Cycles and clustering in multiplex networks

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Abstract: Multiplex networks are a natural extension of the classical model of a single graph. Each multiplex consists of several layers, in which vertices can be connected to each other through edges of different colors. Multiplex networks can model the complex interdependencies of real-world systems, such as social networks, transportation systems, and biological networks. The study of cycles in multiplex networks is important for understanding the structural properties of these systems, as cycles are related to the connectivity and robustness of networks. In this paper, we give an analytical description of the statistics of cycles in multiplex networks. We consider multiplex networks characterized by a given multi-degree distribution (configuration model). As this model is the starting point of any network analysis of a real-world system, it is easy to see how important it is to analytically characterize a structural property such as the statistics of cycles. In a multiplex network, the possibility to switch between layers greatly increases the number of cycle types with respect to the single-layer case. Cycles are then no longer defined simply by their length. We must take into account the proportion of the cycle in each layer, as well as the number of switches between layers. In particular, this leads to a set of different clustering coefficients generated by different cycles of length 3.

I. INTRODUCTION

The realization that many complex systems cannot be understood by representing them as a single network, has led to an explosion of interest in multilayer and multiplex (multiple types of edges) networks. Applications range from infrastructure [1], financial [2], transport [3], and ecological [4].

To properly study multilayer systems, it is essential to understand the fundamental properties of such structures. Many concepts from single-layer networks have already been generalized to multiple layers, from the degree distribution, to connectivity, adjacency, and Laplacian matrices, centrality measures, and so on [5–7]. In many cases, the generalization of concepts from single networks—for example, the meaning of “giant connected component” [8,9]—is not straightforward, and introduces a new dimension to the problem.

In this paper, we give an analytical description of the statistics of cycles in multiplex networks. In particular, we consider multiplex networks characterized by a given multi-degree distribution (configuration model). As this model is the starting point of any network analysis of a real-world system, it is easy to see how important it is to analytically characterize a structural property such as the statistics of cycles. In a multiplex network, the possibility to switch between layers greatly increases the number of cycle types with respect to the single-layer case. Cycles are then no longer defined simply by their length. We must take into account the proportion of the cycle in each layer, as well as the number of switches between layers. In particular, this leads to a set of different clustering coefficients generated by different cycles of length 3.

Even when two cycles contain the same number of edges of each color, they can differ in the way colored edges are arranged within the cycle. We introduce a matrix $s$ characterizing the number of edges of each color, and the number of switches between layers of a cycle. We give formulas for calculating the mean number of cycles corresponding to a given $s$ in a random graph with a given size and degree distribution. As examples, we calculate the distribution of edge colors and switches in cycles of a given length, show the effects of degree correlations between layers, and examine the special case of cycles of length 3, which allows the calculation of the generalized clustering coefficients in multiplex networks.

The statistics of cycles is relevant both from a theoretical and an applicative point of view. From a theoretical perspective, it allows one to understand whether the distribution of cycles observed in a real-world network is significantly different from that in a random graph with similar statistics [10]. Even in the single-layer case, the high concentration of finite cycles in real-world networks has been a formidable barrier to analytic treatment, as mathematical models of large networks are typically based on the local treelike assumption, i.e., the vanishing of density of finite cycles as the size of the network diverges [11–13]. On the other hand, some analytic theories have been quite successful even in real-world networks [14], suggesting that the role played by the detailed architecture of topological correlations and cycles is not easily characterized. In multiplex networks, further types of correlations arise naturally, with implications for the structural properties of the multiplex [15,16]. It is therefore necessary to turn now to multiplex cycles and investigate the statistics of each category based on edge color and layer switches, and the effects of correlations on them. The properties of cycles is also an important topic of graph theory [17,18]. A relatively smaller volume of work has been devoted to cycles on colored edge graphs, mainly involving only theorems of existence [19].

The statistics of cycles in multiplex networks is also relevant for a number of applications. In information technology, the presence of multiple paths of heterogeneous colors is a structural property that improves the robustness of a network [20] and the security of a wireless sensor network from a malicious attack [21]. Multiplex cycles are also relevant when examining commuter behavior on multiple transport networks, as they provide alternative routes [3]. In such applications,
switching between layers may have a time or monetary cost, a consideration that is absent in single-layer networks. The statistics of switches is therefore an essential part of any analysis of cycles in multiplex networks.

