d_i). \tag{5.1.32b}$$

Two checks are then carried out;

$$\|W^{[k]}\| \leqslant tol \quad \text{and} \quad \|F^N V^{[k]}\| \leqslant tol, \tag{5.1.33}$$

for a tolerance *tol*. The system repeats (5.1.31) and (5.1.32) with k replaced by $k+1$ until (5.1.33) is met, in which case the computed solution is defined as

$$U(n_i, 1) = W^{[k]}_{int} - V^{[k]}_{int}, \tag{5.1.34}$$

and

$$U(d_i, 1) = g(d_i). \tag{5.1.35}$$

5.1.3 Results

To solve (2.0.1) with (5.1.1), a mesh was created using MATLAB's *initmesh* function with a maximum side edge of $1/N$ where $N = 20$. This is shown in Figure 5.5.

We first consider the conventional method, i.e., (5.1.31) with $\hat{\varepsilon}_j = \varepsilon$. Using the three initial guesses, $x_1^2 + x_2^2 - 1$, $-1 - x_1$ and 1, computed solutions were found and are presented in Figure 5.6, Figure 5.7 and Figure 5.8. These solutions all differ from one another. From the analysis of §5.1.1 there exists an interior layer solution and the trivial solution $u(x) = -1$. As can be seen in Figure 5.6 - Figure 5.8, none of the solutions found using the conventional method give a solution of either type. Existence of boundary layer solutions has been ruled out in §5.1.1 and so Figure 5.8, the most plausible solution, is incorrect. All computed solutions described here converged within 21 iterations and a tolerance of 10^{-12} was used.

To obtain correct computed solutions the stabilised method of §2.5.2 is used with

$$\hat{\varepsilon}(x_{ij}) = \max\left\{\varepsilon, \frac{\widehat{C}}{N}\right\}. \tag{5.1.36}$$

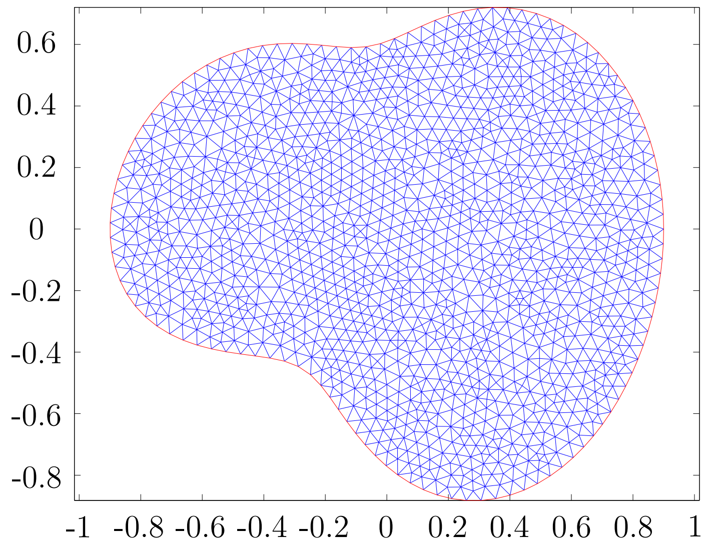


Figure 5.5: Quasiuniform triangulation for the domain Ω described in §5.1.1 created using MATLAB*'s function initmesh with a maximum side edge of* $1/N$ *where* $N = 20$.

In this case we take $\widehat{C} = 1.1$. We again obtain the computed solution using the finite element method as described in §5.1.2. The three initial guesses were again used giving the solution shown in Figure 5.9 or the trivial solution $u(x) = -1$ for all $x \in \bar{\Omega}$. This computation converged within 19 iterations. This solution is of the correct type, the solution $u(x) \approx -1$ in the region between the boundary of the domain and the curve Γ_0, the solution $u(x) \approx \varphi_b(x)$ in the inside of the curve Γ_0 and there is a sharp jump from one solution to the other along the curve Γ_0.

To conclude the conventional method can give incorrect computed solutions, some of which may appear to be plausible without carrying out further analysis. The stabilised method is required to solve this numerical instability as solutions can be of the wrong type without this method.

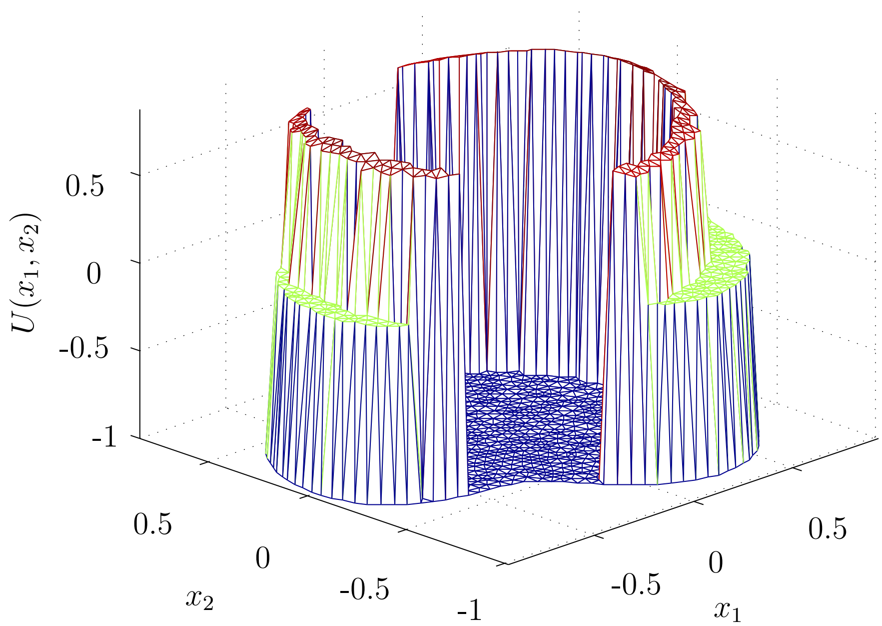


Figure 5.6: Incorrect solution obtained using an initial guess of $x_1^2 + x_2^2 - 1$ with $\varepsilon = 10^{-3}$. Converged in 21 iterations.

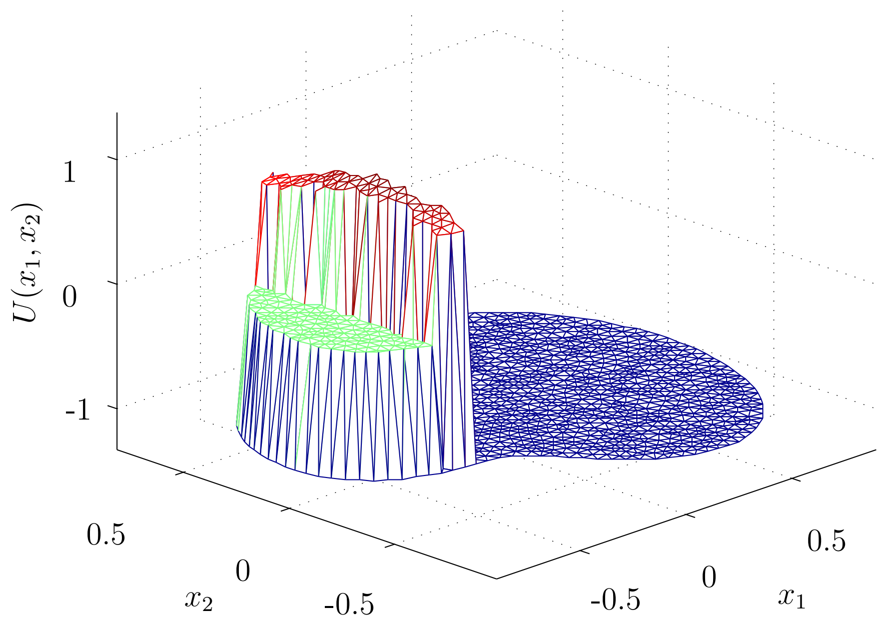


Figure 5.7: Incorrect solution obtained using an initial guess of $-1 - x_1$ with $\varepsilon = 10^{-3}$. Converged in 18 iterations.

