A Purpose-Built Model for the Effective Teaching of Trigonometry: A Transformation of the van Hiele Model

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For the award of Doctor of Philosophy
Submitted to the University of Limerick, March 2015
Abstract

There is widespread acknowledgement that students in second and third-level education have difficulty with trigonometry. This is not only the case domestically in Ireland, but also internationally. Evidence exists that trigonometry is not being taught well at second-level. The fact that many teachers have not studied mathematics to degree level is contributing to this issue. Therefore, students are unprepared in trigonometry upon entering third-level education and fall further behind in their undergraduate mathematical studies. In addition, the teaching and learning of trigonometry is an under-researched issue worldwide.

The purpose of this research was to examine ways of improving the teaching of trigonometry, and to develop a purpose-built model of how to teach it effectively. The author developed a purpose-built model for the effective teaching of trigonometry in two stages. He first extended the van Hiele model of geometric thought to the specific branch of trigonometry, leading to a learning model for trigonometry. The second stage was to elaborate on this learning model to make it applicable to teaching trigonometry. A systematic teaching structure for trigonometry was developed with the use of APOS theory and genetic decomposition. Essentially, the author adapted a model of how people learn geometry, to a model of how to teach trigonometry.

This purpose-built teaching model was applied in the form of a teaching intervention with a group of 19 pre-service secondary mathematics teachers in order to investigate whether or not the model could aid in the development of trigonometric understanding. The research was guided by an Educational Design Research methodology which incorporated a proof-of-concept approach.

The teaching model and its incorporated teaching strategies were shown to have a positive effect on teaching trigonometric concepts for understanding. Pre and post-test findings indicate that the teaching intervention led to significant increases in understanding with reference to the teaching model. Through the proof-of-concept approach, the findings indicate that the teaching model could contribute towards better teaching of trigonometry at second-level.
Author’s Declaration

I certify that this project report is entirely of my own work and that it has not been submitted for any other academic award or part thereof, at this or any other educational institution.

Name: Richard Walsh

Signature:

Date:
Dedication

To Mam, Dad, Nana, Da, and Trevor for everything that they have done for me.
Acknowledgements

• To Olivia for everything that you have done for me over the past 3 to 4 years. It never mattered what you were doing, you always made time to help me when I needed any help or advice. I’m so thankful to you for all the time and patience you have shown me. I should also give a special mention to my ‘third supervisor’ Amber who sat in on many meetings!

• To John for teaching me so much more than just research related matters. You have shown me what it means to be an expert in the field and you are the best leader any of us in the NCE-MSTL could wish for. I’m truly grateful for everything.

• To my parents and family for all the support you give me no matter what I’m doing (and no matter how stressed out I get in the process).

• To Niamh for being there throughout. Calming me down when I need calming, and giving me a good laugh when things got stressful. I can’t thank you enough. You’ll be as glad to see the back of this as I will!

• To Claire, Edith, Rob and Beans for the laughs and nothing more. Yer good eggs like.

• To Jazz and Dragon. Weird but sound people. We’ll be number one in the charts next year Dragon.

• To everybody I’ve worked with in the NCE-MSTL: Aoife, Ciara, Dowdy, Fiona, Gráinne, Kathy, Lisa, Mark, Miriam, Niamho, Paddy and Páraic for making this place enjoyable and for showing me a level of expertise to strive towards.

• To the NCE-MSTL, the Dept. of Mathematics and Statistics in UL, and Cork County Council for all the help, both financial and otherwise.
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Chapter 1

Introduction

“The scientific principles that man employs to obtain the fore-knowledge of an eclipse, or of anything else relating to the motion of the heavenly bodies, are contained chiefly in that part of science that is called trigonometry...In fine, it is the soul of science. It is an eternal truth. It contains the “mathematical demonstration” of which man speaks, and the extent of its uses is unknown.”

(Paine, 1880, p.27)

1.1 Introduction

Developing the mathematical competence of young people is a critical factor for economic development and innovation in many countries (NCCCA, 2005; Government of Ireland, 2008). It is also necessary for a country’s future competitiveness on a global scale. In the coming years, Ireland needs to become more productive through growing its ‘smart’ economy. This is an economy that relies on ideas generated with the help of mathematics (O’Keeffe, 2010a). In order to achieve this, the mathematical knowledge of citizens needs to be developed.

Developing the mathematical understanding of young people is an enterprise that, in general, is undertaken by mathematics teachers in schools. Ireland is currently calling for improvement in the teaching of mathematics, as the mathematical understanding of Irish young people has been consistently substandard for many years (State Examinations Commission, 2000;
Procedural teaching approaches which rely heavily on the actions of carrying out rules and procedures, were the main teaching approaches practiced in Irish classrooms nationwide under the traditional curriculum (NCCA, 2005; Hourigan and O’Donoghue, 2007). Despite the implementation of Project Maths, a curriculum aimed at developing more understanding of mathematics, these teaching approaches have still been found to be widespread (Jeffes et al., 2013). Because of these approaches, Irish students’ understanding of mathematics is suffering at a national level (State Examinations Commission, 2000; 2001; 2005). Ireland is also falling behind other countries in relation to its young peoples’ mathematical knowledge and competence (National Center for Education Statistics, U.S. Department of Education, 2000; Gonzales et al., 2004; Gonzales et al., 2009). This is a cause for concern with respect to the future of our country, and its competitiveness on the world stage.

Trigonometry is an area of mathematics that many young people find difficult. Consistently poor understanding of this topic can be seen from reports on State examinations taken at the end of second-level education (State Examinations Commission, 2000; 2001; 2005). This insufficient grasp of trigonometric concepts travels with these students into third-level institutions and it is apparent that they are unprepared for the mathematics that they encounter there (O’Donoghue, 2002; Gill, 2006; Faulkner, 2012). This alarming trend is not happening just nationally in Ireland, but also internationally (Tickly and Wolf, 2000). A lack of mathematical understanding has negative repercussions for students’ further mathematical development, as deficiencies in understanding are not always rectified at third-level (Furinghetti, 2000; Weber, 2005). Research suggests that ineffective teaching at second-level may be the cause of the aforementioned issues (Lehrer and Franke, 1992; NCCA, 2005; Sherman et al., 2010).

There has been concern in recent years in Ireland with respect to the qualifications of mathematics teachers in the country, with many serving teachers of mathematics having little or no mathematics in their qualifications (Ní Ríordáin and Hannigan, 2009; 2011). In this thesis the author refers to teachers who do not have the required level of mathematics in their training as ‘teachers of mathematics’ and refers to teachers who have the required amount of mathematics in their training as ‘mathematics teachers’.
These issues indicate a significant problem in the mathematics education of Irish students and led the researcher to investigate if any methods/models of effectively teaching trigonometry were provided in the literature. Models of effective teaching of mathematics can have a positive effect on problem solving approaches (Ernest, 1989), cooperative learning (Rogoff, Matsusov and White, 1996), information processing of students (Joyce, Showers and Rolheiser-Bennett, 1987), and overall increased student learning (Joyce, Showers and Rolheiser-Bennett, 1987). However, the author did not find any model for the effective teaching of trigonometry in the literature that was reviewed for this research. Following the review of literature, the author created a teaching model that outlines how to teach trigonometry effectively. The creation of the model was achieved through an adaptation of the van Hiele model, a model for learning geometry. The teaching model that was developed is an extension of the van Hiele model and bears similar characteristics to it, along with suitable adaptations.

The author’s research addresses a gap in the mathematics education literature regarding the teaching of trigonometry by developing a purpose-built model for teaching trigonometry effectively.

1.2 Background to the Research

Irish teenagers (14-15 years old) have demonstrated poor understanding of mathematical concepts on international assessments such as the Programme for International Student Assessment (PISA) (OECD, 2010) and the Trends in International Mathematics and Science Study (TIMSS) (National Center for Education Statistics, U.S. Department of Education, 2000; Gonzales et al., 2004; Gonzales et al., 2009). The research reports that Irish students have demonstrated mathematical deficiencies in each PISA and TIMSS study. When the results of the studies are compared, it can be seen that the quality of Irish students’ understanding of mathematics has also decreased over the years since 2003 (OECD, 2010; 2011). In fact, the 2009 set of figures demonstrated the first time that Irish students scored below the OECD average. The NCCA (2005) stated that, in international terms, Ireland rated very highly in terms of memorisation and procedures. Ireland also trailed behind all other countries that took part in TIMSS studies with respect to applications of mathematics (National Center for Education Statistics, U.S. Department of Education, 2000; Gonzales et al., 2004; Gonzales et al., 2009).
Reports on older Irish students at the end of their secondary education (17-18 years old) have shown that this group exhibit deficiencies in their mathematical understanding. This is according to Chief Examiner Reports for Leaving Certificate mathematics examinations, which are national examinations undertaken at the end of secondary school in Ireland (State Examinations Commission, 2000; 2001; 2005).

Poor understanding of mathematics at second-level has been shown to have negative effects on Irish students’ progression and success in third-level mathematics (O’Donoghue, 2002; Gill, 2006; Gill et al., 2010; Faulkner, 2012). Irish students are unprepared for the mathematics they encounter at third-level due to their inadequate understanding of the foundations of the subject matter. Diagnostic assessments have been in place at the University of Limerick since the academic year of 1997/1998. These tests diagnose students who are at risk of failing first year mathematics modules in the University. Gill (2006) showed that first year students at the University of Limerick needed to be retaught logarithms, algebraic skills, trigonometry, geometry, differentiation, graphing lines and quadratics, and complex numbers after their progression to third-level if they were to succeed in their mathematical studies. Evidence also exists that the failure to understand the foundations of mathematics is not improved at third-level unless specific measures are put in place to improve them (Furinghetti, 2000; Weber, 2005). It is clear that students’ failure to understand mathematics at Leaving Certificate level results in an insufficient knowledge base to engage with mathematics at third-level. Therefore, many students never improve their understanding of mathematics and continue through life with mathematical deficiencies.

Reports demonstrate that issues exist for Irish students in mathematical topics such as geometry and arithmetic. These issues are highlighted in students’ responses to problems on domestic State examinations (State Examinations Commission, 2000; 2001; 2005). Trigonometry, however, is one of the most problematic areas of mathematics at second-level. Trigonometric problems have had a consistently low attempt percentage and consistently low average results in Leaving Certificate examinations (State Examinations Commission, 2000; 2001; 2005). It has been noted that Irish students take a procedural approach to mathematics. Trigonometry is favoured mainly by higher achieving students because lower ability students find it too abstract (State Examinations Commission, 2000; 2001; 2005; Gür, 2009). Chief Examiner Reports for Leaving Certificate examinations show that Irish stu-
Students at the end of second-level education do not understand:

- the basics of trigonometry;
- calculator use for trigonometry;
- trigonometric formulae;
- Pythagoras’ theorem;
- the concept of limits and working with limits;
- trigonometric identities;
- trigonometric functions;
- trigonometric ratios;
- solving trigonometric equations for all solutions;

(State Examinations Commission, 2000; 2001; 2005)

Along with this, it has been shown that Irish second-level students do not develop visualisation skills in trigonometry (State Examinations Commission, 2005; 2001; 2000). Visual representation of mathematics is one of the main reasons why trigonometry should be included in a syllabus (Mathematical Sciences Education Board (MSEB), 1990; Joint Mathematical Council of the United Kingdom, 1997). According to Chief Examiner Reports this is not being achieved in Irish classrooms. Therefore, Irish students are leaving second-level education with significant deficiencies in their understanding of trigonometric concepts as well as an inability to visually represent mathematical content.

Issues with students’ understanding of trigonometry are a cause of persistent problems at third-level (O’Donoghue, 2002; Gill, 2006; Gill et al., 2010; Faulkner, 2012), as students’ lack of understanding through second-level education is not always addressed (Furinghetti, 2000; Weber, 2005). Gill (2006) shows that Irish students entering third-level education were unable to (from a sample of 2121 students):

- solve for a missing angle in a triangle when given the measurements of two angles (3.3% gave wrong answer);
- calculate the area of a triangle when given the base and perpendicular height (37.2% gave wrong answer);
• find the length of a side in a right-triangle when given two sides (13.5% gave wrong answer);

• find the sine of an angle when given a diagram of a right triangle and all angles and sides (32.6% gave wrong answer);

• evaluate $\sin^2 66^\circ + \cos^2 66^\circ$ (73.2% gave wrong answer);

• write $90^\circ$ in radians (68.6% gave wrong answer).

Lecturers at the University have highlighted that a lack of foundational knowledge of trigonometry was an issue for their students’ progression in third-level mathematics (O’Donoghue, 2002). This particularly pertains to mathematical topics where trigonometry is a prerequisite, such as in calculus, which underlines most of third-level mathematics (Herriott and Dunbar, 2009). Many third-level mathematics modules, though they are undertaken in the first year of studies (e.g. Science Mathematics 1 at the University of Limerick), operate under the assumption that students begin with a high level of trigonometric understanding. The level of trigonometry in these modules is high from the outset and students quickly fall behind in their studies as a result of this.

Issues with students’ learning of trigonometry are not confined to Ireland. Studies, though few in number, have shown that problems also exist with trigonometry at second and third-level in other countries. The Cockcroft report (Cockcroft, 1982) shows that the problem with trigonometry in the U.K. is not a new one, dating back over 30 years. Cockcroft (1982) presents similar findings to Ireland, that third-level educators reported inaccuracy and low confidence with their students’ understanding of trigonometry. Blackett and Tall (1991) show that 14-15 year old students in the U.K. also struggle to learn trigonometry. Similar findings were observed at second-level in Turkey (Orhun, 2001; Tatar, Okur and Tuna, 2008; Gür, 2009), at third-level in the United States (Weber, 2005), and at third-level in Australia (Chinnappan, Nason and Lawson, 1996). One study from Turkey also illustrates the difficulty experienced by pre-service and in-service teachers in trigonometry (Topçu et al., 2006). These findings are similar to the findings on Irish students from PISA (OECD, 2010) and TIMSS reports (National Center for Education Statistics, U.S. Department of Education, 2000; Gonzales et al., 2004; Gonzales et al., 2009), as well as Chief Examiner Reports (State Examinations Commission, 2000; 2001; 2005).
The literature shows that the quality of teaching that a child receives is the most important contributor to the development of their mathematical understanding (Cockcroft, 1982; Fennema and Franke, 1992; Sanders, 1999; Wenglinsky, 2000). The National Council for Curriculum and Assessment (NCCA, 2005) has stated that teaching standards have to improve in Ireland. Traditional teaching methodologies in Ireland have been primarily orientated towards teaching procedures and rules (NCCA, 2005). Research shows that the level of subject matter knowledge that a teacher possesses has the strongest influence on the quality of their teaching in the mathematics classroom (Hourigan and O’Donoghue, 2007; Ball et al., 2008). Teachers with higher content knowledge have been shown to teach more for understanding rather than for routine procedures (Lehrer and Franke, 1992; Potari, Zachariades, Christou and Pitta-Pantazi, 2008). This is not to diminish the importance of other teacher skills such as pedagogical knowledge, or knowledge of students (Shulman, 1986; 1987; Rowland, 2007; Ball et al., 2008). Rather this points to subject matter knowledge as the foundation on which other teaching skills are built. In fact, Wu (2005) notes that high quality pedagogical practices can only be based on high quality subject matter knowledge. Recent research in Ireland has indicated that many serving second-level teachers of mathematics are out-of-field, meaning that they are teaching mathematics despite not having a recognised qualification in mathematics for teaching (Ní Ríordáin and Hannigan, 2009; O’Keeffe, 2010b; Ní Ríordáin and Hannigan, 2011).

The issues identified above regarding the difficulties trigonometry poses for students, in conjunction with the problems of insufficient teacher subject matter knowledge, inspired the author to conduct this research study. Trigonometry was highlighted as a particular area of concern in Ireland with second and third-level students having demonstrated a significant lack of understanding of trigonometric concepts in the past decade. The fact that many students find trigonometry an abstract topic means that only higher ability students engage with it. It is noted in the research that the only way for students to engage with and understand abstract content is to provide them with meaningful resources, concrete experiences, and focussed direction (Ausubel, 1973). Research therefore suggests that ineffective teaching is the cause of the issues faced by students as this is clearly not being done by all teachers. Teachers lack direction in the topic and research suggests that many teachers have an insufficient mathematical subject matter knowledge. This thesis explores the issues briefly described here in more detail, as well as critically analysing the causes of such problems. All of this culmi-
nates in the significant challenge of the research: developing a purpose-built model of how to effectively teach trigonometry. The processes involved in the production of the model, as well as proofing the model with a sample of pre-service second-level mathematics teachers and the subsequent findings, are presented and analysed in this thesis.

1.3 Scope and Significance of the Research

Research indicates that effective teaching is the most important factor in student achievement (Sanders, 1999; Wenglinsky, 2000) as it contributes more to learning than all other factors (Markley, 2004). Problems with student understanding of trigonometry have a detrimental effect on students' general mathematical understanding. It also negatively affects their further mathematical studies at third-level. Some remedy is necessary to assist teachers in their teaching of this topic. No purpose-built model for effectively teaching trigonometry was found in the review of literature for this research which would guide the teaching process of the topic. Though some advice has been offered on the teaching of trigonometry (Rajan et al., 1990), no comprehensive model was found. Direction was needed so that concepts are understood by students in the right order, and in the right way. Therefore, a model was needed that not only outlined what concepts to teach and when to teach them, but also how to teach concepts for conceptual understanding.

A model for the effective teaching of trigonometry is particularly important in light of the introduction of the Project Maths syllabus in Ireland in September 2010. This syllabus strives to make mathematics more applicable for pupils, and teachers are encouraged to foster a conceptual understanding of content in their classrooms (Project Maths, 2010). This challenges traditional teaching methodologies employed in Ireland which made mathematics procedural in nature, and made mathematics a highly abstract subject (NCCA, 2005). In order to achieve this, citing sources already mentioned, teachers with strong mathematical knowledge are required (Lehrer and Franke, 1992; Potari, Zachariades, Christou and Pitta-Pantazi, 2008) and methods of teaching topics for conceptual understanding are required.

This research provides a starting point for in-depth research on the trigonometric learning of young people, and ways to improve trigonometric understanding through effective teaching. These are under-researched areas in mathematics education (Weber, 2005; Demir, 2012). This thesis high-
lights that further research on the matter is necessary, not just in Ireland, but internationally. The model constructed in this research is a building block for future work and a valuable tool for researchers, as well as an educational model for teachers upon which their teaching of trigonometry can be based.

1.4 Research Intent

The author pinpointed a particular weakness in the trigonometric understanding of Irish secondary school leavers through a review of the literature. Ineffective teaching of the topic was identified as a major contributor to this problem. This research intended to respond to the issue of ineffective teaching of trigonometry by developing a model that would outline a systematic approach to teaching it at second-level. This model would need to encompass all aspects of trigonometric teaching such as what content should be covered, in what order concepts should be taught, how to teach concepts for a conceptual understanding, as well as allowing for the formative assessment of students. The model would therefore contribute to the mathematical education, and general educational, knowledge of teachers, providing a solid structure on which to base their teaching of trigonometry.