This paper is organized as follows. In Sec. II, we describe a classification of cycles in multiplex networks and explain our formula to calculate the number of cycles within a given class. In Sec. III, we use our formula to calculate multiplex clustering coefficients. Finally, in Sec. IV we state our conclusions.

II. STATISTICS OF CYCLES

We characterize a given cycle in a multiplex network, by the matrix $s = \{s_{ab}\}_{a,b=1,\ldots,M}$, where each element $s_{ab}$ defines the number of nodes in the cycle, which connect an edge of type $a$ with an edge of type $b$. When $a = b$, this counts the number of nodes where the cycle remains in the same layer. When $a \neq b$, $s_{ab}$ counts the number of switches between layer $a$ and layer $b$. In this paper, we identify and study the classes of equivalent cycles with the same $s$. In addition, here we only consider cycles without an orientation, therefore the order of the switches is not important, i.e., $s_{ba} = s_{ab}$. Examples of cycles with a given $s$ are shown in Fig. 1. As a consequence of this definition, the total number of edges in layer $a$ is then $n_a = s_{aa} + \frac{1}{2} \sum_{b \neq a} s_{ab}$. Clearly, the number of switches must be such that $n_a$ is an integer for all $a$. The total length of a cycle $L$ is the sum of all entries of $s$.

We consider a generalization of the configuration model to multiplex networks, i.e., large sparse random multiplex networks with $N$ nodes in $M$ layers, defined by the joint multidegree distribution $P(q_1, q_2, \ldots, q_M)$. This ensemble includes all possible configurations with multidegree sequence sampled from this distribution with equal statistical weight [22,23].

To calculate the mean number of cycles $N(s)$ with a given $s$ in a random graph, we first count the number of ways we can select, from the given multidegree distribution, the nodes that have the connectivity required for each $s_{ab}$; then, we count the number of ways we can connect these nodes to form a cycle. This method is similar to that used in, for example, Ref. [24] but we extend it to account for edges of different colors and switches between layers. The results can be written as the product of several factors:

$$N(s) = G(s)W(N,s)R(N,s,P(q)),$$

where $R(N,s,P(q))$ counts the number of ways one can select pairs of edges connected to $L$ nodes, in the correct numbers to match the elements of matrix $s$, $W(N,s)$ counts the number of graphs in the ensemble containing the cycle, and $G(s)$ counts the number of ways of arranging the selected nodes to form a cycle. In principle one can complete this calculation for an arbitrary cycle in a multiplex with an arbitrary number of layers. However, the calculation of $G(s)$ becomes somewhat complicated for more than two layers, for anything but the shortest cycles.

Let us focus now on a two layer random multiplex (duplex), defined by the joint degree distribution $P(q_1, q_2)$. In the case of two layers, $s$ has three entries: $s_{11}$, $s_{22}$, and $s_{12}$. For a given $L$, $s_{11}$, and $s_{22}$ can take any value from 0 to $L$, while the number of switches $s_{12}$ is even, for integer $p$, to ensure that the number of edges of each color is integral, while all three must satisfy $s_{11} + s_{22} + s_{12} = L$. The formula (1) can be calculated explicitly in the asymptotic case characterized by $L \ll N$.

Without switches, $G(s)$ is simply the number of possible orderings of the $L$ nodes, dividing by $2L$ as each direction and starting point in the ordering is equivalent:

$$G(L,0,0) = \frac{L!}{2L}$$

and similarly for $G(0,L,0)$. When $s_{12} > 0$, $G(s)$ is given by the number of ways of ordering the switches $(s_{12} - 1)!$ multiplied by the number of ways $D(s_{11}, p)D(s_{22}, p)$ of placing the nonswitching nodes in the spaces between the switches, where

