A unified approach to percolation processes on multiplex networks

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Abstract Many real complex systems cannot be represented by a single network, but due to multiple sub-systems and types of interactions, must be represented as a multiplex network. This is a set of nodes which exist in several layers, with each layer having its own kind of edges, represented by different colours. An important fundamental structural feature of networks is their resilience to damage, the percolation transition. Generalisation of these concepts to multiplex networks requires careful def nition of what we mean by connected clusters. We consider two different def nitions. One, a rigorous generalisation of the single-layer def nition leads to a strong non-local rule, and results in a dramatic change in the response of the system to damage. The giant component collapses discontinuously in a hybrid transition characterised by avalanches of diverging mean size. We also consider another def nition, which imposes weaker conditions on percolation and allows local calculation, and also leads to different sized giant components depending on whether we consider an activation or pruning process. This 'weak' process exhibits both continuous and discontinuous transitions.

1 Introduction

Networks are a powerful tool to represent the heterogeneous structure of interactions in the study of complex systems [11]. But in many cases there are multiple kinds of interactions, or multiple interacting sub-systems that cannot be adequately represented by a single network. Examples include f nancial [7, 14], infrastructure [20], informatic [16] and ecological [18] systems.

There are many representations of multi-layer networks, appropriate in different circumstances(see e. g. [15] for a summary). We focus on multiplex networks, which

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are networks with a single set of nodes present in all layers, connected by a different type of edge (which may be represented by different colours) in each layer. See [5] for a recent review of the topic. Some interdependent networks, in which different layers have different sets of nodes as well, but the nodes are connected between layers by interdependency links [6, 13], are able to be captured by this construction [21].

One of the fundamental structural properties of a network is its response to damage, that is, the percolation transition, where the giant connected component collapses. In multi-layer networks, interdependencies between layers can make a system more fragile. Damage to one element can trigger avalanches of failures that spread through the whole system [12, 19]. Typically a discontinuous hybrid phase transition is observed [3], similar to those observed in the network k-core or in bootstrap percolation [2] in contrast to the continuous transition seen in classical percolation on a simplex network.

Under a weaker definition of percolation, a more complex phase diagram emerges, with the possibility for both continuous and discontinuous transitions. When invulnerable or seed nodes are introduced, we can define activation and pruning processes, which have different phase diagrams. The results presented in this Chapter are based on those obtained in [3] and [4].

In a single-layer network (simplex), two nodes are connected if there is at least one path between them along the edges of the network. A group of connected nodes forms a cluster. The giant connected component (GCC) is a cluster which contains a finite fraction of the nodes in the network. The existence of such a giant component is synonymous with percolation. We can study it's appearance by applying random damage to the network. A fraction p of nodes are removed, independently at random, and we check whether the remaining network contains a giant connected component. Typically the GCC appears linearly with a continuous second-order transition, although when the degree distribution is very broad (as in scale-free networks) the nature and location of the transition may be dramatically altered [9].

For multiplex networks, we must generalise these definitions of clusters and percolation. Consider a multiplex network, with nodes $i = 1, 2, ..., N$ connected by m colours of edges labeled $s = a, b, ..., m$. Two nodes i and j are m-connected if for each of the m types of edges, there is a path from i to j following edges only of that type. Let us suppose that the connections are essential to the function of each site, so that a node is only viable if it maintains connections of every type to other viable vertices. A viable cluster is then a cluster of m-connected nodes. This definition is described in Figure 1.

In a large system, we wish to find when there is a giant cluster of viable nodes. From this definition of viable clusters, it follows that any giant viable cluster is a subgraph of the giant connected component of each of the m layers formed by considering only a single colour of edge in the multiplex network. The absence of a giant connected component in any one of the layers means the absence of the giant viable cluster. Note that when $m = 1$, the viable clusters are identical to clusters of connected vertices in ordinary networks with a single type of edges. As we will see, the rigorous requirements for viability in multiplex networks have a profound effect

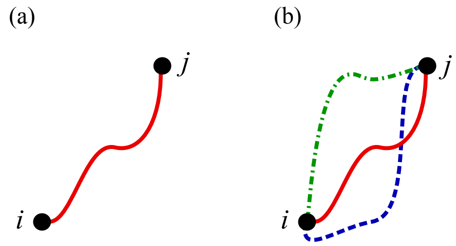

Fig. 1 (a) In an ordinary network, two vertices i and j belong to the same cluster if there is a path connecting them. (b) In a multiplex network, vertices i and j belong to the same viable cluster if there is a path connecting them for every kind of edge, following only edges of that kind. In the example shown, there are $m = 3$ kinds of edges. Vertices i and j are said to be 3-connected.

on the percolation of the network, revealing a discontinuous hybrid transition in the collapse of the giant viable component.

The viable clusters are related to giant mutually connected component in interdependent networks [6, 13]. Consider two networks in which a node in one network may have a mutual dependency on a node in the other network – if one is damaged, the other is automatically damaged. To be part of the giant mutually connected component, a node must be connected to the cluster via links within its own network, and also have any interdependency links intact. This system can be mapped to a multiplex network by simply merging the interdependent nodes into a single node. Nodes without interdependencies then only have (and require) links of a single colour [21]. In this way, the giant viable cluster corresponds to the giant mutually connected component in the case of full interdependency. When the interdependency is only partial, the giant mutually connected component is larger than the giant viable cluster.

If we relax the criterion that a cluster must be connected by all layers, instead requiring only connection via paths of any colour of mixture of colours, we immediately return to ordinary percolation, equivalent to projecting all the layers of the multiplex onto a single layer, that is, ignoring all the colours. If, rather, we were to consider clusters of nodes in which each pair is connected by at least one single coloured path, the resulting giant connected component would be the union of the connected components of the individual layers.

Instead, we may consider a more interesting def nition, which is still weaker than the viable clusters def ned above. To differentiate it from the def nition above, we will call this *w*eak percolation. We continue with the requirement that each node only functions if it is connected to other functioning nodes by every colour of edge. However, it does not need to be connected to every node in the cluster by every kind of edge. Weak percolation can be def ned in the following way: a node i is active if, for each of the m colours, it is connected to at least one active neighbour by an edge of that colour. Weak percolating clusters are then simply connected clusters of active nodes. Examples of the connected clusters resulting from the two different rules can be compared in Fig. 2.

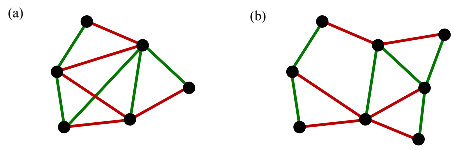

Fig. 2 Examples of small connected clusters in the strong and weak definitions of connectedness in a two-layer multiplex network. (a) In the strong definition of a cluster, every node in a viable cluster can reach every other by every kind of edge. (b) In the weak definition, every node has connections of both colours, but there is not necessarily a path of every colour between every pair of nodes.