1.5 Aims and Objectives of the Research

The main aim of this research is to develop a model for the effective teaching of trigonometry. The model would therefore need to take into account the prerequisite subject matter knowledge of teachers before teaching trigonometry, the analysis of student understanding of trigonometry, factors that aid in developing conceptual understanding, and hence, the improvement of trigonometric understanding through various methods.

The objectives of the research based on the aims are:

• To review current literature in order to gain detailed insights into trigonometry and the teaching and learning of trigonometry, teacher knowledge required and its importance, as well as established models for effective teaching.

• To develop a research-based model for the effective teaching of trigonometry from which methods and materials for improving trigonometric understanding can be derived.
• To create a teaching intervention based on the teaching model.

• To administer this intervention to a sample of pre-service teachers in order to investigate if the model is valid with respect to effectively teaching trigonometry.

• To determine through an analysis of gathered data (both quantitative and qualitative), the impact that the intervention had on the trigonometric understanding of the subjects.

1.6 Research Questions

The research questions are aligned with the phases of the research (see section 1.7.1).

Phase 1: Literature Review

• Why is trigonometry included in second-level syllabi?

• How are Irish students and students in other countries performing in the topic of trigonometry?

• If performance is poor, then what is the cause for the poor performance over time?

• How does a teacher affect student learning? What aspects of a teacher’s knowledge base are most important?

• Do teachers currently have this knowledge base in trigonometry?

Phase 2: Developing a Model for the Effective Teaching of Trigonometry

• What models have been formulated in past research for the assessment of knowledge/understanding of a specific topic in mathematics/outside of mathematics?

• How can the model(s) be adapted/applied by the author for the effective teaching of trigonometry?

• Can a diagnostic assessment instrument for the adapted model be created?
Phase 3: Pre-Test of Intervention Sample

- What concepts that are relevant to junior and senior cycle trigonometry are the selected sample struggling to understand?

- Are any other significant findings emerging from the data relating to knowledge of trigonometry in general (e.g. in solving equations/constructing proofs)?

Phase 4: Intervention

- How can the findings from Phase 3 be addressed in the form of a teaching intervention?

- What approach(es) should be used in the trigonometry teaching intervention?

Phase 5: Post-Test & Evaluation of Intervention

- What changes have occurred after the intervention with regards to the sample’s conceptual understanding of trigonometry?

- Have the findings from the post-test highlighted the usefulness/limitations of any of the teaching strategies employed?

- Have the findings validated the model created?

- Can any of the teaching strategies be employed in a secondary school classroom setting? If so, how might they be employed?

1.7 The Research

1.7.1 Research Design

The phases of the research are presented in figure 1. This figure gives a diagrammatic display of the first cycle of the research which was conducted in 6 phases.
Began with Phase 1 Consisted of A review of literature of general issues related to mathematics education, namely Ireland’s current problems, and international problems. Particular attention given to trigonometry.

An in depth review of some models of teacher knowledge.

A review of research on teachers’ understanding of mathematics.

A detailed analysis of the van Hiele model and some methods of measurement.

Informed Phase 2 Consisted of Creating a purpose-built model for teaching trigonometry, and an assessment for conceptual understanding of trigonometry.

Informed Phase 3 Consisted of Assessment & evaluation of participants’ understanding of trigonometry.

Informed & assisted the development of Phase 4 Consisted of Intervention development & field testing.

Informed Resulted in Phase 5 Consisted of Evaluation of intervention.

Phase 6 Consisted of Discussion of findings.

Figure 1: Research Phases
Each phase of the research was guided by theory. Any tools or methods used were either adapted, or used directly, from the literature. The following table (Table 1) illustrates each phase of the research, the tool/method constructed for the purpose of the research, the main theory that underlined this tool/method, and how the tool/method was used in the research.

Table 1: Main Theories Used in Each Phase of the Research

<table>
<thead>
<tr>
<th>Phase</th>
<th>Tool/Method Constructed</th>
<th>Main Theory</th>
<th>Use in this Research</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Background to research</td>
<td>Bloom et al., 1956; Biggs and Collis, 1982; van Hiele 1984a; 1984b; State Examinations Commission, 2000; 2001; 2005; NCCA, 2006; Gill, 2006</td>
<td>Rationale and background information for the research. Insight into models that exist for developing understanding.</td>
</tr>
<tr>
<td>2</td>
<td>Teaching model and assessment</td>
<td>Usiskin, 1982; Mayberry, 1983; van Hiele, 1984a; 1984b; Burger and Shaughnessy, 1986; Crowley, 1987; Gutiérrez, Jaime, and Fortuny, 1991; Asiala et al., 1997; Dubinsky and McDonald, 2002</td>
<td>Development of teaching model based on the van Hiele model. Necessary adaptations through APOS theory.</td>
</tr>
<tr>
<td>3</td>
<td>Evaluation of assessment method</td>
<td>Gutiérrez, Jaime and Fortuny, 1991</td>
<td>Method of evaluating participants with respect to the van Hiele model and hence the teaching model constructed.</td>
</tr>
<tr>
<td>4</td>
<td>Intervention tools</td>
<td>van Hiele, 1984b; Lesh et al., 2000; Lesh et al., 2003; Moore and Diefes-Dux, 2004</td>
<td>Teaching stages for the intervention. Types of tasks for the intervention (Model Eliciting Activities (MEAs)).</td>
</tr>
</tbody>
</table>

These theories were not just random elements selected for use in the research. They were identified as valuable tools over the course of the research.
The researcher met many new challenges and obstacles which needed to be overcome throughout this study. The next figure (Figure 2) depicts how each of the theories fits in with the narrative of this thesis.
Figure 2: Theory Selected at Each Stage
1.7.2 Research Methodology

The overall methodological approach that was taken was based on the Educational Design Research model. This is a model where the implementation of multiple approaches is done with the “intent of producing new theories, artifacts, and practices that account for and potentially impact learning and teaching in naturalistic settings” (Barab and Squire, 2004, p.2). In relation to the challenge of this research which was developing a model for the effective teaching of trigonometry, Educational Design Research was a methodology that was fit for purpose. This is due to actions performed in this type of research being concerned with constructing theories, not just fine-tuning what has already been shown to work (Cobb et al., 2003). Educational Design Research follows three main phases. The description of the phases of Analysis, Design, and Evaluation, as well as their application to this research, are:

• Analysis - research and learning on the problem at hand and its causes;

  Application: Analysis, as outlined for Educational Design Research, was conducted in this research through an analysis of literature relating to current issues in the teaching and learning of trigonometry at second and third-level. Causes of the issues were investigated which led to the discovery of ineffective teaching as a potential root of the problem.

• Design - review of theory relevant to topic; various options considered; design created to try out in real scenarios;

  Application: Theories for improving learning in mathematics were researched which ultimately led to the van Hiele model being chosen as the most appropriate option (for other options considered see sections 2.13 and 2.14). The model for this research was then adapted from the van Hiele model.

• Evaluation - testing and revision of the design created and the ideas on which it was built.

  Application: The model was tested through the means of a teaching intervention with a sample of pre-service secondary mathematics teachers. The intervention and hence the model were then evaluated from the perspective of their effectiveness in student learning, and from the point of view of the students themselves.
(their opinions, insights etc.) in relation to learning from the teaching model. This led to future considerations for research into the model.

(Educause, 2012)

Educational Design Research consists of a repeating cyclical process of thought experiments and intervention experiments (Gravemeijer and Cobb, 2006). This doctoral thesis deals with the first cycle in this research (Figure 2) and concludes with findings to consider before the second cycle. The first cycle is conducted in 6 phases as stated in section 1.7.1.

General methods developed in the research were in keeping with research conducted into the van Hiele model of the development of geometric thought, in particular with the work of Gutiérrez, Jaime and Fortuny (1991). As the model that was constructed for the effective teaching of trigonometry was adapted from the van Hiele model, methods of assessment and teaching consistent with the van Hiele model were employed. Assessment, and evaluation of assessment strategies were in line with the work of Gutiérrez, Jaime and Fortuny (1991). This was a means of staying relevant to the adapted model. Teaching strategies for the model and hence the teaching intervention in this research were ultimately be taken from the work of the van Hiele’s themselves (van Hiele, 1984a; 1984b).

Under the umbrella of Educational Design Research, a mixed methods approach was adopted. A mixed, or multi-methods approach, is one where both qualitative and quantitative research approaches are undertaken (Cohen, Manion, and Morrison, 2007; Teddlie and Tashakkori, 2011). A mixed methods approach is endorsed for use in educational research due to its positive effect on triangulation (Cohen, Manion and Morrison, 2007). It therefore is influential in establishing concurrent validity, primarily in qualitative research, which is the “degree to which a measurement instrument produces the same results as another accepted or proven instrument that measures the same parameters” (Lipsett and Kern, 2009, p.123). The assessment tool which was administered to the research sample on two occasions (pre and post-intervention), was designed to retrieve both types of data, while focus group and journal entries from intervention proceedings were solely qualitative in nature.

This mixed methods approach informed the evaluation of the teaching intervention which was done through the use of Shapiro’s model for evalu-
ating interventions (1987). Shapiro’s model states that an intervention or treatment must satisfy four conditions in order to be deemed successful:

- Treatment effectiveness;
- Treatment integrity;
- Treatment acceptability; and
- Social validity

The intervention in this research was analysed with respect to these four conditions.

From the evaluation of the intervention the model for the effective teaching of trigonometry could be evaluated through a proof-of-concept approach. In order for a model to be proven effective it must “actually work as proposed” (Dym et al., 2009, p.165). In order to prove that the model produced in this research was effective in the teaching of trigonometry it was therefore necessary for the model to fit a theoretical framework in its design, and also work in situ. The model produced in this research was adapted from the van Hiele model. The performance of the model in situ was shown to be high during the evaluation of the intervention when analysing treatment effectiveness (Shapiro, 1987).

A detailed analysis of the methodological approaches used, as well as the methods employed, is provided in chapter 3.

A note on the sample selected

The sample that was selected for this research was third and fourth year pre-service secondary mathematics teachers at the University of Limerick. The third year group would form the intervention sample in the following year as the fourth year group would have graduated from the University. Pre-service teachers were found to have deficiencies in their understanding of mathematics (Chinnappan, Nason, and Lawson, 1996; Corcoran, 2005; Fi, 2006; Leavy and O’Loughlin, 2006; Delaney, 2010; Hourigan and O’Donoghue, 2013). It is also the case that pre-service mathematics teachers at the University of Limerick do not study a specific module on trigonometry in their course of study. Trigonometry is integrated into certain modules, such as in calculus modules, however, trigonometric concepts are not solely engaged with over a full module of study. Therefore, graduates from this course enter schools for
the first time without having deeply engaged with trigonometry since exiting secondary school. As discussed in section 1.2, students who finish secondary school do not understand trigonometry well. Along with this, subject matter knowledge is the most critical aspect of a teacher’s knowledge base. Therefore pre-service teachers were deemed appropriate subjects for the research due to them potentially having deficiencies in their trigonometric knowledge at the end of secondary school, not engaging deeply with trigonometry at third-level, and needing a high level of trigonometric knowledge for their future teaching.

1.8 Theoretical Framework

Effective teaching of trigonometry is an under-researched area (Weber, 2005; Demir, 2012). However, findings from recent years in Ireland and in other countries indicate that research into this area is warranted, with second and third-level students failing to grasp basic concepts in the topic (Blackett and Tall, 1991; Chinnappan, Nason, and Lawson, 1996; Kendal and Stacey, 1997; Department of Education and Science, 1999; State Examinations Commission, 2000; Orhun, 2001; State Examinations Commission, 2001; 2003; 2005; 2006a; Weber, 2005; Fi, 2006; Topçu, Kertil, Akkoç, Yilmaz, and Onder, 2006; Tuna, 2013). Findings from the research state that students find the topic highly abstract and that poor teaching is a significant cause of the problems that occur in learning trigonometry (Orhun, 2001; Gür, 2009).

The need for a remedy led the author to construct a model of how to effectively teach trigonometry. The theoretical framework integrates two theories: the van Hiele model and APOS theory. The author identified the van Hiele model (van Hiele, 1984a; 1984b) as one which could provide solid direction for the model to be constructed. The van Hiele model is a level-based hierarchial model primarily focused on how people learn geometry. It also gives teaching phases that one should follow in order to move learners up through the levels of understanding. This model would provide the base for the theoretical framework with many properties of the van Hiele model still keeping their place in the author’s model. However the author pinpointed certain deficiencies of the van Hiele model with respect to the model that he wished to construct:

- The van Hiele model was too general in relation to their levels for an application to trigonometry to be achieved. When consideration is paid to the significantly substandard understanding students have in
the topic of trigonometry it is clear that a very direct course of action is required for teachers to implement in their classrooms.

- The teaching phases provided by van Hiele's therefore would be difficult to implement without a systematic structure in place.

These deficiencies led the author to the second pillar of the theoretical framework, APOS theory (Dubinsky and McDonald, 2002). APOS theory is concerned with the ‘Actions’, ‘Processes’, ‘Objects’ and their organisation into ‘Schemas’ that, in tandem, determine the mathematical knowledge that one possesses. The use of this framework allowed the author to perform a ‘genetic decomposition’ on trigonometric schemas. What this means is that the concepts involved in trigonometric schemas were analysed. This in turn permitted a further adaptation of the van Hiele model in order to outline a teaching approach for trigonometry. This provided the direct course of study for trigonometry and the systematic teaching structure for content that the author desired. In other words, the genetic decomposition outlined what concepts to teach and when to teach them, where each concept would be taught through the teaching phases of the van Hiele model. The following figure (Figure 3) illustrates the use of the theoretical framework in order to achieve the desired teaching model.

![Figure 3: Use of Theoretical Framework](image)

1.9 Limitations of the Research

There were a number of limitations to the research conducted:

- Time constraints (assessment)

The assessment in phase 3 of the research (pre-intervention) was bound by time constraints. The author was aware of the commitments of the sample and had to ensure that every student in both
the third and fourth year pre-service teacher groups at the time of the assessment took part. This was to ensure that as much data as possible could be collected. If the assessment lasted longer than 50 minutes then many participants may have left to attend other classes. The assessment was conducted during a timetabled pedagogy lecture (with the third year group) and during a mathematical analysis lecture (with the fourth year group). Otherwise, the author may have compiled a longer assessment which may/may not have been more comprehensive. The author was also very aware of discouraging future participation amongst the third year group (at the time of the assessment) as they would form the group which would be voluntarily participating in the intervention.

- Time constraints (intervention)

Time was also a constraint for the intervention in two different ways. One constraint was in the time of year that the intervention could be conducted. The group with which the intervention was conducted were away from the university on teaching practice for the entire first semester (September 2013 - December 2013). Therefore the intervention could only take place from January 2014. The second way in which time was a constraint was that the participants who were voluntarily taking part had a very busy final semester with respect to course work for their own studies. The author was aware of this, having completed the same degree. Therefore the author did not want to stretch the intervention beyond week 4 of a 12-week semester in order to be mindful of the workload required from the students in their own degree work.

- Response rate

The response rate for the intervention was a concern for the author. As mentioned, the workload required from the potential participants in the intervention in their final semester of university was substantial. A group of 20 people took part in all intervention classes (19 of which had completed a pre-test and 1 had not). Five students who had completed the pre-test did not take part in the intervention classes, however three of this five did complete a post-test. One further students completed the pre-test, half of the required intervention classes, and the post-test.
1.10 Glossary of Terms Used in this Thesis

Some of the terms that are used in this thesis that relate to the Irish education system specifically, or this research specifically, are explained in this section.

- Primary school - Schooling from the approximate ages of 5-12 years.

- Secondary school - Schooling from the approximate ages of 12-18 years. Also referred to as post-primary education.

- Traditional curriculum - Post-primary mathematics curriculum pre-September 2010. This curriculum was strongly focused on procedures and rules. Mathematics tended to be highly abstract in nature.

- Project Maths - Post-primary mathematics syllabus for which implementation began in September 2010. This syllabus aims to increase conceptual understanding, knowledge of applications, and appreciation of mathematics through active learning methods.

- State examinations - Examinations for second-level education prepared by the Irish State Examinations Commission. Every student receives the same examination nationwide.

- Junior Certificate - First set of state examinations taken at the approximate age of 15 years. Examinations are completed after the third year in secondary school in approximately ten subjects. The first three years in secondary school is known as junior cycle.

- Leaving Certificate - Final state examinations taken after the sixth and final year of secondary school at the approximate age of 18 years. Examinations are completed in approximately seven subjects. CAO points are awarded to grades achieved and these points are used as the basis for gaining acceptance to university courses. The last three years of secondary school is known as senior cycle.

- CAO (Central Applications Office) points - Points system used for gaining admission to third-level courses. Points attributed to each grade achieved in the Leaving Certificate. The points for the best six grades achieved (regardless of how many subjects are taken in the examinations) are added to give the total amount of points scored for applying for third-level courses. A maximum of 600 points can be achieved.
• Higher, Ordinary & Foundation level - Students can complete each subject in their Junior and Leaving Certificate examinations at one of three levels: Higher, Ordinary or Foundation. The Higher level examinations are the most difficult out of the three levels. An A1 grade is worth 100 points in the Leaving Certificate if the subject was studied at Higher level and the number of points decreases as the grade gets lower (e.g. An A2 is awarded 90 points). Ordinary level is the next step for students who may find Higher level too challenging. An A1 is worth 60 points in the Leaving Certificate if the subject was studied at Ordinary level. Foundation is the remaining choice if Higher and Ordinary level are too challenging. An A1 is worth 20 points if the subject was studied at Foundation level.

Table 2: Leaving Certificate Points System Based on Grade Achieved at Each Level

<table>
<thead>
<tr>
<th>Leaving Certificate grade</th>
<th>Higher Level</th>
<th>Ordinary Level</th>
<th>Foundation Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>100</td>
<td>60</td>
<td>20</td>
</tr>
<tr>
<td>A2</td>
<td>90</td>
<td>50</td>
<td>15</td>
</tr>
<tr>
<td>B1</td>
<td>85</td>
<td>45</td>
<td>10</td>
</tr>
<tr>
<td>B2</td>
<td>80</td>
<td>40</td>
<td>5</td>
</tr>
<tr>
<td>B3</td>
<td>75</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>C1</td>
<td>70</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>C2</td>
<td>65</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>C3</td>
<td>60</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>D1</td>
<td>55</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>D2</td>
<td>50</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>D3</td>
<td>45</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

If mathematics is studied at Higher level an extra 25 points is awarded regardless of grade received. This is part of the Project Maths initiative to encourage students to study mathematics at Higher level.

• Out-of-field teachers - Defined as “teachers assigned by school administrators to teach subjects which do not match their training or education” (Ingersoll, 2002, p.5). Ní Órdáin and Hannigan (2009) state that this is someone who in general, has a teaching qualification,
but has little to no training with respect to mathematics/mathematics education.