$$D(s, p) = \sum_{n_1=0}^{s} \binom{s}{n_1} \binom{n_1}{p} \sum_{n_2=0}^{s-n_1} \binom{s-n_1}{n_2} \sum_{n_3=0}^{s-n_1-n_2} \binom{s-n_1-n_2}{n_3} \cdots \sum_{n_p=0}^{s-n_1-n_2-\cdots-n_{p-1}} \binom{s-n_1-n_2-\cdots-n_{p-1}}{n_p-1} [s - \sum_{j=1}^{p-1} n_j]$$

$$= (s + p - 1)! / (p - 1)!.$$  

Thus,

$$G(s_{11},s_{22},s_{12}) = \begin{cases} \frac{(s_{12} - 1)!D(s_{11}, p)D(s_{22}, p)}{2(s_{11} + s_{12})}, & s_{12} > 0 \\ \frac{2(s_{11} + s_{12})!}{2(s_{11} + s_{12})}, & s_{12} = 0. \end{cases}$$

The number of ways to form layer 1 is the number of ways to connect $c_1 N$ stubs in pairs: $(c_1 N - 1) (c_1 N - 2) \cdots 1 = (c_1 N - 1)!!$, while the number of ways to connect the edges not forming part of the loop is $(c_1 N - 2s_{11} - s_{12} - 1)!!$. Hence the fraction of layer 1 configurations containing the loop is the ratio of these two numbers. Repeating for layer 2, we find that $W(N,s)$ can be simply written

$$W(N,s) = \frac{(c_1 N - 2s_{11} - s_{12} - 1)!!(c_2 N - 2s_{22} - s_{12} - 1)!!}{(c_1 N - 1)!!(c_2 N - 1)!!}.$$  

A. Formulas for short cycles

When $L \ll N$, the factor $W(N,s)$ can be approximated as

$$W(N,s) \approx \frac{1}{N^L(q_1)^{s_{11}+p}(q_2)^{s_{22}+p}}.$$  

Furthermore, we can treat the selection of nodes as being done with replacement, meaning that $R(N,s,P(q))$ can be simply
written as a product of terms for each node in the cycle:

\[ R[N,s,P(q)] = \frac{N^L(q_1(q_1-1))^{s_{11}}(q_2(q_2-1))^{s_{22}}(q_1q_2)^{s_{12}}}{s_{11}s_{22}s_{12}!}, \]

where \( \langle \ldots \rangle \) indicates averages with respect to the degree distribution (ensemble averages).

When Eq. (7) is combined with \( G(s) \) and \( W(N,s) \) as given by Eqs. (4) and (6) we find that, when none of \( s_{11},s_{22},s_{12} \) is zero,

\[ N(s) = \left( \frac{s_{11} + p - 1}{s_{11}} \right) \left( \frac{s_{22} + p - 1}{s_{22}} \right) \frac{1}{2p} \left[ \frac{\langle q_1(q_1-1) \rangle}{\langle q_1 \rangle} \right]^{s_{11}} \times \left[ \frac{\langle q_2(q_2-1) \rangle}{\langle q_2 \rangle} \right]^{s_{22}} \left[ \frac{\langle q_1q_2 \rangle}{\sqrt{\langle q_1 \rangle \langle q_2 \rangle}} \right]^{2p}. \]

(8)

In the case \( s_{12} = 0 \), the cycle consists of only one color, so \( N(s) = 0 \) unless either \( s_{22} = 0 \) or \( s_{11} = 0 \), in which case

\[ N(L,0,0) = \frac{1}{2L} \left[ \frac{\langle q_1(q_1-1) \rangle}{\langle q_1 \rangle} \right]^L. \]

(9)

This coincides with the single-layer result found in, for example, Ref. [24]. Similarly, the formula for \( N(0,L,0) \) is found simply by exchanging the subscripts.

On the other hand, when \( s_{11} = 0 \), it is still possible to have \( s_{12} > 0 \), when each segment in layer 1 consists of only a single edge (i.e., each switch between layers is immediately followed by another switch). Then

\[ N(0,s_{22},2p) = \left( \frac{s_{22} + p - 1}{s_{22}} \right) \frac{1}{2p} \left[ \frac{\langle q_2(q_2-1) \rangle}{\langle q_2 \rangle} \right]^{s_{22}} \times \left[ \frac{\langle q_1q_2 \rangle}{\sqrt{\langle q_1 \rangle \langle q_2 \rangle}} \right]^{2p}. \]

(10)

and similarly for the case \( s_{22} = 0 \) but \( s_{12} > 0 \) and \( s_{11} > 0 \), by exchanging the subscripts 1 and 2. If we project the two layers onto a single network, we recover the existing result for a single-colored network, which has the same form as Eq. (9).