We can consider an activation process, in which a small number of nodes are initially activated, and activation may spread to neighbouring nodes. This can represent, for example, social mobilisation or the repair of infrastructure after a disaster [12]. This generalizes activation processes such as bootstrap percolation [1] to multiplex networks. Comparing with the counterpart pruning process, we find that the two processes do not result in the same giant active component [4].

In the following Section, we analyse the strong definition of percolation on multiplex networks, identifying the nature of the percolation transition and the associated avalanches of damage. In Section 3, we analyse the weak definition of percolation and explore the activation and pruning processes, showing that they also exhibit hybrid transitions, and outlining the complex phase diagrams that appear.

2 Multiplex Percolation

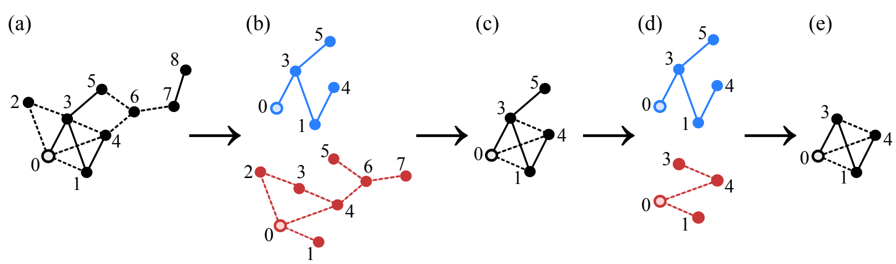

Fig. 3 An example demonstrating the algorithm for identifying a viable cluster in a small network with two kinds of edges. (a) In the original network, in step (i) we select vertex 0 as the test vertex. (b) In step (ii) we identify the clusters of vertices connected to 0 by each kind of edge. (c) Step (iii): the intersection of these two clusters becomes the new candidate set for the viable cluster to which 0 belongs. (d) We repeat steps (ii) using ony vertices from the candidate set shown in (c). Repeating step (iii), we find the overlap between the two clusters from (d), shown in (e). Further repetition of steps (ii) and (iii) does not change this cluster, meaning that the cluster consisting of vertices 0, 1, 3 and 4 is a viable cluster.

The viable clusters in a multiplex network can be identif ed by an iterative pruning process, testing the connectivity in every layer, and removing nodes that fail. Such removals may affect the connectivity of the remaining nodes, so we must repeat the process until an equilibrium is reached. An algorithm for identifying viable clusters is the following:

(i) Choose a test vertex i at random from the network.
(ii) For each kind of edge s, compile a list of vertices that can be reached from i by following only edges of type s.
(iii) The intersection of these m lists forms a new candidate set for the viable cluster containing i.
(iv) Repeat steps (ii) and (iii) but traversing only the current candidate set. When the candidate set no longer changes, it is either a viable cluster, or contains only vertex i.
(v) To f nd further viable clusters, remove the viable cluster of i from the network (cutting any edges) and repeat steps (i)-(iv) on the remaining network beginning from a new test vertex.

Note that this process is non-local: it is not possible to identify whether a node is a member of a viable cluster simply by examining its immediate neighbours. An example of the use of this algorithm in a small network is illustrated in Fig. 3.

We now study in more detail the collapse of the giant viable cluster under damage by random removal of nodes. We use the fraction p of undamaged nodes as a control variable. In uncorrelated random networks the giant viable cluster collapses at a critical undamaged fraction p_c in a discontinuous hybrid transition, similar to that seen in the k-core or bootstrap percolation [10, 1].

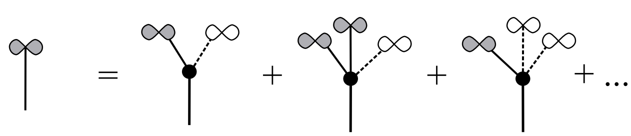

Fig. 4 Diagrammatic representation of Eq. (1) in a system of two interdependent networks a and b. The probability X_a, represented by a shaded inf nity symbol can be written recursively as a sum of second-neighbor probabilities. Open inf nity symbols represent the equivalent probability X_b for network b, which obeys a similar recursive equation. The f lled circle represents the probability p that the vertex remains in the network.

Let us consider sparse uncorrelated networks, which are locally tree-like in the inf nite size limit $N \to \infty$. We take advantage of this locally tree-like property to def ne recursive equations which allow us to f nd the giant viable cluster. We def ne X_s, with the index $s \in \{a, b, \ldots\}$, to be the probability that, on following an arbitrarily chosen edge of type s, we encounter the root of an inf nite sub-tree formed solely from type s edges, whose vertices are also each connected to at least one inf nite subtree of every other type. We call this a type s inf nite subtree. The vector $\{X_a, X_b, \ldots\}$ plays the role of the order parameter. In a two-layer network, for example, the probability X_a can be written as the sum of second-level probabilities in terms of X_a and X_b, as illustrated in Figure 4. In general, writing this graphical representation in equation form, using the joint degree distribution $P(q_a, q_b, ...)$, we arrive at the self consistency equations (for more details, see [3])

$$\begin{aligned} X_s =& p \sum_{q_a, q_b, ...} \frac{q_s}{\langle q_s \rangle} P(q_a, q_b, ...) \left[1-(1-X_s)^{q_s-1}\right] \prod_{l \neq s} \left[1-(1-X_l)^{q_l}\right] \\ \equiv& \Psi_s(X_a, X_b, ...) \,, \end{aligned} \quad (1)$$

where p is the probability that the vertex was not initially damaged. The term $(q_s/\langle q_s \rangle) P(q_a, q_b, ...)$ gives the probability that on following an arbitrary edge of type s, we find a vertex with degrees $q_a, q_b, ...$, while $[1-(1-X_a)^{q_a}]$ is the probability that this vertex has at least one edge of type $a \neq s$ leading to the root of an infinite sub-tree of type a edges. This becomes $[1-(1-X_s)^{q_s-1}]$ when $a = s$.

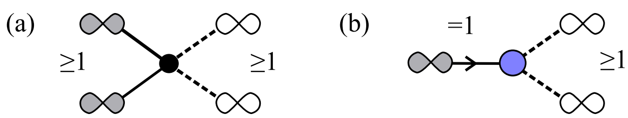

Fig. 5 Viable and critical viable vertices for two interdependent networks. (a) A vertex is in the giant viable cluster if it has connections of both kinds to giant viable subtrees, represented by infinity symbols, which occur with probabilities X_a (shaded) or X_b (open) – see text. (b) A critical viable vertex of type a has exactly 1 connection to a giant sub-tree of type a.

A vertex is then in the giant viable cluster if it has at least one edge of every type s leading to an infinite type s sub-tree (probability X_s), as shown in Fig. 5(a)

$$S = p \sum_{q_a, q_b, ...} P(q_a, q_b, ...) \prod_{s=a,b,...} \left[1-(1-X_s)^{q_s}\right], \quad (2)$$

which is equal to the relative size of the giant viable cluster of the damaged network.

