Weak percolation on multiplex networks

Gareth J. Baxter,1 Sergey N. Dorogovtsev,1,2 José F. F. Mendes,1 and Davide Cellai3

1Department of Physics & I3N, University of Aveiro, Portugal
2A. F. Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia
3MACSI, Department of Mathematics and Statistics, University of Limerick, Ireland

(Received 17 December 2013; published 3 April 2014)

Bootstrap percolation is a simple but nontrivial model. It has applications in many areas of science and has been explored on random networks for several decades. In single-layer (simplex) networks, it has been recently observed that bootstrap percolation, which is defined as an incremental process, can be seen as the opposite of pruning percolation, where nodes are removed according to a connectivity rule. Here we propose models of both bootstrap and pruning percolation for multiplex networks. We collectively refer to these two models with the concept of “weak” percolation, to distinguish them from the somewhat classical concept of ordinary (“strong”) percolation. While the two models coincide in simplex networks, we show that they decouple when considering multiplexes, giving rise to a wealth of critical phenomena. Our bootstrap model constitutes the simplest example of a contagion process on a multiplex network and has potential applications in critical infrastructure recovery and information security. Moreover, we show that our pruning percolation model may provide a way to diagnose missing layers in a multiplex network. Finally, our analytical approach allows us to calculate critical behavior and characterize critical clusters.

DOI: 10.1103/PhysRevE.89.042801 PACS number(s): 64.60.aq, 64.60.ah, 89.75.Hc, 89.20.Ff

I. INTRODUCTION

While network representations of complex systems have proven to be tremendously useful, it is often the case that a single (simplex) network cannot capture the complex interactions between systems or subsystems. Examples include financial [1,2], ecological [3], infrastructure [4], and information systems [5].

Several different types of multilayer networks have been introduced in the past few years (for a review, see, e.g., [6]). In interdependent networks, corresponding nodes on different layers may be linked by special dependency links, meaning the survival of a node in one layer depends on the survival of its partner in another layer [7]. The nature of the interlayer edges may have different properties (dependence, control, etc.) and nodes may not have a corresponding node on every other layer. In multiplex networks, instead, the same nodes exist in every layer and only the types of edges characterize the different layers [6,8,9]. For one-to-one interdependency, multiplex and interdependent networks are equivalent with regard to percolation [10].

These interdependencies between layers can have a profound effect on the behavior of the entire system, behavior which could not be predicted by studying each network in isolation. In particular, in interdependent or multiplex networks, damage to one layer can spread to other layers, leading to a dramatic collapse of the whole system [7,11]. Typically a discontinuous hybrid phase transition is observed [12], in contrast to the continuous transition seen in classical percolation on a simplex network.

Multiplexity ought to have effects on other network processes too. In this paper we introduce an activation model on multiplex networks, inspired by the bootstrap percolation process on a simplex network. This represents activation of vertices on a network, such as in social mobilization, or the repair of infrastructure networks after a disaster [13,14]. We also define its counterpart pruning process.

The ordinary percolation process can be viewed equally as a damage or as an activation process, and the result is the same. However, as we will see, in the case of multiplex (and by extension, interdependent) networks, activation of the network yields a very different phase diagram than a pruning/damage process. We describe a pair of processes, which we call weak bootstrap percolation (WBP) and weak pruning percolation (WPP), which represent activation/repair and deactivation/damage processes, respectively. We refer to such two models as weak to distinguish them from the simple (strong) extensions of classical percolation, where nodes belong to the same cluster if they are connected by homogeneous paths on each layer [12]. Incidentally, for strong percolation, activation and pruning result in the same giant percolating cluster. We also introduce the concept of invulnerable vertices, which are a special category of vertices that are considered to be always active. A small number of these vertices is necessary to stimulate the activation of other vertices in the bootstrapping process. The proportion of seed or invulnerable nodes also affects the nature of the critical transitions observed.

Numerous generalizations of the original interdependent and multiplex models have appeared, examining the effects of reduced coupling strength [15], link overlap [16,17], partial interdependence [18,19], and degree correlations [20,21], among many others. Another theoretical approach involves spectral analysis of interconnected networks [22]. Nevertheless, the focus has remained largely on the effects of damage on such networks. The present work is inspired by the bootstrap percolation process [23,24] on a simplex network, which has recently been shown [25] to be related to the $k$-core process. We present a pair of multiplex models which exhibit complex critical behavior similar to that found in these simplex network models.

In Sec. II, we introduce the models and describe the formalism for a multiplex network with an arbitrary number of layers. In Sec. III, we solve the WPP model on Erdős-Rényi network topologies on two and three layers, while in Sec. IV
we do the same for the WBP model on two Erdős-Rényi layers. In Sec. V, we characterize the critical clusters on both models and show that their sizes diverge at the transition, and in Sec. VI we present our conclusions.

II. THE MODELS

A. Formalism

Before introducing the different models, we review the general notation we are going to use in this paper. We consider a multiplex network with multidegree distribution $P_{\vec{q}}$, where $\vec{q} = (q_1, \ldots, q_M)$. There is growing tendency in the multiplex literature to indicate layers with a superscript and vertices with subscripts; for example $q^m_i$ is the degree of type $m$ edges of vertex $i$ [9], or using Greek letters for the vertices and Latin letters for the layer [8]. In this paper, however, as we will rarely use the vertex indices, we will only use subscript (Latin) indices to avoid confusion with exponents. Once the multidegree distribution is defined, the multiplexes we consider are totally random; therefore they have the treelike property and vanishing edge overlap in the limit of large network size [26].

In the following, we will consider percolation models that can be defined by a pruning or a bootstrap process where vertices are recursively removed or activated if their neighborhood fulfills a certain property $P$. In addition, we allow a fraction $f$ of randomly selected vertices to be considered part of the clusters at the end of the process even if they do not satisfy the property $P$ (invariable or seed vertices). Then, we define the probability $S$ (often called strength in the classical percolation literature) that a randomly chosen vertex is in the resulting largest cluster at the end of the process. We will see that $S$ can be written in terms of two sets of probabilities $\{Z_n\}$ and $\{X_n\}$. $Z_n$ is simply the probability that, following an edge of type $n$, the node encountered is in one of the resulting clusters at the end of the process. $X_n$ is the probability that, following an edge of type $n$, the node encountered is in the infinite resulting cluster at the end of the process. It is often instructive to represent equations for these probabilities in graphical form. We therefore represent these variables by the symbols shown in Fig. 1. Finally, we also allow the multiplex network to be randomly damaged with probability $(1 - p)$. Here damage means that a fraction $(1 - p)$ of the vertices, irrespective of their being among the $f$ invulnerable (seed) nodes or not, are initially removed from the multiplex network together with their edges, of any type. Our aim is to study the behavior of $S$ as a function of $p$ and describe the different critical phenomena that may arise in the different models.