These results are valid for \( L \ll N \), such as in the limit \( N \rightarrow \infty \). The expected number of cycles in other cases, for longer cycles (when terms of \( O(1/N) \) can’t be neglected) can be found by a more precise derivation, which we outline in the Appendix.

B. Representative examples

The number of cycles having exactly \( n_1 = s_{11} + \frac{1}{2}s_{12} \) edges of type 1 for a fixed \( L \) can be found by summing Eq. (8) over each \( s_{12} \). In the absence of interlayer degree correlations, the resulting distributions for \( n_1 \) match the binomial distribution found by selecting \( L \) edges at random from the network, as shown in Fig. 2. The mean number of type 1 edges is \( p_1 = \langle q_1 \rangle/\langle (q_1) + (q_2) \rangle \), and similarly for \( n_2 \). In addition, the number of switches \( s_{12} \), which must always be even, can be found by summing over \( s_{11} \) and \( s_{22} \). The mean number of switches \( \langle s_{12} \rangle \) is well predicted by \( 2Lq_1q_2/\langle (q_1) + (q_2) \rangle^2 \), which is the expected number of mismatches when pairing \( L \) randomly chosen edges, and the distribution is also well matched by a binomial distribution.
III. CLUSTERING

The clustering coefficient of a network is related to the number of triangles, that is, cycles of length three. In such short cycles, the computation of factor $G(s)$ is straightforward, thus we can calculate the number of cycles of length 3 for any number of layers. Such a cycle may be entirely within one layer: $N_m^{(1)} = z_m^2/6c_m^2$; have two edges in one layer ($m$) and one edge in a second layer ($n$): $N_{m,n}^{(2)} = z_m(q_mq_n)^2/2c_m^2c_n$; or have one edge each in three different layers: $N_{m,n,r}^{(3)} = (q_mq_nq_r)/c_mc_rc_r$, where $z_m = \langle q_m(\Delta m - 1) \rangle$ and $c_m = \langle q_m \rangle$.

We can then define a global clustering coefficient by

$$ C = \frac{3\sum_{L=3}^N N(s)}{\mathcal{V}(N)} = \frac{3\sum_{L=m} N_m^{(1)} + 3\sum_{m,n\neq m} N_{m,n}^{(2)} + 3\sum_{m,n,r \neq m,r} N_{m,n,r}^{(3)}}{\mathcal{V}(N)\sum_{L=2}(q_mq_nq_r)}, $$

where $\mathcal{V}(N)$ is the number of adjacent edge pairs in the graph, and the summation is over all cycle matrices $s$ having length $L = 3$.

One may also define partial clustering coefficients for triangles entirely within a given layer, two given layers, or three given layers ($C_m^{(3)}$, respectively):

$$ C_m^{(1)} = \frac{3N_m^{(1)}}{2Nz_m} = \frac{z_m^2}{6c_m^2}, $$

$$ C_{m,n}^{(2)} = \frac{3N_{m,n}^{(2)}}{2Nz_m + 2(q_mq_n)} = \frac{3z_m(q_mq_n)^2}{N[z_m + 2(q_mq_n)]c_m^2}, $$

$$ C_{m,n,r}^{(3)} = \frac{3N_{m,n,r}^{(3)}}{2Nz_m + 2(q_mq_n)} = \frac{3z_m(q_mq_nq_r)}{N[z_m + 2(q_mq_n)]c_m^2c_r}. $$