A hybrid transition appears at the point where $\Psi_s(X_a, X_b, ...)$ first meets X_s at a non-zero value, for all s. This occurs when

$$\det[\mathbf{J} - \mathbf{I}] = 0 \quad (3)$$

where $\mathbf{I}$ is the unit matrix and $\mathbf{J}$ is the Jacobian matrix $J_{ab} = \partial \Psi_b / \partial X_a$. The critical point p_c can then be found by simultaneously solving Eqs. (1) and (3). To find the scaling near the critical point, we expand Eq. (1) about the critical value $X_s^{(c)}$. We find that

$$X_s - X_s^{(c)} \propto (p - p_c)^{1/2}. \quad (4)$$

This square-root scaling is the typical behaviour of the order parameter near a hybrid transition. It results from avalanches of spreading damage which diverge in size near the transition. The scaling of the size of the giant viable cluster, S, immediately follows

$$S - S_c \propto (p - p_c)^{1/2}. \quad (5)$$

2.1 Avalanches

We now examine the avalanches of damage which occur in the system, in order to understand the nature of the transition more completely. We focus on the case of two types of edges. Consider a viable node that has exactly one edge of type a leading to a type a inf nite subtree, and at least one edge of type b leading to a type b inf nite subtree. We call this a critical node of type a. It is illustrated in Fig. 5(b). It is a critical vertex because it will be removed from the viable cluster if it loses its single link to a type a inf nite subtree. The removal of any node from the giant viable cluster, and the edges to which it is connected, therefore also requires the removal of any critical vertices which depend on the removed edges. Removed critical nodes may have edges leading to further critical nodes. This is the way that damage propagates in the system. The removal of a single node can result in an avalanche of removals of critical vertices from the giant viable cluster.

To represent this process visually, we draw a diagram of viable nodes and the edges between them. We mark the special critical edges, that critical viable nodes depend on, with an arrow leading to the critical node. An avalanche can only transmit in the direction of the arrows. For example, in Fig. 6, removal of the vertex labeled 1 removes the essential edge of the critical vertex 2 which thus becomes non-viable. Removal of vertex 2 causes the removal of further critical vertices 3 and 4, and the removal of 4 then requires the removal of 5. Thus critical vertices form critical clusters. Graphically, upon removal of a vertex, we remove all vertices found by following the arrowed edges, which constitutes an avalanche. Note that an avalanche is a branching process. Removing a vertex may lead to avalanches along several edges emanating from the vertex (for example, in Fig. 6, removing vertex 2 leads to avalanches along two edges). As we approach the critical point from above, the avalanches increase in size. The mean size of avalanches triggered by a randomly removed vertex f nally diverges in size at the critical point, which is the cause of the discontinuity in the size of the giant viable cluster, which collapses to zero. These avalanches are thus an inherent part of a hybrid transition. To show this, we use a generating function approach [17] to calculate the sizes and structure of avalanches.

There are three possibilities when following an arbitrarily chosen edge of a given type: i) with probability X_s we encounter a type s inf nite subtree ii) with probability R_s we encounter a vertex which has a connection to an inf nite subtree of the opposite type, but none of the same type. Such a vertex is part of the giant viable cluster if the parent vertex was; or iii) with probability $1 - X_s - R_s$, we encounter a vertex which has no connections to inf nite subtrees of either kind. These probabilities are represented graphically in Fig. 7. We will use these symbols in subsequent diagrams.

The probability R_a obeys

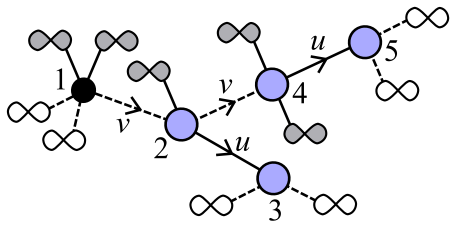

Fig. 6 A critical cluster. Removal of any of the shown viable vertices will result in the removal of all downstream critical viable vertices. Vertices 2-5 are critical vertices. Removal of the vertex labeled 1 will result in all of the shown vertices being removed (becoming non-viable). Removal of vertex 2 results in the removal of vertices 3, 4, and 5 as well, while removal of vertex 4 results only in vertex 5 also being removed. As before, inf nity symbols represent connections to inf nite viable subtrees. Other connections to non-viable vertices or f nite viable clusters are not shown.

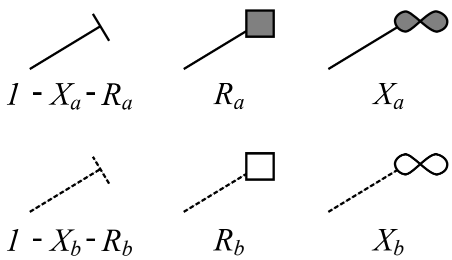

Fig. 7 Symbols used in the diagrams to represent key probabilities. Solid lines represent edges of type *a*, dashed lines represent edges of type *b*.

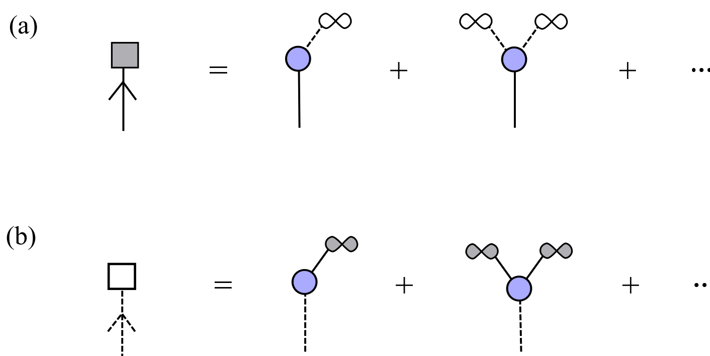

Fig. 8 (a) The probability R_a can be def ned in terms of the second-level connections of the vertex found upon following an edge of type *a*. Note that possible connections to 'dead ends' – vertices not in the viable cluster (probability $1-X_a-R_a$ or $1-X_b-R_b$) are not shown. (b) The equivalent graphical equation for the probability R_b.

$$R_a = \sum_{q_a}\sum_{q_b}\frac{q_a}{\langle q_a\rangle}P(q_a,q_b)(1-X_a)^{q_a-1}\left[1-(1-X_b)^{q_b}\right] \qquad (6)$$

and similarly for R_b. This equation is represented graphically in Fig. 8.

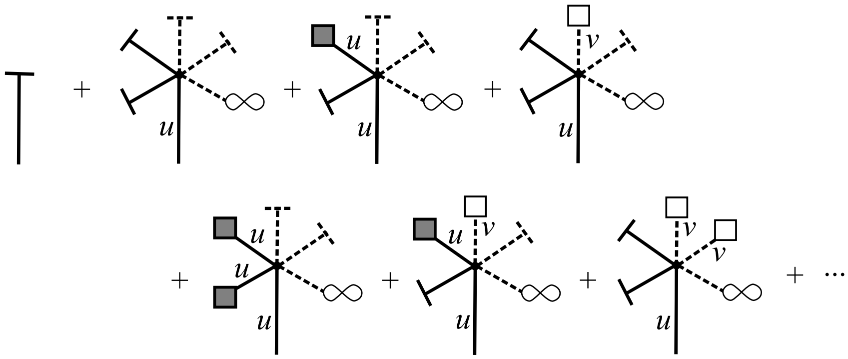

Fig. 9 Representation of the generating function $H_a(x,y)$ (right-hand side of Eq. 7) for the size of a critical cluster encountered upon following an edge of type a.

