Multiplicative AF Kinematic Hardening in Plasticity

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Abstract

The basic innovation proposed in this work is to consider one of the two coefficients of the Armstrong and Frederick (AF) evolution equation for the back stress, function of another dimensionless second order internal variable evolving also according to an AF equation in what can be called a multiplicative AF kinematic hardening rule. Introducing the foregoing modification into some of the components of the back stress additive decomposition model proposed by Chaboche et al (1979), one obtains a refined model with improved performance in partial unloading/reloading and ratcheting. In many respects the multiplicative AF kinematic hardening scheme plays a role equivalent to that of the back stress with a threshold scheme introduced by Chaboche (1991) to improve ratcheting simulations. The basis equations are presented for both uniaxial and multiaxial stress spaces and the calibration of the model constants is addressed in detail. Numerical applications are executed for uniaxial cyclic loading only, and indicate that the proposed refinement can perform quite well in simulating uniaxial experimental data, including ratcheting, while the potential to simulate successfully multiaxial loading data is an issue to be addressed in the future.

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1. Introduction

Kinematic hardening and the associated concept of back stress and its evolution constitute fundamental constitutive ingredients of classical plasticity theory in order to simulate the inelastic material response under stress reversals. Cyclic plasticity addresses such response under a sequence of repeated stress reversals and the ensuing technologically important phenomenon of plastic strain accumulation, called ratcheting. Clearly the success of cyclic plasticity to realistically describe the material response depends on the kind of kinematic hardening used.

The literature on the subject matter is vast and any attempt to cover it in this article is bound to be not complete. Nevertheless, one can at least identify some important building blocks starting with the first proposition of a linear kinematic hardening rule by Ishlinskii (1954) and Prager (1956), referred to as Prager linear kinematic hardening. The linear kinematic hardening was modified to a non linear kinematic hardening by Armstrong and Frederick (1966), also known as the evanescence memory model but referred to here as the AF model for abbreviation. The combination of concepts proposed by Besseling (1958), Mroz (1967) and Iwan (1967), resulted in the so-called multisurface plasticity model. The AF and multisurface plasticity models contained already ideas which were the basis for the next two significant contributions. Firstly, Dafalias and Popov (1974, 1975, 1976) and Krieg (1975) introduced the two-surface model for metals, which was generalized to the more general framework of Bounding Surface plasticity theory for any material (Dafalias, 1987). A form of Bounding Surface theory employing the similar concepts of yield and sub-yield surfaces was introduced initially for soils by Hashiguchi and Ueno (1977), and later expanded to other materials. Secondly, Chaboche at al (1979) introduced the additive decomposition of the back
stress into components each one of which obeyed its own AF rule, often referred to as the Chaboche model. Compared to the two-surface version of bounding surface plasticity, this model has certain features of simplicity while the bounding surface has the advantage of decoupling the plastic modulus from the direction of kinematic hardening (Dafalias, 1984). No reference is made here to the non classical but important contribution to cyclic plasticity by the endochronic theory of Valanis at al (e.g. Valanis and Lee, 1982).

Further development of models for cyclic plasticity has followed a steep increase over the last twenty years in conjunction with (and often because of) an extensive experimental investigation of cyclic plasticity by various researchers, attempting to address the extremely difficult issue of simulating uniaxial and multiaxial ratcheting response under non zero mean stress or strain cyclic loading. Most of the new contributions are very significant refinements of the aforementioned basic models in the area, and often the originality and importance of such refinements compete with that of the basic model which is being refined. No attempt will be made to cover the literature for such refined and improved theories and corresponding experimental investigations because of limited space, with the exception of those works very closely related to the specific scope of the present work.

The focus of this paper is limited to offer one such refinement associated with the model of additive decomposition of the back stress. The new refinement is called the multiplicative AF kinematic hardening rule, or for brevity the multiplicative scheme. The mathematical formulation will clarify exactly the proposed idea, but one can say now the following. The basic overall model will be actually one of an additive back stress decomposition in several components as proposed by Chaboche at al (1979), but with one important difference. For some of the components (usually for only one), instead of considering both coefficients of its
AF rate equation of evolution constants, one coefficient will be variable enhanced by expressions associated with the rate evolution equation of another dimensionless second order internal variable also evolving according to an AF rule, which is not a back stress component itself. Because of this enhancement, the current value of this second dimensionless internal variable multiplies the current value of the corresponding back stress component in the expression for the rate equation of the latter. Since both evolve according to an AF rule and the one “multiplies” in some sense the other, the name multiplicative AF kinematic hardening rule is adopted for the proposed scheme, while the dimensionless variable is called the multiplier.

Note that variability of the coefficients of an AF back stress evolution law has been introduced in the past in various forms for improving ratcheting simulations, e.g. Chaboche(1991), Guionnet(1992), Ohno and Wang (1993), to mention a few important ones. Such modification of the AF rule had mostly to do with non linear dependence on the back stress itself, the concept of thresholds on evolution laws or dependence of coefficients on a cumulative plastic strain measure, while here the dependence of the AF back stress rate coefficients on other variables with AF evolution type appears to be a novel proposition.

It will be shown that such multiplicative AF kinematic hardening rule combined with the underlying additive back stress decomposition can offer improvement in the simulation of the loops created by partial reverse loading/reloading, without sacrificing the ability to model the ratcheting response that is often improved because it is ultimately related to the underlying modelling of partial reverse loading/reloading. While the formulation of the multiplicative AF kinematic hardening rule will be presented in the uniaxial and multiaxial stress space, it is only the uniaxial response that will be compared with available data.
Application of the model reveals an important role the multiplicative scheme can be assigned to play. It can substitute for the refinement proposed by Chaboche (1991) and elaborated by Bari and Hassan (2000), which introduces a back stress with a threshold, within which the back stress evolves according to a linear Prager rule and outside the threshold it behaves like an AF non linear hardening model. It will be shown that one can use a multiplicative scheme instead of the threshold scheme with improved performance, in general, under various loading conditions including ratcheting, for the price of an extra constant but without the necessary need to monitor the excess of the threshold. A systematic calibration procedure of the multiplicative scheme constants is presented in conjunction with the corresponding constants of the threshold back stress scheme, and the simulations are compared with both experimental data and the performance of the model with a threshold back stress.

2. The Armstrong and Frederick (AF) Model

It is instructive to consider first the basic equations of the otherwise well known AF model which constitutes the basis of what follows in order to introduce on the one hand the notation which will used, and on the other hand discuss an issue associated with its saturation. A typical isotropic and kinematic hardening plasticity model for metals has a Mises-type yield criterion given by

\[ f = \frac{3}{2}(s-a):(s-a)-k^2 = 0 \]  \hspace{1cm} (1)

where \( s \) is the deviatoric part of the stress tensor \( \sigma \), \( a \) is the deviatoric back stress tensor whose evolution determines the kinematic hardening, \( k \) measures the size of the yield surface whose evolution determines the isotropic hardening, and the symbol : implies the trace of the
product of the two tensors which are placed left and right of it. With the loading index (or plastic multiplier) $\lambda$ defined in terms of the stress rate $\dot{\sigma}$ by

$$\lambda = \frac{1}{K_p} n : \dot{\sigma}$$

(2)

where the unit traceless normal tensor to the yield surface along the gradient $\partial f / \partial \sigma$ is given by $n = \sqrt{2/3} (s - a)/k$, with $trn = 0$ and $tm^2 = n : n = 1$, and $K_p$ is the plastic modulus to be defined in the sequel. For a state satisfying Eq. (1), i.e. for the stress on the yield surface, the plastic strain rate is given based on the associative flow rule by

$$\dot{\varepsilon}^p = \langle \lambda \rangle n$$

(3)

with the operation of loading/unloading defined by means of the Macauley brackets $\langle \rangle$ which yield $\langle \lambda \rangle = \lambda$ if $\lambda > 0 > 0$ and $\langle \lambda \rangle = 0$ if $\lambda \leq 0$. The isotropic hardening is defined as usual by

$$\dot{k} = c_k (k_s - k) \dot{\varepsilon}_{eq}^p = \langle \lambda \rangle \sqrt{\frac{2}{3}} (k_s - k)$$

(4)

where the equivalent plastic strain rate $\dot{\varepsilon}_{eq}^p = \sqrt{(2/3)\dot{\varepsilon}^p : \dot{\varepsilon}^p}$, $k_s$ is the saturation limit of $k$, $c_k$ is a model constant controlling the pace of evolution of $k$ towards $k_s$, while use of Eq.(3) was made in deriving the third member of Eq.(4).

