

Uniform Pointwise Convergence of Difference Schemes for Convection-Diffusion Problems on Layer-Adapted Meshes*

N. Kopteva, Moscow

Abstract

We consider two convection-diffusion boundary value problems in conservative form: for an ordinary differential equation and for a parabolic equation. Both the problems are discretized using a four-point second-order upwind space difference operator on arbitrary and layer-adapted space meshes. We give ε -uniform maximum norm error estimates $O(N^{-2} \ln^2 N (+\tau))$ and $O(N^{-2} (+\tau))$, respectively, for the Shishkin and Bakhvalov space meshes, where N is the space meshnodes number, τ is the time meshinterval. The smoothness condition for the Bakhvalov mesh is replaced by a weaker condition.

AMS Subject Classifications: 65L10, 65L12, 65L70, 65M06, 65M12, 65M15.

Keywords: Convection-diffusion problems, four-point upwind difference scheme, singular perturbation, Shishkin mesh, Bakhvalov mesh.

1 Introduction

This paper is concerned with ε -uniform numerical methods for the two model boundary value problems: for an ordinary differential equation

$$Lu := -\varepsilon \frac{\partial^2}{\partial x^2} u - \frac{\partial}{\partial x} (p(x)u) = f(x) \quad \text{for } 0 < x < 1, \quad u(0) = g_0, \quad u(1) = g_1, \quad (1.1)$$

and for a parabolic equation

$$\begin{aligned} \frac{\partial}{\partial t} u + Lu &= f(x, t) && \text{for } 0 < x < 1, \quad 0 < t \leq 1, \\ u(x, 0) &= \varphi(x) && \text{for } 0 \leq x \leq 1, \\ u(0, t) &= g_0(t), \quad u(1, t) = g_1(t) && \text{for } 0 < t \leq 1, \end{aligned} \quad (1.2)$$

*This work was supported by the Russian Foundation for Basic Research under grant N 99-01-01056.

where

$$p(x) \geq \beta = \text{const} > 0 \quad (1.3)$$

and $\varepsilon \in (0, 1]$ is a small parameter. Note that the results given in this paper hold for $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is a positive constant depending on the data of the problems. We assume that the data of (1.1) and (1.2) are smooth enough, particularly

$$|p'(x)| \leq P. \quad (1.4)$$

For (1.2) we also assume that $\varphi(0) = g_0(0)$, $\varphi(1) = g_1(0)$ and the compatibility conditions [11] are satisfied so that the solution has no internal layers.

It is well known [13, 15] that as $\varepsilon \rightarrow 0$, the solutions of (1.1) and (1.2) have an exponential boundary layer at $x = 0$ and, as a result, the accuracy of classical numerical methods depends on ε as well as on the space meshnodes number N . One of the approaches to constructing ε -uniform numerical methods is combining classical discretizations of differential equations with layer-adapted highly nonuniform meshes. Bakhvalov [3] was the first to use the approach. The space mesh [3] for problems (1.1) and (1.2) is as follows:

$$x_i = x(i/N), \quad i = 0, 1, \dots, N, \quad (1.5)$$

where $x(\xi)$ is the continuous function defined by

$$x(\xi) = \begin{cases} \begin{cases} \varepsilon \lambda \ln [b/(b - \xi)] & \text{for } \xi \in [0, \theta] \\ 1 - d(1 - \xi) & \text{for } \xi \in [\theta, 1] \end{cases} & \text{if } \varepsilon \leq \bar{\varepsilon}_0 \\ \xi & \text{otherwise,} \end{cases} \quad (1.6)$$

$$d = d(\theta) = (1 - \varepsilon \lambda \ln [b/(b - \theta)]) / (1 - \theta),$$

with constants λ , $0 < \theta < b < 1$, $\bar{\varepsilon}_0 \leq b/\lambda$. Note that the mesh [3] for problems like (1.1) was considered in [12] and [1, 2], ε -uniform accuracy being obtained $O(N^{-1})$ and $O(N^{-2})$ respectively. In the mentioned papers mesh (1.5),(1.6) is assumed to be smooth, i.e. the function $x(\xi)$ is continuously differentiable and $\theta = \bar{\theta}$, defined implicitly by the nonlinear equation

$$\bar{\theta} = b - \varepsilon \lambda / d(\bar{\theta}), \quad (1.7)$$

can be computed using the following iterations [3]

$$\theta^{(0)} = 0, \quad \theta^{(k)} = b - \varepsilon \lambda / d(\theta^{(k-1)}), \quad \lim_{k \rightarrow \infty} \theta^{(k)} = \bar{\theta}, \quad 0 = \theta^{(0)} < \theta^{(1)} < \dots < \bar{\theta}. \quad (1.8)$$

Note that the impossibility of solving the nonlinear equation exactly, when constructing the mesh, can be considered a certain drawback [19, 15]. As in [9], we replace the mesh smoothness condition implying (1.7) by the following weaker condition

$$b - \varepsilon \bar{C} < \theta < b - \varepsilon C_0 \quad (1.9)$$

with arbitrary positive constants C_0 and \bar{C} satisfying $C_0 < \bar{C} < b$. Here the right-hand inequality implies $\max_i h_i = O(N^{-1})$ for mesh (1.5),(1.6), while the left-hand inequality

provides ε -uniform second-order consistency in the negative W_∞^{-1} discrete norm. We point out that the choice $\theta = \bar{\theta}$ is a particular case of (1.9) as well as

$$\theta = \theta^{(1)} = b - \varepsilon\lambda, \quad (1.10)$$

which is the result of the first iteration (1.8), and both the choices generate the meshes satisfying the reasonable condition $h_i \leq h_{i+1}$ (which is provided by $\theta \leq \bar{\theta}$).

Shishkin [17] suggested piecewise uniform layer-adapted meshes, in particular, for problems (1.1) and (1.2) the space mesh [17] is as follows:

$$\Omega = \{x_i \mid x_i = \begin{cases} ih & \text{for } i = 0, \dots, n, \\ x_n + (i - n)H & \text{for } i = n + 1, \dots, N, \end{cases} \quad (1.11)$$

$$h = \delta/n, \quad H = (1 - \delta)/(N - n), \quad n/N = b, \quad \delta = \min(\varepsilon\lambda \ln N, a)\}$$

with constants $a, b \in (0, 1)$ and λ , and the results from [17, 13] lead to ε -uniform error estimate $O(N^{-1} \ln N)$. Recently (see, e.g., the survey [14]) on mesh (1.11) other schemes for problems like (1.1) are studied, ε -uniform accuracy being obtained of order $O(N^{-2} \ln^2 N)$.

It should be remarked that still other layer-adapted meshes were suggested to provide ε -uniform convergence [15].

We shall study difference schemes, using a four-point upwind space difference operator [6] (see also [15, I.2.1.2]), that are second-order consistent and, though do not yield M-matrices, but enjoy certain stability on arbitrary meshes unlike the second-order central-difference scheme. These schemes can be easily extended into two dimensions (unlike, e.g., three-point second-order schemes like [2, 18]). Note also that a similar many-point regularization idea leads, e.g., to the Gontcharov-Frjasinov five-point scheme [5], which works well for the Navier-Stokes equations at high Reynolds numbers.