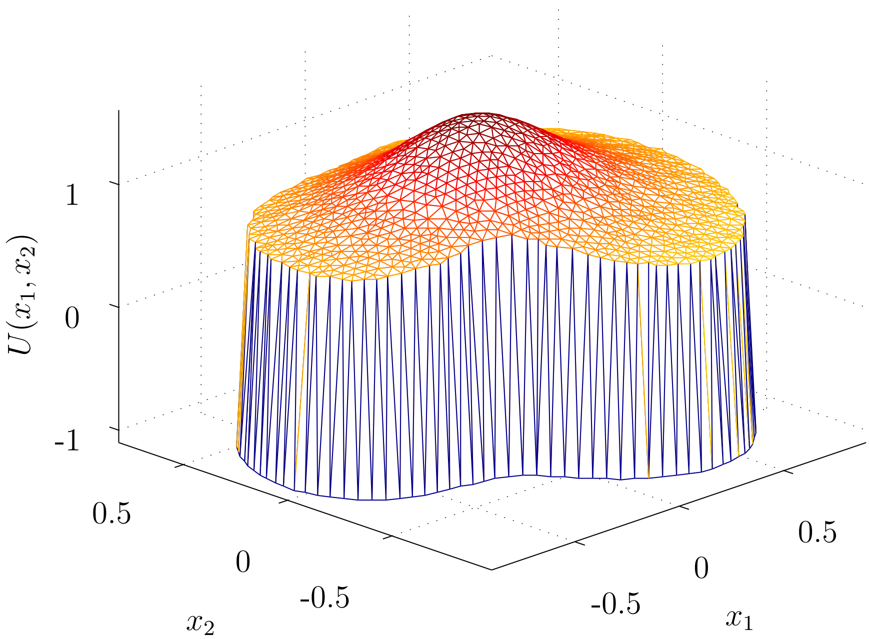


Figure 5.8: Incorrect solution obtained using an initial guess of 1 *with* $\varepsilon = 10^{-3}$*. Converged within* 7 *iterations.*

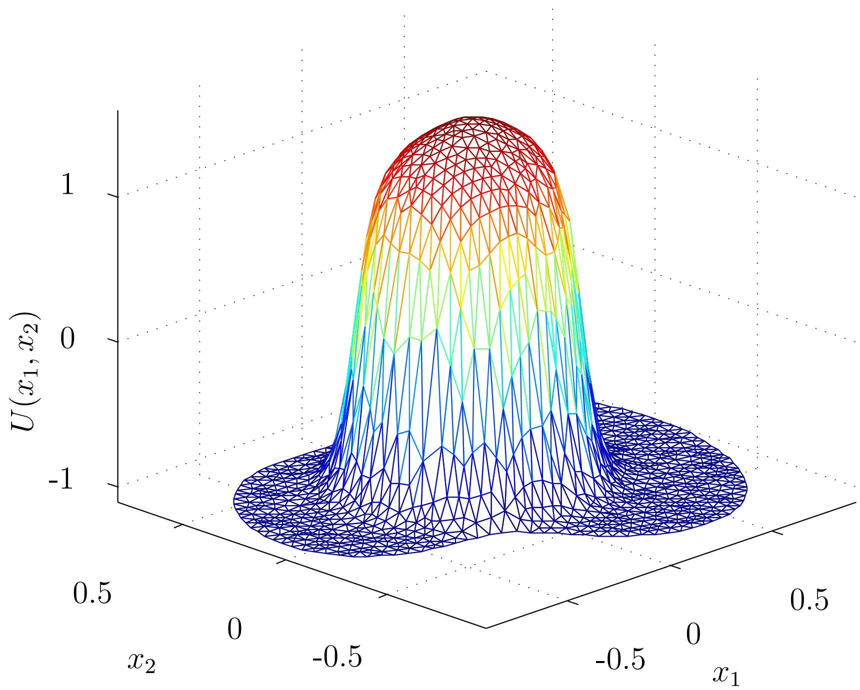


Figure 5.9: Solution of (5.1.31) *using the stabilised method with* $\widehat{C} = 1.1$ *and* $\varepsilon = 10^{-3}$. *Stabilised solution with initial guess of* 1. *Converged within 19 iterations. Other initial guesses give the trivial solution* $u(x) = -1$ *for all* $x \in \bar{\Omega}$.

5.2 Numerical Results for *Chapter 3*

For a test problem of type (3.0.1) we present results using the finite difference method on a uniform mesh, the Shishkin mesh and the Bakhvalov mesh. We give computational rates and error bounds for solutions on the Shishkin and Bakhvalov meshes and again show the need for the stabilised method.

5.2.1 Test Problem for *Chapter 3*

We consider the system (3.0.1) with the following nonlinear function,

$$f(x,t,u) = (2-u)(u-\phi_1)u(u-\phi_2), \tag{5.2.1a}$$

$$\phi_1(x,t) = 1 - 1/2\cos(\pi x - t), \quad \phi_2(x) = -(x^2+1/2), \tag{5.2.1b}$$

with boundary conditions

$$g_0(t) = a_0\sin(t), \quad g_1(t) = a_1\sin(t), \tag{5.2.1c}$$

with $a_0 = 0.4$ and $a_1 = -0.4$ and initial condition

$$\varphi(x) = 1/2. \tag{5.2.1d}$$

We solve this on the domain $\mathcal{D} := \{(x,t) \in [0,1] \times [0,T]\}$ with $T = 1$.

Assumption (B1)*:*

There are four solutions to the reduced problem $f(x,t,u) = 0$; they are $\phi_1(x,t)$, $\phi_2(x)$, 0 and 2. The derivative of f with respect to u is

$$\begin{aligned} f_u(x,t,u) = &-(u-\phi_1)u(u-\phi_2) + (2-u)u(u-\phi_2) \\ &+ (2-u)(u-\phi_1)(u-\phi_2) + (2-u)(u-\phi_1)u. \end{aligned} \tag{5.2.2}$$

By a calculation the derivatives evaluated at the reduced solutions are found

to be

$$f_u(x,t,\phi_1) = (2-\phi_1)\phi_1(\phi_1-\phi_2) > 0, \quad f_u(x,t,\phi_2) = (2-\phi_2)(\phi_2-\phi_1)\phi_2 > 0, \tag{5.2.3a}$$

$$f_u(x,t,0) = 2\phi_1\phi_2 < 0 \quad \text{and} \quad f_u(x,t,2) = -2(2-\phi_1)(2-\phi_2) < 0, \tag{5.2.3b}$$

noting $0.5 \leqslant \phi_1 \leqslant 1.5$ and $-1.5 \leqslant \phi_2 \leqslant -0.5$. Hence both $\phi_1(x,t)$ and $\phi_2(x)$ are stable solutions to the reduced problem while 0 and 2 are unstable solutions. We consider $u(x,t) \approx \phi_1(x,t)$, i.e., let $u_0(x,t) = \phi_1(x,t)$.

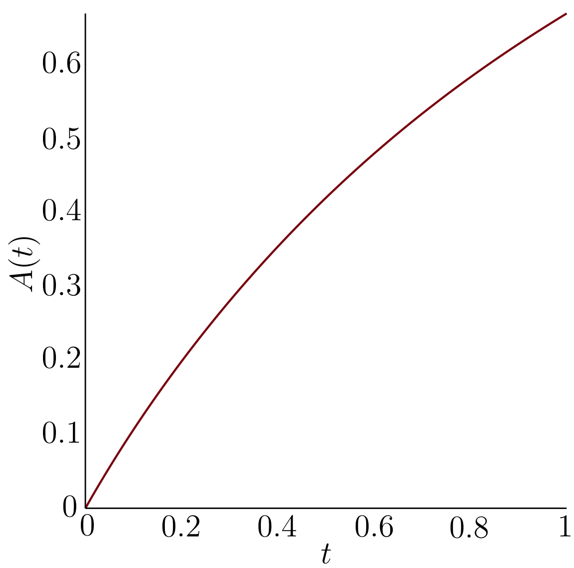


Figure 5.10: Solution $A(t)$ of (5.2.5) *for* $t \in [0,1]$.

Assumption (B2)*:*
Recalling (B2) we aim to find a sufficiently smooth function $A(t)$ that satisfies

$$\int_0^{A(t)} f(0,t,u_0(0,t)+s)ds = \frac{a_0^2 \sin^2(t)}{2}. \tag{5.2.4}$$

Calculating the integral in (5.2.4) we have the following equation for $A(t)$,

$$-48A(t)^5 + [90\cos(t) - 90]A^4(t) + [60\sin^2(t) + 120\cos(t) + 20]A^3(t) \\ +[15\cos^3(t) + 135 + 45\sin^2(t) - 60\cos(t)]A(t)^2 = 120\sin^2(t). \tag{5.2.5}$$

The solution of (5.2.5) is represented in Figure 5.10 where it can be seen that (5.2.5) has a sufficiently smooth solution and so (B2) is satisfied.

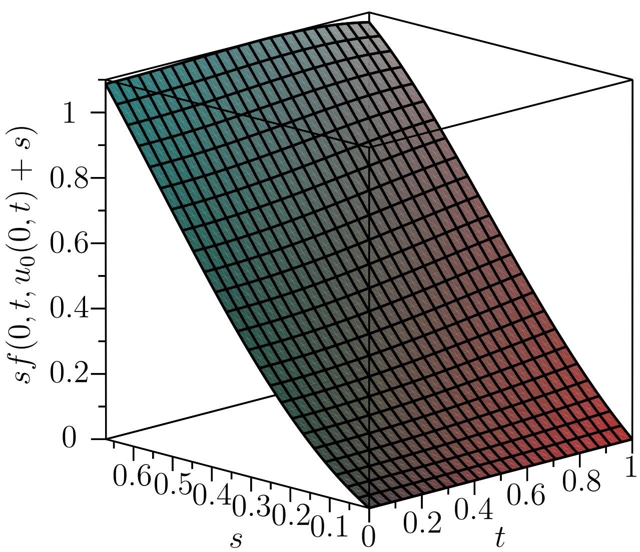


Figure 5.11: Graph showing (B3) *for* $t \in [0,1]$ *and* $s \in [0, \max_t\{A(t)\}]$.