The evolution rate equation for the back stress $a$ characterizes the kind of kinematic hardening associated with the above framework, and it is in this respect that various models differ from each other. In this work we will restrict attention to the so-called evanescent memory non-linear kinematic hardening model introduced by Armstrong and Frederick (1966) in their classical paper, to be referred as the AF model or rule for brevity. According to their proposition one has
\[ \dot{a} = \frac{2}{3} h \varepsilon^p c^p - c \varepsilon^p \varepsilon_{ce} = \lambda \left( \frac{\sqrt{2} h}{3} \right) \left( \frac{\sqrt{2}}{3} c \right) \left( \frac{\sqrt{2}}{3} n - a \right) = \lambda \left( \frac{\sqrt{2} h}{3} \right) \left( \frac{\sqrt{2}}{3} a' n - a \right) = \lambda \left( \frac{\sqrt{2} c}{3} \right) (a' - a) \] (5)

where \( h \) and \( c \) are the two model constants and use of the previous equations was made in deriving the various forms of the rate Eq.(5) for \( a \). The interesting expressions of the third, fourth and fifth members of Eq.(5) reveals that the backstress evolves towards its maximum saturation value defined by the tensor \( a' = \frac{\sqrt{2} h}{3} n = \frac{\sqrt{2}}{3} a' n \), where the constant \( a' = h/c \) controls the saturation level while the constant \( c \) controls the pace at which this level is approached. Thus, instead of the original constants \( h \) and \( c \) one may want to think in terms of the equivalent pair of constants \( a' \) and \( c \) with the foregoing interpretation. We will mostly make use of the latter choice and associated expression of Eq.(5).

It remains finally to obtain the expression for the plastic modulus \( K_p \) entering the definition of the loading index in Eq.(2). This is achieved by the satisfaction of the consistency condition \( \dot{f} = 0 \) which in conjunction with Eqs.(2), (4) and (5) yield after some algebra

\[ K_p = \frac{2}{3} h - \sqrt{\frac{2}{3}} c a : n + \frac{2}{3} c_k (k_s - k) = \frac{\sqrt{2} h}{3} \left( \frac{\sqrt{2}}{3} c a : n \right) + \frac{2}{3} c_k (k_s - k) \]

\[ = \frac{\sqrt{2}}{3} c \left( \frac{\sqrt{2}}{3} a' - a : n \right) + \frac{2}{3} c_k (k_s - k) \] (6)

Notice the dependence of \( K_p \) on the "distance" measure \( \delta = \sqrt{\frac{2}{3}} a' - a : n = (a' - a) : n \), the latter expression based on the definition of \( a' \) from Eq.(5). In essence the \( \delta \) measures the distance between \( a' \) and \( a \) projected along \( n \). This observation reminds the close connection
of the AF model to the bounding surface formulation (Dafalias and Popov, 1974, 1975, 1976, and Krieg, 1975) where the main constitutive ingredient is the dependence of the plastic modulus on a distance in stress space between a current and a bounding value (in this case a saturation value) of a state variable like \( a \). In the foregoing references the distance was measured between stress states rather than back-stress states as is was done in later publications. In the sequel the concept of distance will be used in the presentation of the new model in multiaxial space.

It is instructive at this point to write the uniaxial stress loading counterpart of all the above equations. To achieve this task, one must carefully carry out the algebra accounting for the fact \( \text{tra} = 0 \), \( n_{22} = n_{33} = -1/2 \), and other fine details. Then, the yield surface Eq.(1) becomes

\[
f = (\sigma - a)^2 - k^2 = 0
\]

(7)

where \( \sigma \) is the uniaxial stress and \( a = (3/2)a_{11} \), while the \( \dot{\varepsilon}^p = |\dot{\varepsilon}^p| \) with the uniaxial plastic strain rate \( \dot{\varepsilon}^p = \dot{\varepsilon}_{11}^p \). Eq.(4) becomes

\[
\dot{k} = c_k (k - k) |\dot{\varepsilon}^p|
\]

(8)

and Eq.(5) reads

\[
\dot{a} = h\dot{\varepsilon}^p - c|\dot{\varepsilon}^p| a = c \left( \frac{h}{c} \mp a \right) \dot{\varepsilon}^p = c \left( a' \mp a \right) \dot{\varepsilon}^p
\]

(9)

with \( a' = h/c \) and the \( \mp \) owning its appearance to the relation \( |\dot{\varepsilon}^p| = (\text{sign}\dot{\varepsilon}^p)\dot{\varepsilon}^p \), thus the minus and plus signs appear for positive and negative plastic strain rates, respectively. Notice
that the quantity $c\left(a' \mp a\right)$ is the uniaxial counterpart of the multiaxial quantity
\[
\sqrt{\frac{2}{3}c} \left(\sqrt{\frac{2}{3}} \frac{h}{c} - a : n \right)
\]
which is the distance $\delta = \sqrt{\frac{2}{3}a' - a : n = (a' - a) : n}$ multiplied by $\sqrt{\frac{2}{3}c}$.

Finally Eq.(6) for the plastic modulus, which in the uniaxial case expresses the all important slope of the stress-plastic strain curve $d\sigma / d\varepsilon^p = \frac{E}{\varepsilon} = (d\alpha / d\varepsilon^p) \pm (dk / d\varepsilon^p)$, becomes
\[
E^p = h \mp ca + c_k (k_s - k) = c\left(a' \mp a\right) + c_k (k_s - k)
\]
(10)
where again recall that $a' = h/c$ and observe that Eq.(10) could have been derived also directly from Eqs.(7), (8) and (9) with careful consideration of the combination of plus and minus signs implied by square roots and absolute values. Eq.(10) in conjunction with Fig. 1 shows eloquently the basic characteristics of the AF model. Setting aside the isotropic hardening contribution, which sooner or later drops out when saturation renders asymptotically $k = k_s$, one observes that at $a = 0$ the slope $E^p = h = ca'$, and as $a$ develops along the path $AFBB'$ the $E^p = c\left(a' - a\right)$ until saturation at $a = a'$ yields $E^p = 0$ (not shown in Fig.1). Upon unloading/reverse loading along the path $BCDC'$ one has $E^p = c\left(a' + a\right)$. If such unloading takes place from a saturated state $a = a'$ one has $E^p = 2ca' = 2h$ at initiation of reverse loading. It is exactly this increased value of the $E^p$ upon reverse loading that renders the AF model a much more realistic tool in describing cyclic plasticity than its predecessor, the linear kinematic hardening model by Prager (1956) and Ishlinskii (1954) obtained by setting $c = 0$ in Eqs.(5) and (9), which results in having $E^p = h$ always.

It is now straightforward to integrate Eq.(9) and obtain
\[ a = \pm a' (1 - m \exp[-c \Delta \varepsilon^n]) \] (11)

where \( m = 1 \) when \( a = 0 \) at \( \Delta \varepsilon^p = 0 \) and \( m = 2 \) when \( a = \mp a_s \) at \( \Delta \varepsilon^p = 0 \). The \( \Delta \varepsilon^p \) is the amount of plastic strain variation when a loading process begins form whatever initial value of \( a \) and is always taken to be positive. It follows that \( a = \pm a_s \) asymptotically when \( \Delta \varepsilon^p \to \infty \). The practical question though for purpose of calibration is what would be an estimated amount of plastic strain variation \( \Delta \varepsilon^p \) for which, say, \( a = \pm 0.99 a_s \), where the 99% of saturation is chosen

![Fig. 1. Schematic illustration of the response and deficiency of the Armstrong & Frederick kinematic hardening rule (after Dafalias (1984)).](image)

as a very reasonable level of proximity to saturation. The answer is obtained if one inserts the \( a = \pm 0.99 a_s \) in Eq.(11) and solves for the plastic strain variation \( \Delta \varepsilon^p \) to obtain

\[ \Delta \varepsilon^p = \ln 100 / c = 4.6 / c \] and \( \Delta \varepsilon^p = \ln 200 / c = 5.3 / c \) when \( m = 1 \) and \( m = 2 \), respectively. We
round up the above conclusion and, henceforth, we consider that the back stress reaches about 99% of its saturation level when the induced plastic strain is given by

$$\Delta \mathbf{e}^p = \frac{5}{c}$$  \hspace{1cm} (12)

Eq.(12) will be very useful in controlling the range of application of the new multiplicative scheme and determining the constant $c$. It must be understood though that the 99% of saturation chosen to calculate the corresponding plastic strain, as well as the roundup of 4.6 and 5.3 to obtain the number 5 are quite arbitrary decision and aim at only an approximate consideration of conditions for constant calibration.