B. Strong percolation (SP)

An extension of percolation to multiplex networks has been already studied [7,12]. In this paper, we refer to it as strong percolation (SP) to distinguish it from the weak percolation models we are going to define. For orientation, we briefly recapitulate this percolation model, and introduce the concept of invulnerable nodes. A straightforward way to represent classical percolation on a multiplex network is as a pruning process. First, we assume that a fraction $(1 - p)$ of nodes is randomly removed. Then, we recursively remove a node if at least one of its degrees $q_m$ is zero. As introduced in the previous section, we consider two kinds of vertices: some are invulnerable with probability $f$, and some are vulnerable with probability $(1 - f)$. Only vulnerable vertices can be pruned according to the rule above; the invulnerable ones can always help in building a percolating cluster as soon as they are connected to it by any edge type. We define a mutually connected component as a cluster where each pair of vertices is joined by a full path of edges of each type. If the largest mutual connected component is infinite, we say that there is strong percolation (SP).

Let $Z_n$ be the probability that, following an edge of type $n$, the node encountered is part of a cluster where vertices can be removed no further. The following equation holds:

$$Z_n = pf + p(1 - f)\sum_{\vec{q}} \frac{q^n P_{\vec{q}}}{\langle q^n \rangle} \prod_{m=1}^{M} \left[ 1 - (1 - Z_m)q^m \right].$$

The first term on the right-hand side is the probability that the encountered node is invulnerable ($f$) and undamaged ($p$). The second term calculates the probability that, if the encountered node is vulnerable, it is connected to an unpruned cluster by at least one edge of each type.

At the end of the pruning process, the remaining clusters can be finite or infinite; thus we define $X_n$ as the probability that, following an edge of type $n$, the node encountered is attached to an infinite mutually connected component by one of the outgoing edges and has at least one adjacent edge of each type (for edges of type $n$, the incoming edge is sufficient). $X_n$ satisfies the following equation:

$$X_n = p \sum_{\vec{q}} \frac{q^n P_{\vec{q}}}{\langle q^n \rangle} [1 - (1 - X_n)^{q^m}] \prod_{m=1}^{M} \left[ 1 - (1 - X_m)^{q^m} \right].$$

The right-hand side calculates the probability that the encountered node is connected to an unpruned infinite cluster by at least one edge of each type. Finally, the probability $S$ that a randomly chosen node is in the infinite percolating cluster is

$$S = p \sum_{\vec{q}} P_{\vec{q}} \prod_{m=1}^{M} \left[ 1 - (1 - X_m)^{q^m} \right].$$
It is easy to notice that when all nodes are vulnerable \((f = 0)\), there are not finite surviving clusters \((X_n = Z_n)\), similarly to what occurs in \(k\)-core percolation [25,27].

Alternatively, one could consider a bootstrapping process. Invulnerable nodes in the pruning process correspond to seed nodes in the bootstrap scheme. Any node which has at least one connection of each type to an active (percolating) cluster is added to the cluster. This process is repeated until no more nodes can be added. A strong percolating cluster requires a full path for each edge type connecting each pair in the cluster. Nodes can be added. A strong percolating cluster requires a full path for each edge type connecting each pair in the cluster. This process is repeated until no more nodes can be added. A strong percolating cluster requires a full path for each edge type connecting each pair in the cluster.

### C. Weak pruning percolation (WPP)

We now define a multiplex percolation process that is entirely local and, as we will see, gives rise to distinct processes when we consider either pruning or bootstrapping.

Let us consider a multiplex network with multidegree distribution \(P_q\). Vertices are randomly assigned, with probability \(f\), the property of being invulnerable, the remaining ones being vulnerable instead. We consider a pruning process where only vulnerable vertices can be pruned. More specifically, we define weak pruning percolation (WPP) as the process in which every vulnerable node in a multiplex network is recursively pruned if at least one of its degrees \(q_m\) is zero.

Let \(Z_n\) be the probability that, following an edge of type \(n\), the node encountered is part of a cluster where vertices can be removed no further. At the end of the pruning process, the remaining clusters can be finite or infinite; thus we define \(X_n\) as the probability that, following an edge of type \(n\), the node encountered is attached to an infinite cluster by one of the outgoing edges and has at least one adjacent edge of each type (for edges of type \(n\), the incoming edge is sufficient). Differently from SP, this definition does not require that the encountered node is connected to an infinite cluster by every edge type, but only that there exist at least one outgoing edge to an infinite cluster at each step. That is why we call this model weak pruning percolation.

In a multiplex with \(M\) types of edges and a degree distribution \(P_q\), we can write equations for the variables \(Z_n\) and \(X_n\) for the generic edge type \(n\). The equation for \(Z_n\) is

\[
Z_n = pf + p(1 - f) \sum_q \frac{q_n P_q}{(q_n)} \left[ \prod_{m=1}^{M} (1 - (1 - Z_m)^{q_m}) - \prod_{m=1}^{M} [(1 - X_m)^{q_m} - (1 - Z_m)^{q_m}] \right].
\]

As \(Z_n\) represents the probability that following an edge of type \(n\) an unpruned cluster is reached, this equation consists of two terms. The first term \((pf)\) accounts for the probability that the encountered node is an undamaged invulnerable node. The second term \([\propto p(1 - f)]\) calculates the corresponding probability of not being pruned for a vulnerable node. It is calculated as the product of probabilities that at least one neighbor by each edge type is in an unpruned cluster. We do not need to consider the same edge type \((n)\) we have picked, as the very existence of that edge implies an unpruned neighbor at the other end of the edge we are considering. Note that Eq. (4) is identical to the equivalent equation for strong percolation, (1). We repeat it here to emphasize that these are distinct models.

The equation for \(X_n\), instead, is

\[
X_n = pf \sum_q \frac{q_n P_q}{(q_n)} \left[ 1 - (1 - X_n)^{q_n} \right] \prod_{m=1}^{M} (1 - X_m)^{q_m} + p(1 - f) \sum_q \frac{q_n P_q}{(q_n)} \left\{ \prod_{m=1}^{M} [1 - (1 - Z_m)^{q_m}] - \prod_{m=1}^{M} [(1 - X_m)^{q_m} - (1 - Z_m)^{q_m}] \right\}.
\]
In addition, we allow a fraction (1

\[ f \]

finite fraction of nodes is activated at the end of the process. We say that weak bootstrap percolation (WBP) occurs if at least one of the neighbors connected by each edge type is active. We now propose an extension of this model to multiplex networks. A node becomes active when it has \( k \) active neighbors, one in each layer of the multiplex. Let us consider that vertices are initially active (seed) with probability \( f \), inactive with probability \( 1 - f \). We define weak bootstrap as the process where every inactive node in a multiplex network is activated if at least one of the neighbors connected by each edge type is active. We say that weak bootstrap percolation (WBP) occurs if a finite fraction of nodes is activated at the end of the process. In addition, we allow a fraction \( (1 - p) \) of the multiplex to be randomly damaged and study how the behavior of the percolating cluster depends on the parameters \( f \) and \( p \). As in the simplex bootstrap scheme, this process is monotonic; i.e., active vertices cannot become inactive.

At the end of the activation process, the active clusters are in general not the same as those that would be found through the pruning process. This is because in WPP nodes are considered active until pruned. This means that, for example, a pair of nodes connected by an edge of one type provide the required support of that type for one another, even if neither has another edge of that type. In WBP, on the other hand, such an isolated dimer can never become activated (Fig. 3). The same holds for many larger configurations as well.