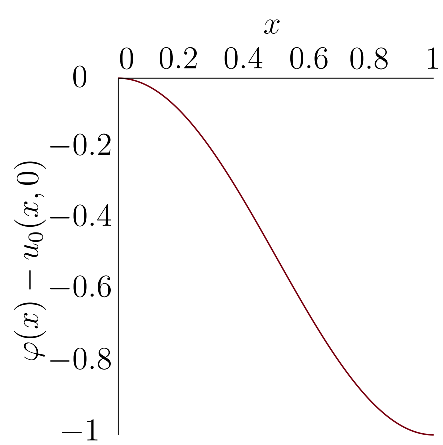
Assumption (B3)*:*

Noting $A(t) \geqslant 0$ we consider $sf(0,t,u_0(0,t)+s)$ for $s \in (0, \max_{t\in[0,T]} A(t)]$ where $\max_{t\in[0,T]} A(t) = 0.67$. This is represented in Figure 5.11. This figure includes values of $s \notin (0, A(t)]$ for the given t. However, as $sf(0,t,u_0(0,t)+s) > 0$ for all values (B3) is satisfied.

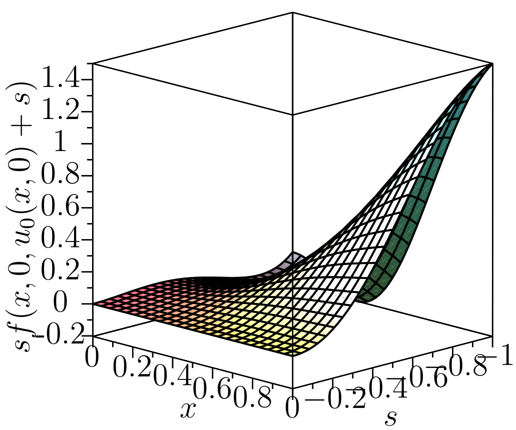
Assumption (B4)*:*
We look to calculate,

$$sf(x,0,\phi_1(x,0)+s) = s^2(2-\phi_1(x,0)-s)(\phi_1(x,0)+s)(s+\phi_1(x,0)-\phi_2(x,0)), \tag{5.2.6}$$

for $s \in (0, \varphi(x) - \phi_1(x,0)]'$. We note the form of $\varphi(x) - \phi_1(x,0)$ given in Figure 5.12a. The solution of (5.2.6) is shown in Figure 5.12b. A small portion of this graph is positive however in this region $s \notin (0, \varphi(x) - \phi_1(x,0)]'$ for the value of x in the region. Hence this region can be ignored and (B4) is met.



(a) Plot of $\varphi(x) - u_0(x,0)$ for $x \in [0,1]$.



(b) Graph showing (B4) *for $x \in [0,1]$ and $s \in (0, \min_{x\in[0,1]}\{\varphi(x) - u_0(x,0)\}]'$. The region in which* (B4) *is positive can be disregarded as $s \notin (\varphi(x) - u_0(x,0)]'$ for $x \in [0,1]$.*

Figure 5.12: Plot of values of $\varphi(x) - u_0(x,0)$ and plot of (B4)*.*

Assumption (B5)*:*
Considering the necessary equations (B5a) is satisfied as

$$\frac{\partial \varphi}{\partial x} = 0, \quad g_0(0) = g_1(0) = 0, \quad \phi(0) - u_0(0,0) = \phi(0) - u_0(0,1) = 0. \tag{5.2.7}$$

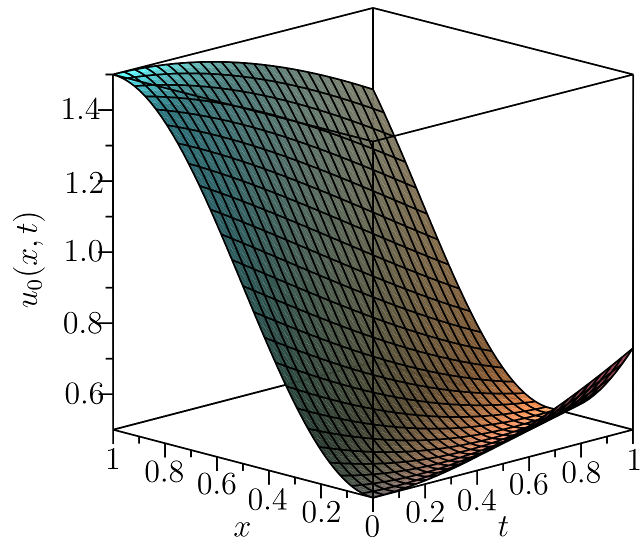


Figure 5.13: Plot of $u_0(x,t)$ *for* $x \in [0,1]$ *and* $t \in [0,1]$.

Finally $\dfrac{\partial u_0}{\partial x} = \pi/2 \sin(\pi x - t)$ and evaluating this at $x = 0$, $t = 0$ gives $\left.\dfrac{\partial u_0}{\partial x}\right|_{x=0,t=0} = 0$. Hence all conditions in (B5) are satisfied.

From this analysis the conditions (B1)-(B5) are satisfied for (3.0.1) with (5.2.1) and a boundary layer solution to the problem exists. We note the form of the function $u_0(x,t) = \phi_1(x,t)$ is represented in Figure 5.13. The solution of the test problem will be close to this solution in the majority of the domain.

5.2.2 Implementation of Finite Difference Method

We use Newton's Method to solve the system (3.0.1) with (5.2.1) and the initial guess set as the solution at the previous time step. Define N as the number of space steps and M the number of time steps.

We aim to solve

$$\mathcal{T}^N(U_j) := SU_j + C_j(U_j - U_{j-1}) + F(U_j) = 0, \tag{5.2.8}$$

for $j = 0, \ldots, M$ where the matrices S, C_j and vector $F(U_j)$ are defined as

$$S := \begin{pmatrix} \frac{\varepsilon}{h_1} & -\frac{\varepsilon}{h_1} & & & & & 0 \\ -\frac{\varepsilon^2}{h_1 \hbar_1} & \varepsilon^2 a_1 & -\frac{\varepsilon^2}{h_2 \hbar_1} & & & & \\ & -\frac{\varepsilon^2}{h_2 \hbar_2} & \varepsilon^2 a_2 & -\frac{\varepsilon^2}{h_3 \hbar_2} & & & \\ & & & \ddots & & & \\ & & & -\frac{\varepsilon^2}{h_{N-1} \hbar_{N-1}} & \varepsilon^2 a_{N-1} & -\frac{\varepsilon^2}{h_N \hbar_{N-1}} \\ 0 & & & & -\frac{\varepsilon}{h_N} & \frac{\varepsilon}{h_N} \end{pmatrix}, \tag{5.2.9}$$

where $a_i := \frac{1}{h_i \hbar_i} + \frac{1}{\hbar_i h_{i+1}}$,

$$C_j := \hat{\varepsilon}^2(t_j) \begin{pmatrix} \frac{h_1}{2\varepsilon k_j} & & & & 0 \\ & \frac{1}{k_j} & & & \\ & & \ddots & & \\ & & & \frac{1}{k_j} & \\ 0 & & & & \frac{h_N}{2\varepsilon k_j} \end{pmatrix}, \tag{5.2.10}$$

$$F(V_0^{[k]}) := \begin{pmatrix} -\varphi(x_0) \\ -\varphi(x_1) \\ \vdots \\ -\varphi(x_{N-1}) \\ -\varphi(x_N) \end{pmatrix}, \quad F(V_j^{[k]}) := \begin{pmatrix} \frac{h_1}{2\varepsilon} f(x_0, t_j, V_{0j}^{[k]}) + g_0(t_j) \\ f(x_1, t_j, V_{1j}^{[k]}) \\ \vdots \\ f(x_{N-1}, t_j, V_{N-1,j}^{[k]}) \\ \frac{h_N}{2\varepsilon} f(x_N, t_j, V_{N,j}^{[k]}) - g_1(t_j) \end{pmatrix} \tag{5.2.11}$$

The solution U_j has the form

$$U_j := \begin{pmatrix} U_{0j} \\ U_{1j} \\ \vdots \\ U_{Nj} \end{pmatrix}. \tag{5.2.12}$$

To solve (5.2.8) we employ Newton's method, giving the initial guess $V_j^{[0]}$ as the computed solution at the previous time step. We then solve the following

$$W_j^{[k+1]} = -J(V_j^{[k]}) \backslash \mathcal{T}^N(V_j^{[k]}), \tag{5.2.13}$$

again using the MATLAB backslash command, where we recall $\mathcal{T}^N$ from (5.2.8) and define $J(V_j^{[k]}) := C_j + S + D_j^{[k]}$ and

$$D_j^{[k]} := \begin{pmatrix} \frac{h_1}{2\varepsilon} f_u(x_0, t_j, V_{0j}^{[k]}) & & & & 0 \\ & f_u(x_1, t_j, V_{1j}^{[k]}) & & & \\ & & \ddots & & \\ & & & f_u(x_{N-1}, t_j, V_{N-1j}^{[k]}) & \\ 0 & & & & \frac{h_N}{2\varepsilon} f_u(x_N, t_j, V_{Nj}^{[k]}) \end{pmatrix}. \tag{5.2.14}$$

Once this iteration is complete, $V_j^{[k+1]}$ is defined as

$$V_j^{[k+1]} := W_j^{[k+1]} - V_j^{[k]}. \tag{5.2.15}$$

When

$$\max |W^{[k+1]}| \leqslant tol, \quad \max |\mathcal{T}^N(V^{[k]})| \leqslant tol, \tag{5.2.16}$$

are met, for a given tolerance *tol*, the system stops iterating. The computed solution U_j is defined as

$$U_j := W^{[k]} - V^{[k]}, \tag{5.2.17}$$

and the system moves on to solving the next time step, $j + 1$.