3. Additive Back Stress Decomposition

The AF model has a hidden deficiency which can be best understood by referring again to Fig.1 and the associated discussion by Dafalias (1984). Consider the path $ABCDEE'$ in Fig. 1. The rapid decrease of $a$ during the partial reverse loading path $CD$ (notice the initial value of $E^p = c \left(a^s + a \right)$ shown in the figure) cannot be compensated fast enough during the subsequent unloading -reloading path $DEE'$ and the predicted elastic-plastic stress strain curve $EE'$ undershoots the actual one which should merge fast with $BB'$ as observed in corresponding experimental data. In fact one can show that the reloading curve $EE'$ is a parallel translation of the original curve $FBB'$ before unloading/reverse loading/reloading. This deficiency can severely over-estimate the ratcheting phenomenon for cyclic stress loading with non-zero mean stress.
3.1 The classical version of the additive decomposition

In attempting to eliminate the foregoing deficiency of the AF model, Chaboche at al (1979) proposed an additive decomposition of the back stress $a$ into components $a_i$ according to $a = \sum a_i$ for the multiaxial and $a = \sum a_i$ for the uniaxial case, where each component obeys an AF kinematic hardening rule with its own constants $h_i$ and $c_i$ or equivalently $a_i^\varepsilon$ and $c_i$ along the lines of Eqs. (5)-(9). It is not necessary to go through the details of the formulation of this very well known model, but it is instructive to write only the expressions equivalent to Eqs. (6) and (10) for the multiaxial and uniaxial plastic moduli, respectively, as

$$K_p = \sqrt{\frac{2}{3}} \sum_i \left[ \sqrt{\frac{2}{3}} h_i - (c_i a_i) : n \right] + \frac{2}{3} c_k (k_s - k)$$

$$= \sqrt{\frac{2}{3}} \sum_i c_i \left[ \sqrt{\frac{2}{3}} h_i - a_i : n \right] + \frac{2}{3} c_k (k_s - k)$$

$$= \sqrt{\frac{2}{3}} \sum_i c_i \left[ \sqrt{\frac{2}{3}} a_i^\varepsilon - a_i : n \right] + \frac{2}{3} c_k (k_s - k)$$

(13)

and

$$E^p = \sum_i h_i + \sum_i c_i a_i + c_k (k_s - k) = \sum_i c_i \left( a_i^\varepsilon + a_i \right) + c_k (k_s - k)$$

(14)

where recall that $a_i^\varepsilon = h_i / c_i$. By distributing the saturation level $a_i^\varepsilon$ among the various components of the additive decomposition and controlling the corresponding pace of saturation...
expressed by $c_i$, it is possible to considerably reduce the aforementioned undershooting deficiency of the original AF model, at the expense of course of an increased number of back stress components, a well known attribute of the Chaboche at al (1979) model. In practical terms this is possible because the modeller has at its disposal the richer Eq.(14) instead of Eq.(10) for the value of the plastic modulus $E''$ which is the key of a successful curve fitting.

### 3.2 The version with a threshold

Although not easily seen, even the additive back stress decomposition model of Chaboche at al (1979) had certain problems with the simulation of partial reverse loading/reloading and simultaneously of the ratcheting response. One can identify this problem with the intrinsic geometry and curvature of the exponential nature of the produced back stress-plastic strain curves in the AF model, which we will have the opportunity to illustrate at a later section for the calibration of model constants. At present it suffices to state that if one wanted an initial stiff slope for a given saturation level $a^*$, then according to Eq.(11) he should choose a high enough value of $c$ that satisfies this requirement but simultaneously induces a fast saturation, i.e. a saturation for a very small value of $e^p = 5/c$ as per Eq.(12). This may not seem at first very important for a monotonic loading and possibly one regular unloading, but when applied to partial reverse loading/reloading and consequently to ratcheting (a series of such partial reverse loading/reloading operations) it was found that even this small effect culminates to a serious deficiency when its cumulative effect is considered. This is because the ratcheting phenomenon is very sensitive to the exact shape of the unloading/reloading curves, and such sensitivity created the need for further modifications of the additive back stress decomposition model.
There are various such modifications addressing the so-called dynamic recovery term, which is the second term of the second member of Eq.(5) associated with the constant \(c\), among them the introduction of a non-linear power dependence on the back stress by Henshall at al (1987) and the non-hardening region by Ohno and Wang (1993). The one we will focus for comparison and reference in regards to our proposition will be the AF model modification by Chaboche (1991) which introduces a threshold for the dynamic recovery term below which it induces a linear response, according to a rate equation that in the uniaxial case reads

\[
\dot{a} = h\dot{e}^p - c|\dot{a}|<\frac{|a|}{|a|} > a = c\left(\frac{h}{c} <\text{sgn } a <|a| - \bar{a}>\right)\dot{e}^p = c\left(a' <\text{sgn } a <|a| - \bar{a}>\right)\dot{e}^p
\]  

(15)

The \(\text{sgn } a\) means the sign of \(a\) (not necessarily identical to the sign of \(\dot{e}^p\) which induces the appearance of the \(\mp\) in Eq.(15)), the \(|a|\) is the absolute value of \(a\), and the \(\bar{a}\) is the threshold. Observe that when \(|a| - \bar{a} \leq 0\) Eq.(15) yields the linear relation \(\dot{a} = h\dot{e}^p = ca'\dot{e}^p\), while when \(|a| - \bar{a} \geq 0\) Eq.(15) yields an AF evolution for the “excess” (i.e. above the threshold) back stress \(a - \bar{a}\) since the rate of it equals the rate of \(a\) alone. In the latter case it easily follows from Eq.(15) that the \(a'\) is the saturation value of the excess back stress \(a - \bar{a}\), thus, the \(a\) saturates at \(a' + \bar{a}\) after reaching the \(\bar{a}\) in a linear way as it was intended to begin with. The form of Eq.(15) is slightly more general than the original proposition by Chaboche (1991) including both loading and unloading and the possibility of different signs for \(a\) and \(\dot{e}^p\).

Employing the scheme of the threshold for one of the four AF back stress components of the additive decomposition, it was shown in Bari and Hassan (2000) that both the partial reverse loading/reloading and the ratcheting improve considerably in comparison not only with a three component decomposition (that was expected), but also in comparison with a four back
stress decomposition of the AF type without the concept of the threshold applied to any one of them. Bari and Hassan (2000) attribute this beneficial effect of the threshold scheme to the particular shape with a “knee” that the curve of the back stress with threshold versus plastic strain acquires, as a result of the combination of linear (at the beginning) and non linear (afterwards) evolution of the back stress that allows for a stiff initial response (the linear part) followed by a not so fast saturation process (when the non linear part is activated). One should also observe that because of the rather stiff initial linear response of the back stress with a threshold, the overall stress-strain curve shows a small but detectable and rather un-physical linear portion at the initiation of loading or reverse loading. Also the threshold term must be monitored in any loading (i.e. if it is exceeded or not) which may become cumbersome for implicit numerical implementation.

4. Multiplicative AF Kinematic Hardening Rule

The threshold scheme and the reasoning for introducing it constitute some of the motivations for introducing the multiplicative AF kinematic hardening rule, or simply the multiplicative scheme for abbreviation. It will be shown that this new scheme will avoid the aforementioned un-physical linear portion of the stress-strain curve which was due to the stiff linear response before exceeding the threshold, and that no need to check the sign of a quantity associated with the threshold arises, while the ratcheting response simulation slightly improves. The price for these improvements will be one additional constant compared to the scheme with the threshold. The basic idea is to achieve for some (usually one) of the components of the additive back stress decomposition a similar response to the one obtained when a threshold is
used, by varying one of the coefficients of its evolution law during loading and unloading in a way which depends on the direction of loading. The details are presented below.

4.1 Uniaxial formulation

For the new kinematic hardening model introduced here, the concept of the back stress additive decomposition presented in the previous section remains, but for some of the back stress components it is altered by the aforementioned variation of one of the coefficients of its AF type evolution equation. The variable coefficient will be enhanced by expressions related to the AF evolution equations of other dimensionless second order internal variables, called the multipliers, in a way specified exactly in the sequel. Henceforth, with a plausible notation convention a multiplier associated to a back stress component will be denoted by the same symbol as the back stress component with the addition of a superscript *. The same notation convention applies to the constants of the rate equation of evolution of the multiplier which is also of the AF type. For example for the back stress component $a_i$ with constants $c_i$ and $a_{i^s}$ entering its rate expression as per Eq.(9), the notation for the associated multiplier will be $a_i^*$, with constants $c_i^*$ and $a_{i^s}^{**}$ in a corresponding equation. It is possible to extend the notation convention to the case of a multiplier of a multiplier, by simply adding a double ** as superscript, and so forth.