- Pre-service teachers - In this thesis pre-service teachers refers to university students who are studying and training to be mathematics teachers.
1.11 Outline of Chapters

This thesis is outlined as follows:

- **Chapter 1: Introduction** gives a brief rationale for the research that was conducted, as well as a brief description of the research itself. The scope and significance of this research to the Irish context as well as the field of mathematics education was also discussed briefly. The chapter also gives a brief description of the methodologies that were employed to guide the research in an attempt to answer the research questions and achieve the overall aim. A brief explanation of the terms used throughout the thesis is also supplied in this chapter.

- **Chapter 2: Part 1 - Current Issues in the Teaching and Learning of Trigonometry** gives a detailed analysis of trigonometry and trigonometric teaching and learning at second-level education in Ireland and abroad as found in a review of literature. The chapter also analyses the effect of teacher knowledge on instruction and on teaching for understanding. Attention is paid in this chapter to findings from literature on the level of subject matter knowledge possessed by teachers and consequences of this.

- **Chapter 2: Part 2 - Models for Understanding Mathematical Development** describes and critically analyses various models such as the van Hiele model, Bloom’s taxonomy, and the SOLO taxonomy, that were found in the review of literature. These models demonstrate an understanding of mathematical development/understanding, or an understanding of learning in general.

- **Chapter 3: Methodology** outlines the methodologies and methods used in this PhD. research. The methodological stances that guided the research as well as the procedures for the creation and analysis of the intervention and assessment are described.

- **Chapter 4: A Purpose-Built Model for the Effective Teaching of Trigonometry** describes in detail the teaching model created by the author, using an adaptation of the van Hiele model. The chapter outlines the purpose of the model, the details of the model and the characteristics of each level within it.

- **Chapter 5: Discussion of Results from the Pre-Intervention Test** presents the findings from the pre-intervention test from Phase 3 of the research. The analysis of the data placed each participant on a level of
the model described in chapter 4, and hence uncovered where an intervention would begin for each participant. The assessment ensured the reliability of the model and assessment through Guttman Scalogram Analysis. This procedure is described in detail.

- **Chapter 6: Intervention Development and Administration** discusses in detail the considerations and actions appraised in the production and development of the intervention resources and classes in relation to the model produced in the research.

- **Chapter 7: Discussion of Results from the Post-Intervention Test** discusses the findings from the post-intervention assessment. It looks at any changes that occurred to the level of understanding of the participants after the intervention was completed.

- **Chapter 8: Thesis Contributions and Future Work** discusses the contribution of this thesis to mathematics education research as well as the potential for applying this research in Ireland and abroad in order to improve the teaching of trigonometry. The potential future work of the author relating to this particular research, and an extension which includes applying the methods produced here to other topics in mathematics, is also described.
Chapter 2

Part 1 - Current Issues in the Teaching and Learning of Trigonometry

2.1 Introduction

This chapter provides a review of literature relevant to this piece of research. A literature review is “a systematic, explicit, and reproducible method for identifying, evaluating, and synthesising the existing body of completed and recorded work produced by researchers, scholars, and practitioners” (Fink, 2010, p.3). This literature review discusses trigonometry at second and third-level, including the performance of Irish students and international students in the topic. Chapter 2 also looks at important contributors to effective teaching. The role that teaching plays in developing students’ mathematical understanding as well as some critical skills of teachers that were found in the literature are examined. Models for understanding conceptual development are also discussed in the second part of the chapter.

Chapter 2 is divided into two parts. The author designed the literature review in this way due to the research focusing on two different areas. The first area was trigonometry at second and third-level. After issues were identified a second field of literature was investigated, namely models for understanding mathematical development. Each of these areas of research were given their own part in this literature review to stress the individual importance of each of them.
Part 1 of chapter 2 demonstrates the problems that students experience in learning trigonometry. This is not just shown to be an issue in Ireland, but also internationally. Part 1 starts with foundational issues such as the rationale for trigonometry being included on second-level syllabi, and moves on to discuss the problems that students have with the topic at second and third-level. The reasons why students find trigonometry so difficult are discussed. A strong focus on poor teaching of the topic arose from the literature, as well as the need for increased subject matter knowledge amongst teachers of mathematics. The chapter makes it clear that a model for effective teaching of trigonometry is needed. No such model was found in the review of literature.

Part 2 of this chapter builds on the need for a model for teaching trigonometry by discussing some models for understanding conceptual development. The van Hiele model is the primary focus of discussion as it was ultimately chosen to be the most appropriate for adaptation in this research. The author discusses a large amount of the literature available on the van Hiele model and reports on aspects of the model such as the levels of the model, properties of the model, measurement techniques for the model, amongst others. Some criticisms of the van Hiele model are also discussed as the author needed to consider all dimensions of the model before an adaptation could be developed. Bloom’s taxonomy and the SOLO taxonomy are also discussed in this chapter as they were other models considered during the research. The author demonstrates why the van Hiele model was chosen instead of these.

2.2 Trigonometry - What is it?

Trigonometry comes from the Greek words ‘trigōnon’ (triangle) and ‘metron’ (measure) and is a topic in mathematics that deals with the relationships between the side lengths and angles in a triangle. The Greek mathematician Hipparchus began the development of trigonometry in his attempts to apply geometry to astronomical studies circa 120 B.C. (Rouse Ball, 2010). Trigonometry, in its roots, is therefore a branch of geometry and is still seen in this way today. It has been extended by historical figures such as Ptolemy (90 A.D.-168 A.D.) and Copernicus (1473-1543), amongst others. Despite most people having heard of Pythagoras, his work predates the beginning of trigonometry in Hipparchus’ time, with Pythagoras having lived from about 569 B.C.-500 B.C.. During the lifetime of Pythagoras trigonometry was still referred to
as geometry. It was not deemed an individual branch of geometry until the circulation of Hipparchus’ work.

The English born revolutionary Thomas Paine wrote that trigonometry is “soul of science. It is an eternal truth. It contains the “mathematical demonstration” of which man speaks, and the extent of its uses is unknown” (Paine, 1880, p.27). This quote demonstrates the power of trigonometry not just to trivial matters, but to the universe as whole. It is a key to understanding aspects of the world we live in, and the cosmos beyond our world. Trigonometry has applications in almost every area of life and work. It is used in areas such as; construction, in finding measurements for ceiling joists; music technology, in creating effects for instruments; and navigation, for pilots.

Though the history of trigonometry is rooted in navigation and astronomy, modern life and work would not function without it. All branches of physics use trigonometry, from the early fields of optics and statics (Joyce, 2013), up to current strides in ‘modern physics’. The importance of trigonometry to physics at third-level can be seen in module descriptors which classify trigonometry as a prerequisite for various physics modules (e.g. Boston University, 2014). Trigonometry also plays a role in modern day music production and music technology, with specific emphasis on sine functions as can be seen with the Adrenalin unit (Anderton and Linn, 2002). Music, Media and Performance Technology students at the University of Limerick study trigonometry and trigonometric functions as part of Year 1 in their course of study (University of Limerick, 2014). The list is endless.

Trigonometry is included in every mathematics syllabus in the world and the next section analyses the rationale for studying trigonometry at second-level.

2.3 Why Study Trigonometry at Second-Level?

Trigonometry is an important part of every mathematics syllabus in the world (Delice, 2002) and there are valid reasons as to why it is included in syllabi. Research indicates different reasons why a mathematical topic should be included in a school syllabus. It is stated that a topic that highlights connections between “different branches of mathematics, and between mathematics and other disciplines” (Leitzel, 1991, p.3) should be included
in a syllabus. The idea of connections is a central issue, as a mathematical topic studied at school level should aim to enhance a young person’s conceptual framework of the subject. This is achieved by linking various topics and concepts to each other. The National Council of Teachers of Mathematics (NCTM, 2014) standards state that primary and secondary geometry curricula should aim to develop analysis skills of characteristics and properties of two and three-dimensional shapes, to specify locations and develop spatial understanding in terms of co-ordinates, analyse transformations and use symmetry to analyse mathematical situations, and to develop visualisation and spatial reasoning. Some other recommendations for the inclusion of mathematical topics include ensuring that the chosen topic develops:

- notation and understanding the importance of notation;
- working to precise definitions, learning the meaning of mathematical language;
- flexibility of viewpoint and a fluency to transform to different representations;
- generalisation;
- concepts of proof.

(Joint Mathematical Council of the United Kingdom, 1997, pgs.20-21)

This is just a selection of recommendations in the document mentioned. Elsewhere, similar recommendations are noted. The Mathematical Sciences Education Board (MSEB, 1990) state that a mathematics syllabus should develop representation, operations, and interpretation skills through the topics of algebra, geometry, data analysis, discrete mathematics, and optimisation. This is done in conjunction with a focus on real applications of the content. It is stated that these skills and content areas would prepare students for the world of work, university, and citizenship. It can be seen that trigonometry not only fulfils the recommendations of Leitzel (1991) and the Joint Mathematical Council of the United Kingdom (1997) but is also a branch of geometry whose inclusion was advocated by the MSEB (1990). The rationale for the inclusion of trigonometry is one which is recognised and upheld worldwide (Delice, 2002), with trigonometry being an important part of syllabi in countries such as the United States (Kirst, 2000), Australia (Australian Curriculum, 2014), and Singapore (Ministry of Education, 2006), amongst others.
Ireland is no different in including trigonometry as a topic of study at second-level. The next two sections analyse the approach to trigonometry in the traditional Irish syllabus, which was in place until September 2010, and the current ‘Project Maths’ syllabus. The content to be covered in schools and the examination procedures for both syllabi are analysed in order to provide insights into trigonometry in the Irish school system.

2.4 Approaches to Trigonometry in the Traditional Irish Curriculum

Trigonometry played a role in the traditional Irish secondary school mathematics syllabus. It was one question on both Junior and Leaving Certificate State examinations, and was placed in the second of two examination papers which were sat separately. In Junior Certificate examinations after 1999 trigonometry was a mandatory question as pupils had to attempt all questions. Trigonometry was not a compulsory topic in the traditional Leaving Certificate syllabus. It only needed to be covered if a teacher wished to cover the entire syllabus with their students (NCCA, 2000). In Leaving Certificate examinations pupils had a choice of five questions to be attempted from seven, one of which was trigonometry. Reports from the time of the traditional curriculum showed that Leaving Certificate students favoured eliminating trigonometry through this choice.

“For years, topics such as trigonometry were left out by candidates - showing their lack of understanding, only to realise they needed to understand it more fully for Leaving Cert.”

(NCCA, 2006, p.12)

“Many students can completely disregard whole sections of the course, e.g. trigonometry, and still enter a third level college where this can be a major component of their required knowledge.”

(NCCA, 2006, p.19)

The nature of these reports suggest that there was a reluctance by teachers to teach pupils to engage with trigonometry in their examinations and in any revision they did. The consequence of this was that many pupils progressed to third level education without the prerequisite knowledge of this
topic (O’Donoghue, 2002).

On State examinations based on the traditional\(^1\) Leaving Certificate curriculum the questions asked of Irish pupils were largely procedural and predictable (NCCA, 2006; Breen, Cleary and O’Shea, 2009; Liston and O’Donoghue, 2010). Teachers focussed on procedural approaches to teaching as they were able to predict what questions were required of students in their examinations (NCCA, 2006). The application of concepts to real-life scenarios was almost non-existent (NCCA, 2006) and an instrumental understanding\(^2\) of content was assessed more than a relational understanding\(^3\) (Skemp, 1976; NCCA, 2006). The traditional focus on instrumental understanding in examinations led to a view by Irish students that instrumental understanding was the most important element in being a good mathematician. This was found to be the case even after they progressed to third-level (Breen et al., 2009; Liston and O’Donoghue, 2010).

Syllabi that are implemented by the means of strongly instrumental approaches have negative effects on student learning, both the learning of content and the acquisition of views of mathematics as a subject (Cockcroft, 1982; Hiebert and Lefevre, 1986; NCCA, 2005; Breen et al., 2009; Liston and O’Donoghue, 2010). The Irish public fears mathematics and believes that the subject is too challenging (Maths Week Ireland, 2011). This is the societal view of those who were instructed in traditional ways where mathematics is:

> “problems to solve, or a method of calculation to explain, or a theorem to prove. The main work will be done in writing, usually on the blackboard. If the problems are solved, the theorems proved, or the calculations completed, then the teacher and the class know they have completed the daily task.”

(Davis and Hersh, 1981, cited in Ball et al., 2001, p.434)

Any issues that arise in the traditional classroom are dealt with by the teacher through the means of explaining the exact same content again at a slower pace (Davis and Hersh, 1981, cited in Ball et al., 2001). Traditional

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\(^1\)Traditional syllabus: Pre-Project Maths

\(^2\)Instrumental understanding relates to procedures in mathematics (Skemp, 1976)

\(^3\)Relational understanding is the understanding of concepts and the links between different concepts in mathematics (Skemp, 1976)
classroom methodologies do not incorporate any form of exposition (Cockcroft, 1982), which is conversation initiated by the teacher with the aim of developing students’ mathematical understanding. Meaningful learning, the learning and understanding of relationships between concepts in mathematics, is not accomplished. Rote learning, the learning of context-specific applications of rules, is the main focus in traditional teaching methodologies (Hiebert and Lefevre, 1986). This contributed to the findings of the NCCA (2005), that in international terms, Ireland rated very highly in terms of memorisation and procedures, and came last place with respect to applications of mathematics.

These findings led to an NCCA-led initiative that implemented strategies that aimed to increase students’ understanding of mathematics at second-level. A national change was implemented in September 2010 with the introduction of the current Project Maths syllabus for secondary schools.

### 2.5 Trigonometry on the New ‘Project Maths’ Irish Curriculum

The Project Maths syllabus strives to make mathematics more applicable to pupils’ lives, and teachers are encouraged to foster a conceptual understanding of content in their classrooms (Project Maths, 2010). This runs counter to traditional teaching methodologies employed in Ireland which made mathematics as a subject highly abstract and procedural in nature (NCCA, 2005). Project Maths embodies an approach which places greater emphasis on developing students’ understanding of mathematical concepts (Project Maths, 2010). As part of trying to develop students’ understanding, teaching methodologies that highlight real-life applications are used to enable students to relate mathematics to everyday experiences (Project Maths, 2010). Similarities can be noted between the goals of Project Maths and the work of the NCTM (2014), Joint Mathematical Council of the United Kingdom (1997), Leitzel (1991), and the MSEB (1990) (section 2.3), as well as adhering to the recommendation of the Office for Standards in Education, Children’s Services and Skills to increase the use and application of mathematics in the U.K. (Ofsted, 2012).

Trigonometry is a significant part of the Project Maths secondary school syllabus currently being implemented in Irish schools. The Project Maths syllabus is composed of five strands:
1. Statistics and Probability;
2. Geometry and Trigonometry;
3. Number;
4. Algebra;
5. Functions.

(Project Maths, 2010)

In State examinations under the Project Maths syllabus, all questions are mandatory. In other words, there are no choices to be made by students when preparing for State examinations as every question must be answered. This requirement means that pupils have to engage with every strand, and ideally, every concept within each strand. This means that trigonometry must be covered in depth by every pupil, a different requirement to that of the traditional pre-Project Maths syllabus (NCCA, 2000). Along with the mandatory nature of trigonometry at second-level, the content that is to be covered in trigonometry under the Project Maths syllabus is greater in quantity compared to that on the traditional syllabus. Most concepts have carried over from the previous syllabus, however the approaches taken in teaching these concepts have changed due to the applied and relational features of Project Maths (Project Maths, 2010). Recent findings indicate however that teachers are still teaching using traditional methods (Jeffes et al., 2013). One extra set of concepts included in the new curriculum however, are those found in trigonometric functions (NCCA, 2011; Treacy and Gallagher, 2013). It is stated that pupils at Leaving Certificate Higher level should be able to:

- graph the trigonometric functions sine, cosine, tangent;
- graph trigonometric functions of type
  \[ f(\theta) = a + b\sin c\theta, \]
  \[ g(\theta) = a + b\cos c\theta, \]
  for \( a, b, c \in \mathbb{R} \)

(NCCA, 2011, p.23)

The point was made that these two aspects of the syllabus are gaining greater importance on examinations than before (S. Clowry, personal communication, 23rd November 2013). The amount of time that needs to be invested
into the concepts involved in these objectives reflects the significant increase in the depth of trigonometric content to be covered. Again, these must be taught in a manner that promotes a conceptual understanding, and teachers and pupils cannot rely on procedures and instrumental understanding alone. A note must be made that the topics of vectors and matrices were removed from the traditional syllabus for Project Maths in order to allow for extra teaching time and extra time spent developing students’ conceptual understanding of content.

Taking the syllabi in Ireland (both past and present) into account, we can look at the performance of Irish students in trigonometry in State examinations over the past few years. First, an analysis of the literature on trigonometric understanding from other countries is conducted. The performance of Irish students is then compared to our international counterparts as a means to view how Irish students are performing in international terms.
2.6 Student Performance in Trigonometry

2.6.1 International Data

There is not a very extensive literature on the teaching of trigonometry per se (Weber, 2005; Demir, 2012). However, some research exists on the trigonometric understanding of students at second and third-level. This research, though small in quantity, supports the findings from Ireland that trigonometry is an area of mathematics that students find difficult. Much of the research that exists however, is based on measuring the effects of different teaching methods in trigonometry through the use of control and experiment groups rather than specifically analysing the problems that students have in trigonometry (Delice, 2002; Cavey and Berenson, 2005; Weber, 2005). The literature states that trigonometry is an area that students struggle with, but often fails to specify what exactly causes problems.

Where research exists it can be seen that trigonometry is not understood to a high degree by pupils in other countries. Similar issues to those found in the Irish context have been noted in Turkey (Orhun, 2001; Gür, 2009) for pupils between 14 and 17 years of age (similar age to Irish secondary pupils), and for teachers (both pre-service and in-service) of mathematics (Topçu et al., 2006). Gür (2009) found that second-level Turkish students find trigonometric concepts highly abstract and rely strongly on rote learning from textbooks. Orhun (2001) states that the fundamental concepts of trigonometry are not understood by second-level students. This does not allow for the learning of more complicated content and also inhibits success in first-year college calculus courses (Orhun, 2001). The concept of a radian is also pointed to in Turkish literature as something that is not understood visually by students (Orhun, 2001 (second-level); Topçu, Kertil, Akkoç, Yilmaz, and Onder, 2006 (pre-service and in-service second-level teachers); Tuna, 2013 (pre-service second-level teachers)). This lack of understanding of the radian concept affects their understanding of more advanced concepts such as trigonometric functions. Some research emanating from Turkey states that trigonometry is one of the most difficult areas of mathematics for second-level students to understand (Tatar, Okur and Tuna, 2008).

In the United States, mathematics education is experiencing difficulties. Problems in the mathematical achievement of US students have been well documented over the years (Stevenson et al., 1986; Peterson et al., 2011;
Vigdor, 2013) and didactic methodologies in the teaching of mathematics have been found to be widespread (Woodward, 2004). Trigonometry in the United States is again an area that students find difficult, even at third-level (Weber, 2005). Weber (2005) found that, in a pre-test before an instructional intervention, that 26 out of 40 undergraduate college students were unable to answer any of four questions based on trigonometric functions:

1. What is the range of \( \sin x \)? Why?
2. What is \( \sin 90^\circ \)? What is \( \cos 270^\circ \)? Why?
3. In what quadrants is \( \cos x \) positive? Why?
4. In what quadrants is \( \cos x \) increasing? Why?