5.2.3 Computational Rates and Maximum Nodal Errors

As we do not know the exact solution to (3.0.1) with (5.2.1) we use a method discussed in [5, *Chapter 8*] and [17] to obtain computation rates and maximum nodal errors for the computed solutions on the Bakhvalov and Shishkin meshes.

The Bakhvalov Mesh

For the Bakhvalov mesh with N and M space and time steps, respectively, two auxiliary meshes are created, the first is created using $2N$ and $4M$ space and time steps, respectively, and the second is created using $4N$ and $16M$ space and time steps, respectively. For the error $u_{ij} - u_{ij}^{N,M}$ assume

$$u_{ij} - u_{ij}^{N,M} \approx C_1 N^{-r} + C_2 M^{-r/2} \tag{5.2.18}$$

for some $r > 0$ and some positive constants C_1 and C_2. For the error $u_{ij}^{2N,4M} - u_{ij}$ we also assume

$$u_{ij} - u_{ij}^{2N,4M} \approx C_1(2N)^{-r} + C_2(4M)^{-r/2}. \tag{5.2.19}$$

Subtracting (5.2.19) from (5.2.18) yields

$$u_{ij}^{2N,4M} - u_{ij}^{N,M} \approx (C_1 N^{-r} + C_2 M^{-r/2})(1 - 2^{-r}), \tag{5.2.20}$$

and using (5.2.18) gives

$$u_{ij}^{2N,4M} - u_{ij}^{N,M} \approx (u_{ij} - u_{ij}^{N,M})(1 - 2^{-r}). \tag{5.2.21}$$

From (5.2.21) we can write,

$$\|u^{N,M} - u\| \approx (1 - 2^{-r})^{-1} \|u^{2N,4M} - u^{N,M}\|, \tag{5.2.22}$$

where $\|\cdot\|$ is the discrete maximum norm $\max|\cdot|$. Solving (5.2.21) for r gives

$$r \approx \log_2 \frac{\|u^{N,M} - u\|}{\|u^{2N,4M} - u\|}. \tag{5.2.23}$$

Following the same argument for $u_{ij}^{4N,16M}$ we get

$$u_{ij} - u_{ij}^{4N,16M} \approx 2^{-r}(u_{ij} - u_{ij}^{2N,4M}), \tag{5.2.24}$$

and so we subtract (5.2.19) from (5.2.24) giving

$$u_{ij}^{2N,4M} - u_{ij}^{4N,16M} \approx (2^{-r} - 1)(u_{ij}^{2N,4M} - u_{ij}). \tag{5.2.25}$$

Calling on (5.2.22) we replace $(1 - 2^{-r})$ in (5.2.25) to give

$$\frac{u_{ij}^{2N,4M} - u_{ij}^{N,M}}{u_{ij} - u_{ij}^{N,M}} \approx \frac{u_{ij}^{2N,4M} - u_{ij}^{4N,16M}}{u_{ij}^{2N,4M} - u_{ij}}. \tag{5.2.26}$$

Rearranging terms gives

$$\frac{u_{ij}^{N,M} - u_{ij}}{u_{ij}^{2N,4M} - u_{ij}} \approx \frac{u_{ij}^{N,M} - u_{ij}^{2N,4M}}{u_{ij}^{2N,4M} - u_{ij}^{4N,16M}}, \tag{5.2.27}$$

and so finally r from (5.2.23) can be written as

$$r \approx \log_2 \frac{\|u^{N,M} - u^{2N,4M}\|}{\|u^{2N,4M} - u^{4N,16M}\|}. \tag{5.2.28}$$

The Shishkin Mesh

For the Shishkin mesh auxiliary meshes are created by bisecting the previous mesh. We obtain a mesh using the method described in §3.4.2 and define the solution on this mesh as u^{NM}. Bisecting the original mesh we denote the solution on this new mesh as $\bar{u}^{2N,4M}$. Bisecting this mesh a second time we denote the solution on this third mesh as $\bar{\bar{u}}^{4N,16M}$. The result is three meshes with the same transition point.

The error in the solution is given as

$$u_{ij}^{N,M} - u_{ij} \approx C_3(N^{-1} \ln N)^r + C_4(M^{-1} \ln M)^{r/2}, \tag{5.2.29}$$

for $r > 0$ and some positive constants C_3 and C_4 and considering the bisected mesh we have

$$\bar{u}_{ij}^{2N,4M} - u_{ij} \approx C_3((2N)^{-1} \ln N)^r + C_4((4M)^{-1} \ln M)^{r/2}, \tag{5.2.30}$$

i.e.,

$$\bar{u}_{ij}^{2N,4M} - u_{ij} \approx (u_{ij}^{N,M} - u_{ij})2^{-r}. \tag{5.2.31}$$

Subtracting (5.2.29) from (5.2.30) gives

$$\bar{u}_{ij}^{2N,4M} - u^{N,M} \approx (u_{ij}^{N,M} - u_{ij})(1 - 2^{-r}). \tag{5.2.32}$$

Bisecting a second time and again using the same transition point, we calculate the error as

$$\bar{\bar{u}}_{ij}^{4N,16M} - u_{ij} \approx C_3((4N)^{-1}\ln N)^r + C_4((16M)^{-1}\ln M)^{r/2}. \qquad (5.2.33)$$

Recalling (5.2.30), (5.2.33) can be written as

$$\bar{\bar{u}}_{ij}^{4N,16M} - u_{ij} \approx (\bar{u}_{ij}^{2N,4M} - u_{ij})2^{-r}. \qquad (5.2.34)$$

Subtracting (5.2.30) from (5.2.34) yields

$$\bar{u}_{ij}^{2N,4M} - \bar{\bar{u}}_{ij}^{4N,16M} \approx [C_3(N^{-1}\ln N)^r + C_4(M^{-1}\ln M)^{r/2}](1-2^{-r})2^{-r}. \qquad (5.2.35)$$

and using (5.2.32) gives

$$\bar{u}_{ij}^{2N,4M} - \bar{\bar{u}}_{ij}^{4N,16M} \approx 2^{-r}(\bar{u}_{ij}^{2N,4M} - u^{N,M}). \qquad (5.2.36)$$

Solving for r gives

$$r \approx \log_2 \frac{\|u^{N,M} - \bar{u}^{2N,4M}\|}{\|u^{2N,4M} - \bar{u}^{4N,16M}\|}. \qquad (5.2.37)$$

This enables us to numerically calculate the computational rate of the system and compare it with that obtained from our analysis.

5.2.4 Results

Solutions are found using (5.2.13) with (5.2.15). For the conventional method $\hat{\varepsilon}_j^2 = \varepsilon^2$ and for the stabilised method the parameter $\hat{\varepsilon}_j^2$ was chosen as in (3.4.6) with $\widehat{C} = 4$. A tolerance of 5×10^{-10} is used with $\gamma = 0.9$, $T = 1$ and $M = N^2$. All figures have been created using $\varepsilon = 10^{-4}$ and $N = 32$.

The solution of the conventional method on a uniform mesh is given in Figure 5.14a. This solution is incorrect and jumps between the solutions to

the reduced problem.

Using stabilised method gives the solution shown in Figure 5.14b. This solution is of the correct type and the numerical instability is resolved. We note that the solution on the uniform mesh does not properly capture the layers in the solution, in particular the initial layer is not present in Figure 5.14b.

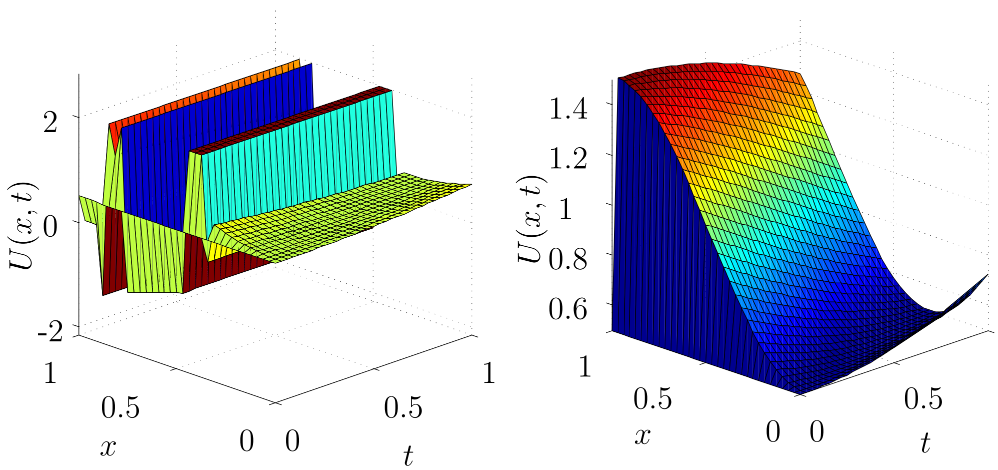
The Shishkin mesh is used in the form described in §3.4.2 and for the Bakhvalov mesh of §3.4.2 we have

$$d_x = \frac{-2\varepsilon/\gamma \ln(1-4\theta_x) - 1/2}{\theta_x - 1/2}, \quad C_3 = 2\gamma^{-1}, \tag{5.2.38a}$$

$$d_t = \frac{-\varepsilon^2/\gamma^2 \ln(1-2\theta_t) - T}{\theta_t - 1}, \quad \text{and} \quad C_4 = (\gamma^2 T)^{-1}. \tag{5.2.38b}$$