The generating function for the size of the avalanche triggered by removing an arbitrary type a edge which does not lead to an inf nite type a subtree can be found by considering the terms represented in Figure 9. The f rst term represents the probability that, upon following an edge of type a (solid lines) we reach a node with no connection to a type b subtree (and hence is not viable),that is, a critical cluster of size 0. The second term represents the probability to encounter a critical cluster of size 1. The node encountered has a connection to the type b inf nite subtree (inf nity symbol), but no further connections to viable nodes. Subsequent terms represent recursive probabilities that the vertex encountered has 1 (third and fourth terms), 2 (f fth, sixth, seventh terms) or more connections to further potential critical clusters. The variables u (for type a edges) and v (type b) are assigned to each such edges.

Considering these terms, we can write the generating function for the number of critical vertices encountered upon following an arbitrary edge of type a (that is, the size of the resulting avalanche if this edge is removed) as

$$H_a(u,v) = 1 - X_a - R_a + uF_a[H_a(u,v), H_b(u,v)] \tag{7}$$

and similarly for $H_b(u,v)$, the corresponding generating function for the size of the avalanche caused by removing a type b edge is

$$H_b(u,v) = 1 - X_b - R_b + vF_b[H_a(u,v), H_b(u,v)], \tag{8}$$

where the functions $F_a(x,y)$ and $F_b(x,y)$ are def ned as:

$$F_a(x,y) \equiv \sum_{q_a} \sum_{q_b} \frac{q_a}{\langle q_a \rangle} P(q_a, q_b) x^{q_a - 1} \sum_{r=1}^{q_b} \binom{q_b}{r} X_b^r y^{q_b - r} \tag{9}$$

and similarly for $F_b(x,y)$, by exchanging all subscripts a and b. While the function $F_a(x,y)$ does not necessarily represent a physical quantity or probability, we can see that it incorporates the probability of encountering a vertex with at least one child edge of type b leading to a giant viable subtree (probability X_b) upon following an edge of type a. All other outgoing edges then contribute a factor x (for type a edges) or y (type b). Here u and v are auxiliary variables. Following through a critical cluster, a factor u appears for each arrowed edge of type a, and v for each arrowed edge of type b. For example, the critical cluster illustrated in Fig. 6 contributes a factor u^2v^2.

The mean number of critical vertices reached upon following an edge of type a, i.e. the mean size of the resulting avalanche if this edge is removed, is given by $\partial_u H_a(1,1) + \partial_v H_a(1,1)$, where ∂_u signifies the partial derivative with respect to u. Unbounded avalanches emerge at the point where $\partial_u H_a(1,1)$ [or $\partial_v H_b(1,1)$] diverges. Taking derivatives of Eq. (7),

$$\partial_u H_a(u,v) = F_a[H_a,H_b] + u\left\{\partial_u H_a \partial_x F_a[H_a,H_b] + \partial_u H_b \partial_y F_a[H_a,H_b]\right\} \tag{10}$$

$$\partial_v H_a(u,v) = u\left\{\partial_v H_a \partial_x F_a[H_a,H_b] + \partial_v H_b \partial_y F_a[H_a,H_b]\right\} \tag{11}$$

with similar equations for $\partial_u H_b(u,v)$ and $\partial_v H_b(u,v)$. Some rearranging gives

$$\partial_u H_a(1,1) = \frac{R_a + \partial_u H_b(1,1)\partial_y F_a(1-X_a,1-X_b)}{1-\partial_x F_a(1-X_a,1-X_b)} \tag{12}$$

and

$$\partial_v H_a(1,1) = \frac{\partial_u H_a(1,1)\partial_x F_b(1-X_a,1-X_b)}{1-\partial_y F_b(1-X_a,1-X_b)} \tag{13}$$

where we have used that $H_a(1,1) = 1-X_a$ and $F_a(1-X_a,1-X_b) = R_a$. From Eqs. (1) and (9),

$$\partial_x F_a(1-X_a,1-X_b) = \frac{\partial}{\partial X_a}\Psi_a(X_a,X_b) \tag{14}$$

$$\partial_y F_b(1-X_a,1-X_b) = \frac{\langle q_a\rangle}{\langle q_b\rangle}\frac{\partial}{\partial X_a}\Psi_b(X_a,X_b), \tag{15}$$

and similarly for $\partial_x F_b$ and $\partial_y F_b$, which when substituted into (12) and (13) give

$$\partial_u H_a(1,1) = \frac{R_a[1-\frac{\partial}{\partial X_b}\Psi_b(X_a,X_b)]}{\det[\mathbf{J}-\mathbf{I}]}. \tag{16}$$

We see that the denominator exactly matches the left-hand side of Eq. (3), meaning that the mean size of avalanches triggered by random removal of vertices diverges exactly at the point of the hybrid transition.

3 Weak Multiplex Percolation

Now we consider, for comparison, the weaker def nition of percolation on multiplex networks. In this case we also f nd a discontinuous hybrid transition, but a continuous second order transition may also occur.

In ordinary percolation, and the strong multiplex percolation considered above, activation and deactivation yield the same giant cluster. In weak percolation, however, activation of the network yields a very different phase diagram than a pruning process. We def ne an activation process, which we call Weak Bootstrap Percolation (WBP) and a deactivation/pruning process, Weak Pruning Percolation (WPP). We also introduce invulnerable vertices, which are always active. These are necessary to seed the activation process, and we include them in the pruning process, for symmetry.

3.1 Weak Pruning Percolation (WPP)

Let us begin with Weak Pruning Percolation. A fraction f of the nodes are randomly assigned as invulnerable, the rest being vulnerable. In the WPP process, the network is then damaged, with a fraction p of all nodes remaining undamaged. Once again, p acts as a control parameter. Each of the remaining vulnerable nodes is pruned if it fails to have at least one connection in each layer to a surviving node (vulnerable or invulnerable). The removal of some nodes may affect the neighbourhoods of other surviving nodes, so the pruning process must be repeated until no more nodes can be removed. Invulnerable nodes cannot be pruned.

Let Z_s be the probability that, upon following an edge of type s, we encounter the root of a sub-tree (whether f nite or inf nite) formed solely from type s edges, whose vertices are also each connected to at least one such subtree of every other type. We def ne X_s as the probability that such a subtree is inf nite. Precisely, X_s is the probability that each member the subtree encountered, as well as meeting the criteria for Z_s, also has at least one edge leading to an inf nite subtree of any type (probability X_a etc.).