At the end of the activation process, let \( Z_n \) be the probability that, following an edge of type \( n \), the node encountered is part of a cluster of active vertices,

\[
Z_n = pf + p(1 - f) \sum_{q} \frac{q_n P_q}{q_n} \prod_{m=1, m \neq n}^{M} \left[ 1 - (1 - Z_m)^{q_m} \right].
\]

The first term in this equation \((pf)\) accounts for the probability that the encountered node is an undamaged seed node.

![Diagram](image_url)

**FIG. 2.** Diagrams describing the second term on the right-hand side of Eq. (5) in the case of a node with multidegree \((q_1,q_2) = (2,2)\): \( A_1 = [1 - (1 - Z_2)^{q_1}] \) and \( B_1 = (1 - X_1)^{q_1 - 1}[(1 - X_2)^{q_2} - (1 - Z_2)^{q_2}] \). \( A_1 \) calculates the probability that, after having followed a type-1 edge, we get to a node from which at least one edge of type 2 leads to an unpruned cluster (finite or infinite). We can display this as a sum of three terms representing all the relevant possibilities in the case of \( q_2 = 2 \). \( B_1 \) is composed of two factors. The first factor represents the probability that no edge of type 1 leads to an infinite unpruned cluster \((\infty = 0)\). The second factor calculates the probability that all the edges of type 2 either lead to finite unpruned clusters \( (\infty = 1 \cap \infty = 0) \) or no clusters at all \( (\infty = 0) \). This is the meaning of the first line of \( B_1 \). The second line shows that \( B_1 \) can be written in a more compact way as a difference between the probability that none of the outgoing edges of type 2 lead to an infinite cluster and the probability that all the outgoing edges of type 2 do not lead to an unpruned cluster (even finite). This second line explains pictorially the way we write \( B_1 \) in Eq. (5).

**D. Weak bootstrap percolation (WBP)**

In simplex networks, a bootstrap process is generally defined by a simple contagion mechanism where a node becomes active as soon as at least \( k \) of its adjacent nodes is active [23]. When \( k = 1 \) this corresponds to ordinary percolation. Bootstrap percolation occurs when a giant fraction of the system becomes active at the end of the process. We now propose an extension of this model to multiplex networks. A node becomes active when it has \( M \) active neighbors, one in each layer of the multiplex. Let us consider that vertices are initially active (seed) with probability \( f \), inactive with probability \( 1 - f \). We define weak bootstrap as the process where every inactive node in a multiplex network is activated if at least one of the neighbors connected by each edge type is active. We say that weak bootstrap percolation (WBP) occurs if a finite fraction of nodes is activated at the end of the process. In addition, we allow a fraction \( (1 - p) \) of the multiplex to be randomly damaged and study how the behavior of the percolating cluster depends on the parameters \( f \) and \( p \). As in the simplex bootstrap scheme, this process is monotonic; i.e., active vertices cannot become inactive.

At the end of the activation process, the active clusters are in general not the same as those that would be found through the pruning process. This is because in WPP nodes are considered active until pruned. This means that, for example, a pair of nodes connected by an edge of one type provide the required support of that type for one another, even if neither has another edge of that type. In WBP, on the other hand, such an isolated dimer can never become activated (Fig. 3). The same holds for many larger configurations as well.

At the end of the activation process, let \( Z_n \) be the probability that, following an edge of type \( n \), the node encountered is part of a cluster of active vertices,

\[
Z_n = pf + p(1 - f) \sum_{q} \frac{q_n P_q}{q_n} \prod_{m=1, m \neq n}^{M} \left[ 1 - (1 - Z_m)^{q_m} \right].
\]

The first term in this equation \((pf)\) accounts for the probability that the encountered node is an undamaged seed node.
The second term [proportional to \( p(1-f) \)] calculates the corresponding probability of being active for a nonseed node. It is calculated as the product of probabilities that at least one neighbor by each edge type is in an active cluster. In the case where we are considering the same edge type we have picked, a further active node must exist by an edge different from the one we came from.

Such clusters of active vertices can be finite or infinite; thus we define \( X_n \) as the probability that, following an edge of type \( n \), the node encountered is attached to an infinite cluster by one of the outgoing edges and has at least one outgoing edge of each type (even for edges of type \( n \), the incoming edge is not sufficient, unlike the WPP model). This definition does not require that the encountered node be connected to an infinite cluster by each edge type, but only that there exist at least one outgoing edge to an infinite cluster at each step. That is the reason why we call this model weak bootstrap percolation.

An argument similar to the one regarding Eq. (5) yields the following equation for \( X_n \):

\[
X_n = pf \sum_{\tilde{q} \neq n} q_n p_{\tilde{q}} \left\{ (1 - (1 - X_n)^{\tilde{q} - 1}) \prod_{m=1, m \neq n}^M (1 - X_m)^{\tilde{q} - 1} - [1 - (1 - Z_n)^{\tilde{q} - 1} - (1 - X_n)^{\tilde{q} - 1}] \right\}.
\]

The meaning of this equation is schematically explained in Fig. 4. Similarly to Eq. (5), the first term calculates the probability that following a randomly chosen \( n \)-type edge we get to a seed node where at least one of the outgoing neighbors is part of a giant activated cluster. The second term contains the difference of two products. The first product represents the probability of having at least one activated cluster by each edge type. Then, we have to subtract to this quantity the second product, which calculates the probability that, for any edge of each type, all the activated clusters are finite. The second sum in Eq. (8) is schematically exemplified in Fig. 4.

While \( Z_n \) and \( X_n \) are different from their WPP counterparts, the equation for \( S \) is the same as Eq. (6).

### III. WPP ON ERDŐS-RÉNYI NETWORKS

To demonstrate the qualitative behavior of these models, we apply the formalism described above for WPP to uncorrelated Erdős-Rényi networks. In the following section we will repeat these calculations for WBP.

#### A. Two identical Erdős-Rényi networks

We first apply the formalism to uncorrelated Erdős-Rényi networks with identical mean degree \( \mu \). The symmetry of the identical layers implies that \( Z_1 = Z_2 \equiv Z \) and \( X_1 = X_2 \equiv X \):

\[
Z = p[1 - (1-f)e^{-\mu Z}], \quad X = p[1 - e^{-2\mu X} - (1-f)e^{-\mu Z}(1 - e^{-\mu X})].
\]
FIG. 4. Diagrams describing the second term on the right-hand side of Eq. (8) in the case of a node with multidegree \((q_1, q_2) = (2, 2)\): \(A_1 = [1 - (1 - Z_1)^{\mu_1-1}][1 - (1 - Z_2)^{\mu_2}]\) and \(B_1 = [(1 - X_1)^{\mu_1-1} - (1 - Z_1)^{\mu_1-1}][(1 - X_2)^{\mu_2} - (1 - Z_2)^{\mu_2}]\). \(A_1\) calculates the probability that, following an edge of type 1, we get to a node from which at least one edge of type 2 leads to an active cluster (finite or infinite). We can display this as a sum of three terms representing all the relevant possibilities in the case of \(q_2 = 2\). \(B_1\) is composed of two factors, one for each type of edge. In the first line, we show that each factor calculates the probability that all the edges of type 1 or 2 either lead to finite active clusters \((\Box = 1 \cap \infty = 0)\) or no clusters at all \((\Box = 0)\). This can be written in a more compact way (second line) as a difference between the probability that none of the outgoing edges of type 2 lead to an infinite cluster and the probability that all the outgoing edges of type 2 do not lead to an active cluster (even finite). This second line explains pictorially the way we write \(B_1\) in Eq. (8).