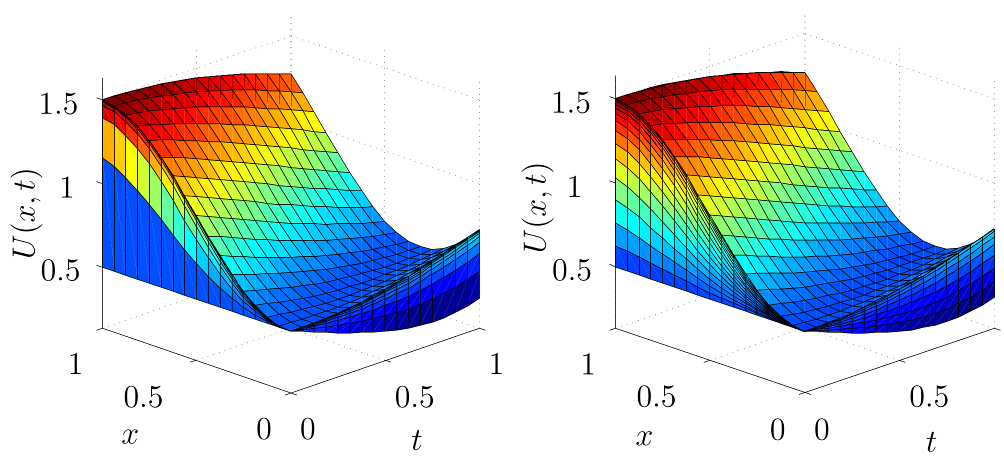
The stabilised solutions for $\varepsilon = 10^{-4}$ and $N = 32$ on the Shishkin mesh and the Bakhvalov mesh are presented in Figure 5.15. On the layer adapted meshes the conventional and stabilised methods both produce correct computed solutions. The convergence rates confirm the results found in Theorem 3.4.1.

The computational rate r and the maximum nodal errors for the conventional and stabilised methods on the Shishkin and Bakhvalov meshes are given in Tables 5.1 - 5.4. These results match the theoretical results of Theorem 3.4.1. Both methods on the Shishkin and the Bakhvalov meshes have ε-uniform convergence.



(a) The conventional method. *(b) The stabilised method with $\widehat{C} = 4$.*

Figure 5.14: Solutions of (5.2.13) *with* (5.2.15) *on a uniform mesh with* $\varepsilon = 10^{-4}$ *and* $N = 32$.



(a) Solutions on the Shishkin mesh with $\varepsilon = 10^{-4}$, $N = 32$ *for the stabilised method.* *(b) Solutions on the Bakhvalov mesh with* $\varepsilon = 10^{-4}$, $N = 32$ *for the stabilised method.*

Figure 5.15: Solutions of (5.2.13) *with* (5.2.15) *with* $\varepsilon = 10^{-4}$ *and* $N = 32$.

N, ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
32	1.97	1.97	1.97	1.97	1.97	1.97	1.97	1.97
64	1.99	1.99	1.99	1.99	1.99	1.99	1.99	1.99
128	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
32	4.49e-2	7.13e-3	6.87e-3	6.84e-3	6.84e-3	6.84e-3	6.84e-3	6.84e-3
64	1.15e-2	1.82e-3	1.75e-3	1.75e-3	1.75e-3	1.75e-3	1.75e-3	1.75e-3
128	2.88e-3	4.57e-4	4.40e-4	4.39e-4	4.39e-4	4.39e-4	4.39e-4	4.39e-4
256	7.22e-4	1.14e-4	1.10e-4	1.10e-4	1.10e-4	1.10e-4	1.10e-4	1.10e-4

Table 5.1: Computational rates r in N^{-r} and maximum nodal errors for the Bakhvalov mesh using the conventional method.

N, ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
32	1.98	1.98	1.98	1.98	1.98	1.98	1.98	1.98
64	1.99	1.99	1.99	2.00	1.99	2.00	1.99	1.99
128	2.00	2.20	2.00	2.00	2.00	2.00	2.00	2.00
32	4.49e-2	8.68e-3	8.39e-3	8.36e-3	8.35e-3	8.35e-3	8.35e-3	8.35e-3
64	1.15e-2	2.21e-3	2.13e-3	2.12e-3	2.12e-3	2.12e-3	2.12e-3	2.12e-3
128	2.88e-3	5.53e-4	5.35e-4	5.33e-4	5.33e-4	5.33e-4	5.33e-4	5.33e-4
256	7.22e-4	1.20e-4	1.34e-4	1.33e-4	1.33e-4	1.33e-4	1.33e-4	1.33e-4

Table 5.2: Computational rates r in N^{-r} and maximum nodal errors on the Bakhvalov mesh using the stabilised method.

N, ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
32	1.96	1.83	1.83	1.83	1.83	1.83	1.83	1.83
64	1.99	1.93	1.93	1.93	1.93	1.93	1.93	1.93
128	2.00	1.97	1.97	1.97	1.97	1.97	1.97	1.97
32	8.70e-3	1.93e-2	1.86e-2	1.85e-2	1.85e-2	1.85e-2	1.85e-2	1.85e-2
64	2.24e-3	9.34e-3	9.00e-3	8.97e-3	8.97e-3	8.97e-3	8.97e-3	8.97e-3
128	6.00e-4	3.80e-3	3.70e-3	3.70e-3	3.70e-3	3.70e-3	3.70e-3	3.70e-3
256	1.41e-4	1.39e-3	1.34e-3	1.34e-3	1.34e-3	1.34e-3	1.34e-3	1.34e-3

Table 5.3: Computational rates r in $(N^{-1} \ln N)^r$ and maximum nodal errors on the Shishkin mesh using the conventional method.

N, ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
32	1.96	1.84	1.84	1.84	1.84	1.84	1.84	1.84
64	1.99	1.93	1.93	1.93	1.93	1.93	1.93	1.93
128	2.00	2.00	1.97	1.97	1.97	1.97	1.97	1.97
32	8.70e-3	2.08e-2	2.01e-2	2.00e-2	2.00e-2	2.00e-2	2.00e-2	2.00e-2
64	2.24e-3	9.73e-3	9.39e-3	9.35e-3	9.35e-3	9.35e-3	9.35e-3	9.35e-3
128	6.00e-4	3.90e-3	3.80e-3	3.70e-3	3.70e-3	3.70e-3	3.70e-3	3.70e-3
256	1.41e-4	1.40e-3	1.37e-3	1.36e-3	1.36e-3	1.36e-3	1.36e-3	1.36e-3

Table 5.4: Computational rates r in $(N^{-1} \ln N)^r$ and maximum nodal errors on the Shishkin mesh using the stabilised method.

Chapter 6

Conclusions

In this work existence and accuracy results were obtained for three nonlinear reaction diffusion problems. These were; a two-dimensional steady state equation with Dirichlet boundary conditions exhibiting interior layer solutions (*Chapter 2*); a time-dependent equation with singularly perturbed Neumann boundary conditions with boundary layer solutions (*Chapter 3*); and a steady state equation with singularly perturbed Neumann boundary conditions exhibiting boundary layer solutions (*Chapter 4*).

For each problem upper and lower solutions were obtained by means of asymptotic analysis. By considering examples of problems for *Chapter 2* and *Chapter 3* using the standard method it was clear that the stabilised method of [16] was necessary to obtain solutions of the correct type. Hence analysis for the standard and stabilised methods was carried out in *Chapter 2* and *Chapter 3*.

By considering appropriate discretisations for each problem discrete upper and lower solutions were set up. In *Chapter 2* and *Chapter 4* this was done by using lumped mass finite elements on a quasiuniform Delaunay triangulation in the outer regions and finite differences in the layer regions. Each side of the interface curve between these regions were discretised using a fictitious

Neumann condition, these were combined to obtain a discretisation for the interface curve itself, eliminating the fictitious Neumann condition. By using these discretisations we were able to employ the theory of Z-fields. For the problem of *Chapter 3* a finite difference discretisation was used again allowing the theory of Z-fields to be used. In all three chapters existence of an exact solution between the upper and lower solutions was proven using this theory. Accuracy results of each system were then found.

In *Chapter 2* we obtained convergence results using the Shishkin mesh. Outside the layer region we obtained second order convergence for the conventional method and first order convergence for the stabilised method. Inside the layer region the truncation error was found to be $O((N^{-1}\ln N)^{2-\varpi})$ for the conventional method and $O((N^{-1}\ln N)^{2-\varpi}+N^{-1})$ for the stabilised method, where $\varpi \in [0,2]$ satisfies $c_0\varepsilon \geqslant (C'N^{-1}\ln N)^{2+\varpi}$ for some c_0 and N. In the case where the relationship between ε and N was stronger, that is $\varepsilon \leqslant CN^{-\varpi'}$ for some $\varpi' \geqslant 4-\lambda$, we found the truncation error to be $O(N^{-(2-\lambda)})$ for the standard method and $O(N^{-1})$ for the stabilised method. As the results found were not ε-uniform, we performed post-processing and found the truncation error of the standard method to be $O(N^{-2}\ln^4 N)$ and the stabilised method to be $O(\max\{N^{-2}\ln^4 N, N^{-1}\})$ for small ε. These results are consistent with the one-dimensional analysis [18].