In a multiplex with m types of edges and a degree distribution $P(q_a, q_b, \dots)$, the equation for Z_s is (see [4] for more details):

$$Z_s = pf + p(1-f) \sum_{q_a, q_b, \dots} \frac{q_s P(q_a, q_b, \dots)}{\langle q_s \rangle} \prod_{n \neq s} [1 - (1 - Z_n)^{q_n}] \equiv \Phi_s(Z_a, Z_b, \dots). \quad (17)$$

The f rst term (pf) accounts for the probability that the encountered node is an undamaged invulnerable node, which is always active, and so its state doesn't depend on any of its neighbours. The second term (proportional to $p(1-f)$) calculates the recursive probability for vulnerable undamaged nodes.

The equation for X_s is

$$\begin{aligned}X_s =& pf \sum_{q_a,q_b,\ldots} \frac{q_s P(q_a,q_b,\ldots)}{\langle q_s\rangle}\left[1-(1-X_s)^{q_s-1}\prod_{n\neq s}(1-X_n)^{q_n}\right] \\ &+ p(1-f)\sum_{q_a,q_b,\ldots} \frac{q_s P(q_a,q_b,\ldots)}{\langle q_s\rangle}\left\{\prod_{n\neq s}[1-(1-Z_n)^{q_n}]-(1-X_s)^{q_s-1}\right. \\ &\left.\times \prod_{n\neq s}[(1-X_n)^{q_n}-(1-Z_n)^{q_n}]\right\} \\ \equiv& \Psi_s(X_a,X_b,\ldots,Z_a,Z_b,\ldots) \qquad . \end{aligned} \tag{18}$$

The first sum on the right hand side calculates the probability that we encounter an undamaged invulnerable node, and which has at least one child edge leading to an infinite subtree of any type. The second sum calculates the same probability but in the case when the encountered node is not invulnerable. This term is written as a difference between the probability of having at least one edge leading to finite or infinite subtrees of each type and another term which removes the possibility that all of the subtrees are finite. This last product must be multiplied by $(1-X_s)^{q_s-1}$ to exclude the possibility of reaching an infinite subtree by a type s edge.

Finally, having given equations for Z_s and X_s, we can use them to find S, the probability that a randomly chosen node is in the giant percolating cluster defined in this model. This is the strength of the giant percolating cluster. It is given by the following formula:

$$\begin{aligned}S =& pf \sum_{q_a,q_b,\ldots} P(q_a,q_b,\ldots)\left[1-\prod_s(1-X_s)^{q_s}\right] \\ &+ p(1-f)\sum_{q_a,q_b,\ldots} P(q_a,q_b,\ldots)\prod_s[1-(1-Z_s)^{q_s}]-\prod_s[(1-X_s)^{q_s}-(1-Z_s)^{q_s}] .\end{aligned} \tag{19}$$

This equation is constructed in a similar way to that for X_s.

A continuous transition appears at the point where a non-zero solution to $X_s = \Psi_s$ first appears. A hybrid transition appears at the point where Ψ_s is first tangent to X_s at a non-zero value, for all s. Because a jump in X_s is always accompanied by a jump in Z_s, it is more simple to look for the point where Φ_s is tangent to Z_s. This occurs when

$$\det[\mathbf{J}-\mathbf{I}] = 0 \tag{20}$$

where $\mathbf{I}$ is the unit matrix and $\mathbf{J}$ is the Jacobian matrix $J_{ab} = \partial\Phi_b/\partial X_a$. Together these criteria allow us to map the phase diagram of the process with respect to the two parameters f and p.

In the case of only two layers, the phase diagram is characterized by a line of continuous phase transitions. An example is shown in Fig. 10, for the case where each of the two layers is an Erdős-Rényi network, with identical mean degree μ. In the limit $f = 0$, the probability of a node being in the giant WPP component is given by the product of the classical percolation probability in each layer. In

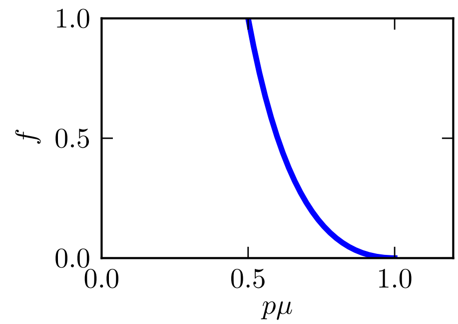

Fig. 10 Phase diagram of the WPP model for two uncorrelated Erdős-Rényi networks with identical mean degree μ.

the Erdős-Rényi example shown in the f gure, this means the percolation point is at $\nu \equiv p\mu = 1$. In the limit $f = 1$, all nodes are invulnerable, and the situation corresponds to classical percolation with the multiplex is treated as a single network. There is no hybrid transition in the two layer case.

In the case of three layers, now a hybrid transition also appears. The line of discontinuous transitions can be calculated by solving Eqs. (17) and (20) together. An example phase diagram is given in Fig. 11. We see that the both continuous and discontinuous transitions are present, with the giant component appearing discontinuously for small f, and having two transitions for slightly larger f: a continuous appearance followed by a discontinuous hybrid transition.

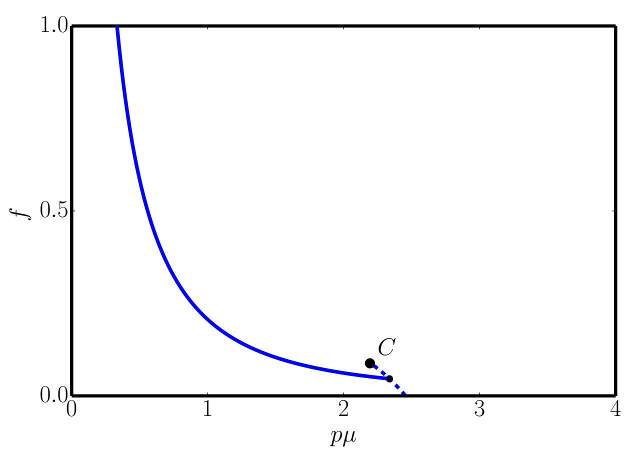

Fig. 11 Phase diagram of the WPP model for three uncorrelated Erdős-Rényi networks with identical mean degree μ. The solid line gives the location of the continuous transition, the dashed line gives the location of the discontinuous transition. The point C is the critical point.

3.2 Weak Bootstrap Percolation (WBP)

Now we consider an activation process called Weak Bootstrap Percolation, which extends the concept of bootstrap percolation [8] to multiplex networks. As for the pruning model, a fraction f of nodes are invulnerable, and are active from the start. Again, a random damage is applied to the network, with the undamaged fraction p acting as the control parameter. Now, however, the vulnerable nodes begin in an inactive state. A node becomes active if it has at least one active neighbour in each of the m layers of the multiplex. The activation of nodes may in turn provide the required active neighbours to more nodes, so the process is repeated until no more nodes can become active.

At the end of the activation process, the active clusters are in general not the same as those that would be found through the pruning process. This is because in WPP nodes are considered active until pruned. This means that, for example, a pair of nodes connected by an edge of one type, provide the required support of that type for one another, even if neither has another edge of that type. In WBP, on the other hand, such an isolated dimer can never become activated (Fig. 12). The same holds for many larger conf gurations as well.