In order to facilitate the introduction of the new concept only one back stress $a = a_i$ and the associated dimensionless multiplier $a_{i^*}$ are considered at the beginning for simplicity. The role of the multiplier is now defined as follows. Instead of considering the coefficient $c_i$
of \( \dot{a}_1 \) constant, an enhancement of \( c_1 \) is introduced by additional terms associated with the expression for \( \dot{a}_1^* \) as shown below, where the full set of equations reads as

\[
\begin{align*}
\dot{a} &= \dot{a}_1 \quad \text{(16a)} \\
\dot{a}_1 &= \left[ c_1 + c_1^* \left( a_1^{*s} \mp a_1^* \right) \right] \left( a_1^* \mp a_1 \right) \dot{\varepsilon}^p \quad \text{(16b)} \\
\dot{a}_1^* &= c_1^* \left( a_1^{*s} \mp a_1^* \right) \dot{\varepsilon}^p \quad \text{(16c)}
\end{align*}
\]

Eqs.(16) are the key equations of the new development. It follows from Eq.(16b) that the additive enhancement of \( c_1 \) is the term \( c_1^* \left( a_1^{*s} \mp a_1^* \right) \) which multiplies the quantity \( a_1^* \mp a_1 \). Thus, a multiplication of \( a_1 \) by \( a_1^* \) occurs in Eq. (16b), and because of this multiplication and the fact that the evolution law for \( a_1^* \) is also of the AF type according to Eq.(16c), the name multiplicative AF kinematic hardening rule was adopted. A most important observation is that despite the introduction of \( a_1^* \) in Eq.(16b), the saturation level \( a_1^* \) of \( a_1 \) remains the same as in the case with no such introduction. It is only the pace of approaching this saturation level that changes because the \( c_1 \) becomes \( c_1 + c_1^* \left( a_1^{*s} \mp a_1^* \right) \) within the framework of allowing variation of only one of the two coefficients \( c_1 \) and \( a_1^* \) of the AF rule for the back stress component. Observe also that the quantity multiplying the \( \dot{\varepsilon}^p \) in Eq.(16b) is in fact the uniaxial plastic modulus \( E^p \) if the isotropic hardening contribution \( c_1^* \left( k_s - k \right) \) is assumed to have been exhausted once \( k = k_s \), which is quite different from the \( E^p \) obtained from Eq.(14) for \( i=2 \).
Given that \( a_1^* \) can be obtained by integration of Eq.(16c) according to Eq.(11), one can also integrate Eq.(16b) to finally obtain in closed analytical form the expression

\[
a_i = \pm a_i^* [1 - m \exp(-c_i \Delta \dot{\varepsilon}_p + ma_i^{**} (1 - \exp(-c_i^* \Delta \dot{\varepsilon}_p)))]
\]

(17)

where \( m = 1 \) when \( a_i = 0 \) at \( \Delta \dot{\varepsilon}_p = 0 \), and \( m = 2 \) when \( a_i = \mp a_i^* \) at \( \Delta \dot{\varepsilon}_p = 0 \). The \( \Delta \dot{\varepsilon}_p \) was defined after Eq.(11) and its always positive sign vis-à-vis the positive or negative sign of the rate of \( \varepsilon^p \) was accounted for in the derivation of Eq.(17). In handling Eq.(17) one has the saturation of \( a_i^* \) occurring before that of \( a_i \).

As already mentioned, the multiplicative scheme can be extended further to a triple or higher multiplication mechanism. For example in Eqs.(16) one can enhance the \( c_i^* \) with dependence on a second multiplier \( a_i^{**} \), i.e. a multiplier of the multiplier evolving according to Eq.(9) with its own constants \( c_i^{**} \) and \( a_i^{***} \), exactly as the \( c_i \) was made to depend on \( a_i^* \). The corresponding equations of evolution can then be written as

\[
\dot{a} = \dot{a}_i
\]

(18a)

\[
\dot{a}_i = \left[ c_i + \left( c_i^* + c_i^{**} \left( a_i^{***} \mp a_i^{**} \right) \right) \left( a_i^* \mp a_i \right) \right] \left( a_i^* \mp a_i \right) \dot{\varepsilon}_p
\]

(18b)

\[
\dot{a}_i^* = \left[ c_i^* + c_i^{**} \left( a_i^{***} \mp a_i^{**} \right) \right] \left( a_i^{**} \mp a_i^* \right) \dot{\varepsilon}_p
\]

(18c)

\[
\dot{a}_i^{**} = c_i^{**} \left( a_i^{***} \mp a_i^{**} \right) \dot{\varepsilon}_p
\]

(18d)
Eqs. (18) contain one only back stress component $a_i$ and two dimensionless multipliers $a_i^*$ and $a_i^{**}$. Observe the cascading degree of multiplicative coupling among the three components. According to Eq. (18b) it follows again that the saturation value $a_i^* = a_i^{**}$ irrespective of the saturation values of the multipliers $a_i^*$ and $a_i^{**}$. In this respect it must be emphasized that the introduction of the multiplicative scheme does not (and should not) eliminate the additive decomposition scheme introduced by Chaboche et al (1979) but rather it complements it. In practical terms a single multiplicative scheme will be needed for only one of the additive components of the back stress.

A typical formulation will include four back stress components, added to yield the total back stress. The first three can either be all of the AF type, or one can be a linear Prager type with very small slope and the other two of the AF type. When all three are of the AF type as per Eq. (9), it is usual to consider one of them almost linear close to a Prager type by proper choice of the constants $a_i^*, c$. Such small variance from a linear to an “almost” linear response has been shown to have significant effect on the simulation of ratcheting. The fourth back stress component will be of the multiplicative scheme having an associated dimensionless multiplier. It will be shown later that this fourth back stress component with its multiplier can successfully substitute for a back stress component with a threshold as introduced in Chaboche (1991) and Bari and Hassan (2000). The above scheme, with the choice of the first three back stress components being of the AF type, is defined by the following set of equations:

\[
\dot{a} = \sum_{i=1}^{4} \dot{a}_i = \sum_{i=1}^{3} c_i \left( a_i^* \mp a_i \right) + \left[ c_4 + c_4^* \left( a_4^* \mp a_4 \right) \right] \left( a_4^* \mp a_4 \right) \dot{\varepsilon}^p
\]  

(19a)
\[ \dot{a}_i = c_i \left( a_i^* \mp \bar{a}_i \right) \dot{\varepsilon}^p \quad (i = 1,2,3) \]  

(19b)

\[ \dot{a}_4 = [c_4 + c_4^* \left( a_4^* \mp \bar{a}_4^* \right)] \left( a_4^* \mp \bar{a}_4 \right) \dot{\varepsilon}^p \]  

(19c)

\[ E^p = \sum_{i=1,3} c_i \left( a_i^* \mp \bar{a}_i \right) + \left[ c_4 + c_4^* \left( a_4^* \mp \bar{a}_4^* \right) \right] \left( a_4^* \mp \bar{a}_4 \right) + c_k \left( k_1 - k \right) \]  

(19d)

It is possible to use a reduced form of Eqs.(19) where two instead of three AF type of back stresses can be employed. Recall also the possibility to have one of the AF back stress substituted by a Prager linear one, by simply setting \( h_i = c_i a_i^* \) and \( c_i = 0 \) for \( i = 1, 2 \) or 3.