Thus problem (1.1) is discretized as follows:

$$L^N u_i^N := -\frac{A^N u_{i+1}^N - A^N u_i^N}{\bar{h}_i} = f_i \quad \text{for } i = 1, \dots, N - 1, \quad (1.12)$$

$$u_0^N = g_0, \quad u_N^N = g_1,$$

where A^N is defined by

$$A^N v_i := \begin{cases} \varepsilon D^- v_i + p_{i-1/2}(v_i - 0.5h_i D^+ v_i) & \text{for } i = 1, \dots, N - 1, \\ \varepsilon D^- v_N + p_{N-1/2}(v_N - 0.5h_N D^+ v_{N-1}) & \text{for } i = N. \end{cases} \quad (1.13)$$

Note that this scheme preserves the conservative form of the differential equation. Here and throughout the paper we use the *notation*

$$D^- v_i = \frac{v_i - v_{i-1}}{h_i}, \quad D^+ v_i = \frac{v_{i+1} - v_i}{h_{i+1}}, \quad Dv_i = \frac{v_{i+1} - v_i}{\bar{h}_i},$$

$$h_i = x_i - x_{i-1}, \quad \bar{h}_i = (h_i + h_{i+1})/2,$$

and $w_i = w(x_i)$, $w_{i-1/2} = w(x_i - h_i/2)$, $w_i^j = w(x_i, t_j)$, $w_i(t) = w(x_i, t)$ for any continuous function $w(x)$ or $w(x, t)$. Thus u_i (or u_i^j) denotes the exact solution at the meshnodes, while u_i^N (or $u_i^{N;j}$) is the computed solution.

Clearly, (1.13) implies

$$A^N v_i = \begin{cases} \varepsilon D^- v_i + p_{i-1/2} [(v_{i-1} + v_i)/2 - (h_i \bar{h}_i/2) DD^- v_i] & \text{for } i = 1, \dots, N-1, \\ \varepsilon D^- v_N + p_{N-1/2} (v_{N-1} + v_N)/2 & \text{for } i = N, \end{cases} \quad (1.14)$$

i.e. A^N is a second-order approximation of the differential operator A defined by

$$Av(x) = \varepsilon \frac{\partial}{\partial x} v + p(x)v(x). \quad (1.15)$$

If $p(x) \equiv 1$ and the mesh is uniform, (1.12) turns into the well-known discretization

$$-\varepsilon DD^- u_i^N + (3u_i^N - 4u_{i+1}^N + u_{i+2}^N)/(2h) = f_i \quad \text{for } i = 1, \dots, N-2, \quad (1.16)$$

with the first-order upwind discretization $-\varepsilon DD^- u_{N-1}^N - D^+ u_{N-1}^N = f_{N-1}$ for $i = N-1$. Solving (1.16) exactly, it can be easily checked that $u_i^N = c_0 + c_1 r_1^i + c_2 r_2^i$ with some constants c_0, c_1, c_2 , where the roots $r_0 = 1, r_1, r_2$ are positive, i.e. the solution u_i^N of (1.16) never oscillates (regarding inverse-monotonicity, see Remark 2).

Note also that in [8] this scheme is studied on the Shishkin mesh (1.11) and proved to converge ε -uniformly in the discrete maximum norm, the accuracy being $O(N^{-2} \ln^2 N)$. In this paper we extend the analysis to more general meshes and our parabolic equation.

Problem (1.2) is discretized using the same four-point space operator L^N , as in (1.12):

$$\begin{aligned} \frac{u_i^{N,j} - u_i^{N,j-1}}{\tau} + L^N u_i^{N,j} &= f_i^j \quad \text{for } i = 1, \dots, N-1, \quad j = 1, \dots, K, \\ u_i^{N,0} &= \varphi_i^N \quad \text{for } i = 1, \dots, N-1, \\ u_0^{N,j} &= g_0(t_j), \quad u_N^{N,j} = g_2(t_j) \quad \text{for } j = 0, \dots, K. \end{aligned} \quad (1.17)$$

To our knowledge the first result of ε -uniform convergence for problems like (1.2) is by Shishkin [17] for the difference scheme with the first-order upwind space operator on the Shishkin space mesh, ε -uniform accuracy being proved $O(N^{-1} \ln^2 N + \tau)$. We also refer to [7], where a time defect-correction approach for (1.2) is considered on the Shishkin mesh, with ε -uniform error bound $O(N^{-1} \ln^2 N + \tau^k)$, $k \geq 2$; and [10], where (1.2) is discretized using the central-difference space operator, with ε -uniform accuracy $O(N^{-2} \ln^2 N + \tau)$.

The main results of this paper (Theorems 1, 2) are ε -uniform maximum norm error estimates $O(N^{-2} \ln^2 N (+\tau))$ and $O(N^{-2} (+\tau))$ for schemes (1.12) and (1.17) on the Shishkin and Bakhvalov space meshes respectively.

Notation: Throughout the paper, C , sometimes subscripted, will denote a generic positive constant that is independent of ε and of the mesh.

Remark 1. All the results given in this paper hold for difference schemes (1.12) and (1.17) with $A^N := \bar{A}^N$ defined by

$$\bar{A}^N v_i = \begin{cases} \varepsilon D^- v_i + p_i v_i - 0.5 h_i D^+(pv)_i & \text{for } i = 1, \dots, N-1, \\ \varepsilon D^- v_N + p_N v_N - 0.5 h_N D^+(pv)_{N-1} & \text{for } i = N \end{cases}$$

(compare with (1.13)).

2 Two point boundary value problem

2.1 Hybrid stability inequality

Let $\omega = \{x_i \mid 0 = x_0 < x_1 < \dots < x_{N-1} < x_N = 1\}$ be an arbitrary nonuniform mesh on $[0, 1]$. Throughout the paper we assume that

$$h := \max_i h_i \leq CN^{-1}, \quad H := h_{N-1} = h_N. \quad (2.1)$$

For any mesh functions v_i and w_i , we assume that $v_0 = v_N = w_0 = w_N = 0$, when these values are not defined explicitly, and use the scalar product

$$(v, w) = \sum_{i=1}^{N-1} \tilde{h}_i v_i w_i \quad (2.2)$$

and the discrete L_∞ , L_2 and W_∞^{-1} norms defined, respectively, by

$$\|v\|_\infty = \max_i |v_i|, \quad \|v\|_2 = \|v\| = \sqrt{(v, v)}, \quad \|v\|_* = \max_i \left| \sum_{j=i}^{N-1} \tilde{h}_j v_j \right|.$$

Note that for any discrete function v_i on an arbitrary nonuniform mesh, we have

$$\|v\|_* \leq \|v\|_2 \leq \|v\|_\infty, \quad \|Dv\|_* \leq 2\|v\|_\infty. \quad (2.3)$$

The key to our analysis of schemes (1.12) and (1.17) is the hybrid stability inequality given by

Lemma 1. *Suppose $p(x)$ satisfies (1.3), (1.4), and $\varepsilon \leq \varepsilon_0 = 0.1\beta^2/P$. Then for any solution v_i of the discrete problem $L^N v_i = f_i$ for $i = 1, \dots, N-1$, $v_0 = v_N = 0$ on an arbitrary nonuniform mesh satisfying (2.1), so that $h \leq h_0 := 0.1\beta/P$, we have*

$$\|v\|_\infty \leq C_0 \|f\|_*. \quad (2.4)$$

Proof. First note that, by (1.13), we have

$$A^N v_i = \begin{cases} -\frac{\varepsilon}{h_i} v_{i-1} + \left[\frac{\varepsilon}{h_i} + \left(1 + \frac{h_i}{2h_{i+1}}\right) p_{i-1/2} \right] v_i - \frac{h_i}{2h_{i+1}} p_{i-1/2} v_{i+1} & \text{for } i = 1, \dots, N-1, \\ -\left(\frac{\varepsilon}{H} - \frac{p_{N-1/2}}{2}\right) v_{N-1} + \left(\frac{\varepsilon}{H} + \frac{p_{N-1/2}}{2}\right) v_N & \text{for } i = N. \end{cases}$$

Since $L^N = -DA^N$, the discrete function v_i admits the representation

$$v_i = W_i - \frac{W_N V_i}{V_N} \quad \text{for } i = 0, \dots, N, \quad (2.5)$$

where V_i and W_i are the solutions of the following discrete problems

$$A^N V_i = 1 \quad \text{for } i = 1, 2, \dots, N, \quad V_0 = 0, \quad (2.6)$$

$$A^N W_i = \eta_i \quad \text{for } i = 1, 2, \dots, N, \quad W_0 = 0 \quad (2.7)$$

with

$$\eta_i = \sum_{j=i}^{N-1} \tilde{h}_j f_j \quad \text{for } i = 1, 2, \dots, N-1, \quad \eta_N = 0.$$

Thus it suffices to prove that $\|v\|_\infty \leq C_0 \|\eta\|_\infty$. Further, we consider the two cases.