In *Chapter 3* and *Chapter 4* we performed analysis using the Shishkin and the Bakhvalov meshes. We obtained ε-uniform convergence, with the problems having second-order convergence in space, and first order convergence in time in *Chapter 3*, both with a logarithmic factor for the case of the Shishkin mesh.

Finally numerical solutions were used to verify the results of *Chapter 2* and *Chapter 3*. For *Chapter 2* the need for the stabilised method was demonstrated. Using the stabilised method we obtained solutions of the correct type on a quasiuniform mesh. The theoretical results of *Chapter 3* were sup-

ported by numerical results on the Shishkin and Bakhvalov meshes for the conventional and stabilised methods.

Appendix A

A.1 Existence of a Solution $\hat{V}_0(\xi, l)$

In this section we prove results for Lemma 2.4.2. This proof follows that of [17] and uses phase plane analysis and dynamical systems. It is included here for completeness. An alternate proof can be found in Fife [6].

To prove existence of a solution $\hat{V}_0(\xi, l)$ we consider the following system,

$$\begin{aligned} \frac{\partial \hat{V}_0}{\partial \xi} &= \nu, \\ \frac{\partial \nu}{\partial \xi} &= b(\xi = 0, l, \hat{V}_0), \end{aligned} \tag{A.1}$$

and

$$\hat{V}_0(\infty, l) = \varphi_1(\bar{x}), \quad \hat{V}_0(-\infty, l) = \varphi_2(\bar{x}), \quad \hat{V}_0(0, l) = \varphi_0(\bar{x}), \tag{A.2}$$

recalling the notation $\bar{x} = x(0, l) \in \Gamma_0$. There are three fixed points to this system; $(\varphi_1(\bar{x}), 0)$, $(\varphi_2(\bar{x}), 0)$ and $(\varphi_0(\bar{x}), 0)$. Consider the Jacobian of (A.1), that is

$$\begin{pmatrix} 0 & 1 \\ b_u(0, l, \hat{V}_0) & 0 \end{pmatrix}, \tag{A.3}$$

From (A.3) the eigenvalues of the system are $\pm\sqrt{b_u(0, l, \hat{V}_0)}$. For the fixed

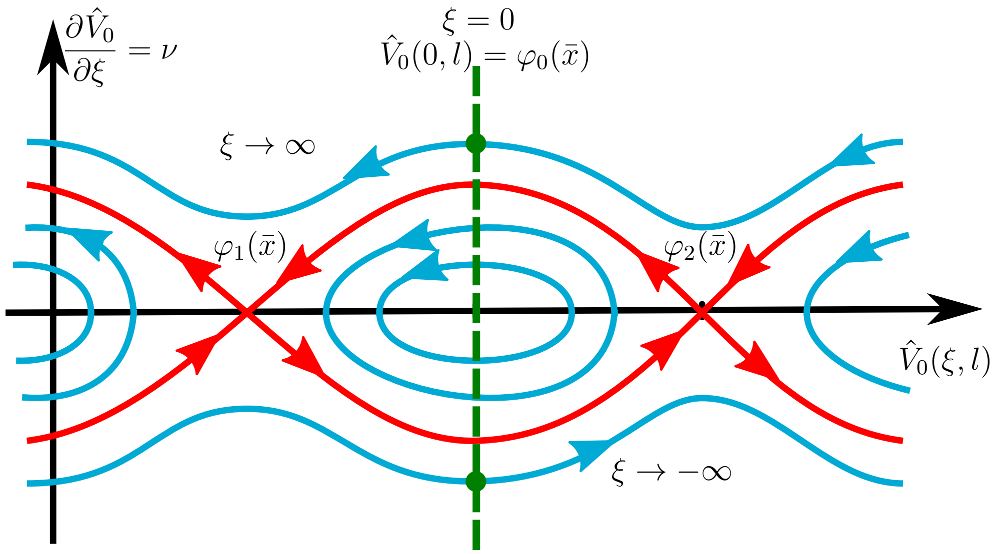


Figure A.1: Phase plane for the system (A.1)*.*

points $(\varphi_1(\bar{x}), 0)$, $(\varphi_2(\bar{x}), 0)$, we have $b_u(0, l, \hat{V}_0) > 0$, and so the eigenvalues are real and of opposite sign. Hence the fixed points $(\varphi_1(\bar{x}), 0)$ and $(\varphi_2(\bar{x}), 0)$ are saddle points. For $(\varphi_0(\bar{x}), 0)$, we have $b_u(0, l, \varphi_0) < 0$, the eigenvalues are complex and of opposite sign. At this fixed point we have a centre as there is no purely real part to the eigenvalue. This is represented in the phase plane in Figure A.1.

We are looking for a trajectory that intersects $\hat{V}_0(0, l) = \varphi_0(\bar{x})$ and either enters the saddle point $\varphi_1(\bar{x})$ as $\xi \to \infty$ or enters the saddle point $\varphi_2(\bar{x})$ as $\xi \to -\infty$. This separatrix will be a solution curve.

Consider the case with $\xi > 0$, with the other case being similar. solving the system $\dfrac{\partial \nu}{\partial \hat{V}_0} = \dfrac{b(0, l, \hat{V}_0)}{\nu}$ we get

$$\nu = \pm\sqrt{2\int_{\varphi_0}^{\hat{V}_0} b(0, l, s)ds + C}. \tag{A.4}$$

As $\nu(\infty,l)=0$ we get $C=0$. Taking $\xi\to\infty$, then $\hat{V}_0(0,l)\to\hat{V}_0(\infty,l)=\varphi_1(\bar{x})$. This trajectory is in the upper half of the plane. Thus $\nu>0$ and so the positive root in (A.4) is chosen. From this we have existence of a solution $\hat{V}_0(\xi,l)$ as a trajectory exists from $\hat{V}_0(0,l)=\varphi_0(\bar{x})$ to $\hat{V}_0(\infty,l)=\varphi_1(\bar{x})$.

We make a change of variables $v_0(\xi,l)=\hat{V}_0(\xi,l)-u_0(\bar{x})$. Now recalling (2.4.22) we have

$$\bar{\gamma}^2\leqslant b_u(0,l,u_0(\bar{x})), \tag{A.5}$$

and there exists $s_0>0$ such that

$$(\bar{\gamma}-\lambda)^2 s\leqslant b_u(0,l,u_0(\bar{x}))\quad \text{for } 0\leqslant s\leqslant s_0,\ \lambda\in(0,\bar{\gamma}). \tag{A.6}$$

Now if for all $\xi>0$, $v_0(\xi,l)>s_0$ then $\dfrac{\partial v_0}{\partial\xi}\geqslant C$ and so $v_0(\xi,l)\geqslant C$ and we get $\lim_{\xi\to\infty}v_0(\xi,l)=\lim_{\xi\to\infty}\hat{V}_0(\xi,l)-u_0(\bar{x})=\infty$ which contradicts (A.2).

Hence we can say there exists ξ_0 such that $v_0(\xi_0,l)\leqslant s_0$ and so for all $\xi\geqslant\xi_0$ then $v_0(\xi,l)\leqslant s_0$. We now fix ξ_0 with $0\leqslant v_0(\xi_0,l)\leqslant s_0$. For $v_0(\xi,l)\in(0,s_0]$, we have

$$\begin{aligned}\nu(\xi,l)\leqslant&\sqrt{2\int_0^{v_0}(\bar{\gamma}-\lambda)^2 s\,ds}\\ \leqslant&(\bar{\gamma}-\lambda)v_0.\end{aligned} \tag{A.7}$$

Recalling $\nu(\xi)=\dfrac{\partial v_0}{\partial\xi}$ we integrate both sides from ξ_0 to ξ to get

$$v_0(\xi,l)\leqslant v_0(\xi_0,l)e^{(\bar{\gamma}-\lambda)\xi_0}e^{-(\bar{\gamma}-\lambda)\xi}\quad\text{for } \xi_0\leqslant\xi\leqslant\infty. \tag{A.8}$$

For $\xi\geqslant\xi_0$ we take $C_\delta\geqslant v_0(\xi_0,l)e^{(\bar{\gamma}-\lambda)\xi_0}$ and get

$$v_0(\xi,l)\leqslant C_\delta e^{-(\bar{\gamma}-\lambda)\xi}. \tag{A.9}$$

For $0 \leqslant \xi \leqslant \xi_0$ we note $s_0 \leqslant v_0(\xi, l) \leqslant v_0(\xi_0, l)$ and taking $C_\delta \geqslant v_0(0, l)e^{(\bar{\gamma}-\lambda)\xi_0}$ we can say

$$v_0(\xi, l) \leqslant v_0(\xi_0, l) \leqslant C_\delta e^{-(\bar{\gamma}-\lambda)\xi_0} \leqslant C_\delta e^{-(\bar{\gamma}-\lambda)\xi}. \tag{A.10}$$

Hence by choosing $C_\delta = \max_{l\in[0,L]}\{\max\{v_0(\xi_0, l)e^{(\bar{\gamma}-\lambda)\xi_0}, v_0(0, l)e^{(\bar{\gamma}-\lambda)\xi_0}\}\}$ we have

$$v_0(\xi, l) \leqslant C_\delta e^{-(\bar{\gamma}-\lambda)\xi} \quad \forall \xi > 0. \tag{A.11}$$