### 4.2 Multiaxial formulation

The multiaxial formulation follows the logic of the uniaxial. Let us again consider first for simplicity only one back stress components \( a = a_i \) and the associated dimensionless multiplier \( a_i^* \) with the same notation convention of a superscript * for the multiplier as in the uniaxial case. The essence of the multiplicative scheme is to enhance the coefficient \( c_i \) entering the rate expression for \( a_i \) as per Eq.(5), by an additional term related to the AF rate evolution law for \( a_i^* \) which measures in multiaxial space the distance of \( a_i^* \) from its saturation value \[ a_i^{*s} = \sqrt{\frac{2}{3}} a_i^{*} n, \] multiplied by \[ \sqrt{\frac{2}{3}} c_i^* \] and projected on \( n \). These “distance” related quantities were discussed after Eqs. (5) and (6) for an AF back stress rate, but they do apply equivalently for the dimensionless multiplier \( a_i^* \) which obeys also an AF rate equation of evolution. The complete set of equations for the so modified rate of \( a_i \) reads...
\dot{a} = \dot{a}_1 \quad (20a)
\dot{a}_1 = \langle \lambda \rangle \left[ \frac{2}{\sqrt{3}} c_1 + \frac{2}{\sqrt{3}} c_1^s \left( \frac{2}{\sqrt{3}} a_1^r - a_1^s : n \right) \right] \left( \frac{2}{\sqrt{3}} a_1^r n - a_1^s \right) \quad (20b)
\dot{a}_1^s = \langle \lambda \rangle \left( \frac{2}{\sqrt{3}} c_i^s \right) \left( \frac{2}{\sqrt{3}} a_i^r n - a_i^s \right) \quad (20c)

Eqs.(20) are the multiaxial counterpart of Eqs.(16), and vice versa. The latter can be derived when uniaxial stress conditions are applied to Eqs.(20), and this is the reason the numerical factor \((2/3)^{1/2}\) appears. It is instructive to state that the generalization of Eqs.(16) to (20) is easily done if one recalls from the discussion after Eq.(9) that \(c \left( a^r + a \right)\) is the uniaxial distance counterpart of the multiaxial distance quantity \(\sqrt{c} \left( \sqrt{c} a^r - a : n \right)\) for any backstress component or dimensionless multiplier following an AF evolution rule.

The multiaxial counterpart of Eqs.(18), referring to the triple multiplicative AF kinematic hardening scheme reads

\dot{a} = a_1 \quad (21a)
\dot{a}_1 = \langle \lambda \rangle \left[ \frac{2}{\sqrt{3}} c_1 + \left( \frac{2}{\sqrt{3}} c_1^s + \frac{2}{\sqrt{3}} c_1^ss \left( \frac{2}{\sqrt{3}} a_1^r - a_1^s : n \right) \right) \left( \frac{2}{\sqrt{3}} a_1^r n - a_1^s \right) \right] \quad (21b)
\[
\dot{a}_i^* = \langle \lambda \rangle \left[ \left( \frac{2}{3} c_i + \frac{2}{3} c_i^* \right) \left( \frac{2}{3} a_i^* n - a_i^* \right) \right] \left( \frac{2}{3} a_i^* n - a_i^* \right)
\]

(21c)

\[
\dot{a}_i^{**} = \langle \lambda \rangle \frac{2}{3} c_i^{**} \left( \frac{2}{3} a_i^{**} n - a_i^{**} \right)
\]

(21d)

Again notice that the saturation value of \( a \) depends only on the saturation value \( \frac{2}{3} a_i^* n \) of the only back stress component \( a_i \). Finally, the multiaxial counterpart of the combined additive and multiplicative (for one only component) scheme portrayed in the uniaxial case by Eqs.(19), is expressed by the equations

\[
\dot{a} = \sum_{i=1,4} \dot{a}_i
\]

(22a)

\[
\dot{a}_i = \langle \lambda \rangle \frac{2}{3} c_i \left( \frac{2}{3} a_i^* n - a_i \right) \quad (i = 1,2,3)
\]

(22b)

\[
\dot{a}_4 = \langle \lambda \rangle \left[ \left( \frac{2}{3} c_4 + \frac{2}{3} c_4^* \right) \left( \frac{2}{3} a_4^* n - a_4^* \right) \right] \left( \frac{2}{3} a_4^* n - a_4^* \right)
\]

(22c)

\[
K_p = \left[ \sum_{i=1,3} \frac{2}{3} c_i \left( \frac{2}{3} a_i^* n - a_i \right) + \left( \frac{2}{3} c_4 + \frac{2}{3} c_4^* \right) \left( \frac{2}{3} a_4^* n - a_4^* \right) \right] + \frac{2}{3} c_k (k_x - k)
\]

(22d)

5. Calibration and Validation of the Model

5.1 Calibration

The multiplicative scheme expressed by one back stress component \( a_i \) and the associated dimensionless multiplier \( a_i^* \) according to Eqs.(16) needs the calibration of four constants,
namely $c_i, a_i$ for the back stress and $c_i^*, a_i^{**}$ for the multiplier. Notice that while $a_i^*$ has the dimensions of stress as the saturation value of a back stress component, the $c_i$ and both $c_i^*, a_i^{**}$ are dimensionless, the latter two for the obvious reason they control the evolution of the dimensionless multiplier $a_i^*$. Such calibration has no meaning before we are able to identify the role the multiplicative scheme must play.

It was mentioned earlier that the final objective is to simulate better the response under partial reverse loading/reloading and ratcheting, vis-a-vis the classical additive decomposition model of Chaboche at al (1979). However it must be stated at the outset that the multiplicative scheme with one back stress component and one associated multiplier cannot do a better job than an equivalent two back stress components additive decomposition scheme for a very small reverse loading/reloading when the back stress component is close to or at saturation. The reason can be easily seen from Eq. (14) with $i=2$ and Eq. (16b). Assume first that the $a_1$ and $a_2$ entering Eq.(14) and the $a_1$ and $a_i^*$ entering Eqs.(16) have been saturated during loading, and that a partial reverse loading activates the fast changing $a_2$ and $a_i^*$, so that they almost saturate again before reloading takes place. At the point of initiation of reloading the slow changing $a_1$ is still almost saturated and the quantity $a_i^{**} - a_1$ is of order $O(\varepsilon) << 1$, thus, the plastic modulus $E^p$ will be given as follows, assuming that the isotropic hardening has been saturated. For the additive decomposition scheme, Eq.(14) yields $E^p = c_i O(\varepsilon) + 2 c_2 a_i^{**} \Delta 2h_2$ since the $a_i^{**} - a_1 \Delta O(\varepsilon)$ and $a_2 = -a_2^{**}$ ($O(\varepsilon)$ means order $\varepsilon$, a very small number). For the multiplicative scheme, Eq.(16b) yields $E^p = \left[ c_i + 2 c_2 a_i^{**} \right] O(\varepsilon) \Delta O(\varepsilon)$ because $a_1^{**} - a_1 \Delta O(\varepsilon)$. In other words no matter how large is the multiplier (within limits of course), if the multiplied
is of $O(\varepsilon)$ so will be the product. Therefore in this case the reloading slope will be very small while in the additive decomposition it would be sufficient to close appropriately the loop of partial reverse loading/reloading.

Having excluded the usefulness of the multiplicative scheme in regards to the above, the question then arises as to where such a scheme is useful. The answer comes in conjunction with the concept of the back stress component with a threshold elaborated in Eq.(15). The reasoning behind the introduction of the threshold scheme was the need to have a back stress which at the beginning has a stiff linear response, followed by a non linear AF saturation process when the non linear response is activated outside the threshold. The idea is to achieve a similar behaviour with the multiplicative scheme without explicitly introducing a threshold.

Before we attempt to organize the calibration procedure towards this goal, it is instructive to present in Fig. 2 the response of a family of AF models with the same saturation level but various values of the constant $c = 1000, 2000, 3000, 4000, 5000$, and compare it with that of a multiplicative AF kinematic hardening model which has the same saturation level as the individual classical AF models. The plots of the AF back stresses, normalized by their common
Figure 2. Comparison of the curves of various AF models and a multiplicative AF kinematic hardening scheme.

saturation level, are obtained from Eq.(11) with $m=1$ and shown in Fig.2 by thin continuous lines. The corresponding plot of the multiplicative back stress scheme, also normalized by the same saturation level denoted by $a_1^p$, is obtained by Eq.(17) with $m=1$ employing the constants $c_1 = 360$, $c_1^* = 2800$ and $a_1^{p*} = 1.3$ ksi and shown in Fig.2 by thick continuous line.

An interesting feature is revealed by the plots of Fig. 2. The AF exponential curves combine necessarily stiff initial slope with fast saturation in accordance with the simple formula of Eq.(12) for the plastic strain at which 99% of the saturation level is reached, and one cannot have the one without the other. To the contrary the multiplicative scheme can have a stiff initial slope followed by a smooth saturation process. This is a result of the curvature of the corresponding curve as it becomes evident from the fact the curve of the multiplicative crosses the curves of the AF models. The threshold modification achieves about the same thing by
having first the stiff linear and then the smooth non-linear response. It is exactly this property of the multiplicative scheme that will be proved useful for partial reverse loading/reloading in cooperation with the other additive back stress components (it cannot do it alone as shown before), and in particular for the description of ratcheting.