(i) If $\varepsilon/H \geq p_{N-1/2}/2$, it can easily be verified that A^N yields an M-matrix. Now, using the barrier functions $V_i^l = 0$, $V_i^u = 1/\beta$, and $W_i^{l,u} = \pm V_i \|\eta\|_\infty$, we get the bounds

$$0 < V_i \leq 1/\beta, \quad |W_i| \leq V_i \|\eta\|_\infty \leq \|\eta\|_\infty / \beta \quad \text{for } i = 1, \dots, N,$$

which, combined with (2.5), yield (2.4) with the stability constant $C_0 = 2/\beta$.

(ii) If $\varepsilon/H < p_{N-1/2}/2$, we set $\bar{p} := p_{N-1/2}$ and, by (2.6),(2.7), have

$$V_N = \left(\frac{\bar{p}}{2} + \frac{\varepsilon}{H}\right)^{-1} \left[1 - \left(\frac{\bar{p}}{2} - \frac{\varepsilon}{H}\right) V_{N-1}\right], \quad W_N = -\left(\frac{\bar{p}}{2} + \frac{\varepsilon}{H}\right)^{-1} \left(\frac{\bar{p}}{2} - \frac{\varepsilon}{H}\right) W_{N-1}.$$

Now, eliminating V_N and W_N from (2.5),(2.6) and (2.7), we obtain

$$v_i = W_i + \frac{\left(\frac{\bar{p}}{2} - \frac{\varepsilon}{H}\right) W_{N-1} V_i}{1 - \left(\frac{\bar{p}}{2} - \frac{\varepsilon}{H}\right) V_{N-1}} \quad \text{for } i = 0, \dots, N-1, \quad (2.8)$$

where V_i and W_i , for $i = 0, \dots, N-1$, are the solutions of the slightly modified problems

$$\tilde{A}^N V_i = 1 \quad \text{for } i = 1, 2, \dots, N-2, \quad \tilde{A}^N V_{N-1} = 1 + \frac{p_{N-3/2}}{2} \left(\frac{\bar{p}}{2} + \frac{\varepsilon}{H}\right)^{-1}, \quad V_0 = 0,$$

$$\tilde{A} W_i = \eta_i \quad \text{for } i = 1, 2, \dots, N-1, \quad W_0 = 0$$

with the slightly modified operator \tilde{A}^N defined by

$$\begin{aligned} \tilde{A}^N V_i &:= A^N V_i \quad \text{for } i = 1, \dots, N-2, \\ \tilde{A}^N V_{N-1} &:= -\frac{\varepsilon}{H} V_{N-2} + \left[\frac{\varepsilon}{H} + \frac{3p_{N-3/2}}{2} + \frac{p_{N-3/2}}{2} \left(\frac{\bar{p}}{2} + \frac{\varepsilon}{H}\right)^{-1} \left(\frac{\bar{p}}{2} - \frac{\varepsilon}{H}\right) \right] V_{N-1}. \end{aligned}$$

Since it can be easily verified that \tilde{A}^N yields an M-matrix, we shall use the barrier functions $V_i^l = 0$, $V_i^u = (5/3)/p_i$, and $W_i^{l,u} = \pm V_i \|\eta\|_\infty$ to get the bounds

$$0 \leq V_i \leq (5/3)/p_i, \quad |W_i| \leq V_i \|\eta\|_\infty \quad \text{for } i = 1, \dots, N-1. \quad (2.9)$$

Here, in particular, we used (1.4) implying $|p(\xi_1)/p(\xi_2) - 1| \leq |\xi_1 - \xi_2|P/\beta$, and also the conditions of the Lemma $\varepsilon \leq \varepsilon_0$ and $h \leq h_0$ implying $\varepsilon |D^-(1/p)_i| \leq 0.1$, and $\tilde{A}^N V_{N-1} \leq 1 + p_{N-3/2}/\bar{p} \leq 2.1$, and $\tilde{A}^N V_{N-1}^u \geq (5/3) [-\varepsilon D^-(1/p)_{N-1} + 1.5p_{N-3/2}/p_{N-1}]$. Combining bounds (2.9) with (2.8), we derive $|v_i| \leq V_i \|\eta\|_\infty [1 - (p_{N-1} V_{N-1})(\bar{p}/p_{N-1})/2]^{-1}$, which yields (2.4) with $C_0 = (40/3)/\beta$. ■

Remark 2. Our analysis for the case (ii) implies that, if $\varepsilon \leq Hp_{N-1/2}/2$, the difference operator L^N is inverse-monotone.

2.2 Truncation error and convergence

Lemma 2. *Let $u(x)$ be the solution of (1.1) with sufficiently smooth $p(x)$ and $f(x)$, and u_i^N be the solution of (1.12),(1.13) on an arbitrary nonuniform mesh. Then, under the conditions of Lemma 1, we have*

$$\|u_i^N - u(x_i)\|_\infty \leq C \left[\max_{i=1,\dots,N} \{h_i \bar{h}_i \max_{\xi \in [x_{i-1}, x_i]} |(pu)''(\xi)|\} + N^{-2} \right], \quad (2.10)$$

$$\|u_i^N - u(x_i)\|_\infty \leq C \left[\max_{i=1,\dots,N} (\min \{h_i \bar{h}_i / \varepsilon^2, 1\} \exp \{-\gamma x_{i-1} / \varepsilon\}) + N^{-2} \right] \quad (2.11)$$

with an arbitrary positive constant γ , satisfying $\gamma < p(0)$, and the notation $h_N := h_N$.

Proof. Let $z_i := u_i^N - u(x_i)$ be the error and $\psi_i := f_i - L^N u_i$ be the truncation error. Then $L^N z_i = \psi_i$ for $i = 1, \dots, N-1$, $z_0 = z_N = 0$, and Lemma 1 implies $\|u_i^N - u(x_i)\|_\infty \leq C_0 \|\psi\|_*$. Further, $\|\psi\|_*$ is estimated as in [2, 9] to derive (2.10),(2.11). ■

Our main result regarding problem (1.1) is given by

Theorem 1. *Let $u(x)$ be the solution of (1.1),(1.3) with sufficiently smooth $p(x)$ and $f(x)$, and u_i^N be the solution of (1.12). Let also our meshnodes be $x_i = x(\xi_i)$ with $\{\xi_i\}$ satisfying $0 = \xi_0 < \xi_1 < \dots < \xi_{N-1} < \xi_N = 1$, $\xi_i - \xi_{i-1} = O(N^{-1})$, and $\xi_N - \xi_{N-1} = \xi_{N-1} - \xi_{N-2}$, where the function $x(\xi)$ is defined by a) (1.6),(1.9) or*

$$b) x(\xi) = \begin{cases} \frac{\delta}{b} \xi & \text{for } \xi \in [0, b], \\ \delta + \frac{1-\delta}{1-b} (\xi - b) & \text{for } \xi \in [b, 1], \end{cases} \quad \text{with } \delta = \min(\varepsilon \lambda \ln N, a)$$

and some constants $a, b \in (0, 1)$, λ . Then, provided that $\lambda > 2/p(0)$, we have

$$a) \|u_i^N - u(x_i)\|_\infty \leq CN^{-2}; \quad b) \|u_i^N - u(x_i)\|_\infty \leq CN^{-2} \ln^2 N.$$