Recall the change of variables gives

$$\hat{V}_0(\xi, l) - u_0(\bar{x}) \leqslant C_\delta e^{-(\bar{\gamma}-\lambda)\xi} \quad \forall \xi > 0. \tag{A.12}$$

By (A.7) we can say

$$\frac{\partial \hat{V}_0}{\partial \xi} \leqslant (\hat{V}_0(\xi) - \varphi_1(\bar{x}))(\bar{\gamma} - \lambda) \tag{A.13}$$

and so using (A.11) gives

$$\frac{\partial \hat{V}_0}{\partial \xi} \leqslant C_\delta e^{-(\bar{\gamma}-\lambda)\xi}. \tag{A.14}$$

Also as $C' \leqslant \dfrac{1}{\sqrt{b_u(0, l, 0)}} \leqslant C''$ and so

$$C' \frac{\partial \hat{V}_0}{\partial \xi} \leqslant \frac{\partial \hat{V}_0}{\partial \xi} \frac{1}{\sqrt{b_u(0, l, 0)}} \leqslant C'' \frac{\partial \hat{V}_0}{\partial \xi}. \tag{A.15}$$

Finally by (A.4), $\dfrac{\partial \hat{V}_0}{\partial \xi} = \sqrt{b_u(0,l,0)}(\hat{V}_0(\xi) - \varphi_1(\bar{x}))$ and so

$$C' \frac{\partial \hat{V}_0}{\partial \xi} \leqslant \hat{V}_0(\xi, l) - \varphi_1(\bar{x}) \leqslant C'' \frac{\partial \hat{V}_0}{\partial \xi}. \tag{A.16}$$

The above analysis is repeated to obtain the results for the case of $\xi < 0$.

A.2 Proof of Lemma 1.0.1

Lemma A.2.1. *[21, p. 7][17, Lemma 3.1] Let $H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be continuous and a Z-field. Let $r \in \mathbb{R}^{n+1}$ be given. Assume that there exists α, $\beta \in \mathbb{R}^{n+1}$ such that $\alpha \leqslant \beta$ and $H\alpha \leqslant r \leqslant H\beta$. Then the equation $Hy = r$ has a solution $y \in \mathbb{R}^{n+1}$ with $\alpha \leqslant y \leqslant \beta$. (The inequalities are understood to hold true componentwise.)*

Proof. Choose v and w such that $Hv \leqslant r$ and $Hw \leqslant r$. Define ζ as $\zeta_i := \max\{v_i, w_i\}$ for $i = 0, ..., n$, and therefore $H\zeta \leqslant r$. Define S to be the set of all lower solutions that lie between α and β, i.e., $S := \{v \in \mathbb{R}^{n+1} : Hv \leqslant r, \alpha \leqslant v \leqslant \beta\}$. S is non empty as $\alpha \in S$. Let $y = \sup\{v : v \in S\}$. We therefore have $\alpha \leqslant y \leqslant \beta$ and $Hy \leqslant r$.

For the remainder of the proof we want to show $Hy = r$. There are two possible cases, $y = \beta$ and $y < \beta$. By the definition of y, we cannot have $\beta < y$.

For the first case, $y_i = \beta_i$, we have $y_k \leqslant \beta_k$ for all $k \neq i$ and as H is a Z-field the mapping is monotonically decreasing therefore $(Hy)_i \geqslant (H\beta)_i$. As $H\beta \geqslant r$, it must hold that $Hy = r$ for this to be true.

For the second case, $y_i < \beta_i$, if $(Hy)_i < r_i$ as H is a Z-field y_i can be increased slightly while retaining the properties $\alpha \leqslant y \leqslant \beta$ and $Hy \leqslant r$.

This new y_i is greater than y which was the supremum of the set S, i.e., $\{v \in \mathbb{R}^{n+1} : Hv \leqslant r, \alpha \leqslant v \leqslant \beta\}$ and so this new y_i contradicts the definition of y. Therefore the result is again $(Hy)_i = r_i$. □

A.3 Proof of Lemma 2.4.1

Lemma A.3.1. *The curvilinear coordinates* (2.4.3) *are orthogonal and for the Laplace operator we have*

$$\Delta u = \eta^{-1} \frac{\partial}{\partial r}\left(\eta \frac{\partial u}{\partial r}\right) + \zeta \frac{\partial}{\partial l}\left(\zeta \frac{\partial u}{\partial l}\right), \tag{A.1}$$

where

$$\eta(r,l) := 1 + \kappa r, \quad \zeta(r,l) := (\tau\eta)^{-1}. \tag{A.2}$$

Proof. The new coordinate system is orthogonal as we have

$$\begin{bmatrix} x_{1,r} & x_{1,l} \\ x_{2,r} & x_{2,l} \end{bmatrix} = \begin{bmatrix} n_1 & q_1' + rn_1' \\ n_2 & q_2' + rn_2' \end{bmatrix}. \tag{A.3}$$

The two-dimensional Laplacian operator becomes

$$\Delta u = \frac{1}{h_1 h_2} \sum_{i=1,2} \frac{\partial}{\partial q_i}\left(\frac{h_1 h_2}{h_i^2} \frac{\partial u}{\partial q_i}\right). \tag{A.4}$$

For our system this is

$$\Delta u = \frac{1}{h_r h_l} \frac{\partial}{\partial r}\left(\frac{h_l}{h_r} \frac{\partial u}{\partial r}\right) + \frac{1}{h_r h_l} \frac{\partial}{\partial l}\left(\frac{h_r}{h_l} \frac{\partial u}{\partial l}\right), \tag{A.5}$$

where $h_r = \sqrt{x_{1,r}^2 + x_{2,r}^2}$ and $h_l = \sqrt{x_{1,l}^2 + x_{2,l}^2}$ are the Lamé coefficients.

As (n_1, n_2) is the unit normal vector, $h_r = 1$. For the remainder of the

proof we show $h_l = (1+\kappa r)\mathcal{T}$.

$$h_l^2 = (q_1' + rn_1')^2 + (q_2' + rn_2')^2 = \mathcal{T}^2 + 2r(q_1'n_1' + q_2'n_2') + r^2(n_2'^2 + n_1'^2). \quad (A.6)$$

Considering the second term here, we have

$$\begin{aligned} 2r(q_1'n_1' + q_2'n_2') &= \frac{2r}{\mathcal{T}^2}\left(q_1'q_2''\mathcal{T} - \mathcal{T}'q_1'q_2' + \mathcal{T}'q_1'q_2' - q_1''q_2'\mathcal{T}\right) \\ &= \frac{2r}{\mathcal{T}}\left(q_1'q_2'' - q_1''q_2'\right) = 2r\mathcal{T}^2\kappa. \end{aligned} \quad (A.7)$$

Next we calculate the final term in (A.6) by showing $\mathcal{T}^4\kappa^2 = \mathcal{T}^2(n_2'^2 + n_1'^2)$. To do this we expand

$$\mathcal{T}^4\kappa^2 = \left(\frac{q_1'q_2'' - q_1''q_2'}{\mathcal{T}}\right)^2 = q_1'\left(n_1' + \frac{\mathcal{T}'q_2'}{\mathcal{T}^2}\right) - q_2'\left(-n_2' + \frac{\mathcal{T}'q_1'}{\mathcal{T}^2}\right) = (q_1'n_1' + q_2'n_2')^2. \quad (A.8)$$

Next,

$$\begin{aligned} (q_1'n_1' + q_2'n_2')^2 &= (\mathcal{T}^2 - q_2'^2)n_1'^2 + (\mathcal{T}^2 - q_1'^2)n_2'^2 + 2q_1'q_2'n_1'n_2' \\ &= \mathcal{T}^2(n_1'^2 + n_2'^2) - ((q_1'n_2')^2 - 2q_1'q_2'n_1'n_2' + (q_2'n_1')^2), \\ &= \mathcal{T}^2(n_1'^2 + n_2'^2) - (q_1'n_2' - q_2'n_1')^2. \end{aligned} \quad (A.9)$$

Finally

$$\begin{aligned} q_1'n_2' - q_2'n_1' &= -q_2'\left(\frac{q_2''\mathcal{T} - q_2'\mathcal{T}'}{\mathcal{T}^2}\right) + q_1'\left(\frac{q_1'\mathcal{T}' - q_1''\mathcal{T}}{\mathcal{T}^2}\right), \\ &= \frac{1}{\mathcal{T}^2}\left(-q_2'q_2''\mathcal{T} + q_2'^2\mathcal{T}' + q_1'^2\mathcal{T}' - q_1'q_1''\mathcal{T}\right), \\ &= \frac{1}{\mathcal{T}^2}\left(\mathcal{T}^2\mathcal{T}' - (q_1'q_1'' + q_2'q_2'')\mathcal{T}\right) \\ &= 0, \end{aligned} \quad (A.10)$$

since $\mathcal{T}' = \dfrac{q_1'q_1'' + q_2'q_2''}{\mathcal{T}}$. Thus (A.10) equals zero.