These formulas give the expected clustering coefficients for large random graphs, taking into account the full degree distribution. This gives a more accurate result than found by simply matching the mean degree to an Erdős-Rényi network [10], for which the clustering coefficients can be calculated by considering the probability for a given edge to be present or absent:

$$ C_{m,(ER)}^{(1)} = \frac{c_m}{N}, $$

$$ C_{m,(ER)}^{(2)} = \frac{3c_mc_n}{N(c_m + 2c_n)}, $$

$$ C_{m,n,(ER)}^{(3)} = \frac{6c_mc_rc_r}{N(c_m + c_m + c_n + c_r)}. $$

FIG. 4. (a) Total clustering coefficient, $C$ (circles, green), single layer, $C^{(1)}$ (triangles, blue) and two layer, $C^{(2)}$ (squares, red), clustering coefficients for a two layer multiplex network with assortative inter layer degree correlations, joint degree distribution $P(q_1, q_2) = \rho[(P(q_1) + P(q_2))\delta_{q_1, q_2} + (1 - \rho)P(q_1)P(q_2)]$, where $P(q) \propto q^{-\gamma}$, with $\gamma = 3.7$, is a power-law distribution, and $L = 80$. Maximum disassortative correlation occurs when the two layers no longer overlap, at which point $n_1 = 0$. The corresponding value of $\rho$ depends on $\gamma$. Figures are qualitatively similar for any value of $\gamma > 3$. (b) Total number of cycles as a function of $r$. 

FIG. 3. (a) Mean for $n_1$ (blue, dashed) and $s_{\delta 2}$ (red, heavy solid) and corresponding standard deviations (blue dash-dot, red light, respectively) as a function of Pearson correlation coefficient $r$ for the degrees of a vertex in different layers. Assortative correlations ($r < 0$) are created using a joint degree distribution of the form $P(q_1, q_2) = \rho[(P(q_1) + P(q_2))\delta_{q_1, q_2} + (1 - \rho)P(q_1)P(q_2)]$. Disassortative correlations ($r > 0$) are of the form $P(q_1, q_2) = \rho[(P(q_1)\delta_{q_1, q_2} + P(q_2)\delta_{q_2, q_1}) + (1 - \rho)P(q_1)P(q_2)]$, where $P(q)\propto q^{-\gamma}$, with $\gamma = 3.7$, is a power-law distribution, and $L = 80$. Maximum disassortative correlation occurs when the two layers no longer overlap, at which point $n_1 = 0$. The corresponding value of $r$ depends on $\gamma$. Figures are qualitatively similar for any value of $\gamma > 3$. (b) Total number of cycles as a function of $r$. 

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which coincide with the results found by inserting uncorrelated Poisson degree distributions into Eqs. (11)–(18).

To illustrate the importance of taking correlations into account, we compared our formulas Eqs. (11)–(18) with measurements of synthetic networks. The results are summarized in Fig. 4. This shows that our method successfully accounts for the effects of broad degree distributions, and interlayer correlations. In comparison, the Erdős-Rényi formulas are generally only accurate for uncorrelated Erdős-Rényi layers and fail completely in the presence of strong correlations between layers.

IV. CONCLUSIONS

In single-layer networks, cycles are characterized by their length. In multiplex networks, there are many more possibilities. In particular, there is the possibility to switch between layers, and this must be accounted for. In this paper we have introduced a classification for cycles in multiplex networks based on the number of edges in each layer, and the number of switches between layers. We further calculated the expected number of each type of cycle in a large random multiplex. Our results are valid for any multidegree distribution, including distributions characterized by arbitrary degree correlations between layers. Interestingly, our formulas show that the first- and second-order moments of the multidegree distribution are sufficient to determine the statistics of cycles in large multiplex networks. The effect of correlations between a vertex’s degrees in different layers affects these statistics through the degree-degree moment \((-q_{mn} q_n)\). Assortative correlations tend to increase the number of switches in cycles of a given length, while also increasing the total number of cycles. Disassortative correlations, on the other hand, may greatly reduce both the total number of cycles and the total number of switches within these cycles.

These results further allow us to give the expected clustering coefficients in multiplex networks. The possibility that a closed triangle may have edges in multiple layers requires a more detailed discrimination of clustering coefficients. We give a complete classification of the various possible clustering coefficients and give their expected values. Interlayer correlations again have a strong effect, significantly increasing or decreasing the mixed-layer clustering coefficients. These results give a much more precise view of cycles and clustering in multiplex networks than using the mean degree alone, and establish the proper baseline for comparison with real-world networks.