Having identified the role we would like to attribute to the back stress multiplicative scheme as that which is equivalent to the threshold modification, allows us to address the calibration process for the four constants $c_1, a_1'$ and $c_1^*, a_1^{**}$. It is assumed that the reasoning for the threshold scheme as presented in Chaboche (1991) and Bari and Hassan (2000) has made already possible to define three things in regards to Eq.(15): the threshold value $\tilde{a}$; the final saturation value $a^*$ of the excess stress $a - \tilde{a}$; and the slope of the linear part $h = ca^*$ or equivalently the $c$ given $a'$. Usually a subscript is given to the above values associated with the fact the back stress with a threshold is still one of the components of the additive back stress decomposition model, but in our case we present them without any subscript since we do not refer to the foregoing model as such, but only to the threshold scheme. It is clear that should we be able to associate the response of the multiplicative scheme with that of the back stress with a threshold, we must account at least approximately for the above three aspects of the threshold scheme, and in addition we need one fourth condition for the four constants of the former. This extra condition is associated with the plastic strain amplitude within which the multiplier $a_1^*$ has been almost saturated, as it will be explained in the following. The foregoing characteristics of the threshold scheme will be related to the following four conditions for the calibration of the four constants of the multiplicative scheme in conjunction with Eqs.(15), (16) and (17).
1. The saturation level \( a_i^s \) of \( a_i \) must equal the sum of the threshold \( \bar{a} \) and \( a^* \), thus,

\[
a_i^s = \bar{a} + a^*
\]  

(23a)

2. The initial slope upon reverse loading of the multiplicative scheme must equal the corresponding initial slope of the threshold scheme, thus,

\[
(c_i + 2c_i^*a_i^{**})a_i^s = h = ca^*
\]  

(23b)

Notice that the initial slope in reverse loading is taken after saturation of both \( a_i \) and \( a_i^* \), because this will be the most common case in the simulations. Thus, according to Eq.(16b) the factor 2 appears at first both in the outside and the inside of the parentheses of the left hand side of Eq.(23b), but so does at the right hand side since the threshold scheme also is saturated before reversal at which it has an initial slope \( 2h = 2ca^* \); hence, the “exterior” factors 2 of the left and right hand sides of the equation are eliminated, but the one inside the parentheses of the left hand side remains.

3. When the multiplier \( a_i^* \) saturates according to Eq.(12) at \( \Delta \varepsilon^p = 5/c_i^* \), we consider that the multiplied \( a_i \), which starts at the saturation level \( \mp a_i^s \), has reached about the value of the threshold \( \pm \bar{a} \) because keeping \( a_i \) high is exactly the role of the multiplicative effect of \( a_i^* \). Thus, referring to Eq.(17) with \( m = 2 \) and making the approximation \( 1 - e^{-5} \approx 1 \), one obtains

\[
\bar{a} = a_i^*[1 - 2\exp[-(5(c_i / c_i^*) + 2a_i^{**})]]
\]  

(23c)
4. The $\alpha_i^*$ must saturate after a $\Delta \varepsilon^p = 5/\alpha_i^*$ according to Eq.(12). This $\Delta \varepsilon^p$ must be within the plastic strain variation range the evolving back stress $\alpha$ with a threshold, starting from its negative saturation level $-(\alpha^s + \bar{\alpha})$, needs in order to reach the positive threshold level $\bar{\alpha}$, but not to exceed the positive saturation level $\alpha^s + \bar{\alpha}$; in the former case the $\alpha$ varied by $2\bar{\alpha} + \alpha^s$ and in the latter by $2(\alpha + \alpha^s)$. Since the threshold scheme implies that the $\alpha$ changes at an almost constant linear slope $h = \alpha^s$ (the initial slope $2h = 2\alpha^s$ at reverse loading becomes very fast equal to the slope $h = \alpha^s$ inside the threshold domain), one can assume that the above two plastic strain variation ranges will be approximately obtained by dividing the aforementioned threshold back stress variations by $h = \alpha^s$. Placing the $\Delta \varepsilon^p = 5/\alpha_i^*$ between these two strain variation ranges and rearranging the terms, one has

$$\frac{5\alpha^s}{2(\alpha + \alpha^s)} < c_i^* < \frac{5\alpha^s}{2\bar{\alpha} + \alpha^s} \quad (23d)$$

The relations (23) can be used judiciously to obtain the values of $c_1, \alpha_1^s$ and $c_1^*, \alpha_1^{**} \alpha^s$ of the multiplicative AF kinematic hardening model, when the back stress with a threshold quantities $\bar{\alpha}, \alpha^s$ and $c$ are known. In fact one can proceed one step further and observe that Eq.(23a) specifies directly the $\alpha_1^s$ in terms of $\bar{\alpha} + \alpha^s$ while inequality (23d) offers the possibility for an educated guess on the value of $c_1^*$. With these two quantities considered known, the process for solving the system of relations (23) can then be focused on Eqs.(23b) and (23c) for
the two unknown quantities $c_i$ and $a_i^{*r}$ in terms of $\bar{a}$, $a^s$, $c$ and $c_i^*$. It is not difficult to work out the solution of the system of Eqs.(23b) and (23c) and obtain

$$c_i = \frac{1}{4} \left[ c_i^* \ln[2(1 + \frac{\bar{a}}{a^s})] - \frac{c}{1 + \frac{\bar{a}}{a^s}} \right]$$  \hspace{1cm} (24a)$$

$$a_i^{*r} = \frac{1}{8} \left[ \frac{5c}{c_i^*} \ln[2(1 + \frac{\bar{a}}{a^s})] - \ln[2(1 + \frac{\bar{a}}{a^s})] \right]$$  \hspace{1cm} (24b)$$

However, recall that the system of the relations (23) and the ensuing system of Eqs.(24) are based on many approximations, as for example the choice of $c_i^*$ and the fact that the number 5 which appears in Eq.(23c) and (24b) is associated with the acceptance of the approximation in Eq.(12), while it could as well be quite different if an assumption other than the 99% saturation level which led to Eq.(12) was adopted. The solution of Eqs.(23) or (24) has as main purpose to provide a first estimate of the relevant constants for the multiplicative scheme. It is important in other words to know the order of magnitude on the sought constants and that is what is given above. It is clear that any such process of calibration will need fine tuning for better results, given the approximate nature of the involved relations and the fact relation (23d) is an inequality, notwithstanding the approximations associated with the threshold scheme on which the calibration of the multiplicative is based.
Finally one can think of the possibility to completely ignore the threshold scheme and, having both the differential and integrated form of the multiplicative scheme, i.e. Eqs.(16) and (17), try to simulate the material response by a direct trial and error approach. Yet in such a trial and error process one indirectly may be guided to mimic the threshold scheme, thus, the use of relations (23) and (24) is still the recommended calibration way to go at present.

5.2 Validation of the Model

The new model was implemented in Matlab for the case of uniaxial loading histories. The choice of forward Euler numerical method of integration was considered to be sufficient in terms of computational simplicity and CPU requirements. Eqs.(19) were used for the uniaxial simulation by the model, often in a reduced form of only three back stress components chosen among the four accordingly. For the first two examples a repetitive routine has been used to determine the starting values of the material parameters, based on the least squares method. Fine-tuning of these starting values has been performed iteratively as the limited number of parameters still allowed for this. For the third and more thorough example, parameter calibration was connected to those of a back stress with a threshold, and the relations (23) were used to guide the calibration which was followed up by a fine tuning. Both strain and stress controlled derived experimental data were used for the validation of the proposed model.
5.2.1 Strain controlled cyclic loading

The data shown in Fig. 3 for a multi-step strain controlled symmetric cyclic loading experiment on 316L steel specimens reported by Chaboche at al (1979) reveal a response with the following basic features. The elastoplastic transition is smooth and the Bauschinger effect is evident. The peak stress increases with the number of cycles for each strain amplitude stabilizing at a level which in turn increases with the subsequent strain amplitude for the next set of cycling. This indicates an increase in the elastic range. One additional feature is that the level of peak stress stabilization for each strain amplitude appears to be independent of previous history as far as this history included stabilization under strain amplitudes smaller than the current one (Chaboche, 1986). The model parameters associated with Eqs. (19) for the simulation shown in Fig.4 are tabulated in Table 1, where $k_m$ is the initial value of $k$. For the isotropic hardening, the saturation value $k_s$ was set to be an increasing function of the multi-step strain amplitude as shown for discrete values of the amplitude in Table 1; clearly an analytical expression for $k_s$ in terms of the amplitude could be constructed easily, but it was not found necessary at this point where the focus is on the multiplicative scheme. The back stress components $a_1$ (Prager type) and $a_4$ in association with its multiplier $a'_4$ are used among the ones entering Eqs. (19), with the AF components $a_2$ and $a_3$ omitted. The model produces fairly acceptable simulation of the overall cyclic response but fails to represent accurately every single cyclic hysteresis loop, as it can be seen from the very first loading curve where the response is not adequately simulated.
Fig. 3. St 316L cyclic loading under piecewise increased strain amplitudes experimental data (experiment and figure after Chaboche et al (1979)).