Proof. These estimates are derived from bound (2.11) of Lemma 2. The right-hand terms in (2.11) for our two meshes are estimated using a slightly modified analysis [2, 9]. ■

Remark 3. *If $\xi_i = i/N$ for $i = 0, 1, \dots, N$, the meshes a) and b) of Theorem 1 turn into (1.5),(1.6),(1.9) and (1.11) respectively, i.e. the meshes a) and b) of Theorem 1 are nonuniform generalizations of the Bakhvalov [3] and Shishkin [17] meshes.*

3 Parabolic problem

3.1 Truncation error

Let K , our time discretization parameter, be a positive integer, and $\tau = 1/K$. We define the tensor-product mesh on $[0, 1] \times [0, T]$

$$\omega \times \omega_\tau = \{ (x_i, t_j), \quad \text{with } t_j = j\tau, \quad \text{for } i = 0, \dots, N, \quad j = 0, \dots, K \},$$

which is uniform in time. It is assumed for the space mesh ω , in addition to (2.1), that

$$h_i \leq h_{i+1} \quad \text{for } i = 1, 2, \dots, N-1. \quad (3.1)$$

which is reasonable for problem (1.2), since its solution has a boundary layer at $x = 0$. On $\omega \times \omega_\tau$ we shall study difference scheme (1.17). For the time difference derivatives we shall use the notation

$$\delta_{\bar{t}} v_i^j = \frac{v_i^j - v_i^{j-1}}{\tau}, \quad \delta_{\bar{t}}^2 v_i^j = \frac{\delta_{\bar{t}} v_i^j - \delta_{\bar{t}} v_i^{j-1}}{\tau} = \frac{v_i^j - 2v_i^{j-1} + v_i^{j-2}}{\tau^2}.$$

Let $z_i^j := u_i^{N,j} - u(x_i, t_j)$ be the error and $\psi_i^j := f_i^j - \delta_{\bar{t}} u_i^j - L^N u_i^j$ be the truncation error. Then

$$\begin{aligned} \delta_{\bar{t}} z_i^j + L^N z_i^j &= \psi_i^j \quad \text{for } i = 1, \dots, N-1, \quad j = 1, \dots, K, \\ z_0^j &= z_N^j = 0 \quad \text{for } j = 0, \dots, K, \quad z_i^0 = \varphi_i^N - \varphi(x_i) \quad \text{for } i = 0, \dots, N. \end{aligned} \quad (3.2)$$

It is easy to check that ψ_i^j can be splitted as

$$\psi_i^j = \Psi_{1,i}^j + \Psi_{2,i}^j = \Psi_{1,i}(t_j) + \Psi_{2,i}(t_j), \quad (3.3)$$

where

$$\begin{aligned} \Psi_{1,i}(t) &:= -L^N u_i(t) + f_i(t) - \frac{\partial}{\partial t} u(x_i, t) \quad \text{for } 0 \leq t \leq 1, \\ \Psi_{2,i}(t) &:= - \left[\delta_{\bar{t}} u_i(t) - \frac{\partial}{\partial t} u(x_i, t) \right] \quad \text{for } \tau \leq t \leq 1, \end{aligned} \quad (3.4)$$

and the obvious notation $\delta_{\bar{t}} v(t) = [v(t) - v(t - \tau)]/\tau$ is used. Note that the corresponding discrete functions $\Psi_{1,i}^j$ and $\Psi_{2,i}^j$ are defined for $i = 1, \dots, N-1$ and $j = 0, \dots, K$ or $j = 1, \dots, K$ respectively.

Integrating (1.2) w.r.t. x over $[x_{i-1/2}, x_{i+1/2}]$ we get

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial}{\partial t} u(x, t) dx = \left[(Au)(x_{i+1/2}, t) - (Au)(x_{i-1/2}, t) \right] + \int_{x_{i-1/2}}^{x_{i+1/2}} f(x, t) dx,$$

which, combined with $L^N = -DA^N$, implies

$$\begin{aligned} \Psi_{1,i}(t) &= D \left[A^N u_i(t) - (Au)(x_{i-1/2}, t) \right] + \left[f_i(t) - \frac{1}{\bar{h}_i} \int_{x_{i-1/2}}^{x_{i+1/2}} f(x, t) dx \right] - \\ &\quad \left[\frac{\partial}{\partial t} u(x_i, t) - \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial}{\partial t} u(x, t) dx \right]. \end{aligned}$$

Now it can be easily verified that

$$\Psi_{1,i}(t) = D\eta_i(t) + [D\bar{\eta}_i(t) + \bar{\mu}_i(t)] + \tilde{\Psi}_i(t), \quad (3.5)$$

where

$$\begin{aligned}\eta_i(t) &:= A^N u_i(t) - (Au)(x_{i-1/2}, t), & \bar{\eta}_i(t) &:= -h_i^2 \frac{\partial}{\partial x} f(x_{i-1/2}, t)/8, \\ \bar{\mu}_i(t) &:= \frac{1}{\bar{h}_i} \left[\int_{x_{i-1/2}}^{x_i} dx \int_x^{x_i} ds \int_{x_{i-1/2}}^s \frac{\partial^2}{\partial x^2} f(\xi, t) d\xi + \int_{x_i}^{x_{i+1/2}} dx \int_{x_i}^x ds \int_s^{x_{i+1/2}} \frac{\partial^2}{\partial x^2} f(\xi, t) d\xi \right], \\ \tilde{\Psi}_i(t) &:= - \left[\frac{\partial}{\partial t} u(x_i, t) - \frac{1}{\bar{h}_i} \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial}{\partial t} u(x, t) dx \right].\end{aligned}\quad (3.6)$$

Thus we proved

Lemma 3. *For the truncation error ψ_i^j , we have (3.3)-(3.5), where $\eta_i(t)$, $\bar{\eta}_i(t)$, $\bar{\mu}_i(t)$ and $\tilde{\Psi}_i(t)$ are defined in (3.6). Also $\tilde{\Psi}_i(t)$ can be represented as*

$$\tilde{\Psi}_i(t) = - [D\tilde{\eta}_i(t) + \tilde{\mu}_i(t)], \quad (3.7)$$

where

$$\begin{aligned}\tilde{\eta}_i(t) &:= -h_i^2 \frac{\partial^2}{\partial x \partial t} u(x_{i-1/2}, t)/8, \\ \tilde{\mu}_i(t) &:= \frac{1}{\bar{h}_i} \left[\int_{x_{i-1/2}}^{x_i} dx \int_x^{x_i} ds \int_{x_{i-1/2}}^s \frac{\partial^3}{\partial x^2 \partial t} u(\xi, t) d\xi + \int_{x_i}^{x_{i+1/2}} dx \int_{x_i}^x ds \int_s^{x_{i+1/2}} \frac{\partial^3}{\partial x^2 \partial t} u(\xi, t) d\xi \right].\end{aligned}\quad (3.8)$$

3.2 Stability inequalities

Note that our four-point space difference operator L^N does not yield an M-matrix, which makes our stability analysis more difficult (we shall follow, partly, the analysis [10]). The main result of this Subsection is the hybrid stability inequality given by Lemma 5. But to prove it, we need a weaker L_2 stability stated in

Lemma 4. *Suppose $p(x)$ satisfies (1.3), (1.4), and our mesh $\omega \times \omega_\tau$ satisfies (2.1), (3.1) and $\tau \leq \tau_0 := 0.5/(1 + 3P)$; then for the discrete function y_i^j , satisfying*

$$\delta_{\bar{t}} y_i^j + L^N y_i^j = f_i^j \quad \text{for } i = 1, \dots, N-1, \quad j = j_0 + 1, \dots, K, \quad (3.9a)$$

$$y_0^j = y_N^j = 0 \quad \text{for } j = j_0, \dots, K, \quad (3.9b)$$

we have

$$\|y^j\| \leq C \left(\|y^{j_0}\| + \sqrt{\sum_{l=j_0+1}^j \tau \|f^l\|^2} \right) \quad \text{for } j = j_0, \dots, K.$$

This Lemma is proved in Appendix A.