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APPENDIX: FACTOR R FOR LONGER CYCLES

The derivation given in the main body of this paper is valid for \(L \ll N\). If one wants to consider longer cycles, a more complex derivation is required, which we sketch here.

1. Cycles in only one layer

For orientation, we first outline the derivation for cycles within a single layer, which is identical to that for a single network, and follows the method used in Ref. [24]. Consider selecting an arbitrary set of nodes from the network, of which \(n_q\) have degree \(q\). There are \({N_q \choose n_q}\) ways to select \(n_q\) nodes, where \(N_q = N P(q)\). We sum over all possible such sets and use a \(\delta\) function to count only sets containing exactly \(L\) nodes. There are \(q(q-1)\) ways to select the edges which connect a node of degree \(q\) in the cycle. Together these considerations mean we can write

\[
R[N,L,P(q)] = N^L \sum_{[n_q]} \delta \left( \sum_q n_q - L \right) \prod_q \left( \frac{N_q}{n_q} \right)^{q(q-1)} N^q \frac{e^{ixL}}{N^{q-1}}.
\]

We can write the \(\delta\) function in integral form, and after replacing the sum over sets \([n_q]\) with a sum over \(n_q\) after the product, we have

\[
R[N,L,P(q)] = \frac{N^L}{2\pi} \int dx \prod_q \sum_{n_q=0}^{N_q} \left( \frac{N_q}{n_q} \right)^{q(q-1)} e^{ixL} N^q \left( 1 + \frac{e^{ixL}(q-1)}{N} \right)^{N_q} e^{-ixL}.
\]

Writing the product as a sum within an exponential, and recognizing that as \(N_q = N P(q)\), this gives an average over degree, we have

\[
R[N,L,P(q)] = \frac{N^L}{2\pi} \int dx \exp \left\{ -ixL + N \log \left( 1 + \frac{e^{ixL}(q-1)}{N} \right) \right\}.
\]

To evaluate this integral, we assume the integrand to be strongly peaked around a certain value \(x^*\). Let

\[
\phi(x) = \frac{-ixL}{N} + \log \left( 1 + \frac{e^{ixL}(q-1)}{N} \right).
\]

A local maximum of this function occurs at \(x^*\), which is the solution of

\[
\frac{q(q-1)}{Ne^{-ix^*} + q(q-1)} = \frac{L}{N}.
\]
The second derivative evaluated at \( x^* \) is
\[
\frac{d^2\phi}{dx^2}|_{x=x^*} = -\frac{q(q-1)}{[N e^{-ix^*} + q(q-1)]^2}N^2 e^{-ix^*}.
\] (A6)

These results can be used to replace \( \phi(x) \) with a Taylor expansion about \( x^* \) in Eq. (A3). It has a simple Gaussian form that can easily be evaluated to give
\[
R[N,L,P(q)] = \frac{N^L}{\sqrt{2\pi}} e^{N\phi(x^*)} \times \left\{ \frac{q(q-1)}{[N e^{-ix^*} + q(q-1)]^2} \right\}^{-1/2}. \tag{A7}
\]

To find an explicit expression, we must solve Eq. (A5) and evaluate \( \phi(x^*) \) and hence Eq. (A7).

In the case \( L \ll N \), \( N e^{-ix} \) dominates \( q(q-1) \), so that Eq. (A5) becomes
\[
\langle q(q-1) \rangle = Le^{-ix^*}
\] (A8)
so that
\[
N\phi(x^*) = L + \log \left( \frac{q(q-1)}{L} \right) \tag{A9}
\]

and
\[
\left\{ \frac{q(q-1)}{[N e^{-ix^*} + q(q-1)]^2} \right\}^{-1/2} \approx \frac{1}{\sqrt{L}}
\] (A10)
giving
\[
R[N,L,P(q)] = \frac{N^L}{\sqrt{2\pi L}} e^{L\left( \frac{q(q-1)}{L} \right)} = \frac{N^L}{L!} \langle q(q-1) \rangle^L.
\] (A11)

where when combined with Eqs. (2) and (6) gives exactly Eq. (9).