(Weber, 2005, p.98)

Weber (2005) notes comparisons to the work of Kendal and Stacey (1997) who demonstrated that 172 of 178 second-level students were unable to successfully answer any question on trigonometric concepts, one year after they had completed a course in trigonometry. Another study that was conducted on pre-service secondary school teachers’ knowledge of trigonometry in the United States (Fi, 2006) found that these pre-service teachers had an inadequate knowledge of trigonometry. The participants’ conceptual organisation of co-functions (i.e. sine & cosine, tangent & cotangent) was clouded by other concepts such as inverse trigonometric functions.

Reports from England (Blackett and Tall, 1991) have also shown the difficulties faced by 14-15 year old students in the learning of trigonometry, as demonstrated by student scores on a pre-test before an instructional intervention based on the use of computers. However, no conclusions as to what students find difficult in trigonometry can be drawn as the research of Blackett and Tall (1991) provides no sample of pre-test questions or any discussion on the difficulties students had with particular concepts. Ofsted (2012) reported more recently that second-level U.K. students do not possess strong conceptual understanding or fluency in trigonometry, and that they have low confidence in trigonometric calculations.

Chinnappan, Nason, and Lawson (1996) found that pre-service secondary teachers in Australia failed to notice the link between trigonometric concepts such as the relationship between tangent, sine, and cosine ratios, and the
relationship between Pythagoras’ theorem and the coordinate system.

The available information from the literature demonstrates that issues for students on the topic of trigonometry stem from inadequate understanding of the basic concepts of trigonometry (Tuna, 2013), as well as algebraic issues (Parish and Ludwig, 1994). Research indicates that poor teaching of the topic is often the root of the issues mentioned (Orhun, 2001). The research notes that trigonometry needs to be taught through the use of visual aids and graphs which are related to numerical relationships, and that this is not occurring in classrooms (Weber, 2005).

It must again be stated that research explicitly based on trigonometric knowledge/understanding is sparse. In fact, not much more applicable research was found that relates to the understanding of trigonometry at second-level in other countries. Nevertheless, though the research into the teaching and learning of trigonometry is sparse, it can be seen that issues do exist in other countries. Now the focus is turned back to Ireland with respect to Irish students’ performance in trigonometry in State examinations.

2.6.2 The State of Trigonometry in Irish Schools

Chief Examiner Reports are reports published every year by Chief Examiners in which they analyse the performance of Irish pupils in various subjects on State examinations. Each year certain subjects are reported on, i.e. not every subject is analysed every year. Chief Examiner Reports on mathematics were published in 2006, 2003 and 1999 at Junior Certificate level and in 2005, 2001, and 2000 for mathematics at Leaving Certificate level. This section will discuss the findings of these reports in order to give an insight into the trigonometric understanding of Irish second-level students. Though the following sections rely more on discussion than on the critical analysis of literature, the discussions are important as they display the issues that exist at second-level in Ireland with respect to students’ understanding of trigonometry.

Junior Certificate Examinations (Lower Secondary)

Three Chief Examiner Reports that report on Junior Certificate mathematics are available. The reports from 2006 (State Examinations Commission, 2006a) and 2003 (State Examinations Commission, 2003) are based on find-
ings from Higher, Ordinary and Foundation level examinations. The report from 1999 (Department of Education and Science, 1999) is based on Higher and Ordinary level examinations only. These reports show that trigonometry is an area that young Irish teenagers (approximately 15 years old) find difficult. Note that all of the questions on the Junior Certificate examinations from 2006 and 2003 were mandatory, but students had a choice in 1999 as it was based on an older syllabus. In 1999 students had to do the first question on each of the two examination papers and had a choice of four questions to be completed from the remaining five.

The Chief Examiner’s Report from 2006 (State Examinations Commission, 2006a) specifies trigonometry as one of the main areas of concern for Higher level Junior Certificate mathematics. The Chief Examiner’s concern was that students demonstrated significantly low levels of trigonometric knowledge and understanding. The trigonometry question was question 5 on the second Higher level paper (total of two examination papers). Students achieved an average mark of 29.3 out of 50, making it the worst answered question on the second paper and the second worst answered question overall. An algebra question on the first examination paper had an average mark of 27.1. Students’ understanding of basic concepts in trigonometry was one of the most concerning findings from the reports. Over half of Higher level students were not able to complete the following problem:

- Without using a calculator or the tables, construct the angle $A$ such that
  \[ \tan A = \frac{3}{4} \]

  Students were unable to infer the existence of a right angle. This is the fundamental idea of trigonometric ratios. If this is not understood, then any further knowledge on trigonometric ratios is built on a weak foundation, and is probably rote learned (Skemp, 1987). Other findings from Higher level in 2006 include that students:

  - incorrectly used the area formula ($A = \frac{1}{2}ab \sin C$) by only using one side in their calculations;
  
  - assumed the existence of right angles in triangles where they did not exist;
  
  - did well on procedural questions, but many could not do problems where any problem-solving approaches were needed;
The trigonometry question at Ordinary level in 2006 was question 6 in the second paper (total of two examination papers). It was the second worst answered question on that paper with students achieving an average mark of 30.4 out of 50. Geometry was the worst answered question on the paper (average of 29.2). Despite the fact that the trigonometry question was poorly answered, the Chief Examiner states that the question was student friendly. The findings indicate that any problem requiring any relational understanding caused issues. Most of the issues that occurred in this question pertained to one particular problem:

![Diagram](image)

Figure 4: 2006 Ordinary Level Junior Certificate: Paper 2 Question 6(b) (State Examinations Commission, 2006b, p.14)

Students displayed inaccuracies in their understanding of trigonometric ratios. Many students found the correct answer when asked to write down the value of \( \cos 60^\circ \), however they proceeded to perform unnecessary calculations after the correct answer was obtained. The Chief Examiner reports that students were not used to being asked a question that had such a simple solution, and tried to find a more complicated problem when one did not exist. This echoes the findings of research which notes that students are too familiar with certain types of questions and expect certain questions in examinations (NCCA, 2006; Breen, Cleary and O’Shea, 2009; Liston and O’Donoghue, 2010). When the question asked was too easy with respect to more familiar questions, students tried to make it more complicated. Though the \( \cos 60^\circ \) problem in part (i) of this question had links to part (ii) it was shown that students were unable to relate the value across. Relational understanding of the trigonometric ratios concept was not of a
Pythagoras' theorem was the only trigonometric concept to appear on the 2006 Foundation level paper (which has one paper in total). It appeared on one part of a geometry question (question 3). This geometry question was the second-worst answered question on the paper in that year with students achieving an average mark of 30.2 out of 50. Students were required to find the length of the hypotenuse when given the length of the other two sides (lengths of 9 and 12). It is noted that this problem was a considerable challenge for a lot of students. According to the Chief Examiner Report, many students attempted to measure the length of the hypotenuse using a ruler. The other unacceptable answers given by students involved either the random use of the figures given (e.g. $9 + 12 = 21$), or just a statement of the formula without any further working out of the problem. The report notes that many students successfully found the squares of the two sides given, but were unable to progress to the square root stage of the problem.

The conclusions of the Chief Examiner for 2006 state that trigonometry is only grasped by the very best students. The Examiner recommends that teachers and students need to pay more attention to trigonometry and study it in more depth.

One part of the geometry question in the 2003 (State Examinations Commission, 2003) Foundation level Junior Certificate examination paper was based on trigonometry (Question 6 (c)). This question was the worst answered question in that year (average mark of 31 out of 50). The trigonometry problem asked students to construct a triangle when given the lengths of two sides and the measure of one angle. The Chief Examiner reports that students were comfortable with practical work and recommended that teachers focus on using practical and visual methods in their teaching when possible. The main issue reported was that some students drew a triangle with all equal sides.

The trigonometry problem was the second worst answered problem at Junior Certificate Ordinary level in 2003. Students averaged 21.8 out of 50 in the trigonometry question (Question 6 Paper 2). Almost half of students (47.9%) achieved less than 20 marks on the question. Transformational geometry was the worst answered question (average mark of 20.3 out of 50). The main issue in this question was students’ lack of knowledge of trigonometric ratios and their correct application. Each part of the ques-
tion involved using trigonometric ratios, and each part of the question was poorly answered. The Examiner reports that trigonometric ratios were not understood and needed more attention from teachers and students.

The author of this thesis cannot discuss the Higher level report from 2003 as there are clear errors in the Chief Examiner’s report. Confusion about which question was trigonometry and which question was statistics can be seen in the report and the average marks reported also change on different pages.

Reading of the mathematical tables was shown to be the most frequent approach to trigonometry problems in the 1999 Junior Certificate (Department of Education and Science, 1999) at both Ordinary and Higher levels. At Higher level the average mark achieved by students was 31.4 out of 50 while the average mark at Ordinary level was 17.1 out of 50. The Chief Examiner reports that many students tried to look up values for cosine in the mathematical tables even though it is clear that they do not understand the tables themselves. The low popularity of the trigonometry question at Ordinary level was a cause for concern. Only 46% of students attempted this question.

**Summary of Junior Certificate Trigonometry**

The Chief Examiner Reports for Junior Certificate mathematics demonstrate that students do not have a deep understanding of the basics of trigonometry. A synopsis of the findings include that:

- trigonometry is one of the most concerning areas of junior cycle mathematics at Higher level;
- junior cycle students do not have a good understanding of the basics/foundations of trigonometry;
- trigonometric ratios are very poorly understood. The application of these to problems has also been highlighted as an area of weakness;
- there is an overreliance from students on procedures and anything that requires some relational understanding causes difficulties;
- questions in Junior Certificate examinations have become familiar to students and certain types of questions are expected;
trigonometry is seen as an elitist topic;

there is an overreliance on the use of mathematical tables even though the tables themselves are not understood.

The only positive point emanating from the review of Chief Examiner reports on trigonometric problems was that practical problems appear to be well attempted by students. Though the early reports such as 2003 show that the level of students’ trigonometric knowledge was increasing, this was dashed in 2006 with the introduction of questions that required an understanding of the content and not just procedures. Though trigonometry has not always been the worst answered question in Junior Certificate examinations, it has been the most consistent in terms of the low marks achieved by students.

The trigonometry performance of Leaving Certificate Irish students is also poor. Leaving Certificate examinations are those taken at the end of secondary education. The understanding of trigonometry of 17-18 year old Irish students is a further cause for concern as is shown in the next section.

Leaving Certificate Examinations (School Leaving Examination)

The following table (Table 3) illustrates the percentage of pupils in the 2005 Leaving Certificate examinations who attempted the question on trigonometry across the three levels (Higher, Ordinary and Foundation), as well as the average mark achieved in the question (out of a total of 50 marks).

<table>
<thead>
<tr>
<th>Trigonometry question</th>
<th>Higher Level</th>
<th>Ordinary Level</th>
<th>Foundation Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attempt %</td>
<td>85</td>
<td>58</td>
<td>67</td>
</tr>
<tr>
<td>Avg. mark (out of 50)</td>
<td>40</td>
<td>29</td>
<td>23</td>
</tr>
</tbody>
</table>

In the findings of the 2005 Chief Examiner Report (State Examinations Commission, 2005) it was shown that, at Foundation level\(^4\), 67% of students attempted the trigonometry question. These students averaged 23 marks out

\(^4\)Foundation Level examinations cater for pupils who find subjects very difficult and struggle to compete academically with their peers.
of 50 in this section which made it the most poorly answered question in
the Leaving Certificate Foundation paper in that year, along with geometry
and arithmetic which also had averages of 23. Findings include that:

- basics of trigonometry were poorly understood;
- calculators were set in wrong mode.

The fact that pupils at Foundation level had a poor understanding of the
basics of trigonometry is a similar finding to that in Turkey (Orhun, 2001).
One common mistake at Foundation level was that pupils had calculators
set in the wrong mode. Though this would lead to unrealistic solutions to
problems, pupils were unable to infer that there may be a calculation issue.
Basic errors such as this highlight the problems some pupils have with not
just trigonometry, but mathematics in general. It can be very abstract for
some pupils.

At Ordinary level in 2005, 58% of students attempted the trigonometry
question, with an average score of 29 marks out of 50, making it the fourth
lowest in terms of marks achieved. Some findings from the Chief Examiner
Report that can be analysed are that pupils:

- had an inability to use formulae;
- applied Pythagoras’ theorem to triangles that did not have a right
  angle;
- had an inability to use trigonometric ratios fully.

An inability to use a formula correctly can stem from learning procedural
approaches (Skemp, 1976). If one understands why the formula is used in
the way that it is then using the formula is not a problem (Skemp, 1976)
as in the majority of cases at this level it will require simple substitution of
figures. The application of Pythagoras’ theorem to non-right triangles stems
from a failure to understand the theorem itself. The theorem is not under-
stood if a student applies it to a problem that does not involve working with
a right angled triangle. Again, this points to a procedural approach from
students. The inability to use trigonometric ratios was a finding noted as
many pupils did not obtain the inverse function of a value to obtain an an-
gle (e.g. pupils final answer when asked for an angle was \( \sin \angle abc = 0.829 \)).

The Higher level trigonometry question had a far greater percentage of
attempts with 85% of students attempting the question. The average mark
achieved at Higher level was 40 marks out of 50 which was significantly greater than the marks achieved at Foundation and Ordinary levels. A source of problems at Higher level was the concept of a limit. Many pupils had enough experience at procedures to evaluate limits correctly but the workings shown were incorrect. This finding demonstrates the procedural approach used by pupils at Higher level. The main problem that arose was:

• the concept of limits and working with limits was poorly understood.

Understanding of trigonometric concepts was found to be poor across all levels in 2005. Of particular concern is that the basics of trigonometry were poorly understood with students proceeding to combine numbers in random ways and this according to the Chief Examiner highlights a severe lack of trigonometric understanding. The Chief Examiner notes that little meaningful work was accomplished in trigonometric problems.

The Chief Examiner Report in 2001 (State Examinations Commission, 2001) was based on findings from the Leaving Certificate mathematics examination at Ordinary level only.

<table>
<thead>
<tr>
<th>Trigonometry question</th>
<th>Ordinary Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attempt %</td>
<td>48</td>
</tr>
<tr>
<td>Avg. mark (out of 50)</td>
<td>32</td>
</tr>
</tbody>
</table>

Table 4: 2001 Trigonometry Performance

Trigonometry was the second least favoured option in this paper in 2001. Misconceptions appeared in the work provided by pupils. Issues were present in:

• visualisation;
• trigonometric identities;
• trigonometric functions.

Pupils had difficulty in selecting the required information to solve the problem even though a diagram with all the necessary information included was supplied to them. Visualisation errors were common. Findings indicate that pupils were unable to respond correctly to the question as it was unfamiliar and not procedural in nature. The findings also show that pupils appeared to come up with their own trigonometric identities for problems. Pupils
equated \cos 2\theta with 2 \cos \theta, even though the correct identity was given as a hint in the questions asked.

The 2000 Chief Examiner Report (State Examinations Commission, 2000) was based on findings across all three levels (Higher, Ordinary and Foundation) and showed:

<table>
<thead>
<tr>
<th>Trigonometry question</th>
<th>Higher Level</th>
<th>Ordinary Level</th>
<th>Foundation Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attempt %</td>
<td>82.5</td>
<td>72</td>
<td>52</td>
</tr>
<tr>
<td>Avg. mark (out of 50)</td>
<td>37</td>
<td>27</td>
<td>23</td>
</tr>
</tbody>
</table>

At Foundation level, only 52% of students attempted the trigonometry question, making it the second least favoured question on that paper. Students who attempted the question achieved an average score of 23 out of 50 which meant that the trigonometry section of the Foundation level examination in 2000 was the 2nd worst answered question overall, i.e. the second lowest out of papers 1 and 2. It was clear from the findings that trigonometric ratios were misunderstood by many of the students. The Chief Examiner notes that students must not have studied this topic in preparation for their examinations. The errors that occurred with trigonometric ratios lay in the failure to understand what a ratio (e.g. \( \tan A = \frac{3}{4} \)) actually means. Students were unable to make inferences related to other ratios (sine and cosine). As this question assessed something of a relational nature, students were unable to respond appropriately.

For the Ordinary level paper, it was reported that 72% of students attempted the trigonometry question, making it the second least favoured question on section A of paper 2. These students averaged a score of 27 out of 50, which was the third worst average on that section. The findings that arose from this question was that it was only attempted by the higher achieving pupils and that ‘routine’ applications were handled well. The fact that procedures were handled well was a product of the procedural and instrumental teaching approaches traditionally applied by teachers in Ireland (NCCA, 2005). Students were familiar with applying a procedure and especially in “predictable” questions, as noted in section 2.4. According to the research (section 2.4) teachers had students prepared for certain
questions that could be attempted using procedural approaches. The point that trigonometry was only attempted by higher achieving pupils suggests that trigonometry is seen as a difficult topic by weaker students. This is also consistent with the findings from other countries (section 2.6.1). Only higher achieving pupils may feel able enough to engage with trigonometry. This again highlights the pitfall in the topic choices that were available at Leaving Certificate level. Trigonometry was one of the most common topics not completed in Leaving Certificate examinations.

At Higher level there were two trigonometry questions in paper 2 in 2000. Findings included that:

- understanding how to solve a trigonometric equation for all solutions was not accomplished by pupils;
- deriving formulae was well attempted;
- the concept of limits and working with limits was poorly understood.

The first of the questions (question 4 paper 2) had a high attempt percentage as 90% of students attempted this question and achieved a good average of 40 marks out of 50. The Chief Examiner’s primary concern was with the fact that almost one-third of students were unable to find all solutions to a trigonometric equation. Quadrants, and how they relate to solving a trigonometric equation, was not understood. Students performed well on deriving a formula, however it is a fact that deriving this formula was prescribed work for students and could be rote learned from Leaving Certificate textbooks such as Concise Maths 4 (Humphrey, 2003). The second trigonometry question (question 5 paper 2) had a smaller attempt percentage with 75% of students attempting this question and obtaining an average mark of 34 out of 50. Again, limits seemed to cause problems for students, similar to 2005.

**Summary of Leaving Certificate Trigonometry**

From the findings of the Chief Examiner’s Reports a list of conceptual difficulties for Irish pupils in trigonometry can be compiled, as well as the finding that trigonometry is favoured by higher achieving pupils.

- basics of trigonometry poorly understood;
- calculators were set in wrong mode;
• had an inability to use formulae;
• applied Pythagoras’ theorem to triangles that did not have a right angle;
• the concept of limits and working with limits was poorly understood;
• visualisation difficulties;
• trigonometric identities not understood;
• trigonometric functions not understood;
• trigonometric ratios misunderstood and misused;
• understanding how to solve a trigonometric equation for all solutions was not accomplished by pupils;
• trigonometry favoured mainly by higher achieving pupils.

(State Examinations Commission, 2000; 2001; 2005)

This demonstrates that students found the entire trigonometry syllabus difficult. Every learning objective on the traditional pre-Project Maths syllabus is involved in this list as a source of problems for students.