Lemma 5. *Let y_i^j satisfy (3.9) with $j_0 = 0$, and let f_i^j be splitted arbitrarily as $f_i^j = f_{1,i}^j + f_{2,i}^j$ for $i = 1, \dots, N-1, j = 1, \dots, K$ with $f_{1,i}^0$ also defined (arbitrarily) for $i = 1, \dots, N-1$; then, under the conditions of Lemma 4, we have*

$$\|y^j\|_\infty \leq C \left(\|f_1^0 - L^N y^0\| + \|f_1^0\|_* + \|\delta_{\bar{t}} f_1^1\|_* + \|f_2^1\|_\infty + \max_{j=2, \dots, K} \{ \|\delta_{\bar{t}}^2 f_1^j\|_* + \|\delta_{\bar{t}} f_2^j\|_\infty \} \right). \quad (3.10)$$

Remark 4. Though $f_{1,i}^0$ is defined arbitrarily, since there is $\delta_{\bar{t}}f_{1,i}^1$ on the right-hand side of (3.10), we need $f_{1,i}^0$ close to $f_{1,i}^1$ to get a sharp estimate. Note that we prove this Lemma to estimate the error z_i^j satisfying (3.2), where $f_i^j := \psi_i^j$ implies, by Lemma 3, the natural definition of $f_{1,i}^0 := \Psi_{1,i}^0$.

Proof. It follows from (3.9) with $f_i^j = f_{1,i}^j + f_{2,i}^j$ that y_i^j admits the representation

$$y_i^j = v_i^j + w_i^j,$$

where v_i^j and w_i^j are the solutions of the following discrete problems:

$$L^N v_i^j = f_{1,i}^j \quad \text{for } i = 1, \dots, N-1, \quad v_0^j = v_N^j = 0 \quad \text{for } j = 0, \dots, K, \quad (3.11)$$

$$\begin{aligned} L^N w_i^j &= f_{2,i}^j - \delta_{\bar{t}}v_i^j - \delta_{\bar{t}}w_i^j \quad \text{for } i = 1, \dots, N-1, \quad j = 1, \dots, K, \\ w_i^0 &= y_i^0 - v_i^0 \quad \text{for } i = 0, \dots, N, \quad w_0^j = w_N^j = 0 \quad \text{for } j = 0, \dots, K. \end{aligned} \quad (3.12)$$

Then, applying Lemma 1 to (3.12) and recalling (2.3), we have

$$\|y^j\|_\infty \leq \|v^j\|_\infty + C(\|\delta_{\bar{t}}v^j\|_\infty + \|W^j\| + \|f_2^j\|_\infty) \quad \text{for } j = 1, \dots, K, \quad (3.13)$$

where $W_i^j := \delta_{\bar{t}}w_i^j$, defined for $i = 0, \dots, N$, $j = 1, \dots, K$, is the solution of the problem

$$\delta_{\bar{t}}W_i^j + L^N W_i^j = \delta_{\bar{t}}f_{2,i}^j - \delta_{\bar{t}}^2v_i^j \quad \text{for } i = 1, \dots, N-1, \quad j = 2, \dots, K, \quad (3.14a)$$

$$W_i^1 + \tau L^N W_i^1 = (f_{1,i}^0 - L^N y_i^0) + f_{2,i}^1 - \delta_{\bar{t}}v_i^1 \quad \text{for } i = 1, \dots, N-1, \quad (3.14b)$$

$$W_0^j = W_N^j = 0 \quad \text{for } j = 1, \dots, K. \quad (3.14c)$$

Note that (3.14b), which serves as an initial condition here, is derived from (3.12) for $j = 1$.

We claim that

$$\|W^1\|^2 \leq C(\|f_{1,i}^0 - L^N y_i^0\| + \|f_{2,i}^1 - \delta_{\bar{t}}v_i^1\|) \leq C(\|f_1^0 - L^N y^0\|^2 + \|\delta_{\bar{t}}v^1\| + \|f_2^1\|). \quad (3.15)$$

This claim is proved in Appendix B.

Further, it follows from (3.11), by Lemma 1, that

$$\|v^j\|_\infty \leq C\|f_1^j\|_* \quad \text{for } j \geq 0, \quad \|\delta_{\bar{t}}v^j\| \leq C\|\delta_{\bar{t}}f_1^j\|_* \quad \text{for } j \geq 1, \quad \|\delta_{\bar{t}}^2v^j\| \leq C\|\delta_{\bar{t}}^2f_1^j\|_* \quad \text{for } j \geq 2.$$

Now, applying Lemma 4 to problem (3.14a),(3.14c) for W^j with $j_0 = 1$ and recalling (3.13),(3.15), we derive

$$\|y^j\|_\infty \leq C\left(\|f_1^0 - L^N y^0\| + \max_j \left\{\|f_1^j\| + \|\delta_{\bar{t}}f_1^j\| + \|\delta_{\bar{t}}^2f_1^j\| + \|f_2^j\|_\infty + \|\delta_{\bar{t}}f_2^j\|_\infty\right\}\right).$$

Since for any discrete function Y^j and any norm $\|\cdot\|$ we have $\|Y^j\| \leq \|Y^{j_0}\| + \max_{j>j_0} \|\delta_{\bar{t}}Y^j\|$ for $j \geq j_0$, we get (3.10). \blacksquare

3.3 Convergence

Theorem 2. Let $u(x, t)$ be the solution of (1.2) with sufficiently smooth $p(x)$, $f(x, t)$ and $\varphi(x)$, and $u_i^{N,j}$ be the solution of (1.17) with the initial condition φ_i^N defined by the solution of

$$L^N \varphi_i^N = (L\varphi)(x_i) \quad \text{for } i = 1, \dots, N-1, \quad \varphi_0^N = \varphi(0), \quad \varphi_N^N = \varphi(1), \quad (3.16)$$

on the mesh $\omega \times \omega_\tau$, where the space meshnodes $x_i = x(\xi_i)$ are defined by a) (1.5), (1.6) and (1.7) or (1.10); b) (1.11). Then, provided that the mesh parameter $\lambda > 2/\beta$, we have

$$\begin{aligned} a) \max_j \|u_i^{N,j} - u(x_i, t_j)\|_\infty &\leq C(N^{-2} + \tau); \\ b) \max_j \|u_i^{N,j} - u(x_i, t_j)\|_\infty &\leq C(N^{-2} \ln^2 N + \tau). \end{aligned} \quad (3.17)$$

Remark 5. Our initial condition φ_i^N defined by (3.16) is artificial and caused by our analysis. On the other hand, since the analysis of Section 1 applied to problem (3.16) implies $|\varphi_i^N - \varphi_i| \leq CN^{-2}$, our initial condition is only slightly different from the natural initial condition $\tilde{\varphi}_i^N := \varphi_i$.

Remark 6. Theorem 2 also holds for the space meshes defined as in Theorem 1 and satisfying (3.1), i.e. for meshes that can, in general, be essentially nonuniform.