If instead we expand the left-hand side of Eq. (A5) we find
\[
\frac{s}{N} = \frac{\langle q(q-1) \rangle}{N e^{-ix^*}} - \frac{q^2(q-1)^2}{N^2 e^{-2ix^*}} + \ldots
\] (A12)
and keeping the two leading terms, the solution is
\[
e^{-ix^*} \approx \frac{\langle q(q-1) \rangle}{s} - \frac{q^2(q-1)^2}{N(q(q-1))},
\] (A13)
which can be substituted into Eqs. (A4) and (A7) to find the expected number of cycles when \( L \) is not small compared with \( N \) but \( L \ll N^2 \). The result will include dependence in \( N \) and on higher moments of the degree distribution.

## 2. Cycles in two layers

Now we outline the derivation for general cycles in a two-layer multiplex. We proceed in the same way, but now there are three sets of nodes \( \{n^{(1)}_{q_1,q_2}\} \) for the nodes connecting two edges in layer 1, \( \{n^{(2)}_{q_1,q_2}\} \) for two edges in layer 2, and \( \{n^{(3)}_{q_1,q_2}\} \) for the switches. We sum over all possible such sets, and use three \( \delta \) functions to count only those whose total number of nodes (of all degrees) match \( s_{ij} \). The combinatorial factor for the number of ways to select the three sets is
\[
B_{s_1,s_2,s_3}^N = \frac{N!}{s_1! s_2! s_3!(N - s_1 - s_2 - s_3)!}.
\] (A14)

There are \( q_1(q_1-1) \) ways to select the two edges connecting each of the \( n^{(1)}_{q_1,q_2} \) nodes, and similarly for \( n^{(2)}_{q_1,q_2} \), while for the \( n^{(3)}_{q_1,q_2} \) switches the factor is \( q_1 q_2 \). These considerations give us
\[
R(N,s,P(q)) = \sum_{n^{(1)}_{q_1,q_2}} \delta \left( \sum_{q_1,q_2} n^{(1)}_{q_1,q_2} - s_{11} \right) \sum_{n^{(2)}_{q_1,q_2}} \delta \left( \sum_{q_1,q_2} n^{(2)}_{q_1,q_2} - s_{22} \right) \sum_{n^{(3)}_{q_1,q_2}} \delta \left( \sum_{q_1,q_2} n^{(3)}_{q_1,q_2} - s_{12} \right)
\]
\[
\times \prod_{q_1,q_2} B_{n^{(1)}_{q_1,q_2},n^{(2)}_{q_1,q_2},n^{(3)}_{q_1,q_2}}^{N,q_1,q_2} \left[ \frac{q_1(q_1-1)}{N} \right]^{n^{(1)}_{q_1,q_2}} \left[ \frac{q_2(q_2-1)}{N} \right]^{n^{(2)}_{q_1,q_2}} \left[ \frac{q_1 q_2}{N} \right]^{n^{(3)}_{q_1,q_2}}.
\] (A15)

We can represent the \( \delta \) function as an integral. Replacing the sums over sets \( \{n^{(j)}_{q_1,q_2}\} \) with sums over \( n^{(j)}_{q_1,q_2} \), after the product,
\[
R(N,s,P(q)) = \frac{N^L}{(2\pi)^3} \int dx_1 dx_2 dx_3 \exp \left\{ i x_1 \left( \sum_{q_1,q_2} n^{(1)}_{q_1,q_2} - s_{11} \right) + i x_2 \left( \sum_{q_1,q_2} n^{(2)}_{q_1,q_2} - s_{22} \right) + i x_3 \left( \sum_{q_1,q_2} n^{(3)}_{q_1,q_2} - s_{12} \right) \right\}
\]
\[
\times \prod_{q_1,q_2} B_{n^{(1)}_{q_1,q_2},n^{(2)}_{q_1,q_2},n^{(3)}_{q_1,q_2}}^{N,q_1,q_2} \left[ \frac{q_1(q_1-1)}{N} \right]^{n^{(1)}_{q_1,q_2}} \left[ \frac{q_2(q_2-1)}{N} \right]^{n^{(2)}_{q_1,q_2}} \left[ \frac{q_1 q_2}{N} \right]^{n^{(3)}_{q_1,q_2}}.
\] (A16)