The research indicates that trigonometry at Leaving Certificate level is understood quite well by Higher level students, and poorly by Ordinary level and Foundation level students. Trigonometric functions appear to be a problem for Foundation and Ordinary level pupils, while limits and finding all solutions to trigonometric equations are misunderstood by Higher level candidates. Some research has highlighted that a lack of understanding is not always resolved at university/third-level (Weber, 2005). From the findings of the Chief Examiner Reports, as well as reports from university researchers (Gill et al. 2000; O’Donoghue, 2002), it is clear that Irish students are ill-prepared in the area of trigonometry upon entering third-level. Weber (2005) suggests that a student’s lack of knowledge may not be resolved during their course of study. Ultimately, this will lead to many students falling behind in their third-level mathematical studies which are mainly calculus based (Herriott and Dunbar, 2009) due to a lack of foundational understanding of trigonometry on which to build new knowledge.
It is clear that issues exist regarding second-level Irish students’ performance/understanding of trigonometry. The next section looks at the effect these issues have on students’ progression to third-level and how it affects their prerequisite knowledge for the mathematics they encounter there.

Third-Level

Section 2.4 mentioned that third-level lecturers noticed deficiencies in undergraduates’ understanding of trigonometry (O’Donoghue, 2002). O’Donoghue (2002) states that Irish students enter third-level education without the prerequisite knowledge of trigonometry because they do not understand it at Leaving Certificate level. Gill (2006) researched the mathematical knowledge of entering undergraduate students through her analysis of the University of Limerick database of diagnostic test results.

The diagnostic test at the University of Limerick has six questions which can fall under the umbrella of trigonometry. The questions require students to:

• calculate the size of an angle in a triangle when given the measure of the other two angles;
• calculate the length of a side in a right-triangle when given the length of a side and the hypotenuse (i.e. Pythagoras’ theorem);
• calculate the area of an equilateral triangle when given the sides lengths and the perpendicular height;
• evaluate the sine of an angle when given a right-triangle diagram (measurements of angles and sides included);
• evaluate \( \sin^2 66^\circ + \cos^2 66^\circ \);
• write down 90° in radians.

(O’Donoghue, 1999; Gill, 2006)

Gill (2006) found that entering undergraduates pursuing degrees in technology and science were unable to answer the majority of these questions. The exceptions were the first two problems above. She found that 96.7% were able to find the missing angle in a triangle when given the other two angles and that 86.5% could find a right-triangle’s missing side-length. The most common error made was in the use of Pythagoras’ theorem. The other
problems related to trigonometry were not answered well.

Gill (2006) found that only 62.8% of students could find the area of a triangle. 79.1% of students who completed Leaving Certificate Higher level mathematics and 47.9% of those who completed Ordinary level got this correct. Not knowing the formula, incorrect multiplication, and poor algebraic skills were common amongst the sample.

Finding the sine of an angle when given a diagram was only achieved by 67.4% of the sample. 82.6% of Higher level and 53.3% of Ordinary level got it right. Gill (2006) states that the proportion of students that got this problem wrong is alarming as it is a very basic trigonometry problem. She states that a lot of answers were guesswork. This highlights the insufficient understanding that students have of trigonometric ratios. These have been shown to cause issues at junior cycle and senior cycle in the previous sections.

Only 51.5% of beginning third-level students who completed Higher level mathematics in their Leaving Certificate examinations were able to correctly answer the problem which asked to evaluate $\sin^2 66^\circ + \cos^2 66^\circ$ (only on Higher level Leaving Certificate course). Gill (2006) states that these students were unable to recall the identity $\sin^2 x + \cos^2 x = 1$ without the mathematical tables, despite being required to derive this identity for their Leaving Certificate examinations. Another reason given is that students could not relate the question asked to the formula as it was given in a different form. This is similar to that noted by the NCCA (2006), Breen, Cleary and O’Shea (2009), and Liston and O’Donoghue (2010).

The final problem mentioned in the list above was for students to write $90^\circ$ in radians. Students did not have to explain their reasoning but had to just write the answer. Gill (2006) again reports that only students who completed Higher level at Leaving Certificate level would be able to answer this question as Ordinary level students do not learn about radians. Only 53.4% of Higher level students could answer this correctly.

Reasons why students leave second-level education with a poor grasp of the basics of mathematics are numerous. The literature indicates that the quality of teaching that a child receives has the largest effect on the learning that they accomplish (Nye, Konstantopoulos and Hedges, 2004; Center for Public Education, 2005). The next section deals with the problems of poor teaching and some skills/knowledge of teachers highlighted in the literature.
as being imperative to student learning.

2.7 Poor Teaching and Critical Skills of Teachers

Many reasons why students find mathematics difficult have been identified. These reasons range from the environment in which learning takes place (Brekelmans et al., 1997), to the curriculum from which material is studied (Sherman et al., 2010). Some reasons include:

- teacher instruction that relies heavily on memorisation and maintains an abstract ideology of mathematics (Cockcroft, 1982; Sherman et al., 2010);
- the attention span of learners (Sherman et al., 2010);
- the complex nature of cognition in the thinking processes in mathematics (Duval, 2006);
- difficulties grasping the language used in mathematics (Aiken, 1972).

The literature is unanimous in saying that one of the most detrimental things to student learning and hence one of the primary causes for difficulties experienced in mathematics is that of poor teaching (Cockcroft, 1982; Fennema and Franke, 1992). Poor teaching of mathematics leads to inaccurate views of mathematics as a discipline by students, and negatively effects their acquisition of mathematical knowledge (Schoenfeld, 1988). The quality of mathematics teaching that a student receives has a greater effect on their learning than does their ethnicity, parental income, school attended, or the size of their class (Nye, Konstantopoulos and Hedges, 2004; Center for Public Education, 2005). With respect to the teaching of trigonometry, Weber (2005) notes Hirsch, Weinhold, and Nichols (1991) and NCTM standards (1989, 2000) in stating that the teaching of trigonometry needs to move away from methodologies that rely on memorisation, facts, and procedures and move towards emphasising conceptual understanding, relations and connections, mathematical modelling, and problem solving.

The literature notes multiple characteristics of an effective teacher that can help students to learn mathematics. The skills which are characteristic of a teacher that provides high quality instruction of mathematics are that he/she has strong:
• subject matter content knowledge (Shulman, 1986; Shulman, 1987; Ernest, 1989; Fennema and Franke, 1992; Rowland, 2007; Ball et al., 2008; O’Meara, 2011);

• pedagogical content knowledge (Shulman, 1986; Shulman, 1987; Ernest, 1989; Fennema and Franke, 1992; Rowland, 2007; Ball et al., 2008; O’Meara, 2011);

• curricular knowledge (Shulman, 1986; Shulman, 1987; Ernest, 1989; Ball et al., 2008; O’Meara, 2011);

• knowledge of education (Ernest, 1989; Fennema and Franke, 1992);

• transformational knowledge (Rowland, 2007; Ball et al., 2008; O’Meara, 2011);

• conception of the nature of mathematics (Ernest, 1989; Fennema and Franke, 1992; Ball et al., 2008; O’Meara, 2011);

• knowledge of learners’ cognitions in mathematics (Ernest, 1989; Fennema and Franke, 1992; Ball et al., 2008);

• connectional knowledge in mathematics (Fennema and Franke, 1992; Rowland, 2007; Ball et al., 2008; O’Meara, 2011);

• contingency ability (Rowland, 2007);

• knowledge of the mathematical horizon (Ball et al, 2008; O’Meara, 2011);

• knowledge of real-life applications (O’Meara, 2011);

• knowledge of the history of mathematics (O’Meara, 2011).

This list originated from a critical analyses of models of teacher knowledge. In each of these models it is clear that teachers’ knowledge of subject matter is essential for everything else that is to follow in the name of good teaching. Shulman (1986) notes that ‘subject matter content knowledge’ is the basis for good teaching. Shulman’s (1987) further work notes ‘comprehension’ as the beginning of a cycle of pedagogical reasoning. The Knowledge Quartet of Rowland (2007) highlights ‘foundation’ as the core of a teacher’s skill in teaching mathematics. Foundation refers to the mathematical background and beliefs that a teacher has in mathematics. Finally, Ball et al. (2008) and O’Meara (2011) both note subject matter knowledge
as one of the main elements of a teacher’s knowledge base that is critical for everything else to work. Subject matter knowledge, or foundation, or comprehension, has remained as a pillar in the teacher’s knowledge base throughout the research, highlighting its importance and how a strong understanding of content can influence everything from the type of questions asked to the type of activities organised for learning.

The author acknowledges the importance of the other skills and knowledge mentioned. Research has shown all to be necessary components of a teacher’s knowledge base and skill set, however, a close examination of the research indicates that without subject matter knowledge, then a teacher cannot teach effectively in the classroom.

Though the power of subject matter knowledge is evident from the models of teacher knowledge, links exist between strong/weak subject matter knowledge and teaching performance. Consequences of such knowledge and the teaching approaches that one employs are discussed in the next section.

### 2.8 Teacher Content Knowledge and Teaching

As depicted in the models of teacher knowledge it is a common view that a high level of content knowledge on behalf of the teacher is essential for the high quality teaching of mathematics. Wu (2005) notes that without a firm mastery of the subject matter, good pedagogy is impossible.

Potari, Zachariades, Christou and Pitta-Pantazi (2008), in their study on the relationship between teachers’ mathematical knowledge and quality teaching of calculus showed that teachers with higher levels of conceptual understanding (the ‘why’ or the importance of the ideas related to the implementation of procedural mathematical tasks (Berenson et al., 1997)) of the topic gave opportunities for pupils to investigate the reasoning behind certain mathematical facts. These teachers also encouraged pupils to investigate different methods of solving problems. Teaching strategies such as these have been widely advocated in the literature for use in the teaching of mathematics (NCCA, 2005 (Ireland); Ofsted, 2012 (U.K.); NCTM, 2014 (U.S.A.)). A point that arose in Potari et al.’s (2008) analysis of one teacher who had a high level of mathematical understanding of calculus (as he had a completed a postgraduate course in this topic) was that this teacher felt that students should understand calculus concepts more than geometry concepts.
He felt that while his students should understand calculus, they could learn geometry by rote. Here, his conceptual understanding of different topics dictates the teaching approaches he employs and his ideas about the nature of those mathematical topics.

Knowledge Management and Dissemination (2010) also note that teachers with strong content knowledge are less likely to make mathematical or language errors in the classroom and that they also respond to pupils' mathematical ideas more appropriately than teachers with weaker content knowledge. In fact, Knowledge Management and Dissemination (2010) state that a lack of content knowledge limits a teacher’s instruction and may cause a teacher to augment instruction using inappropriate representations or vocabulary when they have to deviate from their lesson plan. These findings show that the characteristics of an effective teacher (section 2.7), such as contingency (Rowland, 2007), though not specifically mentioned as subject matter knowledge, are related to a teacher’s content knowledge in a direct way.

Lehrer and Franke (1992) carried out a study to demonstrate the differences in teaching between a teacher with a very high conceptual understanding of fractions and a teacher who is less proficient in the content. The teacher with superior content knowledge started with a fraction which was familiar to pupils, used knowledge of related fractions, provided pictures and symbols, encouraged verbal interaction, focused on process as she asked the pupils how they solved their problems, and focused on the understanding of larger versus smaller fractions (though some of these interlink with the teacher’s knowledge of pupils and pedagogy). The teacher in this study who was less proficient in the content confined her teaching to what was in the textbook and significantly limited her teaching strategies. This is similar to the statement of Mewborn (2001) who notes that mathematics teachers with a low level of subject matter knowledge can often perform calculations, but are unable to explain the concepts behind the procedural tasks they carry out.

Other pieces of research have provided similar findings related to the teaching of mathematics by teachers who have a strong or weak subject matter knowledge. Some findings included that teachers of mathematics with a strong subject matter knowledge:
• analyse the accuracy of definitions in textbooks, choose appropriate and helpful examples for students, and interpret student explanations on concepts and problems (Ball and Bass, 2003);

• use new materials from the curriculum, encourage wider participation and make participation in the mathematics classroom more accessible, and help students to succeed on more demanding and challenging assessments (Hill and Ball, 2004);

• make mathematical sense of student work and choose powerful ways to represent concepts in order to provide opportunities for students to understand the content (Ball, Thames, and Phelps, 2008);

• guide students to an interconnected understanding of mathematical concepts, as well as promoting reflective learning, critical thinking, and ultimately lead the students to mathematical understanding and achievement (Saritas and Akdemir, 2009).

In contrast to this are some findings that teachers of mathematics with a weak subject matter knowledge:

• confuse their students, and get confused by their students’ questions and problems (Ball and Bass, 2003);

• worry that they might get asked questions of a conceptual nature by students and will be unable to answer correctly (Ball, 1988);

• are unable to think of their feet and unable to deviate from the lesson plan (Rowland, 2007).

This section has shown that teachers with strong subject matter knowledge practice better pedagogy than those with weaker knowledge. However, it remains to be seen what effect strong teacher subject matter knowledge has on student achievement. The next section discusses research related to this issue and the findings that were identified.
2.9 Teacher Content Knowledge and Student Achievement

Though determining the link between teacher content knowledge and student achievement is a highly complex matter, literature which is concerned with Education Production Function\(^5\) studies provides some evidence to consider. Education Production Function studies report on “the relationship between educational resources and outcomes” (Hill, Rowan and Ball, 2005, p.374). In relation to this matter the ‘educational resource’ which was researched by the author was teacher knowledge, much of which was quantified/explained in the literature from teacher scores on assessments. The author reviewed the literature to analyse how the resource of teacher knowledge affects the outcome of student achievement. Findings in the literature on student achievement were also produced from test scores.

In order to gain valid insights into the relationship between teacher content knowledge and student achievement in schools, the research states that one must first be confident that assessment measures used in classifying/quantifying a teacher’s level/amount of content knowledge are valid. This also holds true for student achievement which is primarily viewed from the perspective of test scores (Hill, Rowan and Ball, 2005). Even if these measures are valid, the tests do not take into account the life of the student, for example, what he/she was experiencing in their personal life on the day, or days, before the testing or what his/her attendance rate was in school (Hanushek, 1986). Reviewers of Education Production Function literature (Nye, Konstantopoulos and Hedges, 2004) agree that correlating between a teacher’s characteristic (such as subject matter knowledge) and the achievement of their students is an extremely difficult task, even if there are controls that take in various characteristics of both the teacher and student (e.g. past performance of student or the class the student was assigned to based on their results) (Nye et al., 2004). Nevertheless, findings have been reported that indicate that a teacher’s subject matter knowledge has positive effects on student achievement (Greenwald, Hedges and Laine, 1996), whilst others have noted that it is not possible to infer a relationship due to the countless number of potential inputs (e.g. the mental state of the student at the time of testing) and outputs (e.g. reliability of marking scheme) of the research (Hanushek, 1986).

\(^5\)Education Production Function: Studies of how different educational resources affect student achievement (or other educational outcomes) (Hill, Rowan and Ball, 2005)
Rivkin, Hanushek and Kain (2005) state definitively that teacher quality plays a prominent role in student achievement. Knowledge Management and Dissemination (2010) also shows that this statement is backed up by other research (Mullens, Murnane and Willett, 1996; Hill et al, 2005; Clotfelter, Ladd and Vigdor, 2007). Mullens et al. (1996) showed in their research that teacher knowledge of the subject matter is the most important contributor to student achievement out of three teacher characteristics, of which they examined teacher subject matter knowledge, teacher training, and teacher high school completion. Hill et al. (2005) also showed that teacher mathematical content knowledge for teaching positively affects student learning, not just in higher level classes at second and third-level, but also at primary-level. Clotfelter et al. (2007) found extreme differences between a teacher with an above average subject matter knowledge and a teacher with a less than average subject matter knowledge. They state:

“that teachers who scored 2 or more standard deviations above the average boosted student gains by 0.068 standard deviations relative to the average teacher, and teachers who scored 2 or more standard deviations below the average reduced achievement gains by 0.062 standard deviations. The overall difference between teachers at the two extremes is a whopping 0.130”

(Clotfelter et al., 2007, p.36)

However, each of these researchers (Mullens, Murnane and Willett, 1996; Hill et al, 2005; Clotfelter, Ladd and Vigdor, 2007) have highlighted the limitations of their own respective studies. Hill et al. (2005) note the limitations of a small sample, missing data, and most notably a lack of alignment between teacher subject matter knowledge and student achievement. This final limitation echoes the warning of Hanushek (1986) which he says results in an inability to infer. Though Hill et al. (2005) concur with this point of Hanushek, they nevertheless state that the positive effects observed in their research were so substantial that it allowed for inference. Mullens et al. (1996) also had a problem of missing data. The data that they possessed on teacher subject matter knowledge (based on test scores) was missing for 23 teachers, or 32% of their sample. However, conclusions were made based on the analysis conducted with the remainder of the sample. Clotfelter et al. (2007) based their results on data provided in teachers’ personnel files which were available to the researchers. The data used was rich and extensive, however, anything related to teacher content knowledge was based on
test scores, where the test was not provided to the reader which, as noted, raises validity concerns. It also provides only a snapshot in time, disregarding personal issues on the day(s) of assessment, as noted by Hanushek (1986).

Teachers need strong subject matter knowledge. An analysis of Irish teachers can be conducted with this in mind. The next section analyses the qualification requirements in Ireland to teach mathematics. It also investigates if the requirements are being upheld nationwide. Comparisons between better and weaker performing countries on international assessments are carried out in order to view any relationship that exists between teacher qualification and student performance.

2.10 Irish Second-Level Mathematics Teaching and the Need for Improvement

According to the Teaching Council of Ireland’s original requirements (Teaching Council, 2011a) for a teacher to be adequately qualified to teach mathematics, they stipulated that the teacher should have:

- studied mathematics as a major subject in their degree which lasted for a minimum of three years, and which was comprised of at least 30% mathematics studies.

- details of their degree to show that the breadth and depth of the syllabi undertaken in their degree is sufficient to be certain they can teach mathematics to the highest possible level at post-primary education level.

- evidence that they have achieved a minimum pass result in all mathematics examinations undertaken.

As of November 2011, the Teaching Council (2011b) have updated their requirements for the mathematical qualifications of teachers. These requirements are in line with Teaching Council [Registration] Regulations (Regulation Four) (2009). They stipulate that a second-level mathematics teacher should, as part of their qualifications, have studied mathematics with courses of mathematical studies accumulating at least 60 ECTS credits. They state that at least 40 ECTS credits must acquired from:

- Analysis - minimum of 10 ECTS credits
- Algebra - minimum of 10 ECTS credits
• Geometry - minimum of 5 ECTS credits

• Probability and Statistics - minimum of 5 ECTS credits

(Teaching Council, 2011b, p.60)

The remaining minimum of 20 ECTS credits must be acquired either through the above areas, or may be acquired from the optional areas of:

• Dynamical Systems and Chaos

• Numerical Analysis or Computational Mathematics

• Mathematical Modelling

• Discrete Mathematics

• Statistical Inference

• History or Philosophy of Mathematics

• Mathematical Logic

• Set Theory and Cardinality

(Teaching Council, 2011b, p.61)

This list has been documented again in the last Teaching Council publication of requirements (Teaching Council, 2013).