Proof. Applying Lemma 5 to problem (3.2) and recalling Lemma 3, we get

$$\|z^j\|_\infty \leq C \left(\|\Psi_1^0 - L^N(\varphi_i^N - \varphi_i)\| + \|\Psi_1^0\|_* + \|\delta_{\bar{t}}\Psi_1^1\|_* + \|\Psi_2^1\|_\infty + \max_{j=2, \dots, K} \{ \|\delta_{\bar{t}}^2\Psi_1^j\|_* + \|\delta_{\bar{t}}\Psi_2^j\|_\infty \} \right).$$

The first right-hand term, by (3.16) and (1.1) at $t = 0$, vanishes:

$$\Psi_{1,i}^0 - L^N(\varphi_i^N - \varphi_i) = [-L^N\varphi_i + f_i(0) - \frac{\partial u}{\partial t}(x_i, 0)] - [L^N\varphi_i^N - L^N\varphi_i] = (L\varphi)_i - L^N\varphi_i^N = 0.$$

Further, using the Mean Value Theorem, we obtain

$$\|z^j\|_\infty \leq C \max_t \left\{ \|\Psi_1(t)\|_* + \left\| \frac{\partial}{\partial t} \Psi_1(t) \right\|_* + \left\| \frac{\partial^2}{\partial t^2} \Psi_1(t) \right\|_* + \|\Psi_2(t)\|_\infty + \left\| \frac{\partial}{\partial t} \Psi_2(t) \right\|_\infty \right\}. \quad (3.18)$$

To estimate this, we shall use the following decomposition of $u(x, t)$ [17, p. 221]

$$u(x, t) = U(x, t) + V(x, t), \quad \left| \frac{\partial^{k+l}}{\partial x^k \partial t^l} U \right| \leq C, \quad \left| \frac{\partial^{k+l}}{\partial x^k \partial t^l} V \right| \leq C\varepsilon^{-k} \exp(-\gamma x/\varepsilon), \quad (3.19)$$

for $k, l = 0, 1, 2, 3$, with any positive constant γ satisfying $\gamma < \beta$. Then, by (3.4), Taylor series expansions yield

$$\max_{t \in [\tau, 1]} \left\{ \|\Psi_2(t)\|_\infty + \left\| \frac{\partial}{\partial t} \Psi_2(t) \right\|_\infty \right\} \leq C\tau. \quad (3.20)$$

The terms with $\Psi_{1,i}(t)$ in (3.18) are estimated, by (3.5),(2.3), as

$$\max_{t \in [0,1]} \left\| \frac{\partial^l}{\partial t^l} \Psi_1(t) \right\|_* \leq C \left[\left\| \frac{\partial^l}{\partial t^l} \eta_i(t) \right\|_\infty + \left\| \frac{\partial^l}{\partial t^l} \bar{\eta}_i(t) \right\|_\infty + \left\| \frac{\partial^l}{\partial t^l} \bar{\mu}_i(t) \right\|_\infty + \left\| \frac{\partial^l}{\partial t^l} \tilde{\Psi}_i(t) \right\|_* \right] \quad (3.21)$$

for $l = 0, 1, 2$. Now we shall split $\tilde{\Psi}_i(t)$ as $\tilde{\Psi}_i(t) = \tilde{\Psi}_i^U(t) + \tilde{\Psi}_i^V(t)$, where the right-hand terms are defined as $\tilde{\Psi}_i(t)$ in (3.6) and admit the representations as (3.7),(3.8) with $U(x, t)$, $\tilde{\eta}_i^U(t)$, $\tilde{\mu}_i^U(t)$ and $V(x, t)$, $\tilde{\eta}_i^V(t)$, $\tilde{\mu}_i^V(t)$ instead of $u(x, t)$, $\tilde{\eta}_i(t)$, $\tilde{\mu}_i(t)$ respectively. By (2.3), this yields

$$\begin{aligned} \left\| \frac{\partial^l}{\partial t^l} \tilde{\Psi}_i(t) \right\|_* &\leq 2 \left[\left\| \frac{\partial^l}{\partial t^l} \tilde{\eta}_i^U(t) \right\|_\infty + \left\| \frac{\partial^l}{\partial t^l} \tilde{\mu}_i^U(t) \right\|_\infty \right] + \\ &2 \max_{i \leq \bar{i}} \left\{ \left| \frac{\partial^l}{\partial t^l} \tilde{\eta}_i^V(t) \right| + \left| \frac{\partial^l}{\partial t^l} \tilde{\mu}_i^V(t) \right| \right\} + \max_{i \geq \bar{i}} \left| \frac{\partial^l}{\partial t^l} \tilde{\Psi}_i^V(t) \right|, \end{aligned} \quad (3.22)$$

with the number \bar{i} defined by the condition $h_{\bar{i}} \leq \varepsilon < h_{\bar{i}+1}$. Further, combining (3.21) with (3.22), recalling (3.19) and using a slightly modified analysis [2, 9], we derive, by Taylor series expansions, that

$$\max_{t \in [0,1]} \left\| \frac{\partial^l}{\partial t^l} \Psi_1(t) \right\|_* \leq C \left[\max_i \left(\min \{ \hbar_i^2 / \varepsilon^2, 1 \} \exp(-\gamma x_{i-1} / \varepsilon) \right) + N^{-2} \right] \quad (3.23)$$

for $l = 0, 1, 2$. Finally, combining (3.18),(3.20) and (3.23), we get the bound

$$\max_j \| u_i^{N,j} - u(x_i, t_j) \|_\infty \leq C \left[\max_i \left(\min \{ \hbar_i^2 / \varepsilon^2, 1 \} \exp(-\gamma x_{i-1} / \varepsilon) \right) + N^{-2} + \tau \right],$$

which, as in the proof of Theorem 1, yields (3.17). ■

4 Numerical results

Table 1: Two point boundary value problem, maximum nodal error and computational rate of convergence

N	$\varepsilon = 1$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$	\max_ε
16	4.96e-3	2.82e-2	3.11e-2	3.12e-2	3.12e-2	3.12e-2
	2.00	1.86	1.88	1.88	1.88	1.88
32	1.24e-3	7.78e-3	8.42e-3	8.44e-3	8.44e-3	8.44e-3
	2.00	1.94	1.94	1.94	1.94	1.94
64	3.09e-4	2.02e-3	2.19e-3	2.20e-3	2.20e-3	2.20e-3
	2.00	1.99	1.97	1.97	1.97	1.97
128	7.71e-5	5.10e-4	5.59e-4	5.60e-4	5.60e-4	5.60e-4
	2.00	2.01	1.99	1.99	1.99	1.99
256	1.92e-5	1.26e-4	1.41e-4	1.41e-4	1.41e-4	1.41e-4
	2.00	2.04	1.99	1.99	1.99	1.99
512	4.80e-6	3.07e-5	3.54e-5	3.55e-5	3.55e-5	3.55e-5

We consider test problems (1.1) and (1.2) with $p(x) = (x + 1)^3$ and the other data such that their solutions are

$$u(x) = \frac{1}{p(x)} \exp \left(-\frac{1}{\varepsilon} \int_0^x b(s) ds \right) + \exp(-x/2)$$

(this example is from [4]) and

$$u(x, t) = \frac{1}{p(x)} \exp\left(-\frac{1}{\varepsilon} \int_0^x p(s) ds\right) \sin 2t + \exp(-x/2) \sin t,$$

respectively.

The problems were solved numerically on the Bakhvalov space mesh (1.5),(1.6),(1.10) with $C = 2.3$, $b = 0.5$, $\bar{\varepsilon}_0 = b/\lambda$.

In table 1 for test problem (1.1), solved using difference scheme (1.12),(1.13), we give the error in the discrete L_∞ norm in the odd lines and the numerical rate of convergence, computed by the formula $\log_2(\|u_i^{2N} - u(x_i)\| / \|u_i^N - u(x_i)\|)$, in the even lines. The numerical tests confirm ε -uniform second-order convergence claimed by Theorem 1. Note that similar results for a steady problem on the Shishkin mesh are given in [8].