Fig. 4. Simulation of the data of Fig. 2 for St 316L cyclic loading by the multiplicative AF kinematic hardening model. The model parameters are given in Table 1.

5.2.2 Stress controlled cyclic loading with ratcheting

A typical uniaxial stress-strain response from a ratcheting experiment on SS 304 is shown in Fig.5 (after T. Hassan, private communication). The cyclic mean stress is 5.2 ksi and the stress amplitude is 32.025 ksi. The back stress components $a_1$ (Prager type), $a_2$ (AF) and $a_4$ in
Table 1. Parameters for St 316L

<table>
<thead>
<tr>
<th>Elastic modulus</th>
<th>$E = 180$ GPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic hardening</td>
<td></td>
</tr>
<tr>
<td>Strain amplitude</td>
<td>$k$ (MPa)</td>
</tr>
<tr>
<td>-1.0 to 1.0</td>
<td>177.78</td>
</tr>
<tr>
<td>-1.5 to 1.5</td>
<td>222.22</td>
</tr>
<tr>
<td>-2.0 to 2.0</td>
<td>277.78</td>
</tr>
<tr>
<td>-2.5 to 2.5</td>
<td>333.33</td>
</tr>
<tr>
<td>-3.0 to 3.0</td>
<td>355.56</td>
</tr>
</tbody>
</table>

Kinematic hardening

<table>
<thead>
<tr>
<th>Prager</th>
<th>$h_1 = 3000$ MPa</th>
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</thead>
<tbody>
<tr>
<td>Multiplicative</td>
<td></td>
</tr>
<tr>
<td>Back stress</td>
<td>$a_4^s = 250$ MPa</td>
</tr>
<tr>
<td>Multiplier</td>
<td>$a_4^* = 3$</td>
</tr>
</tbody>
</table>

association with its multiplier $a_4^*$ are used among the ones entering Eqs. (19) with the AF components $a_3$ omitted, in order to obtain the simulations shown in Figs. 6 and 7, with parameters tabulated in Table 2. With no isotropic hardening, the constant value of $k$ is shown in Table 2. The model simulates accurately the shape of the cyclic curves, except the first loading curve, but it steadily under-predicts the ratcheting rate. Particularly it is noticed from Fig. 7, which shows the ratcheting in terms of plastic strain at positive peak stress per cycle versus number of cycles, that the model predicts a plastic strain which is approximately 0.05% to 0.13% (average of 0.07%) lower than the experimental one. This is deemed acceptable given
that the maximum plastic strain is approximately 1.55% and the error is much lower than the margin which derives from the applicable safety factor. The observed reduction in the rate of

Fig. 5. SS 304 uniaxial cyclic loading experiment, T. Hassan (private communication).

Fig. 6. SS 304 uniaxial cyclic loading simulation of data in Fig. 5 by the multiplicative AF kinematic hardening model. Model parameters are given in Table 2.
Fig. 7 Experimental and simulated ratcheting in terms of plastic strain at positive peak stress per cycle versus number of cycles for SS 304. Model parameters are given in Table 2. Data after T. Hassan (private communication).

<table>
<thead>
<tr>
<th>Table 2. Parameters for SS304</th>
</tr>
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<tbody>
<tr>
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<td>$k_s = k_m = k = 16$ ksi</td>
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<tr>
<td>$c_k = 0$</td>
</tr>
<tr>
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</tr>
<tr>
<td>Prager</td>
</tr>
<tr>
<td>$h_1 = 380$ ksi</td>
</tr>
<tr>
<td>AF</td>
</tr>
<tr>
<td>$a_2^s = 21.24$ ksi</td>
</tr>
<tr>
<td>$c_2 = 14$</td>
</tr>
<tr>
<td>Multiplicative</td>
</tr>
<tr>
<td>Back stress</td>
</tr>
<tr>
<td>$a_4^* = 33.5$ ksi</td>
</tr>
<tr>
<td>$c_4 = 14$</td>
</tr>
<tr>
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</tr>
<tr>
<td>$a_4^{*s} = 0.49$</td>
</tr>
<tr>
<td>$c_4^{*} = 2000$</td>
</tr>
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</table>

Ratcheting is due to the cyclic hardening feature of the material as explained in Hassan and Kyriakides (1994). While the material exhibited a small negative ratcheting during the first
cycle, in subsequent cycles strain ratcheting was positive. Given that this material exhibits significant cyclic hardening it can be concluded that the interaction between ratcheting and hardening in this case is relatively weak. Finally, we observe that the shape of the loops remained relatively unchanged as plastic strain increases.

5.2.3 Stress controlled cyclic loading with partial unloading/reloading and ratcheting

The final example is the most complete because almost all important features of a uniaxial cyclic experiment are considered, such as symmetric cyclic strain stabilized curves, partial unloading/reverse loading and ratcheting under variable mean and amplitude values of the stress. The corresponding data are taken form Bari and Hassan (2000). Most importantly the constants used by Bari and Hassan (2000) in order to simulate the data using an additive back stress decomposition with one back stress component having a threshold, are also used in order to find a first approximation of the constants of the multiplicative AF kinematic hardening model according to Eqs.(23) and (24).

Since the back stress with a threshold model is instrumental for the calibration of the multiplicative scheme, it is instructive to provide all relevant information for the former. Bari and Hassan (2000) used a four back stress components additive decomposition. The first three were of the AF type, one of them very close to a Prager linear model but not quite; the fourth had a threshold. The reasoning and methodology, including fine tuning, that one follows for calibration of such a model can be found in Chaboche (1991) and Bari and Hassan(2000), thus, they will not be repeated here. The corresponding constants for the threshold model, code-named C-H4T, modified to match the symbols used in this paper are as follows:

Saturation level constants (ksi): \( a_1^s = 3, \ a_2^s = 8.07, \ a_3^s = 41.4, \ a_4^s = 3 \)
Rate of approaching saturation constants: \( c_1 = 20,000, \ c_2 = 400, \ c_3 = 11, \ c_4 = 5,000 \)

Threshold associated with \( a_4 \) (ksi): \( \overline{a}_4 = 5 \)

In Bari and Hassan (2000) the \( h_i = c_i a_i^* \) and \( c_i \) symbolized by \( C_i \) and \( \gamma_i \), respectively, were given instead of the \( a_i^* \) and \( c_i \).

In the simulation with the multiplicative scheme that follows, all back stress components appearing in Eqs.(19) will be used, the first three being of the AF type and the fourth being of the multiplicative scheme with its associated multiplier. For the first three AF back stress components, the corresponding constants are taken identical to the ones of the first three components of the C-H4T model shown above, with a very small fine tuning modification; for the third back stress component the value of \( h_3 = 455 \)ksi is kept, but the value of \( c_3 \) is changed from 11 to 10, thus, also the value of \( a_3^* = h_3 / c_3 \) is changed from 41.4 ksi to 45.5 ksi.

In order to estimate now the values of the multiplicative scheme, we consider Eqs.(23) and (24) where one must simply substitute the subscript 4 for 1 because according to Eqs.(19) it is the fourth back stress component \( a_4 \) and its associated multiplier \( a_4^* \) which constitute the multiplicative scheme. One more important point of notation must be clarified associated with the threshold back stress. This is the fourth back stress in the C-H4T model and all relevant constants given above bear the subscript 4, while in Eqs.(23) and (24) no subscript was assigned to the three constants of the threshold back stress. Thus, Eqs.(23) and (24) must be considered with the following association: \( a' = a_4' = 3 \)ksi, \( c = c_4 = 5,000 \) and \( \overline{a} = \overline{a}_4 = 5 \)ksi.

The reader must also not confuse the above threshold back stress constants \( a_4^* \) and \( c_4 \), which are substituted notation-wise by \( a' \) and \( c \) in Eqs.(23) and (24), with the corresponding
identical symbols for the constants of the back stress $a_4$ of the multiplicative scheme which are to be calibrated in the following.