B. Two nonidentical Erdős-Rényi networks

When two uncorrelated Erdős-Rényi networks have different mean degrees \(\mu_1\) and \(\mu_2\), equations as (4) become

\[
Z_1 = p[1 - (1 - f)e^{-\mu_2 Z_2}],
\]

(15)

\[
Z_2 = p[1 - (1 - f)e^{-\mu_1 Z_1}],
\]

(16)

while (5) becomes

\[
X_1 = p[1 - e^{-\mu_1 X_1 - \mu_2 X_2} - (1 - f)e^{-\mu_1 Z_1}(1 - e^{-\mu_1 X_1})],
\]

(17)

\[
X_2 = p[1 - e^{-\mu_1 X_1 - \mu_2 X_2} - (1 - f)e^{-\mu_1 Z_1}(1 - e^{-\mu_1 X_2})].
\]

(18)

These equations can be rescaled with the variables \(x_1 = X_1/p\), \(x_2 = X_2/p\), \(z_1 = Z_1/p\), \(z_2 = Z_2/p\), \(v_1 = p\mu_1\), \(v_2 = p\mu_2\) and rewritten as

\[
z_1 = 1 - (1 - f)e^{-v_2 z_2}, \quad z_2 = 1 - (1 - f)e^{-v_1 z_1},
\]

(19)

which can be reduced to a single equation by substituting one into the other. For \(x_1\) and \(x_2\) we have

\[
x_1 = z_1 - e^{-v_1 x_1}(e^{-v_2 x_2} + z_1 - 1),
\]

(20)

\[
x_2 = z_2 - e^{-v_2 x_2}(e^{-v_1 x_1} + z_2 - 1).
\]

(21)

Assuming that there is a continuous transition in \(X_1\), and that the transition point \(f_c\) is the same for both networks, these become, in the limit \(x_1, x_2 \to 0\),

\[
1 = v_2 x_2 + v_1 z_1, \quad 1 = v_1 x_1 + v_2 z_2.
\]

(22)

Eliminating \(x_1, x_2\), we find that at the critical point

\[
v_1 v_2 = (1 - v_1 z_1)(1 - v_2 z_2).
\]

(23)

Simultaneous solution of (20), (21), and (23) allows us to find the line of the transition (Fig. 6). In the limit \(f = 0\), the solution is \(z_1 = z_2 = 0\) and \(v_1 v_2 = 1\). As in the identical mean degree case, in this limit the probability of a node being in the giant WPP component is given by the product of the classical percolation probability in each layer. Therefore, even if one layer does not percolate on its own \((v_1 < 1)\), the other one may provide more edges to support the weak pruning percolating cluster \((v_2 = 1/v_1 > 1)\). In the other limit \(f = 1\), we find \(z_1 = z_2 = 1\) and \(v_1 + v_2 = 1\). Note that this corresponds to the classical percolation threshold, if the multiplex is treated as a single network with mean degree \(\mu_1 + \mu_2\) (no distinction between kinds of edges). Intermediate values can be found by numerical solution.

C. Three identical Erdős-Rényi networks

As there is no discontinuous transition in the 2-layer case, to show that such a transition can indeed occur in the WPP model, we now consider WPP on three Erdős-Rényi networks with identical mean degree \(\mu\). From Eqs. (4) and (5), and observing that from the symmetry of the problem we have

FIG. 5. (Color online) Phase diagram of the WPP model for two uncorrelated Erdős-Rényi networks with identical mean degree \(\mu\).
\[ Z_1 = Z_2 = Z_3 \text{ and } X_1 = X_2 = X_3, \text{ we have} \]
\[ z = 1 - (1 - f)e^{-\nu z}(2 - e^{-\nu z}), \]  
\[ x = 1 - e^{-3\nu x} - (1 - f)e^{-\nu z}(2 - e^{-\nu z} + e^{-\nu(x+z)}) - 2e^{-2\nu x}, \]  
\[ \frac{S}{p} = 1 - e^{-3\nu x} - 3(1 - f)e^{-\nu z}(1 - e^{-\nu z} + e^{-\nu(x+z)} - e^{-2\nu x}), \]  
(24)  
(25)  
where \( z = Z/p, x = X/p, \) and \( \nu = p\mu. \)

As in the 2-layer case, the continuous transition can be found by imposing \( x = 0, \) yielding
\[ z = 1 - (1 - f)e^{-\nu z}(2 - e^{-\nu z}), \]
\[ 1 - 3\nu + 4\nu(1 - f)e^{-\nu z} - \nu(1 - f)e^{-2\nu z} = 0. \]  
(26)  
(27)  
Once again, for \( f = 1 \) the system behaves as a simplex network with mean degree \( 3\mu \) and therefore the transition occurs at \( p\mu = 1/3. \) At \( f = 0, \) instead, the system collapses at any value of \( p\mu. \) As can be seen analytically, the equations can be manipulated in the case of \( f = 0 \) so that \( \nu \) must satisfy the equation
\[ 1 + \frac{1}{\nu} \ln \left( 2 - \sqrt{1 + \frac{1}{\nu}} \right) = \left( 2 - \sqrt{1 + \frac{1}{\nu}} \right)^2 - \left( 2 - \sqrt{1 + \frac{1}{\nu}} \right)^2. \]  
(28)  

which has no solution for \( \nu > 0. \)

Unlike in the 2-layer case, however, here we also have a line of discontinuous transitions. From Eq. (24), we can write
\[ \Phi_{f,\nu}(z) = 1, \text{ where} \]
\[ \Phi_{f,\nu}(z) = \frac{1 - (1 - f)e^{-\nu z}(2 - e^{-\nu z})}{z}. \]  
(29)

This function is not monotonic in \( z, \) and therefore we have a critical point when \( \Phi_{f,\nu}(z) = 1, \Phi_{f,\nu}''(z) = \Phi_{f,\nu}'(z) = 0, \) which yield
\[ f_{CP} = \frac{2\ln 2 - 1}{2\ln 2 + 3} = 0.088 \ldots, \]  
(30)  
\[ \nu_{CP} = \ln 2 + \frac{3}{2} = 2.193 \ldots, \]  
(31)  
\[ z_{CP} = \frac{2\ln 2}{2\ln 2 + 3} = 0.316 \ldots. \]  
(32)

The expansion of \( S \) [Eq. (26)] around the critical point at \( f = f_c \) yields \( S = S_0 \sim (\nu - \nu_c)^\beta \) with \( \beta = 1/3, \) as in the critical points observed in simplex heterogeneous k-core percolation [25,28]. The line of discontinuous transitions can be calculated by imposing the conditions
\[ \Phi_{f,\nu}(z) = 1, \]
\[ \Phi_{f,\nu}'(z) = 0, \]  
(33)  
\[ \Phi_{f,\nu}''(z) < 0, \]

because we are looking for the point where the maximum of the function encounters the line at 1. The line of continuous transitions intersects the line of discontinuous transitions at a triple point that can be found by imposing the conditions
\[ \Phi_{f,\nu}(z_d) = 1, \]
\[ \Phi_{f,\nu}'(z_d) = 0, \]  
(34)  
\[ \Phi_{f,\nu}''(z_c) = 1, \]
\[ \Psi_{f,\nu}(0, z_c) = 1, \]

where \( z_d \) is the value of \( z \) at the discontinuous transition, \( z_c \) the value at the continuous transition, and \( \Psi_{f,\nu} \) is defined, from Eq. (25), as
\[ \Psi_{f,\nu}(x, z) = \frac{1}{x} \left[ 1 - e^{-3\nu x} - (1 - f)e^{-\nu z}(2 - e^{-\nu z}) + e^{-\nu(x+z)} - 2e^{-2\nu x} \right]. \]  
(35)

Those four equations yield the position of the triple point:
\[ f_t = 0.0462253 \ldots, \nu_t = 2.33666 \ldots. \]  
The phase diagram of this model is illustrated in Fig. 7.