Table 2: **Parabolic problem, maximum nodal error**

τ^{-1}	N	$\varepsilon = 1$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$
16	16	9.28e-4	1.40e-2	1.54e-2	1.54e-2	1.54e-2
	32	4.48e-3	1.04e-2	1.41e-2	1.43e-2	1.43e-2
	64	5.39e-3	1.30e-2	1.58e-2	1.59e-2	1.59e-2
	128	5.61e-3	1.42e-2	1.62e-2	1.63e-2	1.63e-2
	256	5.67e-3	1.45e-2	1.63e-2	1.64e-2	1.64e-2
	512	5.68e-3	1.46e-2	1.64e-2	1.64e-2	1.64e-2
1024	16	4.79e-3	2.40e-2	2.54e-2	2.55e-2	2.55e-2
	32	1.14e-3	6.60e-3	6.88e-3	6.89e-3	6.89e-3
	64	2.21e-4	1.59e-3	1.65e-3	1.66e-3	1.66e-3
	128	1.55e-5	2.77e-4	2.95e-4	2.96e-4	2.96e-4
	256	7.13e-5	1.75e-4	2.41e-4	2.43e-4	2.43e-4
	512	8.52e-5	2.10e-4	2.55e-4	2.57e-4	2.57e-4

Table 2 shows the maximum nodal error $\max_j \|u_i^{N,j} - u(x_i, t_j)\|_\infty$ for test problem (1.2) solved by (1.17). The numerical results correspond with the ε -uniform error estimate given by Theorem 2.

References

- [1] Andreev, V. B.: The uniform convergence with respect to a small parameter of the modified A.A.Samarskii's monotone scheme on a smoothly condensing grid. *Comp. Math. Math. Phys.* *38*, 101–114 (1998).
- [2] Andreev V. B., Kopteva N. V.: On the convergence, uniform with respect to a small parameter, of monotone three-point finite-difference approximations. *Differential Eqns.* *34*, 921–929 (1998).

- [3] Bakhvalov, N. S.: On the optimization of methods for boundary-value problems with boundary layers. *Zh. Vychisl. Mat. Mat. Fiz.* 9, 841–859 (1969) (in Russian).
- [4] Berger A. E.: A conservative uniformly accurate difference method for a singular perturbation problem in conservative form. *SIAM J. Numer. Anal.* 23, 1241–1253 (1986).
- [5] Gontcharov A.L., Frjazinov I.W.: A ten-point difference scheme for the solution of the Navier-Stokes equations. *Zh. Vychisl. Mat. Mat. Fiz.* 28, 867–878 (1988) (in Russian).
- [6] Gromov B. F., Petrishchev V. S.: On numerical solution to two-dimensional incompressible viscous fluid problems. *Proc. of All-Union Seminar on Comp. Viscous Fluid Mechanics*. Novosibirsk: Nauka 1969 (in Russian).
- [7] Hemker P. W., Shishkin G. I., Shishkina L.P.: An ε -uniform defect-correction method for a parabolic convection-diffusion problem. *Recent Advances in Numerical Methods and Applications II*, Iliev O.P., Kaschiev M. S., Margenov S. D., Sendov B.H. and Vassilevski P.S. eds.: *Proceedings of the Fourth International Conference*, Sofia, Bulgaria, August 19-23 (1998). Singapore: World Scientific 521-529 (1999).
- [8] Kopteva, N. V.: Uniform convergence with respect to a small parameter of a four-point scheme for the one-dimensional stationary convection-diffusion equation. *Differential Eqns.* 32, 958–964 (1996).
- [9] Kopteva, N. V.: On the uniform with respect to a small parameter convergence of the central difference scheme on condensing meshes *Comp. Math. Math. Phys.* 39, 1594–1610 (1997).
- [10] Kopteva, N. V.: On the uniform in small parameter convergence of a weighted scheme for the one-dimensional time-dependent convection-diffusion equation. *Comp. Math. Math. Phys.* 37, 1173–1180 (1997).
- [11] Ladyzhenskaya, O. A., Solonnikov V.A., Ural'tseva N. N. *Linear and Quasilinear Equations of Parabolic Type*. *Transl. of Math. Monographs*, 23, American Math. Soc., Providence, RI 1968.
- [12] Liseikin, V. D., Yanenko, N. N.: On a uniformly convergent algorithm for a second order differential equation with a small parameter. *Numer. Meth. Mechs. Continuous Media.* 12, 45–56 (1981) (in Russian).
- [13] Miller, J. J. H., O’Riordan, E., Shishkin, G. I.: *Fitted numerical methods for singular perturbation problems*. Singapore: World scientific 1996.
- [14] H.-G. Roos. Layer-adapted grids for singular perturbation problems. *Z. Angew. Math. Mech.* 78:291–309, 1998.
- [15] Roos, H.-G., Stynes, M., Tobiska, L.: *Numerical methods for singularly perturbed differential equations*. Springer Series in Computational Mathematics. Berlin Heidelberg New-York Tokyo: Springer-Verlag 1996.

- [16] Samarski, A. A.: Theory of Difference Schemes. Moskow: Nauka 1989 (in Russian).
- [17] Shishkin G. I.: Grid Approximation of Singularly Perturbed Elliptic and Parabolic Equations. Ekaterinburg: Ur. O. RAN 1992 (in Russian).
- [18] Stynes, M., Roos, H.-G.: The midpoint upwind scheme. Appl. Numer. Math. 23 361-374 (1997).
- [19] R. Vulanović. *Mesh Construction for Discretization of Singularly Perturbed Boundary Value Problems*. Doctoral Dissertation, University of Novi Sad, 1986.

A Appendix: Proof of Lemma 4

Without loss of generality we shall only prove the Lemma for $j_0 = 0$. Multiplying (3.9a) by y^j as in (2.2), by simple calculations, we get

$$\|y^j\|^2 = (y^j, y^{j-1}) + \tau [- (L^N y^j, y^j) + (f^j, y^j)] = (y^j, y^{j-1}) + \tau [S^j + 1.5P\|y^j\|^2 + (f^j, y^j)]$$

with

$$S^j := - (L^N y^j, y^j) - 1.5P\|y^j\|^2 = - \sum_{i=1}^N h_i (A^N y_i^j) (D^- y_i^j) - 1.5P\|y^j\|^2.$$

Here we used $L^N = -DA^N$. Further, by the Schwarz inequality for the terms (y^j, y^{j-1}) and (f^j, y^j) , we have $\|y^j\|^2 \leq (1 - \bar{\tau})^{-1} [\|y^{j-1}\|^2 + \tau (2S^j + \|f^j\|^2)]$ with $\bar{\tau} := (1 + 3P)\tau$, and consequently

$$\|y^j\|^2 \leq (1 - \bar{\tau})^{-j} \left[\|y^0\|^2 + \tau \sum_{l=1}^j \|f^l\|^2 + 2\tau S^j \right] \quad \text{for } j = 1, \dots, K, \quad (\text{A.1})$$

where

$$S = (1 - \bar{\tau})^{j-1} S^j + (1 - \bar{\tau})^{j-2} S^{j-1} + \dots + S^1. \quad (\text{A.2})$$

Note that $\tau \leq \tau_0 = 0.5/(1 + 3P)$, i.e. $\bar{\tau} \leq 0.5$, implies

$$1 \leq (1 - \bar{\tau})^{-j} \leq (1 - \bar{\tau})^{-1/\tau} \leq (1 - \bar{\tau})^{-1/\bar{\tau}} \leq 1/\bar{C} \quad \text{with } \bar{C} = 1/4. \quad (\text{A.3})$$

Now, by (1.14), we get

$$\begin{aligned} S^j = & -\varepsilon \sum_{i=1}^N h_i |D^- y_i^j|^2 - 0.5 \sum_{i=1}^N h_i p_{i-1/2} (y_{i-1}^j + y_i^j) (D^- y_i^j) + \\ & 0.5 \sum_{i=1}^N h_i^2 p_{i-1/2} (D^- y_{i+1}^j - D^- y_i^j) (D^- y_i^j) - 1.5P\|y^j\|^2. \end{aligned}$$