The requirements listed above illustrate the importance that the Teaching Council places on knowledge of mathematics as a prerequisite for the teaching of mathematics at secondary level. Research indicates however, that large numbers of ‘out-of-field’ teachers are teaching mathematics in Irish schools. Out-of-field teachers are defined as teachers who are assigned to teach subjects by a school’s administration which do not match their teacher training or education (Ingersoll, 2002).

In a study conducted in Ireland by Cosgrove, Shiel, Oldham and Sofroniou (2004) it was shown that while 90% of 856 teachers of junior cycle mathematics had a higher diploma in education, almost 28% had studied degree courses which did not have mathematics as a major component. They did therefore not meet Teaching Council requirements for mathematics but
were teaching the subject nonetheless. More recently, Ní Riordáin and Hannigan (2011) found that 48% of second-level mathematics teachers in their study in Ireland (sample size of 324) did not have a teaching qualification in mathematics (i.e. were out-of-field). They provide further information on the qualification type of the teachers in their survey regarding the percentage of teachers that lay in each qualification type.

Figure 5: Percentage of Teachers that Lay in Each Qualification Type (Ní Riordáin and Hannigan, 2011, p.297)

Ní Riordáin and Hannigan (2011) also provide statistics as to where teachers, who did or did not have a teaching qualification in mathematics, are deployed in schools.
Table 6: Deployment of Teachers in Schools Through the Six Year Groups (Ní Riordáin and Hannigan, 2011, p.298)

<table>
<thead>
<tr>
<th>Teaching qualification in mathematics</th>
<th>1st year</th>
<th>2nd year</th>
<th>3rd year</th>
<th>4th year</th>
<th>5th year</th>
<th>6th year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes ( (n = 168) )</td>
<td>85 ( (51%) )</td>
<td>100 ( (60%) )</td>
<td>134 ( (80%) )</td>
<td>95 ( (57%) )</td>
<td>133 ( (79%) )</td>
<td>131 ( (78%) )</td>
</tr>
<tr>
<td>No ( (n = 156) )</td>
<td>81 ( (52%) )</td>
<td>94 ( (60%) )</td>
<td>79 ( (51%) )</td>
<td>18 ( (12%) )</td>
<td>45 ( (29%) )</td>
<td>38 ( (24%) )</td>
</tr>
</tbody>
</table>

It is clear from Table 6 that the qualified teachers are deployed to the examination years, i.e. 3rd, 5th and 6th years. There is a significant difference in the deployment of qualified and out-of-field teachers in these years which contributes greatly to the learning experiences of pupils (Stockard and Mayberry, 1992; Rice, 2003).

Furthermore Ní Riordáin and Hannigan (2011) display the percentage of teachers teaching Higher level mathematics based on whether they are qualified to teach mathematics or not.

Table 7: Percentage of Qualified/‘Out-of-Field’ Teachers Teaching Higher Level Mathematics (Ní Riordáin and Hannigan, 2011, p.299)

<table>
<thead>
<tr>
<th>Teaching qualification in mathematics</th>
<th>2nd year</th>
<th>3rd year</th>
<th>5th year</th>
<th>6th year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes ( (n = 168) )</td>
<td>52 ( (31%) )</td>
<td>71 ( (42%) )</td>
<td>75 ( (45%) )</td>
<td>68 ( (40%) )</td>
</tr>
<tr>
<td>No ( (n = 156) )</td>
<td>7 ( (4.5%) )</td>
<td>4 ( (3%) )</td>
<td>0 ( (0%) )</td>
<td>2 ( (1%) )</td>
</tr>
</tbody>
</table>

These statistics, from a school management point of view, depict the higher levels of confidence that management have in mathematics teachers who have a higher amount of mathematical knowledge. Clearly there is a far greater percentage of qualified mathematics teachers teaching Higher level class groups and a very small number of out-of-field mathematics teachers teaching Higher level mathematics, especially at Leaving Certificate level.

From this data it can be concluded that those in Higher level Junior Certificate classes have a greater chance of obtaining a qualified mathematics teacher than Ordinary and Foundation level classes.
The benefits of adhering to qualification requirements are made clear by briefly analysing student performance in other countries where teacher qualification requirements are more rigourously observed. A possible link to the reasons why Asian countries do better on assessments such as the Trends in International Mathematics and Science Study (TIMSS)\(^6\) (National Center for Education Statistics, U.S. Department of Education, 2000; Gonzales et al., 2004; Gonzales et al., 2009) are the teacher qualifications in these countries (Ingersoll, 2007). Ingersoll (2007) (Table 8) shows that Asian countries have a higher percentage of fully certified teachers\(^7\) than others (in this case the United States).

Table 8: School Teachers by Highest Degree Earned, and by Teaching Certificate (Ingersoll, 2007, p.11)

<table>
<thead>
<tr>
<th>Education Qualifications</th>
<th>Professional Qualifications</th>
<th>Degree and Certification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than Bachelor's Degree</td>
<td>Bachelor's Degree or Higher</td>
<td>No Certification</td>
</tr>
<tr>
<td>China Elementary School</td>
<td>95%</td>
<td>5%</td>
</tr>
<tr>
<td>L. Sec.</td>
<td>71</td>
<td>29</td>
</tr>
<tr>
<td>U. Sec.</td>
<td>21</td>
<td>79</td>
</tr>
<tr>
<td>Hong Kong Elementary School</td>
<td>27</td>
<td>73</td>
</tr>
<tr>
<td>Secondary</td>
<td>8</td>
<td>92</td>
</tr>
<tr>
<td>Japan Elementary School</td>
<td>15</td>
<td>82</td>
</tr>
<tr>
<td>Secondary</td>
<td>3</td>
<td>82</td>
</tr>
<tr>
<td>Korea Elementary School</td>
<td>14</td>
<td>70</td>
</tr>
<tr>
<td>Secondary</td>
<td>0.5</td>
<td>70</td>
</tr>
<tr>
<td>Singapore Elementary School</td>
<td>52</td>
<td>46</td>
</tr>
<tr>
<td>Secondary</td>
<td>11</td>
<td>82</td>
</tr>
<tr>
<td>Thailand Elementary School</td>
<td>8</td>
<td>88</td>
</tr>
<tr>
<td>Secondary</td>
<td>2</td>
<td>65</td>
</tr>
<tr>
<td>USA Elementary School</td>
<td>1</td>
<td>54</td>
</tr>
<tr>
<td>Secondary</td>
<td>3</td>
<td>49</td>
</tr>
</tbody>
</table>

From this table it can be concluded that Asian countries have more fully certified teachers in their classrooms than the USA (who placed 9th in TIMSS 2007, 15th in TIMSS 2003, and 19th in TIMSS 1999). The only

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\(^6\)“TIMSS is used to measure over time the mathematics and science knowledge and skills of fourth and eighth graders. TIMSS is designed to align broadly with mathematics and science curricula in the participating countries.” (Gonzalez et al., 2009, p.iii)

\(^7\)Fully certified teachers are those that have completed professional preparation requirements in the subject matter and in pedagogical preparation (Ingersoll, 2007).
country that may appear to have a lower percentage is China, however as their secondary schooling is divided into lower secondary and upper secondary, an average can be taken from their total secondary teachers score. This would lead them to have an average percentage of 87% fully certified teachers. Singapore is a stand out country in this table as it can be seen that 100% of teachers are fully certified. This is due to the fact that mathematics or engineering have to be studied at undergraduate level before gaining entrance into a postgraduate teacher training course (Kaur, Lee and Fwe, 2007). The degree also has to be recognised by the education service of Singapore. Most teacher training in Singapore is done at postgraduate level.

Outside of the area of certification, the percentages of teachers who have a major area of study in mathematics must also be looked at. Table 9 was documented in Ní Ríordáin and Hannigan (2009) which shows the percentage of students in different countries who were taught by teachers with a major in mathematics (both certified and uncertified):
Table 9: Teachers Having Both Teacher Certification and Mathematics as a Major Area of Study (Ní Riordáin and Hannigan, 2009, p.10)

<table>
<thead>
<tr>
<th>Selected Countries</th>
<th>% of students taught by teachers with major in mathematics</th>
<th>% of students taught by certified teachers with major in mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Australia</td>
<td>72</td>
<td>72</td>
</tr>
<tr>
<td>Canada</td>
<td>28</td>
<td>25</td>
</tr>
<tr>
<td>Chile</td>
<td>78</td>
<td>77</td>
</tr>
<tr>
<td>England</td>
<td>90</td>
<td>85</td>
</tr>
<tr>
<td>Finland</td>
<td>75</td>
<td>68</td>
</tr>
<tr>
<td>Hungary</td>
<td>99</td>
<td>99</td>
</tr>
<tr>
<td>Japan</td>
<td>93</td>
<td>93</td>
</tr>
<tr>
<td>Korea, Rep. of</td>
<td>97</td>
<td>97</td>
</tr>
<tr>
<td>Malaysia</td>
<td>72</td>
<td>66</td>
</tr>
<tr>
<td>Netherlands</td>
<td>91</td>
<td>87</td>
</tr>
<tr>
<td>New Zealand</td>
<td>51</td>
<td>49</td>
</tr>
<tr>
<td>Philippines</td>
<td>87</td>
<td>81</td>
</tr>
<tr>
<td>Singapore</td>
<td>84</td>
<td>84</td>
</tr>
<tr>
<td>South Africa</td>
<td>82</td>
<td>72</td>
</tr>
<tr>
<td>United States</td>
<td>61</td>
<td>-</td>
</tr>
<tr>
<td>International Mean</td>
<td>84</td>
<td>83</td>
</tr>
</tbody>
</table>

From this data it can be seen that the five countries with the highest percentage of teachers teaching mathematics, whose major area of study was mathematics, are Hungary (99%), Republic of Korea (97%), Japan (93%), Netherlands (87%) and Singapore (84%). A possible link exists here between these statistics and how each of these countries performed in the TIMSS studies in 2007, 2003, 1999 (Gonzales et al., 2009; Gonzales et al., 2004; National Center for Education Statistics, U.S. Department of Education, 2000):

<table>
<thead>
<tr>
<th>Country</th>
<th>TIMSS 2007</th>
<th>TIMSS 2003</th>
<th>TIMSS 1999</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hungary (99%)</td>
<td>6th</td>
<td>9th</td>
<td>9th</td>
</tr>
<tr>
<td>Korea, Rep. of (97%)</td>
<td>2nd</td>
<td>2nd</td>
<td>2nd</td>
</tr>
<tr>
<td>Japan (93%)</td>
<td>5th</td>
<td>5th</td>
<td>5th</td>
</tr>
<tr>
<td>Netherlands (87%)</td>
<td>Did not take part</td>
<td>7th</td>
<td>7th</td>
</tr>
<tr>
<td>Singapore (84%)</td>
<td>3rd</td>
<td>1st</td>
<td>1st</td>
</tr>
</tbody>
</table>

Each of these countries were placed in the top 10 in the world in TIMSS studies. The high percentage of fully qualified teachers with a major area of study in mathematics in each of these countries indicates that teachers in these countries have a high level of subject matter knowledge. The literature suggests that this fact reflects in the performance of secondary pupils in these countries as subject matter knowledge has such a strong positive influence on the effective teaching of mathematics (section 2.8).

2.11 Conclusion

Trigonometry is important to the areas of work and education. It is included as a topic of study in every second-level mathematics syllabus in the world (Delice, 2002) and Ireland is no exception (NCCA, 2013). This is not just due to the importance of the topic itself, but the benefits it provides to the development of mathematical cognition (MSEB, 1990; Joint Mathematical Council of the United Kingdom, 1997; NCTM, 2014). Trigonometry also forms a foundation for mathematical studies at third-level (O’Donoghue, 2002; NCCA, 2006) and studies in other areas such as physics.

Student understanding of trigonometry is not good despite it being included in mathematics syllabi worldwide. It has been shown in this chapter that Irish second-level students have performed poorly in trigonometry on State examinations (State Examinations Commission, 1999; 2000; 2001; 2003; 2005; 2006a). Some Irish students have even ignored the topic completely at Leaving Certificate level (O’Donoghue, 2002; NCCA, 2006). Third-level lecturers have noted the lack of understanding that Irish students
have in trigonometry (O’Donoghue, 2002; Gill, 2006). They have provided evidence that third-level students are falling behind in their mathematical studies due to a lack of foundational knowledge in areas such as trigonometry (O’Donoghue, 2002). They also note that students do not realise how much understanding of trigonometry is necessary at Leaving Certificate level before undertaking further mathematical studies (NCCA, 2006).

The quality of teaching that a student receives has the greatest effect on students’ success or failure to understand mathematics (Cockcroft, 1982; Schoenfeld, 1988; Fennema and Franke, 1992; Nye et al., 2004; Center for Public Education, 2005). This chapter has looked at important contributors to effective teaching. The most important element of a teacher’s knowledge base is their knowledge of the subject matter (Rowland, 2007). Teachers cannot provide high quality instruction of mathematics without having strong subject matter knowledge, regardless of how much pedagogical knowledge they have. Subject matter knowledge is seen by the models of teacher knowledge as a prerequisite for any other teaching skills (Rowland, 2007). Despite the importance of subject matter knowledge, reports indicate that there is an issue with out-of-field teaching in Ireland (Cosgrove et al., 2004; Ñí Riordáin and Hannigan, 2011). This means that many in-service mathematics teachers have not studied the subject to degree level. However, the only evidence that exists on Irish teachers’ knowledge of specific mathematical topics is from research conducted with pre-service teachers. Irish pre-service primary teachers have demonstrated a lack of understanding of concepts in geometry (Corcoran, 2005; Delaney, 2010), statistics (Leavy and O’Loughlin, 2006), and decimals (Hourigan and O’Donoghue, 2013). Pre-service teachers in the United States (Fi, 2006) and Australia (Chinnappan et al., 1996) have demonstrated a lack of understanding of trigonometry. Trigonometry was not found to be researched with any group of teachers in Ireland.

The author identified that a purpose-built model for the teaching of trigonometry could be developed in response to the issues in the teaching and learning of the topic. This model would respond to the gap in the literature with respect to the effective teaching of trigonometry. The review of literature required a change of focus in order to begin the process of developing this purpose-built model. The second part of this literature review looks at models for understanding mathematical development. This is done in an attempt to find a model that could help the author to develop a purpose-built teaching model. Where Part 1 of the chapter provides the context and the issues pertaining to the research, Part 2 serves the purpose
of providing a basis with which to progress in developing a theory/model. Part 2 of this chapter analyses the van Hiele model of geometric thought, Bloom’s taxonomy of learning domains, and the SOLO taxonomy.
Part 2 - Models for Understanding Mathematical Development

2.12 Introduction

It was shown in chapter 2 part 1 that Irish students struggle with trigonometry at the end of secondary school. However, it was also shown that this problem is not confined to Ireland, with students in other countries such as Turkey and the United States also having issues with trigonometry. There is a paucity of research into the trigonometric understanding of adolescents, either into the quality of understanding that they possess, or ways to improve it. Though the topic of trigonometry has been identified as one of concern, and one on which little research exists, the author was unable to find any model for effectively teaching trigonometry in his review of literature.

Chapter 2 part 2 analyses and discusses models for understanding mathematical development and cognitive development that guided the development of the author’s purpose-built model. The author considered three models that he encountered in his research. These models were Bloom’s taxonomy of learning domains (Bloom, Englehart, Furst and Hill, 1956), the Structure of Observed Learning Outcomes (SOLO) taxonomy (Biggs and Collis, 1982), and the van Hiele model of the development of geometric thought (van Hiele, 1984a; 1984b).

Bloom’s taxonomy (section 2.13) of learning domains classifies three categories of educational aims that educators set for their students. The three categories are the cognitive, affective and psychomotor domains. Each of these domains is hierarchial in structure. Therefore higher order learning in
the cognitive domain is dependant on having achieved lower order knowledge within this domain. Bloom’s taxonomy is not mathematics specific.

The SOLO taxonomy (section 2.14) demonstrates how learning outcomes flourish in complexity from a surface understanding to a deep understanding. The types of responses provided by students to problems demonstrate where they are on a five level interval (prestructural-unistructural-multistructural-relational-extended abstract). The SOLO taxonomy is not mathematics specific.

The van Hiele model (section 2.15) is a constructivist theory that describes five qualitatively different levels of thought development in geometry. It is primarily a learning theory for geometry but it has been used as a basis for instruction in the past. However, the instructional methods proposed by van Hiele (1984b) have received some criticism (discussed in section 2.15.8). The van Hiele model is mathematics specific. It is specific to the topic of geometry.

This chapter will discuss each of the models of development in more detail. The chapter concludes with the selection of the most appropriate model for the purpose of this research, which was to guide the development of the author’s purpose-built model.

2.13 Bloom’s Taxonomy of Learning Domains: The Cognitive Domain

The word ‘taxonomy’ created some problems in the initial use of Bloom’s taxonomy in education, as this word had not been used in the context of education before. It is important that there is an awareness that the word ‘taxonomy’ is synonymous with ‘classification’ when discussing Bloom’s taxonomy (Forehand, 2005). The taxonomy classifies six cognitive levels, in a hierarchical structure, based on the level of complexity of the cognitive levels. Bloom’s updated taxonomy (Pohl, 2000) for the levels of the cognitive domain were as follows:

- Remembering
- Understanding
- Applying
• Analysing
• Evaluating
• Creating

(Bloom et al., 1956)

The least demanding cognitive level of these is ‘remembering’. Remembering is concerned with the basic recall of information from long-term memory (Bloom et al., 1956; Forehand, 2005). An example here could be for a pupil to recall Pythagoras’ theorem or to recall an anecdote for remembering the trigonometric ratios.

The next cognitive level is ‘understanding’. This requires students to understand what is being communicated (Bloom et al., 1956). There is increased cognitive demand as students are no longer required to recall basic facts, but to investigate understanding instead.

‘Applying’ is the next level of Bloom’s taxonomy and requires individuals to use previously gained knowledge to solve problems in new situations. Application requires both knowledge and comprehension in order to problem-solve. An individual would need to use his/her acquired procedural knowledge of Pythagoras’ theorem in order to solve a problem. For example, in the application of Pythagoras’ theorem to find the height of a tree or a similar type of question.

‘Analysing’ is concerned with formulating inferences and supporting them with factual information as well as the identification of patterns. A learner should be able to reorganise information and have the ability to compare and contrast information (Weil and Kincheloe, 2004). One of the important notes from this level is that an individual should have the ability to pose questions himself/herself. It is an understanding of the content which allows these problems to be devised and posed (Weil and Kincheloe, 2004).

The next level is ‘evaluating’. At this level an individual can critique or judge their own work based on a set of standards or criteria (Huitt, 2011). A pupil at this level could judge whether or not a mathematical proof they have constructed actually proves what it was supposed to prove. For example, they know whether their proof is for a specific case or the general case.
At the final ‘creating’ level an individual should be able to create something on his/her own that illustrates an understanding of a concept or a skill (Weil and Kincheloe, 2004). An example here would be an individual explaining a concept such as radians in their own language in order to demonstrate an understanding of the concept.