Finally, we need to check whether Eq. (25) can have multiple solutions in \( x \) at fixed \( z. \) If that is the case, the model may be characterized by another discontinuous transition, not captured by Eq. (24). If a critical point exists, then it must satisfy the equations \( \Psi_{f,\nu}(x, z) = 1, \frac{\partial \Psi_{f,\nu}}{\partial x} = \frac{\partial^2 \Psi_{f,\nu}}{\partial x^2} = 0, \) that yield
\[ 1 - x - e^{-3\nu x} - (1 - f)e^{-\nu z}(2 - e^{-\nu z} + e^{-\nu(x+z)} - 2e^{-2\nu x}) = 0, \]
\[ 1 - \nu(1 - f)e^{-2\nu z}e^{-\nu z} + 4\nu(1 - f)e^{-\nu z}e^{-2\nu z} - 3\nu e^{-3\nu x} = 0, \]
\[ (1 - f)e^{-2\nu z} - 8(1 - f)e^{-\nu z}e^{-\nu z} + 9e^{-2\nu z} = 0. \]  
(36)
From the third equation, we can work out a relationship between \( x \) and \( z \):

\[
x = z - \frac{1}{\nu} \ln \left[ \frac{1}{9} (1 - f) \left( 4 \pm \sqrt{\frac{7 - 16f}{1 - f}} \right) \right].
\]

(37)

but this is impossible because it implies that \( x > z \) for any value of \( f \). Therefore, Eq. (25) does not have solutions associated with extra discontinuous phase transitions.

IV. WBP ON ERDŐS–RÉNYI NETWORKS

Now let us compare these results with the WBP model.

A. Two identical Erdős–Rényi layers

First we consider two uncorrelated Erdős–Rényi networks with identical mean degree \( \mu \). The symmetry of the identical layers implies that \( Z_1 = Z_2 \) and \( X_1 = X_2 \), so that we can simply define two variables \( Z \equiv Z_1 = Z_2 \) and \( X \equiv X_1 = X_2 \). Then

\[
Z = pf + p(1 - f)(1 - e^{-\mu Z^2})
\]

and

\[
X = p(1 - e^{-2\mu X}) - p(1 - f)2e^{-\mu X}(1 - e^{-\mu X}).
\]

(39)

These equations can be rescaled with the variables \( x = X/p, z = Z/p, \nu = p\mu \) and rewritten as

\[
\Phi_{f,\nu}(z) = \frac{1 - (1 - f)e^{-\nu z}(2 - e^{-\nu z})}{z} = 1,
\]

(40)

\[
\Psi_\nu(x) = \frac{1 - e^{-2\nu x} - x}{2(1 - e^{-\nu x})} - (1 - f)e^{-\nu x}.
\]

(41)

Figure 8 displays the phase diagram of this model. As \( \Psi_\nu(x) \) is always a monotonic decreasing function, the maximum occurs at \( \Psi_\nu(0) = 1 - \frac{1}{\nu} \). Therefore, the following equation defines the line of continuous transitions between a percolating

\( (x > 0) \) and a nonpercolating phase \( (x = 0) \):

\[
(1 - f) = e^{\nu z} \left( 1 - \frac{1}{2\nu} \right).
\]

(42)

However, as the function \( \Phi_{f,\nu}(z) \) is not always monotonic, the phase diagram also contains a line of discontinuous transitions. This line is defined by the conditions

\[
\Phi_{f,\nu}(z) = 1,
\]

\[
\Phi_{f,\nu}''(z) = 0,
\]

(43)

The last condition captures the lower branch of the solutions (which is a minimum of \( \Phi_{f,\nu} \)), as the higher branch is unphysical in a bootstrap model [25]. In fact, the unstable branch corresponds to the stable branch of the WPP model on three layers that we have calculated in Sec. III C. The line of discontinuous transitions ends at a critical point defined by the conditions \( \Phi_{f,\nu}(0) = 1, \Phi_{f,\nu}'(0) = 0, \) and \( \Phi_{f,\nu}''(0) = 0 \).

(44)

In order for this equation to hold, for \( \nu \to \infty \) it must be \( \nu z(\nu) \to 0 \) and \( f(\nu) \to 0 \), which is our statement. The line of discontinuous transitions never intersects the line of continuous transitions. This can be shown by imposing the conditions \( \Phi_{f,\nu}(z) = 1, \Phi_{f,\nu}'(z) = 0, \) and \( \Phi_{f,\nu}''(z) = 0 \), which yield \( \nu z = 0 \); i.e., the intersection cannot occur at finite \( \nu \). Comparing Figs. 5 and 8, it transpires that the most remarkable difference between WPP and WBP in two layers is the occurrence of a discontinuous transition in the latter. At low values of seed fraction \( f \), we observe a decoupling between
a percolating phase driven by the cascade of activations and a smaller percolating phase driven by the percolation of seeds.

B. Two nonidentical Erdős-Rényi networks

We apply the above formalism to two uncorrelated Erdős-Rényi networks with mean degrees $\mu_1$ and $\mu_2$. In this case, it so happens that $Z_1$ and $Z_2$ obey identical equations, so that Eq. (7) becomes

$$Z_1 = Z_2 = pf + p(1 - f)(1 - e^{-\mu_1Z_1})(1 - e^{-\mu_2Z_2}),$$

so we can define $Z \equiv Z_1 = Z_2$, giving

$$Z = pf + p(1 - f)(1 - e^{-\mu_1Z})(1 - e^{-\mu_2Z}).$$

Similarly, we also find that $X_1 = X_2 \equiv X$, with

$$X = pf(1 - e^{-\mu_1X}) + p(1 - f)(1 - e^{-\mu_1X}) - e^{-\mu_1Z}(1 - e^{-\mu_2X}) - e^{-\mu_2Z}(1 - e^{-\mu_1X}).$$

This symmetry, which applies only in the Erdős-Rényi case, simplifies the calculations significantly.