With the above clarifications on change of notation and the values of the threshold related quantities given as $a^* = 3 \text{ksi}$, $c = 5,000$ and $\tilde{a} = 5 \text{ksi}$, inequality (23d) yields a range of variation for $c_4^*$ (recall change of subscript from 1 to 4) as $4,687 < c_4^* < 5,769$. With the choice of the value $c_4^* = 5,000$, at about the average of the previous range, and the values of $a^*$, $c$ and $\tilde{a}$ as given above, Eqs.(24a) and (24b) yield the values $c_4 = 1,624$ and $a_4^{**} = 0.025$. A first attempt to use these values yielded reasonably good simulations. However, recall that the system of the relations (23) and the ensuing system of Eqs.(24) are based on many approximations and their solution attempts to obtain an estimate of the relevant constant values, in particular their order of magnitude. A fine tuning was done vis-à-vis some important experimental data for ratcheting (to be exactly specified in the sequel), based on which the values $c_4 = 1,800$ and $a_4^{**} = 0.16$ were decided which together with the $a_4^* = a + a^* = 8 \text{ksi}$ and $c_4^* = 5,000$ fully specify the four constants of the multiplicative AF 4th back stress component.

These constants together with the ones associated with the first three AF back stress components discussed earlier are tabulated in Table 3 and used to simulate the data shown in the multiple Figs. 8 and 9. In these figures also the simulations by the aforementioned threshold C-H4T model taken from Bari and Hassan (2000) are shown next to the multiplicative scheme for comparison. The details of the experimental procedure and conditions can be found in the aforementioned reference. The following observations can be made. The stress-strain curve of the threshold scheme in Figs. 8a and 8c shows a strongly linear portion at the initiation of a loading process as a result of the linear response within the threshold, while the multiplicative
scheme in the corresponding Figs. 8b and 8d has no such linear portion. This is due to the
different response of the component \( a_4 \) in the two schemes as seen in Figs. 8a and 8b where the
contribution of each back stress component is shown separately. Notice that the component \( a_4 \)

Fig. 8. Uniaxial experimental data for CS1026 specimens and simulations by the back stress
with threshold scheme (C-H4T model) and the multiplicative AF kinematic hardening scheme
for (a) and (b): symmetric strain controlled loading; (c) and (d): partial reverse
loading/reloading. Multiplicative model parameters are given in Table 3. Data and simulations
by the C-H4T model after Bari and Hassan (2000).
Fig. 9. Uniaxial experimental data for CS1026 specimens and simulations by the back stress with threshold scheme (C-H4T model) and the multiplicative AF kinematic hardening scheme for (a) and (b) ratcheting for fixed stress amplitude and various mean stress levels; (c) and (d): ratcheting for fixed mean stress level and various stress amplitudes. Multiplicative model parameters are given in Table 3. Data and simulations by the C-H4T model after Bari and Hassan (2000).

of the threshold scheme has a linear portion at loading or reloading initiation as sown in Fig. 8a, contrasting the smoother response of the component $a_4$ of the multiplicative scheme in Fig. 8b. Observe the similar improvement of the undershooting seen in the reloading curves in Figs.
8c and 8d achieved by the two models, one of the main reasons for introducing them; still the multiplicative does it in a smoother way. Finally the ratcheting data and corresponding simulations by the two models expressed in terms of plastic strain at peak of cycles versus number of cycles are shown in Figs. 9a and 9b for fixed stress amplitude $\sigma_{xa}$ and various non-zero mean stress levels $\sigma_{xm}$, and vice-versa in Figs. 9c and 9d.

<table>
<thead>
<tr>
<th>Table 3. Parameters for CS 1026</th>
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<tbody>
<tr>
<td>Elastic modulus               $E = 26300$ ksi</td>
</tr>
<tr>
<td>Isotropic hardening           $k_s = 18.8$ ksi $c_k = 0$</td>
</tr>
<tr>
<td>Kinematic hardening           $A_{F1}$ $a_1^* = 3$ ksi $c_1 = 20000$</td>
</tr>
<tr>
<td>$A_{F2}$ $a_2^* = 8.07$ ksi $c_2 = 400$</td>
</tr>
<tr>
<td>$A_{F3}$ $a_3^* = 45.5$ ksi $c_3 = 10$</td>
</tr>
<tr>
<td>Multiplicative                $\text{Back stress}$ $a_4^* = 8$ ksi $c_4 = 1800$</td>
</tr>
<tr>
<td>$\text{Multiplier}$ $a_4^{<strong>} = 0.16$ $c_4^{</strong>} = 5000$</td>
</tr>
</tbody>
</table>

It is worth mentioning here that the data from the middle curve of Figs. 9a or 9b (the data are identical in the two figures), i.e. the curve for $\sigma_{xa} = 32.0$ ksi and $\sigma_{xm} = 6.52$ ksi, were the ones used for the aforementioned fine tuning of the constants $c_4$ and $a_4^{**}$ of the multiplicative scheme model. All other ratcheting curves shown in both Figs. 9b and 9d are pure predictions and they were not used for fine tuning at all. It is interesting that some of the predicted curves are more accurately simulated than the one used for fine tuning, as for example the curve in Fig. 9d for $\sigma_{xm} = 6.5$ ksi and $\sigma_{xa} = 33.28$ ksi.
Comparing the two models one observes a slightly better simulation capability of the multiplicative scheme compared to the threshold scheme, with the exception of the case in Figs. 9c and 9d for $\sigma_{xm} = 6.5$ ksi and $\sigma_{xa} = 28.29$ ksi. This comes at the price of one additional constant, since a back stress with a threshold requires three constants (two for the AF model and one for the threshold) while the multiplicative scheme requires four constants, two for the back stress component and two for the corresponding multiplier. On the other hand the advantage of the multiplicative scheme is that it does not need to check whether or not a threshold has been exceeded, an issue of importance for implicit numerical implementation.

6. Conclusion

The multiplicative AF scheme is one refinement proposed for the classical Armstrong and Frederick (1966) non linear kinematic hardening model used in conjunction with the additive decomposition of the back stress proposed by Chaboche at al (1979). The scheme consists of enhancing the coefficient of the AF evolution rule which controls the pace at which a back stress component approaches its saturation level, by terms associated with the AF evolution rule of another dimensionless internal variable, called the multiplier. These enhancement terms depend on the direction of loading and the distance from saturation. The word multiplicative is adopted because such enhancement terms result in a multiplication of these two AF types of variables in the expression for the rate equation of the former (the back stress component). The second coefficient of the AF rule for the back stress component which defines the saturation level remains fixed and unchanged, thus, the multiplicative scheme does not alter the saturation level but only the pace of approaching it. It usually applies to one only of the three or four AF additive back stress components which are normally required. Such
variation of the coefficient allows for a special form of back stress – plastic strain curve that cannot be obtained by the summation of simple AF components. This special form provides a more abrupt change of the stress-strain slope than the one obtained with additive AF components without a simultaneous fast saturation. The relatively abrupt change of the slope occurs when the multiplier is saturated, and the multiplied back stress remains now a simple AF one. Upon reverse loading the multiplicative scheme activates again the multiplier and so forth. The formulation is presented in both the uniaxial and multiaxial stress space. The latter case is obtained by generalizing the uniaxial concept of stress “distances” between current and saturated states implied by the AF elements, the back bone of Bounding Surface Plasticity.

The multiplicative scheme is closely connected to the back stress with a threshold scheme proposed by Chaboche (1991) and elaborated further by Bari and Hassan (2000). In fact the calibration of constants for the multiplicative scheme can be based on the values of constants obtained for the threshold scheme. When this is the case, a systematic procedure for such calibration involving explicit analytical expressions helps to obtain a first but good estimate of the parameters for the multiplicative AF model, which upon fine tuning prove to be able to provide simulations of uniaxial cyclic experimental data, including ratcheting, that are slightly better than the ones obtained by the corresponding threshold back stress model as shown in Bari and Hassan (2000).

The presented multiaxial formulation is straightforward and its implementation follows standard procedures applied to other similar models, without the extra requirement to check the excess of a threshold. While no multiaxial examples have been worked out, it is expected that the response will be as successful as that of other models with possibly slight improvement in ratcheting, but with all relevant problems associated with the direction of kinematic hardening
of the AF type of back stresses it utilizes. The multiplicative concept can be in principle used in other formulations which do not necessarily use the additive back stress decomposition, because in essence it is a scheme that allows for a realistic variation of coefficients depending on the direction of loading.

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References