The second term on the right, by (1.4), is estimated as

$$\left| \sum_{i=1}^N h_i p_{i-1/2} (y_{i-1}^j + y_i^j) (D^- y_i^j) \right| = \left| \sum_{i=1}^{N-1} (p_{i-1/2} - p_{i+1/2}) (y_i^j)^2 \right| \leq P\|y^j\|^2.$$

Now, noting that $(a - b)b = [(a^2 - b^2) - (a - b)^2] / 2$, with $a = D^- y_{i+1}^j$ and $b = D^- y_i^j$, we get

$$S^j \leq -\varepsilon h_N |D^- y_N^j|^2 + \frac{1}{4} \sum_{i=1}^{N-1} h_i^2 p_{i-1/2} (|D^- y_{i+1}^j|^2 - |D^- y_i^j|^2) - \frac{1}{4} \sum_{i=1}^{N-1} h_i^2 p_{i-1/2} |D^- y_{i+1}^j - D^- y_i^j|^2 - P \|y^j\|^2. \quad (\text{A.4})$$

Setting $v_i = D^- y_i^j$, we observe, by (3.1), that

$$\sum_{i=1}^{N-1} h_i^2 p_{i-1/2} (v_{i+1}^2 - v_i^2) \leq h_N^2 p_{N-1/2} v_N^2 + \sum_{i=2}^N h_{i-1}^2 (p_{i-3/2} - p_{i-1/2}) v_i^2 \leq h_N^2 p_{N-1/2} v_N^2 + 4P \|y^j\|^2.$$

Further, combining this with (A.4), omitting some of the nonpositive terms and recalling (2.1), we derive

$$S^j \leq -\varepsilon H |D^- y_N^j|^2 + \frac{p_{N-1/2}}{4} H^2 |D^- y_N^j|^2 - \frac{p_{N-1/2}}{4} H^4 |DD^- y_{N-1}^j|^2. \quad (\text{A.5})$$

If $\varepsilon \geq p_{N-1/2} H / 4$, then $S^j \leq 0$, which implies $S \leq 0$. Combining this with (A.1) and (A.3), we complete the proof.

Otherwise, if $\varepsilon < p_{N-1/2} H / 4$, omitting the first term on the right in (A.5) and combining (A.5) with (A.2), (A.3), we obtain that

$$S \leq \frac{p_{N-1/2}}{4} \sum_{l=1}^j \left(|y_{N-1}^l|^2 - \bar{C} H^4 |DD^- y_{N-1}^l|^2 \right). \quad (\text{A.6})$$

Here we also used that $y_N^j = 0$ implies $D^- y_N^j = -y_{N-1}^j / H$. It follows from (3.9a) for $i = N - 1$ that

$$|y_{N-1}^j| \leq (1 + \tau \tilde{p} / H)^{-1} (|y_{N-1}^{j-1}| + \varepsilon \tau |DD^- y_{N-1}^j| + \tau |f_{N-1}^j|)$$

with the notation $\tilde{p} := 1.5p_{N-3/2} - 0.5p_{N-1/2}$. Set $\delta := \tau \tilde{p} / H$, $q := (1 + \delta)^{-1}$. Then, by

$$(a + b + c)^2 \leq (1 + \delta) a^2 + (1 + 1/\delta) (b + c)^2 \leq (1 + \delta) [a^2 + (2/\delta) (b^2 + c^2)],$$

we have

$$|y_{N-1}^j|^2 \leq q |y_{N-1}^{j-1}|^2 + q (2/\delta) R^j \quad \text{with} \quad R^j = \varepsilon^2 \tau^2 |DD^- y_{N-1}^j|^2 + \tau^2 |f_{N-1}^j|^2. \quad (\text{A.7})$$

Further, using that $q + q^2 + \dots + q^j \leq q / (1 - q) = 1/\delta = H / (\tau \tilde{p})$, we derive that

$$\sum_{l=1}^j |y_{N-1}^l|^2 \leq \frac{q}{1 - q} |y_{N-1}^0|^2 + \frac{q}{1 - q} \cdot \frac{2}{\delta} \sum_{l=1}^j R^l = \frac{H}{\tau \tilde{p}} |y_{N-1}^0|^2 + 2 \left(\frac{H}{\tau \tilde{p}} \right)^2 \sum_{l=1}^j R^l.$$

Combining this with (A.6) and (A.7), we get

$$S \leq \frac{H p_{N-1/2}}{4 \tau \tilde{p}} |y_{N-1}^0|^2 + \frac{p_{N-1/2}}{4} \left(\frac{2 H^2 \varepsilon^2}{\tilde{p}^2} - \bar{C} H^4 \right) \sum_{l=1}^j |DD^- y_{N-1}^l|^2 + C H^2 \sum_{l=1}^j |f_{N-1}^l|^2.$$

Now we recall that $\bar{C} = 1/4$ and, by (1.4), $(p_{N-1/2}/\tilde{p}) \leq 4/3$, which implies that $\varepsilon < (\tilde{p}H/4)(p_{N-1/2}/\tilde{p}) \leq \tilde{p}H/3$. Then the second right-hand term is negative and consequently

$$2\tau S \leq \frac{2}{3}\|y^0\|^2 + C\tau H \sum_{l=1}^j \|f^l\|^2.$$

Combining this with (A.1) and (A.3), we complete the proof.

B Appendix: Proof of (3.15)

Setting $F_i := (f_{1,i}^0 - L^N y_i^0) + f_{2,i}^1 - \delta_{\bar{t}} v_i^1$, we prove that, under the conditions of Lemma 5, (3.14c),(3.14b) imply $\|W^1\|^2 \leq C\|F\|^2$. Multiplying (3.14b) by W^1 , we have

$$\|W^1\|^2 = (F, W^1) - \tau (L^N W^1, W^1).$$

The similar argument, as used in the proof of Lemma 4 (Appendix A) to derive (A.5), gives

$$S := - (L^N W^1, W^1) \leq -\varepsilon H |D^- W_N^1|^2 + \frac{p_{N-1/2}}{4} H^2 |D^- W_N^1|^2 - \frac{p_{N-1/2}}{4} H^4 |DD^- W_{N-1}^1|^2 + 1.5P \|W^1\|^2.$$

If $\varepsilon \geq p_{N-1/2}H/4$, then $S \leq 0$, and (3.15) is obvious. Otherwise, if $\varepsilon < p_{N-1/2}H/4$, i.e., by (1.4), $\varepsilon < \tilde{p}H/2$ with $\tilde{p} := 1.5p_{N-3/2} - 0.5p_{N-1/2}$, omitting the first term on the right and taking into consideration that $W_N^1 = 0$ implies $D^- W_N^1 = -W_{N-1}^1/H$, and that (3.14b) for $i = N - 1$ yields $W_{N-1} = [H/(H + \tau\tilde{p})] [\varepsilon\tau DD^- W_{N-1}^1 + F_{N-1}]$, we get

$$\begin{aligned} \tau S &\leq \tau \frac{p_{N-1/2}}{4} \left(|W_{N-1}^1|^2 - H^4 |DD^- W_{N-1}^1|^2 \right) + \tau C \|W^1\|^2 \leq \\ &\tau \frac{p_{N-1/2}}{4} \left(\frac{2\varepsilon^2}{\tilde{p}^2} H^2 - H^4 \right) |DD^- W_{N-1}^1|^2 + \tau \frac{p_{N-1/2}}{4} \frac{2H^2}{(H + \tau\tilde{p})^2} |F_{N-1}|^2 + \tau C \|W^1\|^2 \leq \\ &C (H |F_{N-1}|^2 + \tau \|W^1\|^2) \leq C (\|F\|^2 + \tau \|W^1\|^2), \end{aligned}$$

which again yields (3.15).

Natalia Kopteva
 Department of Computational Mathematics and Cybernetics
 Lomonosov Moscow State University
 Vorob'evy gory
 RU-119899 Moscow
 Russia
 email: kopteva@cs.msu.su