Once again we can define rescaled variables $x \equiv X/p$, $z \equiv Z/p$, and $v_1 = \mu_1/p$, $v_2 = \mu_2/p$, giving

$$\Phi_{f,v_1,v_2}(z) \equiv \frac{1}{z}[f + (1 - f)(1 - e^{-v_1z})(1 - e^{-v_2z})] = 1$$

and

$$x = 1 - e^{-(v_1 + v_2)x} - (1 - f)[e^{-v_1z}(1 - e^{-v_2X}) + e^{-v_2z}(1 - e^{-v_1X})].$$

We look for a continuous appearance of the giant weakly percolating cluster. From Eq. (49) in the limit of small $x$, we find the line of continuous transitions given by

$$1 - f \approx \frac{v_1 + v_2 - 1}{v_1e^{-v_1z} + v_2e^{-v_2z}}$$

where on this line, $z$ solves

$$1 - z = \frac{(v_1 + v_2 - 1)(e^{-v_1z} + e^{-v_2z} - e^{-(v_1 + v_2)z})}{v_1e^{-v_1z} + v_2e^{-v_2z}}.$$  (51)

The function $\Phi_{f,v_1,v_2}(z)$ is not always monotonic, so the phase diagram also contains a line of discontinuous transitions. This line is defined by the conditions

$$\Phi_{f,v_1,v_2}(z) = 1,$$

$$\Phi_{f,v_1,v_2}(z) = 0.$$  (52)

The line ends at the critical point defined by these two conditions in combination with a third condition

$$\Phi'_{f,v_1,v_2}(z) = 0.$$  (53)

The three conditions can be written as

$$1 - z = (1 - f)(e^{-v_1z} + e^{-v_2z} - e^{-(v_1 + v_2)z}),$$

$$1 = (1 - f)[v_1e^{-v_1z} + v_2e^{-v_2z} - (v_1 + v_2)e^{-(v_1 + v_2)z}],$$

$$0 = v_1^2e^{-v_1z} + v_2^2e^{-v_2z} - (v_1 + v_2)^2e^{-(v_1 + v_2)z}.$$  (54)

FIG. 9. (Color online) Phase diagram of the WBP model for two uncorrelated Erdős-Rényi networks with mean degree $\mu_1$ and $\mu_2$. Horizontal axis is $v_2 = p\mu_2$. Each solid curve shows the location of the continuous transition for a particular value of $v_1$, from top to bottom $v_1 = \{1.5,2,1.93,5,10\}$. Dashed curves show the corresponding location of the discontinuous transition (which is always above the continuous transition), with circles marking the critical end point.

The phase diagram is plotted in Fig. 9. The critical final point occurs generally at small values of $f$, with the maximum $f_C$ occurring when $v_1 = v_2 \approx 2.193$ (see figure). When $v_1$ becomes large, numerical solution suggests that $f_C \to 0$ at a finite value of $v_2$. To check this, we consider Eqs. (54)–(56) in the large $v_1$ limit. Suppose that $z$ remained finite in this limit. Then $e^{-(v_1 + v_2)} \to 0$, and Eq. (56) implies that $v_2 \to 0$ which would violate Eq. (55). We conclude that $z \to 0$ such that $e^{-v_1z}$ remains nonzero. Under this assumption, and using that $v_2/v_1 \ll 1$, (56) gives us that $v_1z \to 2$. Substituting back into (55) and using this equation to eliminate $f$ from (54), we find that

$$v_2 \to \frac{1}{1 + e^{-2}} \approx 0.88.$$  (57)

This in turn gives that, indeed, $f_C \to 0$, in the limit $v_1 \to \infty$. This result is confirmed by numerical solution of Eqs. (54)–(56).

This analysis of the behavior of two different Erdős-Rényi networks shows that the critical point found in the identical case is indeed robust when we break the symmetry of the two layers.

V. CRITICAL CLUSTERS

To understand the discontinuous transitions which we observe, we now analyze clusters of critical vertices, through which the avalanches of damage or activation propagate. Diverging avalanche sizes lead to the discontinuous transitions. A critical vertex is a vertex that only just meets the criteria for inclusion in the percolating cluster (in the case of WPP), or only just fails to meet the criteria (in the case of WBP).
In the case of WPP, to examine these avalanches, we define the probability $R_m$ to be the probability that, on following an edge of type $m$, we encounter a vulnerable vertex (probability $1 - f$), which has not been removed due to random damage (probability $p$) and has at least one child edge of each type $n \neq m$ leading to a member of the percolating cluster (probability $Z_n$), and zero of type $m$. That is

$$R_m = p(1 - f) \sum_{\vec{q}} \frac{q_m P_2}{\langle q_m \rangle} (1 - Z_m)^{\delta_m}$$

$$\times \prod_{n = 1, n \neq m}^{M} [1 - (1 - Z_n)^{\delta_n}].$$

This probability, in the case of two layers, is represented graphically in Fig. 11. We can then define a generating function for the size of the critical subtree encountered upon following an edge of type $m$ (and hence resulting activation avalanche should the parent vertex of that edge be activated) in a recursive way by

$$H_m(\vec{u}) = Z_m - R_m + u_m F_m[H_1(\vec{u}), H_2(\vec{u}), \ldots, H_M(\vec{u})].$$

FIG. 10. A representation of a cluster of critical vertices in WPP. Hatching indicates that vertices are members of the WPP percolating cluster. Because critical vertices are in the percolating cluster for WPP, a critical vertex may be linked to the percolating cluster via another critical vertex. That is, external edges of type $Z_i$ are not necessarily required. Furthermore, this means that critical dependencies can be bidirectional: it is possible for avalanches to propagate in either direction along such edges. Note that outgoing critical edges must be of the opposite type to the incoming one. The boxes containing crosses represent the probability $1 - Z_m$. A vertex may be critical with respect to more than one outgoing edges of this vertex are critical edges for other critical vertex. That is, external edges of type $Z_m$ to the node at one end of a vertex is transmitted along arrowed direction along such edges. Note that outgoing critical edges must be of the opposite type to the incoming one. The boxes containing crosses represent the probability $1 - Z_m$. A vertex may be critical with respect to more than one outgoing edges of this vertex are critical edges for other critical vertices, these vertices will also be removed. Chains of such connections therefore delineate the paths of avalanches of spreading damage. An example is shown in Fig. 10. Damage to the node at one end of a vertex is transmitted along arrowed vertices.

To examine these avalanches, we define the probability $R_m$ to be the probability that, on following an edge of type $m$, we encounter a vulnerable vertex (probability $1 - f$), which has not been removed due to random damage (probability $p$) and has at least one child edge of each type $n \neq m$ leading to a member of the percolating cluster (probability $Z_n$), and zero of type $m$. That is

$$R_m = p(1 - f) \sum_{\vec{q}} \frac{q_m P_2}{\langle q_m \rangle} (1 - Z_m)^{\delta_m}$$

$$\times \prod_{n = 1, n \neq m}^{M} [1 - (1 - Z_n)^{\delta_n}].$$

This probability, in the case of two layers, is represented graphically in Fig. 11. We can then define a generating function for the size of the critical subtree encountered upon following an edge of type $m$ (and hence resulting activation avalanche should the parent vertex of that edge be activated) in a recursive way by

$$H_m(\vec{u}) = Z_m - R_m + u_m F_m[H_1(\vec{u}), H_2(\vec{u}), \ldots, H_M(\vec{u})].$$

A. WPP

The functions $F_m(\vec{x})$ are defined to be

$$F_m(\vec{x}) = p(1 - f) \sum_{\vec{q}} \frac{q_m P_2}{\langle q_m \rangle} (1 - Z_m)^{\delta_m - 1}$$

$$\times \prod_{n = 1, n \neq m}^{M} \sum_{l = 1}^{q_n} \left(\frac{q_l}{l}\right)(1 - Z_n)^{\delta_n} x_l.$$

FIG. 11. Representation of the probability $R_1$ that, on following an edge of type 1, we encounter a (nondamaged and nonseed) vertex that has $\geq 1$ child edge of type 2 leading to a member of the percolating cluster and zero of type 1. The barred line represents the probability $1 - Z_1$. The arrow in the edge is to illustrate the direction of propagation of activation.