Bloom’s taxonomy was considered by the author as a potential model for adaptation as it provides levels of learning that lead to the summit of learning within a programme of study. It was rejected as it is not mathematics specific. Though Bloom’s taxonomy could potentially have been used, a more suitable model was eventually found (i.e. the van Hiele model). More on the selection of the van Hiele model as opposed to Bloom’s taxonomy is discussed in section 2.16.

The next section discusses another model that was considered for the purpose of guiding the author’s purpose-built model, the SOLO taxonomy.

2.14 The SOLO Taxonomy

The Structure of Observed Learning Outcomes (SOLO) taxonomy was devised by Biggs and Collis (1982). They state that this taxonomy outlines a simple and powerful way of depicting how learning outcomes flourish in complexity from a surface understanding to a deep understanding of content. The literature states that the responses children give to problems elaborate how they are progressing through Piaget’s modes of cognitive development (sensory motor stage, pre-operational stage, concrete operational stage, formal operational stage) (Pegg and Davey, 1989; Pusey, 2003).

These types of answers were classified under the SOLO taxonomy. The SOLO taxonomy assesses learning outcomes through students’ responses to problems, rather than through some levels of understanding (as seen in van Hiele and Bloom’s taxonomy). It therefore gives an assessment of how children are progressing through Piaget’s modes of cognitive development from their answers to problems.

The SOLO taxonomy was considered by the author because it permits the identification of a student’s level of thinking based on their answers to problems. A researcher can identify how far students have progressed through Piaget’s modes of cognitive development and can plan their teach-
ing accordingly.

In summary there are five response types to problems under the SOLO taxonomy:

- Prestructural
- Unistructural
- Multistructural
- Relational
- Extended abstract

(Pegg, 1992)

The ‘prestructural’ level is the foundation level of this model and entails individuals having an inadequate level of knowledge required to complete a task/answer a question. Pegg (1992) gives examples of this level such as students’ refusal to answer a question or students repeating themselves in different ways when providing an answer.

The ‘unistructural’ level is the next level and is reflected by an individual’s use and focus on one particular relevant aspect. In mathematics pupils may select one cue and apply a relevant operation to that cue (e.g. they may differentiate once when asked to find critical points on a curve). This level has clear increases in knowledge over the previous level, however Pegg (1992) notes that inconsistencies are common due to the individual’s focus on one aspect.

An individual at the ‘multistructural’ level of the SOLO taxonomy has the ability to carry out various different operations, however, he/she does not recognise the interrelationships or connections between them. Each of the operations/concepts are seen as independent structures (Pegg, 1992). At the multistructural level, drawing from the previous example of critical points, a student may differentiate the function and let the derivative equal zero. However, the student at the multistructural level cannot explain the connection between these two steps and sees them as independent structures in finding critical points.
At the ‘relational’ level an individual is able to generalise their knowledge from their experiences in context similar to those asked of them (familiar questions/settings). They may not be able to generalise in new contexts and this results in inconsistencies when dealing with unfamiliar problems. Students at this level can generalise the steps to find critical points for familiar functions. Less familiar functions (e.g. exponentials/trig functions) would still cause issues as the deep understanding of the concept is not yet achieved. They do not see that the methods of finding critical points are the same for all functions.

The final level of ‘extended abstract’, when reached, enables individuals to generalise information and apply it to new, unfamiliar situations. A student at this level has a deep understanding of the content. They can generalise from their understanding how critical points are found, and why they are found in this way. They can therefore generalise for any function, whether it is familiar or unfamiliar.

The SOLO taxonomy was rejected as it is not subject specific and a more suitable model was found. The author is not stating that the SOLO taxonomy could not be used, similar to Bloom’s taxonomy. He only states that a more suitable candidate was found.

The last model for discussion is the van Hiele model of the development of geometric thought. An in-depth discussion is required on the van Hiele model as it was ultimately chosen as the basis for the author’s purpose-built model.
2.15 The van Hiele Model of the Development of Geometric Thought

2.15.1 The Model

The van Hiele model arose from the 1957 dissertations of Dina van Hiele-Geldof (1984a) and Pierre Marie van Hiele (1984b) at Utrecht University in the Netherlands. This pair of researchers noticed that students in their classes struggled to learn geometry. Their research produced a model of the progression of one’s understanding of geometry (Mason, 2002).

The van Hiele model outlines five qualitatively different levels of thought development in geometry (van Hiele-Geldof, 1957; Hoffer, 1981; Mayberry, 1983; Burger and Shaughnessy, 1986; Crowley, 1987):

- **Level 0 (Visualisation).** The pupil is aware of shapes, names of shapes (square, circle etc.) but is unconcerned with the properties of them. For example, a pupil could recognise a square shape on a piece of paper but would not recognise the existence of right angles or that opposite sides are parallel (Crowley, 1987).

- **Level 1 (Analysis).** The pupil informally analyses the components of geometric concepts (Burger and Shaughnessy, 1986), for example by colouring the angles within a rectangle to see that the angles are equal (Crowley, 1987). At this level, definitions, relationships between properties, or interrelationships between figures are still not understood and cannot be explained (Crowley, 1987).

- **Level 2 (Abstraction (Burger and Shaughnessy, 1986)/Informal Deduction (Crowley, 1987)).** Pupils at level 2 have the ability to construct interrelationships between and within figures. They can draw from their experience to understand properties of figures and recognise classes of figures. Furthermore, definitions are understood and proofs can be constructed. However, the pupil can construct proof only from a familiar starting point and cannot do so in new, unfamiliar ways.

- **Level 3 (Deduction).** A pupil at this level relies less on memorisation as a way of constructing proofs, as undefined terms, axioms, definitions etc. become understood. Proving something in one particular way is no longer a problem as the pupils can see different ways of proving something.
• **Level 4 (Rigor).** At the final level geometry is seen in the abstract. In the absence of concrete models pupils can continue to study various geometries (such as non-Euclidean geometries).

The van Hiele model is based on a constructivist approach (Faucett, 2007; Ness and Farenga, 2007). Ness and Farenga (2007) state that it is based on constructivism in two ways:

1. an individual’s success in a later level is dependant on prior knowledge, experience with, and mastery of geometric knowledge in earlier levels; and

2. individuals do not merely engage in geometric activity in a passive way; instead, they actively manipulate, transform, and build upon notions of shape and measure.

(Ness and Farenga, 2007, p.50)

The levels of the van Hiele model are appropriate for use with both male and female students (Haviger and Vojkůvková, 2014) and aim to identify a student’s level of geometric maturity (Crowley, 1987). Van Hiele (1984b) states that the idea of levels is inherent in an individual’s thought processes and development. Levels exist regardless of the teaching one receives. What is important is that a teacher guides students towards a correct level structure. In other words the teacher must ensure that the building of levels is done in the right sequence and that a conceptual grasp of the subject is achieved through a systematic approach. The van Hiele levels have recently been found to be continuous (i.e. the levels are not separate from each other) (Perdikaris, 2011). From a mathematics education point of view, this means that students may obtain a certain amount of understanding of one level and start thinking at the next level, even though the preceding level has not been fully acquired.

Pierre Marie van Hiele once pondered the idea of dropping one or two of the higher levels of his model, and even failed to mention level 4 in his address to the NCTM Research Presession in Seattle (1980). However, removing two levels would make it difficult for the ‘new’ three level model to be accepted in the area of mathematics education as it may have been viewed as “too simplistic” (Usiskin, 1982, p.14).
2.15.2 Abilities and Skills at Each Level of the Model

The literature reports various abilities and skills exhibited by students who are at various different levels of the van Hiele model. Most of the research is concerned with levels 0 through 3. Not a lot of research has dealt with the final level of the model, level 4. This section illustrates the behaviours of students at the different levels of geometric thought.

Level 0

Students at level 0 of the van Hiele model can learn the vocabulary associated with a shape. Formal definitions of the vocabulary are not understood and would not be understood if taught to them. Students at level 0 can only recognise a shape by its physical appearance and not by any of its properties (Pandiscio and Knight, 2010). Burger and Shaughnessy (1986) provide many behaviours that are characteristic of a student’s thinking at the various levels. They state that students at level 0 use inaccurate properties to identify and name shapes. Students also refer to visual examples to name shapes (e.g. a door is a rectangle). They frequently mistake the orientation of a shape as meaning that it is a different shape. For example a triangle that does not have an apex at the northern-most point is no longer a triangle to students at this level. Students cannot process varieties of a type of shape. They cannot identify a shape by its given properties, as properties are not understood. (Burger and Shaughnessy, 1986)

In summary students at this level recognise shapes by their appearance. They analyse them visually without reference to their parts or properties and by seeing a shape they can classify it as a rectangle or circle or other shape.

Level 1

Pandiscio and Knight (2010) state that students at level 1 can identify the parts of figures so as to create a class system of figures in their mind. However, they state that students cannot explain any relationships between properties of figures. They also cannot understand formal definitions. Burger and Shaughnessy (1986) provide the following list as that which characterises the behaviour of students at this level.

- Comparing shapes explicitly by means of properties of their components.
• Prohibiting class inclusions among general types of shapes, such as quadrilaterals.

• Sorting by single attributes, such as properties of sides, while neglecting angles, symmetry, and so forth.

• Application of a litany of necessary properties instead of determining sufficient properties when identifying shapes, explaining identifications, and deciding on a mystery shape.

• Descriptions of types of shapes by explicit use of their properties, rather than by type names, even if known. For example, instead of rectangle, the shape would be referred to as a four-sided figure with all right angles.

• Explicit rejection of textbook definitions of shapes in favour of personal characterizations.

• Treating geometry as physics when testing the validity of a proposition; for example, relying on a variety of drawings and making observations about them.

• Explicit lack of understanding of mathematical proof.

(Burger and Shaughnessy, 1986, p.44)

In summary everything that the student does is based on visual representation. When a problem is posed, students attempt various diagrams and illustrations to come to a conclusion. The language that they use is based on whatever they see it to be and is not formal or exact.

**Level 2**

Students begin to form and establish relationships between the properties of figures and the classes of figures which were already processed at level 1 (Pandiscio and Knight, 2010). Students at level 2 can define types of shapes where the definitions recognise the correct role of properties. Students can also adapt these definitions in order to use mathematical concepts in various contexts. When problems are asked students at level 2 often use definitions in their explanations of how they solved a problem or why they implemented a particular course of action. They are not confined to specific definitions as they can recognise how various formal definitions can define the same concept or figure. A conceptual framework is developed in the mind of the
student where a logical order of figures and concepts exists. From the conceptual framework developed a student can use conditional statements ("if, then") and use them to solve problems. However, a student at level 2 cannot differentiate between higher-order mathematical terms such as ‘axiom’ and ‘theorem’. (Burger and Shaughnessy, 1986) They can follow a formal proof on paper or on the board but cannot develop a proof from an unfamiliar starting point (Senk, 1989; Pandiscio and Knight, 2010).

**Level 3**

Students develop an understanding of the, as yet, undefined terms that they have been dealing with. Theorems are also understood and students have the ability to construct proofs from unfamiliar starting points (Pandiscio and Knight, 2010). Students can also identify what they need to solve a problem and what they do not need (Pandiscio and Knight, 2010). It is clear from the literature that a more in-depth relational understanding is obtained at this level. Students have the ability to translate or rephrase problems into equations or an appropriate mathematical form. A behaviour that is also exhibited is the hypothesising on problems. Students hypothesise from their relational understanding and attempt to verify or disprove their hypothesis using mathematics. There is also a recognition at this level that mathematical proof is the only means of establishing proof. Not all proofs can be constructed but the idea is grasped. Simple proofs can be completed. The students also use further language to describe the mathematics they are doing and studying (e.g. theorem, proof, definitions, axioms). Euclid’s postulates are no longer abstract ideas for students and are accepted as fact (Burger and Shaughnessy, 1986).

**Level 4**

Most of the research conducted on the van Hiele model does not consider this level. It is often left outside of the area of research. For example, Burger and Shaughnessy (1986) left out this level because their research was focused on formal reasoning in geometry, or in other words the use of logic or deduction. The researchers state that level 4 requires the comparison of different geometries and was therefore excluded from their study. Abdullah and Zakaria (2013) ignored the final two levels of the van Hiele model as their study was conducted with lower second-level students (similar to junior cycle students in Ireland). They state that students of this age only have an understanding up to level 2 of the van Hiele model and therefore the last
two levels were excluded from their study. Most of the studies that exist on
the van Hiele model with young people have ignored this level. Even Pierre
Marie van Hiele (1984b) himself in his discussion on childrens’ understand-
ing relating to the van Hiele model excluded a discussion of this level.

Mayberry (1983) included level 4 in her study. Though she does not ex-
licitly characterise students’ behaviour at this level, she does note certain
things that they should be able to do. She states that students can ex-
plain why they make assumptions about a contradiction when constructing
an indirect proof. They can also explain the difference between axioms and
theorems, as well as having the ability to deduct facts from statements about
a finite geometry. Mason (2002) says that students at this level understand
formal deduction, the use of indirect proof, and proof by contradiction. She
also notes that non-Euclidean systems are understood.

Level 4 is the summit for understanding geometry according to the van
Hiele model. If a person operates at this level of the model then they see
geometry in a relational way. They can also explain why certain aspects
behave in the way they do and can use deductive reasoning fluently. They
are no longer confined to the area of Euclidean geometry and can engage
with non-Euclidean geometries.

2.15.3 Properties of the van Hiele Model

Though the levels of the van Hiele model have been described, another factor
which a teacher needs to account for is the properties of the model (Crowley,
1987; Fuys, Geddes, and Tischler, 1988). These properties are said to be
“generalities that characterize the model” (Crowley, 1987, p.4). Therefore,
as the levels characterise the thinking that is involved at each level of geo-
metric thought, the properties that will be outlined below provide guidance
for instructional decisions that are made in order to allow for progression
through levels of thinking (Crowley, 1987). Hence, these properties are some
of the pragmatic elements of the van Hiele model. The properties provided
in the literature are listed below.

- Sequential;
- Advancement;
- Intrinsic/Extrinsic Objects;
- Linguistics;
• Mismatch.

The van Hiele model is sequential. This means that pupils must progress through the levels of the model in order. Pupils must be proficient at one level and understand its content and strategies to proceed to the next. This sequential, or hierarchial, property of the model has been proven (Mayberry, 1983). De Villiers (1987) states that hierarchial classification systems have three roles. The first role is that they make definitions more economical, i.e. they take away the need for over-elaborate definitions. The second role is that hierarchial structures allow for a simpler application of deduction. De Villiers (1987) gives the example of “defining a rectangle as a special kind of parallelogram implies that all parallelogram theorems are immediately applicable to rectangles, without having to prove them anew” (p.23). The final role of hierarchial classification systems is that they allow for a systematic conceptual framework development. A teacher may be able to use the system to teach and build students' conceptual framework of a topic.

Advancement through the levels of the model depends more on the teaching and instruction provided than on the age of the pupils.

Certain objects at one level are intrinsic, but may be extrinsic in the proceeding level (van Hiele, 1984b, p.246). For example, a pupil notices a rectangle by its shape at level 0, but at level 1 the properties of the rectangle, such as the angles within it or the relationships between the sides of it, are analysed. This is to say that the rectangle is part of the study (or intrinsic) at level 0 as the rectangle is a concept to learn by shape, while at level 1 the rectangle is outside of the area of study (extrinsic) as it is the properties of it which are the focus.

Linguistics (Crowley, 1987) or language structure (Fuys et al., 1988) are essential in the teaching of geometry. The teacher must use language which is appropriate for each level when teaching pupils who are at this level. Van Hiele (1984b) notes that each level has its own set of linguistic symbols and its own set of relations to connect these symbols. Van Hiele also notes that one of the main failures in the teaching of geometry is in the language barrier (i.e. the teacher uses language of a higher order than the children understand) (Fuys et al., 1988).

The final property of the van Hiele model is mismatch (Crowley, 1987). This relates to instruction that is at a different level to the level a pupil may
be at. Crowley (1987) highlights that if instruction (including vocabulary, content, materials etc.) is at a higher level than a pupil is able for, then the pupil will not be able to understand the thought processes being used.

2.15.4 Instructional Phases for Students’ Improvement

Some teaching methods make the higher levels of thought in the van Hiele model inaccessible (van Hiele, 1984b). According to van Hiele (1984b), there are five phases that allow for the transition from one level to the next (Fuys et al., 1988). It is important to note a point made by Crowley (1987) that correct progress through the levels is more dependant on instruction rather than age, and that the materials, content, and methods of instruction used are still of the utmost importance. These phases are again elements of the pragmatic facet of the van Hiele model. The phases proposed by van Hiele to aid transition are again sequential (Crowley, 1987) and are given as follows:

- **Phase 1: Inquiry**
- **Phase 2: Directed Orientation**
- **Phase 3: Explicitation**
- **Phase 4: Free Orientation**
- **Phase 5: Integration**

In the *inquiry* phase pupils are presented with the topic/concept they are studying (van Hiele, 1984b). The teacher and pupils actively converse about the topic and take part in activities about the topic of study (Crowley, 1987). The activities at this stage serve two purposes. The first is that the activities give the teacher information on what pupils already know about the topic/concept of study, and the second is that the pupils themselves learn what direction their learning will take (Crowley, 1987).

In the *directed orientation* phase, the pupil explores the area of study. They already are aware of what direction the study is going in from the first phase, and the teacher should select appropriate materials so the aims are presented to and learned by the pupil gradually (van Hiele, 1984b). Crowley (1987) notes from this that tasks done in class should be short and purposeful in that they should be done to elicit specific responses from pupils.

Systems of relations begin to form in the *explicitation* phase (van Hiele, 1984b). In this phase the pupils relate their knowledge acquired so far to
The teacher must ensure that pupils use proper language and terminology in any discussion or task that takes place (van Hiele, 1984b). The teacher’s role of observation here is the main role and any other work for the teacher is minimal (Crowley, 1987).

The free orientation phase introduces pupils to tasks of higher complexity. Tasks provided to pupils here must be achievable in a variety of ways (van Hiele, 1984b) and be open-ended (Crowley, 1987). This allows pupils to find their own way to solve the task/problem (Crowley, 1987).

The final phase to progression to the next level is integration. As the word may suggest, this phase deals with the pupils’ acquisition of a full overview of that which he/she has studied up to this point (van Hiele, 1984b). Van Hiele (1984b) stresses the importance that a teacher should not present new material to pupils at this phase, as the only purpose is to summarise their learning. Once this phase has been completed the pupil should now have a new level of knowledge and have at his/her disposal a new system of relations, or a network, related to the area that has just been explored.

These phases of learning are repeated at the next level.

Abdullah and Zakaria (2013) display the learning phases for the van Hiele model at each level of understanding (Figure 6).

![Figure 6: Phases of Learning Geometry (Abdullah and Zakaria, 2013, p.255)](image)

A teacher can organise and structure his/her lessons using this information to give pupils the best opportunity to progress through the van Hiele levels. Along with the diagnostic elements of the levels of the model (sec-
tion 2.15.2), this section has also elaborated on the pragmatic features of the work of the van Hiele’s. Though the levels have been described, methods of evaluating the van Hiele level of an individual have not been discussed. It is important that a teacher knows which van Hiele level an individual is at so they can teach/instruct at an appropriate level for the learner. Failure to do this results in mismatched instruction.