Each sum can be completed successively using the binomial identity. Finally, converting the product into a sum within an exponential, and writing it as an expectation value we find
\[
R(N,s,P(q)) = \frac{N^L}{(2\pi)^3} \int dx_1 dx_2 dx_3 \exp[N\phi(x)],
\] (A17)

where
\[
\phi(x) = \frac{1}{N} \left[ \log \left( \frac{y_1}{N} \right)^{s_{11}} + \log \left( \frac{y_2}{N} \right)^{s_{22}} + \log \left( \frac{y_3}{N} \right)^{s_{12}} \right]
\]
\[
+ \log \left[ \frac{1 + b_1}{y_1} + \frac{b_2}{y_2} + \frac{b_3}{y_3} \right],
\] (A18)
and \( y_j = Ne^{-i\omega_j} \) and \( b_1 = q_1(q_1 - 1) \), \( b_2 = q_2(q_2 - 1) \), and \( b_3 = q_1q_2 \).

As before, we expand \( \phi(x) \) about the local maximum \( x^* \), which is the simultaneous solution of

\[
\begin{align*}
\frac{s_{11}}{N} &= \left( \frac{b_1}{y_1^*[1 + b_1/y_1^* + b_2/y_2^* + b_3/y_3^*]} \right) \\
\frac{s_{22}}{N} &= \left( \frac{b_2}{y_2^*[1 + b_1/y_1^* + b_2/y_2^* + b_3/y_3^*]} \right) \\
\frac{s_{32}}{N} &= \left( \frac{b_3}{y_3^*[1 + b_1/y_1^* + b_2/y_2^* + b_3/y_3^*]} \right)
\end{align*}
\]

(A19)

(A20)

(A21)

The Taylor expansion requires the second-order derivatives:

\[
\phi_{ij} = \left. \frac{\partial^2 \phi}{\partial x_i^* \partial x_j^*} \right|_{x^*} = -\frac{1}{y_j^*} \left( \frac{b_j(1 + b_1/y_1^* + b_2/y_2^* + b_3/y_3^*)}{(1 + b_1/y_1^* + b_2/y_2^* + b_3/y_3^*)^2} \right)
\]

(A22)

\[
\phi_{jk} = \frac{\partial^2 \phi}{\partial x_j \partial x_k} \bigg|_{x^*} = \frac{1}{y_j^* y_k^*} \left( \frac{b_j/b_k}{(1 + b_1/y_1^* + b_2/y_2^* + b_3/y_3^*)^2} \right)
\]

(A23)

where the subscripts \( j, k, l \) each take one of the values 1, 2, 3. Substituting the second-order Taylor expansion of \( \phi(x) \) around \( x^* \) into Eq. (A17) yields a more complex integral than in the single-layer case, but its evaluation is still straightforward,

\[
R[N, s, P(q)] = \frac{N^L e^{N\phi(x^*)}}{(2\pi N)^{L/2}} \left[ \phi_{11}\phi_{22}\phi_{33} - \phi_{11}\phi_{23}^2 - \phi_{22}\phi_{33}^2 - \phi_{33}\phi_{12}^2 + \phi_{12}\phi_{33}^2 \right]^{-1/2}
\]

(A24)

To evaluate \( R[N, s, P(q)] \) for a given distribution, one solves Eqs. (A19)–(A21), evaluates Eqs. (A22) and (A23) and substitutes into Eq. (A24).

For \( L \ll N \) we can neglect \( O(1/N) \), giving \( y_i^* = N(b_i)/s_i \), and similar expressions for \( y_2^* \) and \( y_3^* \). The cross derivatives \( \phi_{12}, \phi_{13}, \) and \( \phi_{23} \) vanish, and we recover Eq. (7). For larger \( L \), dependence on \( N \) will remain, and there will be dependence on higher moments of the degree distribution.