Notice that $F_m$ has no dependence on $x_m$. This method is very similar to that used in [12]. A factor $u_m$ appears for every critical edge of type $m$ appearing in the subtree. The first terms $Z_m - R_m$ give the probability that zero critical nodes are encountered. The second term, with factor $u_m$, counts the cases where the first node encountered is a critical one. This node may have outgoing edges leading to further critical nodes. These edges are counted by the function $F_m$, and the use of the generating functions $H_n$ as arguments recursively counts the size of the critical subtree reached upon following each of these edges.

The mean size of the avalanche caused by the removal of single vertex is then given by

$$\sum_{m = 1}^{M} \partial_{x_m} H_m(\vec{1}),$$

where $\partial_{z}$ will be used henceforth to signify the partial derivative with respect to variable $z$.

Let us first examine the mean avalanche size in the case of two layers. Taking partial derivatives of Eqs. (59) and (60), and after some rearranging, we arrive at

$$\partial_{x_1} H_1(1, 1) = \frac{R_1}{1 - \partial_{x_1} F_1(Z_1, Z_2) \partial_{x_1} F_2(Z_1, Z_2)},$$

where we have used that $F_1(Z_1, Z_2) = R_1$ and also that $H_1(1, 1) = Z_1$ and $H_2(1, 1) = Z_2$.

Let us define the right-hand side of Eq. (4) to be $\Psi_1(Z_1, Z_2)$. From Eq. (4), and comparing with Eq. (60), the partial derivatives of $\Psi_1(Z_1, Z_2)$ are

$$\frac{\partial \Psi_1}{\partial Z_1} = 0,$$

$$\frac{\partial \Psi_1}{\partial Z_2} = p(1 - f) \sum_{\vec{q}_1, \vec{q}_2} \frac{P_{\vec{q}_1, \vec{q}_2}}{\langle \vec{q}_1 \rangle} q_1 q_2 (1 - Z_2)^{\delta_2 - 1}$$

$$= \frac{\langle q_2 \rangle}{\langle q_1 \rangle} \frac{\partial}{\partial z_2} F_2(Z_1, Z_2).$$

042801-10
where for compactness we have written \( \partial m / \Psi_1 n \) only ever propagate in one direction along a given edge. Also note will in turn be activated. Note that, unlike for WPP, in WBP it is not activation propagate through the cluster following the arrowed edges.

exactly zero neighbors in the WBP cluster of type \( m \). That is, it has inclusion criterion for a single type of edge. That is, it has

\[ R_m = p(1 - f) \sum q_m P_{\vec{q}} (1 - Z_m)^{\delta_m - 1} \]

\[ \times \prod_{n=1 \atop n \neq m}^{M} [1 - (1 - Z_n)^{\delta_m}]. \tag{66} \]

Note that this is identical to (58), but the following argument is different.

Because critical vertices are outside the WBP cluster, the probabilities \( Z_m \) and \( R_m \) are mutually exclusive. This means that, upon following an edge of type \( m \), there are 3 nonoverlapping possibilities: we encounter a percolating vertex (probability \( Z_m \)), we encounter a critical vertex (probability \( R_m \)), or we encounter neither (probability \( 1 - Z_m - R_m \)). We can then define a generating function for the size of the critical subtree encountered upon following an edge of type \( m \) (and hence resulting activation avalanche should the parent vertex of that edge be activated) in a recursive way by

\[ H_m(\vec{u}) = 1 - Z_m - R_m + u_m F_m(1, H_1(\vec{u}), H_2(\vec{u}), \ldots, H_M(\vec{u})). \tag{67} \]

The functions \( F_m(\vec{x}) \) are defined to be

\[ F_m(x, y) = p(1 - f) \sum q_m P_{\vec{q}} x^{\delta_m - 1} \]

\[ \times \prod_{n=1 \atop n \neq m}^{M} \sum_{l=1}^{q_n} \left( \frac{q_m}{q_n} \right) Z_n^l x_n^{l-1}. \tag{68} \]

Note that \( F_m(1-Z_1,1-Z_2,\ldots,1-Z_M) = R_m \) and \( H_m(1) = 1 - Z_m \).

The mean size of the avalanche caused by the activation of a single vertex is again given by

\[ \sum_{m=1}^{M} \partial_n H_m(\vec{1}). \tag{69} \]

Let us consider the case of WBP in a 2-layer multiplex. Taking partial derivatives of (67) and (68) and after some rearranging, we find

\[ \partial_m H_1(1,1) = \frac{R_1 \left[ 1 - \partial_x F_1(1 - Z_1, 1 - Z_2) \right]}{[1 - \partial_x F_1(1 - Z_1, 1 - Z_2)] \left[ 1 - \partial_x F_2(1 - Z_1, 1 - Z_2) \right] - \partial_x F_1(1 - Z_1, 1 - Z_2) \partial_x F_2(1 - Z_1, 1 - Z_2)}, \tag{70} \]

where we have used that \( F_1(1 - Z_1, 1 - Z_2) = R_1 \) and also that \( H_1(1,1) = 1 - Z_1 \) and \( H_2(1,1) = 1 - Z_2 \).

FIG. 12. An example of a critical cluster in WBP. Avalanches of activation propagate through the cluster following the arrowed edges. If an upstream vertex is activated, all downstream critical vertices will in turn be activated. Note that, unlike for WPP, in WBP it is not possible for an edge to be arrowed in both directions. Activation can only ever propagate in one direction along a given edge. Also note that in the WBP case outgoing critical edges must be of the same type as the incoming one.

and similarly for \( \partial \Psi_2 / \partial Z_1 \) and \( \partial \Psi_2 / \partial Z_2 \). Substituting back, we find that

\[ \partial_n H_1(1,1) = \frac{R_1 \left[ 1 - \partial_x \Psi_1 \left[ \partial_x \Psi_2 + \partial_y \Psi_3 \partial_y \Psi_2 \right] - \partial_x \Psi_3 \partial_y \Psi_2 \right]}{1 - \partial_x \Psi_2 \partial_y \Psi_2} \]

\[ = \frac{R_1}{1 - \frac{\partial \psi_1}{\partial Z_1}}, \tag{65} \]

where for compactness we have written \( \partial_m \Psi_n \) for \( \partial \Psi_n / \partial Z_m \). Now, an alternative form for the condition for the location of the discontinuous transition is \( \frac{\partial \psi_1}{\partial Z_1} = 1 \). We see immediately that this implies that the mean avalanche size diverges at the critical point. In other words the avalanches diverge in size as the discontinuous transition approaches, just as the susceptibility does for an ordinary second-order transition.

B. WBP

In the case of WBP, a critical vertex is one that fails the inclusion criterion for a single type of edge. That is, it has exactly zero neighbors in the WBP cluster of type \( m \), for example, and at least one of every other type. Such a vertex is related to avalanches because, if it gains a single connection of type \( m \) to an active node, it will itself join the active WBP cluster. If, in turn, other outgoing edges of this vertex are the critical edges for other critical vertices, these vertices will also become active. Chains of such connections therefore delineate the paths of avalanches of spreading activation. An example of a small critical cluster is shown in Fig. 12.