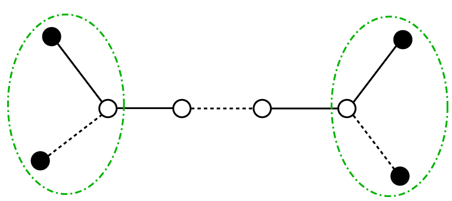

Fig. 12 Example of clusters in a multiplex with two types of edges. Black nodes are invulnerable/seed vertices, white nodes are vulnerable vertices. In WPP, all the nodes are unprunable (remain active), because each white node is connected to another node by each edge type. In WBP, only the nodes inside the green dot-dashed lines become active, while the remaining two nodes have only one active neighbor, by one edge type only, so they cannot become active.

Let Z_s be the probability that, upon following an edge of type s, we encounter the root of a sub-tree (whether f nite or inf nite) formed solely from type s edges, whose vertices are also each connected to at least one such subtree of every type. This obeys the equation

$$\begin{aligned} Z_s &= pf + p(1-f)\sum_{q_a,q_b,\dots}\frac{q_s P(q_a,q_b,\dots)}{\langle q_s\rangle}\left[1-(1-Z_s)^{q_s-1}\right]\prod_{n\neq s}\left[1-(1-Z_n)^{q_n}\right] \\ &\equiv \Phi_s(Z_a,Z_b,\dots). \end{aligned} \tag{21}$$

This differs from the equivalent equation for WPP, (17) because now each node must have connections of every type, not just of the types different from s.

Similarly, we def ne X_s as the probability that such a subtree is inf nite. Precisely, X_s is the probability that each member the subtree encountered, as well as meeting the criteria for Z_s, also has at least one edge leading to an inf nite subtree of any type.

An argument similar to the one for Eq. (18) leads us to the equation:

$$X_s = pf \sum_{q_a,q_b,\dots} \frac{q_s P(q_a,q_b,\dots)}{\langle q_s \rangle} \left[1-(1-X_s)^{q_s-1}\prod_{n\neq s}(1-X_n)^{q_n}\right]$$
$$+\,p(1-f)\sum_{q_a,q_b,\dots} \frac{q_s P(q_a,q_b,\dots)}{\langle q_s \rangle} \left\{[1-(1-Z_s)^{q_s-1}]\prod_{n\neq s}[1-(1-Z_n)^{q_n}]\right.$$
$$\left. -\,[(1-X_s)^{q_s-1}-(1-Z_s)^{q_s-1}]\prod_{n\neq s}[(1-X_n)^{q_n}-(1-Z_n)^{q_n}]\right\}. \quad (22)$$

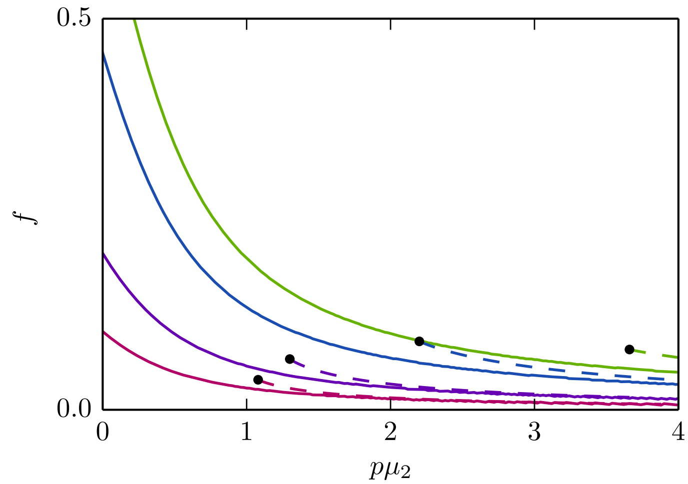

Fig. 13 Phase diagram of the WBP model for two uncorrelated Erdős-Rényi networks with mean degree μ_1 and μ_2. Horizontal axis is $\nu_2 = p\mu_2$. Each solid curve shows the location of the continuous transition for a particular value of ν_1, from top to bottom $\nu_1 = \{1.5, 2.193, 5, 10\}$. Dashed curves show the corresponding location of the discontinuous transition (which is always above the continuous transition), with circles marking the critical end point. Color online.

While Z_n and X_n are different from their WPP counterparts, the equation for S is the same as Eq. (19). In the case of WBP, a hybrid transition appears already in a two layer multiplex. A typical phase diagram is plotted in Figure 13, for the case of two Erdős-Rényi layers with different mean degrees. Now we see that the giant component always f rst appears continuously, with a second, discontinuous hybrid transition occurring afterwards, for small f. The line of discontinuous transitions is found using the conditions

$$\begin{cases} \Phi_{f,v_1,v_2}(z) = 1 \\ \Phi'_{f,v_1,v_2}(z) = 0 \end{cases} \tag{23}$$

The line ends at the critical point defined by these two conditions in combination with a third condition

$$\Phi''_{f,v_1,v_2}(z) = 0. \tag{24}$$

3.3 Avalanches

To understand the discontinuous transitions which we observe in the two weak percolation models, we again analyze avalanches, which propagate through clusters of critical vertices. Diverging avalanche sizes lead to the discontinuous transitions. As before, in the pruning process, WPP, a critical vertex is a vertex that only just meets the criteria for inclusion in the percolating cluster (in the case of WPP). However, in the activation process, WBP, the avalanches which diverge in mean size at the discontinuous transition are of activations of nodes, not of pruning, so critical nodes are those that just fail to meet the criteria for activation.

In the case of WPP, a critical node of type s has exactly one connection to an infinite subtree of type s, and at least one of all the other types. A vertex may be critical with respect to more than one type, if it simultaneously has exactly one connection to infinite subtrees of different types. Such a vertex is related to avalanches because it has one (or possibly more) edge(s) which, if lost, will cause the vertex to be pruned from the cluster. If, in turn, other outgoing edges of this vertex are critical edges for other critical vertices, these vertices will also be removed. Chains of such connections therefore delineate the paths of avalanches of spreading damage. An example is shown in Fig. 14. Damage to the node at one end of an edge is transmitted along arrowed edges.

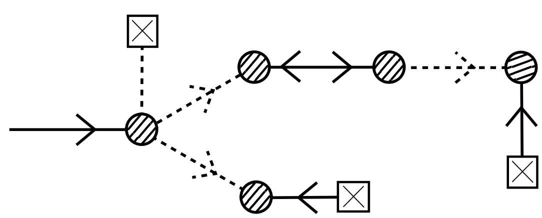

Fig. 14 A representation of a cluster of critical vertices in WPP. Hatching indicates that vertices are members of the WPP percolating cluster. Because critical vertices are in the percolating cluster for WPP, a critical vertex may be linked to the percolating cluster via another critical vertex. That is, external edges of type Z_s are not necessarily required. Furthermore, this means that critical dependencies can be bi-directional: it is possible for avalanches to propagate in either direction along such edges. Note that outgoing critical edges must be of the opposite type to the incoming one. The boxes containing crosses represent the probability $Z_n - R_n$.