The coefficient h_l is now

$$\sqrt{\tau^2(1+2\kappa r+\kappa^2 r^2)} = \tau(1+\kappa r), \tag{A.11}$$

giving

$$\Delta u = \frac{1}{(1+\kappa r)}\frac{\partial}{\partial r}\left((1+\kappa r)\frac{\partial u}{\partial r}\right) + \frac{1}{(1+\kappa r)\tau}\frac{\partial}{\partial l}\left(\frac{1}{(1+\kappa r)\tau}\frac{\partial u}{\partial l}\right). \tag{A.12}$$

$\square$

Nomenclature for Chapter 2

Constants

α Takes the value 2 for the standard method and 1 for the stabilised method, page 76

c_0 Sufficiently small positive parameter such that $\hat{h}^4 \leqslant c_0\varepsilon$, page 87

c_1 Small positive parameter, later defined as $c_1 \geqslant \tau$, page 40

c_2 Sufficiently small positive parameter such that $N \geqslant c_2^{-1}$, page 87

$\bar{C}'$ Parameter in smooth cut-off function $\omega(x_{ij})$, page 76

C' Parameter in analysis of upper and lower solutions, page 59

C^* Fixed positive constant, lower bound for $\partial_n \mathcal{I}(x)$, page 34

C_0 Parameter in β, page 60

C_1 Parameter in analysis of upper and lower solutions, page 58

C_2 Parameter in analysis of β, page 61

C_3 Parameter in analysis of β, page 61

C_4 Parameter in analysis of β, page 64

C_4' Parameter in analysis of upper and lower solutions, page 87

C_λ Parameter in bounds for derivatives of boundary layer functions, page 52

C_τ Shishkin mesh parameter, page 74

C'_τ Bound for Shishkin mesh parameter C_τ, page 87

$\widehat{C}$ Stabilisation parameter, page 75

ε Small positive parameter, $\varepsilon \ll 1$, page 28

ε^* Upper bound for ε, page 64

$\hat{\varepsilon}(x_{ij})$ Stabilised version of ε, page 75

$\bar{\gamma}$ Bound related to the nonlinear function, page 45

γ Bound related to the nonlinear function, page 33

h Mesh diameter, page 73

$\hat{h}$ Scaled mesh diameter and parameter in β, page 75

κ Curvature of Γ_0 at $(q_1(l), q_2(l))$, page 39

L Endpoint of l, page 39

λ Positive constant with $\lambda \in (0, \bar{\gamma})$, page 52

μ Parameter such that $\mu \in [0,1]$ and $\hat{h}^2 \leqslant C\varepsilon^\mu$, page 67

p Small positive perturbation parameter, page 45

p' Perturbation parameter, $p' := \varepsilon \dfrac{C_1}{2C_3} p$, page 75

p'' Perturbation parameter, $p'' := \bar{C}' N^{-\alpha}$, page 76

$\bar{p}$ Perturbation parameter with $\bar{p} \geqslant \dfrac{C_2 \varepsilon^2}{2C_1}$, page 70

$\bar{p}'$ Perturbation parameter, $\bar{p}' := \dfrac{C_1 \varepsilon \bar{p}}{2C_3}$, page 70

T Magnitude of the tangent vector at $(q_1(l), q_2(l))$, page 39

τ Mesh transition point, page 73

θ Remainder parameter, $\theta \in (0,1)$, page 26

Functions

$\beta(x, p, p', \hat{h})$ Form of upper and lower solution, page 60

$\beta_*(x)$ A particular case of β, page 75

$C_I(l)$ Integral terms used to find $t_1(l)$, page 57

$C_{II}(l)$ Integral terms used to find $t_1(l)$, page 57

$\hat{\chi}(\xi, l)$ $\dfrac{\partial \hat{V}_0}{\partial \xi}$, page 46

$\chi(\xi, l; p)$ $\dfrac{\partial V_0}{\partial \xi}$, page 46

$\eta(r, l)$ Function in the Laplace operator, $\eta = 1 + \kappa r$, page 40

$I(x)$ Indicator function, page 75

$\nu(\xi, l)$ Auxiliary function, page 49

$\omega(x_{ij})$ Smooth cut-off function, page 76

$\phi(x)$ Fictitious Neumann condition, page 91

$\psi(\xi, l)$ Right hand side of auxiliary equation for $\nu(\xi, l)$, page 49

$(q_1(l), q_2(l))$ Parametrisation of the curve Γ_0, page 39

$\hat{V}_0(\xi, l)$ Specific $O(1)$ solution in the interior layer, page 44

$V_0(\xi, l; p)$ Perturbed $O(1)$ solution in the interior layer, page 45

$\varphi_0(x)$ Unstable reduced solution, page 33

$\varphi_1(x)$ Stable reduced solution, $O(1)$ solution in $\Omega^{(1)}$, page 33

$\varphi_2(x)$ Stable reduced solution, $O(1)$ solution in $\Omega^{(2)}$, page 33

$\varpi'(x)$ Parameter in the analysis when ε and N have a stronger relationship, page 108

$\vartheta(x)$ Smooth cut off function, page 46

$\zeta(r, l)$ Function in the Laplace operator, $\zeta = (\tau\eta)^{-1}$, page 40

$B(x, s)$ Representation of the nonlinear function, $B(x, s) = b(x, u_0 + s)$, page 46

$b(x, u)$ Nonlinear function, page 28

$g(x)$ Given boundary condition, page 28

$n(l)$ Outward normal vector, page 43

$t_1(l)$ $O(\varepsilon)$ part of the curve Γ , page 43

$t_2(l)$ $O(\varepsilon^2)$ part of the curve Γ, page 43

$u_0(x)$ Reduced solution, page 41

$u_2(x)$ Second order solution outside the layer region, page 41

$u_{as}(x;p)$ Asymptotic expansion of $u(x)$, page 46

$v(x)$ A suitable function, depending on the context, page 26

$v_*(\xi, l)$ Function in β, page 60

$v_0(\xi, l)$ Zeroth order boundary layer function, page 46

$v_1(\xi, l)$ First order boundary layer function, page 46

$v_2(\xi, l)$ Second order boundary layer function, page 46

$w(x_{ij})$ Function in upper and lower solutions, page 76

$z(\xi, l)$ Function in β, page 60

Operators

∂_n Outward normal derivative, page 34

$\delta_r^2 U$ Terms involving derivatives with respect to r in $F^N_{\Omega_{[-\tau,\tau]}} U_{\pm N,j}$, page 92

F^N Standard numerical scheme, page 75

$F^N_{\mathring{\Omega}^N}$ Standard numerical scheme on the outer side of the interface curve, page 100

$\hat{F}^N$ Stabilised numerical scheme, page 75

$F^N_{\Omega_{[-\tau,\tau]}}$ Standard numerical scheme on the layer side of the interface boundary, page 91

$\mathcal{L}_\xi[v(\cdot)]$ Differential operator used for boundary layer functions, page 48

$\Phi[v(\cdot)]$ Difference between the derivatives of functions across the curve Γ_0, page 49

$\psi_1(\xi, l)$ Right hand side of the equation for v_1, page 48

$\psi_2(\xi, l)$ Right hand side of the equation for v_2, page 48

F The original system, page 28

Regions

$\Gamma_{\pm\tau}$ Discrete interface curves, page 73

Γ — Asymptotic expansion of the transition curve, page 34

Γ_0 — Zeroth order location of the transition curve, page 34

$\Gamma_0^{\pm}$ — The interior and exterior side of the transition curve Γ_0, page 43

Γ_a — Curve of distance a from Γ_0 in the outward normal direction, $\{x(r,l) : r = a\}$, page 42

$\Gamma_{\pm c_1}$ — Interface curves, $\{x(r,l) : r = \pm c_1\}$, page 42

$\Omega_{(0,c_1)}$ — The interior layer region of $\Omega^{(1)}$, page 41

$\Omega_{(-c_1,0)}$ — The interior layer region of $\Omega^{(2)}$, page 41

Ω — Smooth closed bounded domain, page 28

$\Omega^{(1)}$ — Region where $u(x) \approx \varphi_1(x)$, page 41

$\mathring{\Omega}^{(1)}$ — The outer region where $u(x) \approx \varphi_1(x)$, page 41

$\Omega^{(2)}$ — Region where $u(x) \approx \varphi_2(x)$, page 41

$\mathring{\Omega}^{(2)}$ — The outer region where $u(x) \approx \varphi_2(x)$, page 41

$\Omega_{(0,\tau)}$ — Discrete layer region where $0 < r < \tau$, page 73

$\Omega_{(-\tau,0)}$ — Discrete layer region where $-\tau < r < 0$, page 73

$\mathring{\Omega}^{N}$ — Discrete outer regions, page 73

$\bar{\Omega}\backslash\Omega_{[-c_1,c_1]}$ — Union of $\mathring{\Omega}^{(1)}$ and $\mathring{\Omega}^{(2)}$, outer regions of the domain Ω, page 41

$\mathring{\Omega}$ — Union of $\mathring{\Omega}^{(1)}$ and $\mathring{\Omega}^{(2)}$, outer regions of the domain Ω, page 41

$\partial\Omega$ — Smooth closed curve, the boundary of Ω, page 28

Variables

(r,l) — Curvilinear coordinate system, page 39

x — Original coordinates in Ω, $x = (x_1, x_2)$, page 33

x_{0j} — Representation of Γ_0, page 73

$x_{\pm N,j}$ — Representation of $\Gamma_{\pm\tau}$, page 73

$\bar{x}$ Piecewise version of x to the left and right of Γ_0, $\bar{x} = x(0^{\pm}, l)$, page 43

$\bar{x}_0$ x along Γ_0, $\bar{x}_0 = x(0, l)$, page 43

ξ Rescaled r, $\xi := r/\varepsilon$, page 44

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