To examine these avalanches, we define the probability \( R_m \) to be the probability that, on following an edge of type \( m \), we encounter a vertex which is not a seed vertex (probability 1 – \( f \)), has not been removed due to random damage (probability \( p \)), and has at least one child edge of all other types \( n \neq m \) leading to members of the percolating cluster (probability \( Z_n \)), and zero of type \( m \). That is

\[ R_m = p(1 - f) \sum q_m P_{\vec{q}} (1 - Z_m)^{\delta_m - 1} \]

\[ \times \prod_{n=1 \atop n \neq m}^{M} [1 - (1 - Z_n)^{\delta_m}]. \tag{66} \]
Let us define the right-hand side of Eq. (7), in the two-layer case, to be \(\Psi_1(Z_1, Z_2)\). Then
\[
\frac{\partial \Psi_1}{\partial Z_1} = p(1 - f) \sum_{q_1, q_2} \frac{P_{q_1, q_2}}{\langle q_1 \rangle} q_1(1 - Z_1)^{\psi_1 - 2}[1 - (1 - Z_2)^{\psi_1}] = \partial_{x_1} F_1(1 - Z_1, 1 - Z_2)
\]
and
\[
\frac{\partial \Psi_1}{\partial Z_2} = p(1 - f) \sum_{q_1, q_2} \frac{P_{q_1, q_2}}{\langle q_1 \rangle} q_2(1 - Z_2)^{\psi_2 - 2}[1 - (1 - Z_1)^{\psi_2}] = \langle q_2 \rangle \partial_{x_1} F_2(1 - Z_1, 1 - Z_2),
\]
and a similar procedure is followed for \(\Psi_2\). This means that the equation for \(\partial_{x_1} H_1(1, 1)\) can be written
\[
\partial_{x_1} H_1(1, 1) = \frac{R_1[1 - \partial \Psi_2 / \partial Z_2]}{\det[J - I]},
\]
where the Jacobian matrix \(J\) has elements \(J_{ij} = \partial \Psi_i / \partial Z_j\), and \(I\) is the identity matrix. The condition \(\partial \Psi_1 / \partial Z_1 = 1\) for the location of the discontinuity in \(Z_1\) (and \(Z_2\)) can be rewritten
\[
\det[J - I] = 0,
\]
meaning that \(\partial_{x_1} H_1(1, 1)\) diverges, and hence the mean avalanche size diverges precisely at the critical point. A similar analysis can be performed for three (and in principle more) layers.

VI. CONCLUSIONS

In this paper we have introduced weak bootstrap percolation (WBP) and weak pruning percolation (WPP) in multiplex networks. These are natural extensions of percolation on simplex networks, and are somewhat analogous to bootstrap percolation and the \(k\)-core pruning algorithm on simplex networks. We have shown that, unlike the case of a single layer, these two models are distinct and give origin to different critical behaviors. We further introduced the concept of invulnerable nodes in multiplex percolation, and showed their effect on the critical transitions associated with the emergence of a giant percolating cluster. We have explicitly calculated the critical phenomena characterizing multiplex networks made of Erdős-Rényi networks on each layer. The WBP model includes both continuous and discontinuous hybrid transitions for two or more layers, while the WPP model has only continuous transitions in 2 layers, but a discontinuous hybrid transition appears when there are 3 or more layers. The discontinuous transition in the 3-layer WPP disappears at the same critical point as that in 2-layer WBP. A cursory examination of the relevant equations reveals why there should be such a connection between the 2-layer WBP and 3-layer WPP. In \(M\)-layer WPP, the probability \(Z_n\) that an edge of type \(n\) leads to a member of the percolating cluster requires that the node reached have connections of the other \(M - 1\) types to percolating nodes. Compare this with the \((M - 1)\)-layer WBP, where the criterion is that the node reached have \(M - 1\) connections (which in this case is all types of links).

In a broader context, the present 3-layer WPP model and 2-layer WBP model show phase diagrams similar to that seen in the (1,3) heterogeneous \(k\)-core model and (1,2) bootstrap percolation models, respectively [25]. There are no tricritical points in these systems, but it was shown in [29] that a tricritical point does appear in the (2,3) heterogeneous \(k\) core. Noting that a similar tricritical point appears in partial interdependence models [18,19], it seems natural to consider WPP and the partial interdependence model as specific cases of a broader class of mixed-rule multiplex percolation models. Recently another work [30] has appeared, where two very similar models have been investigated in the context of infrastructure management.

The WPP model, besides filling a gap in our understanding of percolation on multilayer networks, also provides an interesting diagnostics to evaluate the presence of a missing layer. The absence of critical points and discontinuous transitions in the 2-layer case, in fact, is remarkable as it qualitatively differs from the 3-layer case, and it might be used to determine the layer structure of multiplex networks with limited information. A similar qualitative behavior has been recently observed in the case of classical percolation with edge overlap [17].

The WBP model constitutes the simplest activation process which may occur on a multiplex. Differently from other more complex models, here we provide a simple analytical method which allows not only exact calculation of the critical behavior in locally treelike networks, but also the calculation of critical exponents and the characterization of critical clusters. Moreover, the WBP model has potential applications in infrastructure recovery and information security. Wireless sensor networks require key distribution schemes able to guarantee overall secure communication even in the case where some sensors are compromised by an external attack (see, e.g., [31] for a review). The first key distribution scheme specifically designed for sensor networks was introduced by Eschenauer and Gligor [32]. This scheme is based on assigning to each sensor a random set of keys taken from a large key pool. Sensors sharing the same key can communicate directly. As more secure developments of this scheme, several protocols have been proposed. One of them prescribes a distribution scheme where each node shares a unique pairwise key with each of \(O(\sqrt{N})\) other nodes in the network [33]. This implies that a sensor \(A\) may need an intermediary sensor \(C\) to establish communication with another sensor \(B\). Let us consider now a network where different edge types correspond to a key shared among two sensors. If we define a sensor to be compromised when all its keys have been captured by an intruder, then we can view this problem as a bootstrap process, where either a cascade of compromised nodes occurs, or the attack remains confined to a few nodes.

In the case of infrastructure, the same interaction between infrastructure layers that can lead to dramatic collapse can lead to dramatic recovery. After a large disruption, the reactivation
of a small number of nodes may allow further nodes which have lost one of their essential dependencies to resume functioning. Once a certain threshold is reached, this may lead to dramatic gains in the functioning of the entire system [13].

Finally, our paper shows that extensions of simple traditional models to multiplex networks generate a wealth of new possibilities, both in terms of model definitions and of new critical behaviors, with implications mostly to be understood.

ACKNOWLEDGMENTS

We acknowledge useful interactions with James Gleeson and Adrian Perrig. This work has been partially funded by Science Foundation Ireland, Grant No. 11/PI/1026; the FET-Proactive project PLEXMATH (FP7-ICT-2011-8; Grant No. 317614) and the FET IP Project MULTIPLEX 317532; and the Portuguese Science and Technology Foundation (FCT) Project PEst-C/CTM/LA0025/2011 and Grant No. SFRH/BPD/74040/2010.