There are three possibilities when following an arbitrarily chosen edge of a given type: i) with probability X_s we encounter a type s inf nite subtree ii) with probability R_s we encounter a vertex which has a connection to an inf nite subtree of the opposite type, but none of the same type. Such a vertex is part of the giant viable cluster if the parent vertex was; or iii) with probability $1-X_s-R_s$, we encounter a vertex which has no connections to inf nite subtrees of either kind. These probabilities are represented graphically in Fig. 7. We will use these symbols in subsequent diagrams.

To examine these avalanches, we def ne the probability R_s, to be the probability that, on following an edge of type s, we encounter a vulnerable vertex (probability $1-f$), which has not been removed due to random damage (probability p) and has at least one child edge of each type $n \neq s$ leading to a subtree def ned by the probability Z_n, and zero of type s. That is

$$R_s = p(1-f)\sum_{q_a,q_b,\ldots}\frac{q_s P(q_a,q_b,\ldots)}{\langle q_s\rangle}(1-Z_s)^{q_s-1}\prod_{n\neq s}\left[1-(1-Z_n)^{q_n}\right]. \tag{25}$$

We can then def ne a generating function for the size of the critical subtree encountered upon following an edge of type s (and hence resulting pruning avalanche should the parent vertex of that edge be removed) in a recursive way by

$$H_s(\mathbf{u}) = Z_s - R_s + u_s F_s[H_1(\mathbf{u}),H_2(\mathbf{u}),\ldots,H_m(\mathbf{u})]. \tag{26}$$

Where the functions $F_s(\mathbf{x})$ are def ned to be

$$F_s(\mathbf{x}) = p(1-f)\sum_{q_a,q_b,\ldots}\frac{q_s P(q_a,q_b,\ldots)}{\langle q_s\rangle}(1-Z_s)^{q_m-1}\prod_{n\neq s}\sum_{l=1}^{q_s}\binom{q_s}{l}(1-Z_n)^{q_n-l}x_n^l. \tag{27}$$

Notice that F_s has no dependence on x_s. This method is very similar to that used in [3]. A factor u_s appears for every critical edge of type s appearing in the subtree. The f rst terms $Z_s - R_s$ give the probability that zero critical nodes are encountered. The second term, with factor u_s, counts the cases where the f rst node encountered is a critical one. This node may have outgoing edges leading to further critical nodes. These edges are counted by the function F_s, and the use of the generating functions H_n as arguments recursively counts the size of the critical subtree reached upon following each of these edges.

The mean size of the avalanche caused by the removal of single vertex is then given by

$$\sum_s \partial_{u_s} H_s(\mathbf{1}). \tag{28}$$

Where ∂_z signif es the partial derivative with respect to variable z.

Let us f rst examine the mean avalanche size in the case of two layers. Taking partial derivatives of Eqs. (26) and (27), and after some rearranging, we arrive at

$$\partial_{u_1} H_1(1,1) = \frac{R_1}{1 - \partial_{x_2} F_1(Z_1,Z_2) \partial_{x_1} F_2(Z_1,Z_2)}. \tag{29}$$

where we have used that $F_1(Z_1,Z_2) = R_1$ and also that $H_1(1,1) = Z_1$, and $H_2(1,1) = Z_2$.

Let us define the right-hand side of Eq. (17) to be $\Psi_1(Z_1,Z_2)$. From Eq. (17), and comparing with Eq. (27), the partial derivatives of $\Psi_1(Z_1,Z_2)$, are

$$\begin{aligned}
\frac{\partial \Psi_1}{\partial Z_1} &= 0 \\
\frac{\partial \Psi_1}{\partial Z_2} &= p(1-f) \sum_{q_1,q_2} \frac{P_{q_1,q_2}}{\langle q_1 \rangle} q_1 q_2 (1-Z_2)^{q_2-1} \\
&= \frac{\langle q_2 \rangle}{\langle q_1 \rangle} \frac{\partial}{\partial_{x_1}} F_2(Z_1,Z_2).
\end{aligned} \tag{30}$$

and similarly for $\partial \Psi_2 / \partial Z_1$ and $\partial \Psi_2 / \partial Z_2$. Substituting back, we find that

$$\partial_u H_1(1,1) = \frac{R_1}{(\partial \Psi_1 / \partial Z_2)(\partial \Psi_2 / \partial Z_1)}. \tag{31}$$

The denominator remains finite, and the numerator does not diverge, so this quantity remains finite everywhere in the 2-layer WPP model. This confirms that a discontinuous transition does not occur when there are only two layers.

Following the same procedure in the case of three layers reveals that

$$\partial_{u_1} H_1(1,1) = R_1 \left\{ 1 - \frac{\partial_2 \Psi_1 [\partial_1 \Psi_2 + \partial_1 \Psi_3 \partial_3 \Psi_2]}{1 - \partial_2 \Psi_3 \partial_3 \Psi_2} - \partial_1 \Psi_3 \partial_3 \Psi_1 \right\}^{-1} = \frac{R_1}{1 - \frac{d\Psi_1}{dZ_1}}. \tag{32}$$

where for compactness we have written $\partial_m \Psi_n$ for $\partial \Psi_n / \partial Z_m$. Now, an alternative form for the condition for the location of the discontinuous transition is $\frac{d\Psi_1}{dZ_1} = 1$. We see immediately that this implies that the mean avalanche size diverges at the critical point. In other words the avalanches diverge in size as the discontinuous hybrid transition approaches, just as the susceptibility does for an ordinary second-order transition.

In the case of the activation process, WBP, a critical vertex is one that fails the activation criterion for a single type of edge. That is, it has exactly zero edges leading to the root of type s subtrees (probability Z_s), and at least one of every other type. If such a node gains a single connection to the root of a type s subtree, it will itself become the root of such a subtree. Chains of such connections therefore delineate the paths of avalanches of spreading activation. An example of a small critical cluster is shown in Figure 15.

To examine these avalanches, we now define the probability R_s, to be the probability that, on following an edge of type s, we encounter a vertex which is not a seed vertex (probability $1-f$), has not been removed due to random damage (probability p) has at least one child edge of all other types $n \neq s$ leading to the appropriate

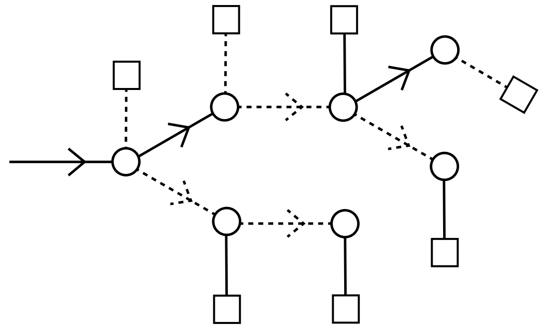

Fig. 15 An example of a critical cluster in WBP. Avalanches of activation propagate through the cluster following the arrowed edges. If an upstream vertex is activated, all downstream critical vertices will in turn be activated. Note that, unlike for WPP, in WBP it is not possible for an edge to be arrowed in both directions. Activation can only ever propagate in one direction along a given edge. Also note that in the WBP case outgoing critical edges must be of the same type as the incoming one.

subtrees (probability Z_n), and zero of type s. That is

$$R_s = p(1-f)\sum_{q_a,q_b,\ldots}\frac{q_s P(q_a,q_b,\ldots)}{\langle q_s\rangle}(1-Z_s)^{q_s-1}\prod_{n\neq s}[1-(1-Z_n)^{q_n}] \tag{33}$$

Note that this is identical to (25), but the probabilities Z_n are different, as is the following argument.

Because critical vertices are outside the WBP cluster, the probabilities Z_s and R_s are mutually exclusive. This means that, upon following an edge of type s, there are three mutuall exclusive possibilities: i) we encounter a subtree of type s (probability Z_m)ii) we encounter a critical vertex (probability R_s)or iii) we encounter neither (probability $1-Z_s-R_s$). We can then def ne a generating function for the size of the critical subtree encountered upon following an edge of type s (and hence resulting activation avalanche should the parent vertex of that edge be activated) in a recursive way by

$$H_s(\mathbf{u}) = 1-Z_s-R_s+u_sF_s[H_1(\mathbf{u}),H_2(\mathbf{u}),\ldots,H_m(\mathbf{u})]\,. \tag{34}$$

The functions $F_s(\mathbf{x})$ are def ned to be

$$F_s(x,y) = p(1-f)\sum_{q_a,q_b,\ldots}\frac{q_s P(q_a,q_b,\ldots)}{\langle q_s\rangle}x_s^{q_s-1}\prod_{n\neq s}\sum_{l=1}^{q_n}\binom{q_n}{l}Z_n^l x_n^{q_n-l}. \tag{35}$$

Note that $F_s(1-Z_1,1-Z_2,\ldots,1-Z_m) = R_s$ and $H_s(\mathbf{1}) = 1-Z_s$.

The mean size of the avalanche caused by the activation of a single vertex is again given by

$$\sum_s \partial_{u_s}H_s(\mathbf{1}). \tag{36}$$

Let us consider the case of WBP in a 2-layer multiplex. Taking partial derivatives of (34) and (35) and after some rearranging, we f nd

$$\partial_{u_1} H_1(1,1) = \frac{R_1\,[1-\partial_{x_2}F_2]}{[1-\partial_{x_1}F_1]\,[1-\partial_{x_2}F_2] - \partial_{x_2}F_1\partial_{x_1}F_2} \tag{37}$$

where for brevity we have not written the arguments of the derivatives of the functions F_1 and F_2, but they should be taken to be evaluated at $(1-Z_1, 1-Z_2)$, and where we have used that $F_1(1-Z_1, 1-Z_2) = R_1$ and also that $H_1(1,1) = 1-Z_1$, and $H_2(1,1) = 1-Z_2$.

Remembering that we have def ned $\Phi_s(Z_1, Z_2)$, in the two layer case, to be the right-hand side of Eq. (21),

$$\begin{aligned}\frac{\partial \Phi_1}{\partial Z_1} &= p(1-f)\sum_{q_1,q_2}\frac{P_{q_1,q_2}}{\langle q_1\rangle} q_1(q_1-1)(1-Z_1)^{q_1-2}[1-(1-Z_2)^{q_2}] \\ &= \partial_{x_1}F_1(1-Z_1, 1-Z_2)\end{aligned} \tag{38}$$

and

$$\begin{aligned}\frac{\partial \Phi_1}{\partial Z_2} &= p(1-f)\sum_{q_1,q_2}\frac{P_{q_1,q_2}}{\langle q_1\rangle} q_1 q_2(1-Z_2)^{q_2-1}[1-(1-Z_1)^{q_1-1}] \\ &= \frac{\langle q_2\rangle}{\langle q_1\rangle}\partial_{x_1}F_2(1-Z_1, 1-Z_2)\end{aligned} \tag{39}$$

and a similar procedure is followed for Φ_2. This means that the equation for $\partial_{u_1} H_1(1,1)$ can be written

$$\partial_{u_1} H_1(1,1) = \frac{R_1[1-\partial\Phi_2/\partial Z_2]}{\det[\mathbf{J}-\mathbf{I}]}\,. \tag{40}$$

where the Jacobian matrix $\mathbf{J}$ has elements $J_{ij} = \partial\Phi_i/Z_j$, and $\mathbf{I}$ is the identity matrix. The condition $\frac{d\Phi_1}{dZ_1} = 1$ for the location of the discontinuity in Z_1 (and Z_2) can be rewritten

$$\det[\mathbf{J}-\mathbf{I}] = 0 \tag{41}$$

meaning that $\partial_{u_1} H_1(1,1)$ diverges, and hence the mean avalanche size, diverges precisely at the critical point. This indicates that indeed a discontinuous hybrid transition, with accompanying avalanches of activations, appears even in the two-layer multiplex. A similar analysis can be performed for three or more layers.

4 Conclusions

In conclusion, the study percolation in multiplex networks requires new def nitions of connectivity. We have studied the robustness of multiplex networks to damage under two different def nitions of connectedness. In the f rst, a natural generalization of the concept of single network connectedness, we f nd a strong criterion which leads to an abrupt collapse of the giant component of a multiplex network having two or more layers. In contrast to ordinary networks, where two vertices are connected if there is a path between them, in multiplex network with *m* types of edges, two vertices are *m*-connected if for every kind of edge there is a path from one to another vertex. The transition is a discontinuous hybrid transition, similar to that found, for example, in the network *k*-core problem. The collapse occurs through avalanches which diverge in size when the transition is approached from above. We described critical clusters associated with these avalanches. The avalanches are responsible for both the critical scaling and the discontinuity observed in the size of the giant viable cluster.

We compared this with a weaker def nition of connectedness, but one which can be calculated locally. In this def nition, nodes are members of a cluster if they have at least one edge of each type leading to another member of the cluster. This means that two nodes can belong to the same cluster even when there are no paths of a single colour connecting them. We also introduced the concept of invulnerable nodes. In the pruning process form of this model, we f nd that a two-layer multiplex network no longer exhibits a hybrid transition in the collapse of the giant component, but in three layers such a transition can occur. Finally, we introduced an activation process on multiplex network, dual to the weak pruning process, in which a small number of seed (invulnerable) nodes are initially activate and further nodes activate if they have connections by every type of edge to active neighbours. The two processes have related phase diagrams, but we f nd that a discontinuous hybrid transition can occur even when there are only two layers.

This work was partially supported by FET IP Project MULTIPLEX 317532 and by the FCT projects EXPL/FIS-NAN/1275/2013 and PEst-C/CTM/LA0025/2011, and post-doctoral fellowship SFRH/BPD/74040/2010.

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