2.15.5 Measuring the van Hiele Levels of Pupils

A teacher must be able to evaluate what level students are at, and how far they are progressed in that level. Without a firm knowledge of what geometric understanding students have, a teacher’s instruction could be mismatched (section 2.15.3). Evaluating students’ level on the van Hiele model is therefore necessary before instruction can begin.

Numerous studies have noted various ways of measuring an individual’s van Hiele level (Usiskin, 1982; Mayberry, 1983; Burger and Shaughnessy, 1986; Gutiérrez and Jaime, 1987; Fuys et al., 1988; Gutiérrez et al., 1991; Jaime and Gutiérrez, 1994; Pandiscio and Knight, 2010). Each of these studies propose different ways of doing this. Some of the differences between the studies are slight, whilst other studies are very different. For example, some researchers used pen and paper tests, while others used interviews. This section provides various evaluation methods for the van Hiele model which have been proposed in previous studies.

The Usiskin test (1982) is based on a pencil and paper examination (multiple choice questions and proofs). The problem with this type of test is a pupil’s reasoning cannot be assessed (i.e. their mindframe in working out a question/task) (Crowley, 1990; Wilson, 1990).

The Mayberry (1983, p.61) method of assessing van Hiele levels is based on interviews in which pupils sat two sessions of one hour interviews. These pupils were given a pencil and paper and given the questions to be answered. These interviews were also audiotaped. A resource intensive problem is evident in the Mayberry assessment for teachers in a school (i.e. having to interview and assess every pupil they teach individually over such a long time could be difficult). The Usiskin (1982) and Mayberry (1983) assessments both centered around pupils giving correct answers to written tasks.
Burger and Shaughnessy (1986) conducted an assessment similar to Mayberry’s (1983). They used interview sessions with pupils of various ages and abilities and again audiotaped the sessions while the pupils used pencils and paper to work out problems. Interviewers also took notes. The data that could be analysed from this was therefore the audio recording, the pupil’s paper, and the interviewer’s notes. The purpose here was to analyse the pupil’s level of thinking.

Fuys et al. (1988) made use of videotaping to analyse pupils’ level of thinking. Similar to the previous studies that were mentioned, this gave the researchers a chance to hear what the pupils were saying about the content, but also allowed them to analyse non-verbal communications and a pupils attitude (Fuys et al., 1988, p.2). The purpose of this Fuys et al. (1988) study was again to analyse a pupil’s thinking level.

Gutiérrez et al. (1991) thought differently to the other researchers mentioned so far. Up to this point, criteria used in research conducted placed a pupil in one of the five van Hiele levels (Gutiérrez et al., 1991). Gutiérrez et al. (1991) concluded from elements of previous research that the van Hiele levels are not discrete, i.e. that they are continuous. As already stated this has recently been found to be the case (Perdikaris, 2011). Gutiérrez et al. (1991) proposed that each van Hiele level has ‘degrees of acquisition’. Each van Hiele level was judged on a scale of 0 to 100 and this scale was divided into five intervals as in Figure 7.

![Figure 7: Degrees of Acquisition of a van Hiele Level (Gutiérrez, Jaime, and Fortuny, 1991, p.238)](image)

Gutiérrez et al. (1991) analysed pupil responses to open-ended tasks in terms of their type of reasoning rather than their ability to answer the tasks correctly, with the hope that a partially correct answer would have provided them with useful information. They identified eight “types” of answers to
aid them in assigning a score to each pupil on the scale of 0-100, or in other words the type of answers given by pupils enabled the researchers to see what degree of acquisition each pupil had of a certain van Hiele level (see Figure 7). The types of answers were as follows:

- **Type 0.** No answer provided.

- **Type 1.** Answer provided shows that pupil has not achieved a certain level, but fails to give information about lower levels.

- **Type 2.** Incorrect answers that show some level of reasoning. Incorrect reasoning processes.

- **Type 3.** Correct but insufficiently worked out answers that give a sense of some level of reasoning. Very few explanations provided and incomplete results.

- **Type 4.** Correct/incorrect answers that reflect two van Hiele levels (e.g. levels 2 & 3 or levels 1 & 2). Clear reasoning and sufficient justifications provided.

- **Type 5.** Wrong answers that clearly show a level of reasoning. Answers show reasoning processes that are either complete but incorrect, or correct reasoning processes that do not lead to the correct solution of the task/problem.

- **Type 6.** Correct answers to the task that show a level of reasoning but that are incomplete or insufficiently justified.

- **Type 7.** Complete, correct answers that are fully justified and show a level of reasoning.

(Gutiérrez et al., 1991, p.240)

One of these types of answers were assigned to every answer provided by a pupil. Gutiérrez et al. (1991) weighted each of these types of answers as in Table 11.
Table 11: Weighting of Different Types of Answers (Gutiérrez et al., 1991, p.241)

<table>
<thead>
<tr>
<th>Type</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>0</td>
<td>0</td>
<td>20</td>
<td>25</td>
<td>50</td>
<td>75</td>
<td>80</td>
<td>100</td>
</tr>
</tbody>
</table>

The researchers were then able to obtain a pupil’s average over the full study to get their score on the 0-100 scale, therefore assessing their degree of acquisition of each van Hiele level.

Many studies, such as some of those mentioned already, have been conducted into the van Hiele model. Most of the studies have been based specifically on the analysis of student understanding with reference to the van Hiele levels. The next section looks at the findings from these studies. This is done to give an exploratory look at findings related to the van Hiele model itself, as well as student understanding with reference to the van Hiele levels.

2.15.6 Findings from van Hiele Studies

Early van Hiele studies mainly conducted through the 1980’s and early 1990’s produced findings related to the model itself (solidifying the properties of the model, investigating the model). Later studies (2000’s) were more focussed on results from exploratory studies (groups’ understanding relative to the model) than findings relating to the development/precision of the model. The findings discussed in this section are predominantly based on the earlier studies as the author wished to know as much as possible on the model itself. The more recent studies are briefly discussed at the end of this section to give recent insights into findings on various groups’ understandings of geometry related to the model.

Burger and Shaughnessy (1986) characterised the van Hiele levels based on American students’ responses to interview tasks. They had a sample of 14 participants - 13 students from grades 1 (age 6 years) through 12 (age 18 years), and 1 college mathematics major. Burger and Shaughnessy found that the van Hiele model is useful for describing a student’s thinking process on different geometric tasks. They also provided characteristics of student reasoning at levels 0-3 of the van Hiele model based on students’ reasoning in the interviews. Level 4 of the model was outside the area of their re-
search. They were unable to confirm that the levels of the van Hiele models were discrete. This is unsurprising as the levels were recently found to be continuous by Perdikaris (2011), as noted in section 2.15.1.

Usiskin (1982) conducted his study with 2699 American high-school, or second-level, students. The study’s aim was to measure the geometric abilities of second-level students with reference to the van Hiele levels. The study was conducted with a 25-item multiple choice assessment. 5 questions were included for each level of the van Hiele model. Usiskin found that 70% of students graduate from second-level education at levels 0, 1 or 2 of the model. The curricular intention is that students have acquired either of the final two levels before they graduate. Usiskin states that almost half of all students finish second-level education with a level of geometric understanding that they should have after primary school.

Senk (1989) showed that students’ ability to write geometric proofs is positively related to their level on the van Hiele model. She assessed 241 American second-level students and found that those who entered a geometry course at level 0 would be unable to construct proofs at the end of the course. As abilities get higher upon entry to the course there is a greater and greater chance that the student will be able to construct proofs. Students beginning at level 2 have a 50-50 chance of writing proofs after the course is completed. Though not explicitly stated by Senk, this reflects the hierarchial structure of the van Hiele model, as well as the fact that students are unable to operate at a level above their abilities. These two points were also demonstrated by Mayberry (1983).

Mayberry (1983) audiotaped interviews with 19 American pre-service primary teachers where she asked them 128 questions. A check sheet was used to rate student responses to problems. Mayberry states that her research was based on the first four levels of the model but it is clear that she conducted research into all five, referring to level 0 as the ‘base level’. She showed that a student will not demonstrate the same level of understanding of every concept tested. Mayberry also showed that a student at a particular level will answer questions on concepts at this level but will not answer questions on concepts above this level. The hierarchial structure of the van Hiele model was therefore demonstrated. This was shown through Guttman Scalogram analysis. This analysis determines if response patterns to an assessment forms a scale, or in this case a hierarchy. The pre-service teachers who participated in Mayberry’s study were found to be below level 3 of the
model. She states that high-school geometry textbooks assume that you are above level 2 before you begin. She notes that this is alarming because if primary educators are below level 3 then they cannot instruct their students to level 3. This reflects the work of Rowland (2007) who notes that subject matter knowledge is the foundation for effective teaching. A teacher who does not know the content they are teaching cannot instruct in a way that allows for the student to understand the content.

Pandiscio and Knight (2010) also illustrate a problem with U.S. pre-service teachers’ understanding of geometry with reference to van Hiele levels. They found that before the geometry course taken in the student teacher preparation programme, that pre-service teachers (of 7-12 grades/secondary school) did not have a level of understanding of geometric content at, or above, the level required of their future pupils. After the geometry course was conducted statistically significant gains of at least one van Hiele level were achieved by the pre-service teachers. However, the gains were not enough to raise the sample populations’ van Hiele level to the level expected of their future pupils.

Studies on the van Hiele model have shown it to be useful in describing the thinking of students of any age (Burger and Shaughnessy, 1986). The hierarchial structure of the model has been demonstrated (Mayberry, 1983; Senk, 1989). Despite this, it has been shown that instruction can be mismatched, as students enter geometry courses at a level that is lower than the level of instruction (Mayberry, 1983). Almost half of U.S. students have been shown to finish second-level education with a level of geometry that is equivalent to what they should know when they finish primary education (Mayberry, 1983). Furthermore, pre-service teachers cannot instruct their students to the level of geometry desired in the curriculum. This is because teachers do not understand geometry to this level themselves (Pandiscio and Knight, 2012). At the higher levels of the model it has been shown that proof-writing in geometry is directly related to one’s level on the van Hiele model (Senk, 1989). If students are not at an appropriate level on the model before the commencement of a geometry course, they will not be able to construct proofs after it has finished (Senk, 1989).

More recent studies based on the van Hiele model have been exploratory studies rather than studies based on properties of the model.
Van der Sandt and Nieuwoudt (2003) showed that South African Grade 7 (final year of primary school) teachers and pre-service teachers do not have enough knowledge of the geometry subject matter that they have to teach. Erdogan, Akkaya and Akkaya (2009) showed that teaching based on the van Hiele model developed creativity and geometric fluency amongst their sample of primary students in Turkey. Halat (2008a) showed that most Turkish middle and high school teachers are at level 2 (abstraction)\(^8\) of the van Hiele model and that no gender differences were found in the findings regarding male/female van Hiele levels of thinking. Watson (2012) showed that results from the van Hiele test (Usiskin, 1982) are a good predictor of student success and failure in a university mathematics module that has a geometric component. Feza and Webb (2005) showed that no Grade 7 South African student in their sample of 30 had achieved the curricular requirement of van Hiele level 2. Wu and Ma (2006) further support the evidence that the van Hiele levels form a hierarchy. They show that Taiwanese primary school children have different levels of thought for different concepts, with 51.9\% of the students in 1st Grade not achieving level 0 of the van Hiele model (visualisation) (Wu and Ma, 2006). Halat (2008b) showed that there was no statistical difference in van Hiele reasoning levels between pre-service primary and pre-service secondary teachers in Turkey, with the vast majority of both being at levels 2 (36.25\% of sample) or 3 (40\% of sample) of the model.

The majority of studies on the van Hiele model have been with reference to its use in geometry. However, the principles of the van Hiele model have been applied to other areas and frameworks. The Realistic Mathematics Education movement is a learning theory (van Groenestijn, 2001) that was partly formed using the van Hiele model as a framework.

2.15.7 The van Hiele Levels Applied to General Mathematics Education

Though the majority of research into the van Hiele levels has dealt specifically with geometry, the work of the van Hiele’s has been accredited with part of the framework upon which the Realistic Mathematics Education movement was built (van Hiele, 1973; de Lange, 1996). This was primarily focused on van Hiele’s categorisation of the process of three levels of general learning:

- That a student attains the first level of thinking once he/she can

\(^8\)Referred to as level 3 (ordering) by Halat (2008a)
manipulate the known characteristics of a pattern that is known to
him/her.

- Once he/she can manipulate the interrelatedness of the characteristics
  he/she has reached the second level.
- He/she will reach the third level of thinking when he/she starts ma-
  nipulating the intrinsic characteristics of relations.

(de Lange, 1996, p.58)

Along with this, the structural idea of the van Hiele levels (i.e. that one must
have an understanding of one level before progression to the next) is itself
an educational idea that dates back to the work of Bruner (1966). Treffers
(1987) refers to this as ‘vertical planning’. This section shows that the van
Hiele model is not limited to a specific use within the area of geometry, and
has been applied to mathematics education in a wider context. This shows
that though the van Hiele model is based on geometry, the potential exists
to apply its principles to other areas.

2.15.8 Criticisms of the van Hiele Model

Though the van Hiele model has proven to be a useful tool in describing
students’ learning of geometry, some criticisms of the model have been re-
ported in the literature.

Pegg (1997) states that the van Hiele levels are too broad. He explains
that because of this, the van Hiele model is not easily generalised to a lot of
questions that school students encounter in geometry. What this means is
that a teacher cannot get an idea of what level of the van Hiele model their
students are at from their students’ responses to many problems found in
school textbooks. This is because it is hard to relate the model to a lot of
the geometry students encounter in schools. The levels of the model accord-
ing to Pegg (1997) need to be more defined and related more to problems
frequently encountered in school. He found it difficult to infer a student’s
thinking level from problems they were asked.

Though it is not specifically stated as a criticism, Crowley (1987) notes
apparent deficiencies in the model. She notes that the phases of learning are
a guideline for content sequencing and teaching. Guidelines like these may
or may not be followed or implemented correctly. They are not direct or
conclusive and leave a lot to the initiative of the teacher. The properties of the van Hiele model also only provide pedagogical advice. These points are similar to those of Pegg (1997). The similarities exist in the nature of the van Hiele model. It is a very broad model, where the brief level descriptors encompass a large spectrum of content. Pusey (2003) states that a teacher would need to invest a lot of time and effort into figuring out teaching strategies if they wanted to use the van Hiele model as a basis for instruction. It can be seen from the points of Crowley (1987) and Pegg (1997) that the van Hiele model does not give a structure or sequence for the teaching of geometry. It only provides advice and considerations to take into account. Therefore the thought that the van Hiele is an established teaching model is nullified. It is a learning model first and foremost, with only supplementary advice for teachers in terms of pedagogy.

The broad nature of the van Hiele model is where the criticism stems from. Nevertheless, the van Hiele model has been shown to be effective in determining a student’s level of geometric thinking and this is highly valuable information for a teacher. The positive points of the model significantly outweighed the negative points in the literature that was reviewed.

2.15.9 Summary of the van Hiele Model

The van Hiele model is a 5-level hierarchial model predominantly based on how students learn geometry. It has been proven to be an effective model for classifying a student’s level of geometrical thinking and understanding (Burger and Shaughnessy, 1986). Various assessment strategies has been devised that place students on a level of the van Hiele model (Usiskin, 1982; Mayberry, 1983; Burger and Shaughnessy, 1986; Gutiérrez et al., 1991). Each of these assessments has positive and negative points, with written assessments failing to allow for students’ thought processes to be elaborated upon, and interviews taking a lot of time to conduct. Second-level students and pre-service teachers have been shown to have low levels of geometric understanding with reference to the van Hiele model (Usiskin, 1982; Mayberry, 1983; Pandisco and Knight, 2010). Many American students have been shown to graduate from second-level with an understanding of geometry that coincides with the proposed requirement at the end of primary schooling (Usiskin, 1982). The van Hiele’s work also provides instructional advice for teaching geometry. There are five phases for instruction, as well as properties of the model itself, which are guidelines for the teaching of geometry (van Hiele, 1984b). However, research has found these to be too
broad (Pegg, 1997; Pusey, 2003). The research states that the model is not easily applicable to the types of problems that students face in school. In other words, it is difficult to pinpoint what level students are at from the problems they meet in school due to the broad nature of the levels. The literature also states that the model is not defined for teaching and is too broad for an application to teaching (Pegg, 1997; Pusey, 2003).
2.16 Bloom’s taxonomy vs. the van Hiele Model

The first comparison that can be made between both models is the hierarchical structure that is present in each of them. In each of the models one level is required before the next level can be understood/acquired.

If the two models are compared a surjective relationship can be observed (figure 8).

Figure 8: Comparison of Bloom’s Taxonomy and the van Hiele Model of the Development of Geometric Thought (House, 2011)

The remembering level of Bloom’s taxonomy maps to the visualisation level of the van Hiele model. This is the case as both of these levels call for basic recall of information, either the recall of facts/information (Bloom) or the recall of the visual appearance of shapes (van Hiele).

Understanding (Bloom) can also be mapped to visualisation (van Hiele) as an understanding/comprehension is formed through the analysis of figures/rules. For example, an individual could understand that there are 180° in a triangle as a result of analysing various triangles with the same result for the summation of their angles. Though House (2011) maps understand-
ing (Bloom) to visualisation (van Hiele), it appears that a more accurate mapping would be comprehension (Bloom) to analysis (van Hiele). This is because of the increased emphasis on understanding, however, the student still does not truly understand the concepts that he/she is working with. The surface understanding that exists on the levels of ‘understanding’ of Bloom’s taxonomy and ‘analysis’ of the van Hiele model make these a more accurate pairing.

Applying (Bloom) is mapped to analysis (van Hiele) as both require the use of previous knowledge to either solve problems or reason inductively about shapes. In Bloom’s ‘applying’ an individual must use what they have learned in a different context (e.g. to solve problems). An individual at van Hiele’s level of analysis must use what they have learned to reason about various shapes. This aids in the solution of problems which require such skills.

Analysing (Bloom) maps to abstraction (van Hiele). Somebody at the level of abstraction (van Hiele) relies less on memorisation as he/she can create proofs etc. from their understanding of the topic and concepts. The level of ‘analysing’ (Bloom) requires individuals to make inferences and support them from their understanding of the content. Overall, these levels of Bloom and van Hiele are similar due to their focus on understanding of content and formulation of certain things (inferences etc.) which demonstrate that understanding. There also exists an ability at both of these levels to break down what an individual has learned into smaller parts (e.g. illustrate properties of figures to classify them).

Evaluating (Bloom) maps to deduction (van Hiele) as a student at the level of evaluating (Bloom) can critique their own work and judge whether a mathematical proof is valid and proves what it set out to prove, as with the level of deduction (van Hiele). Students at each of these levels can also evaluate what they need and do not need to solve a problem. At evaluation of Bloom’s Taxonomy and deduction of the van Hiele model it is clear from the literature that a higher order relational understanding of content is achieved by students (Pandiscio and Knight, 2010; Sections 2.13 and 2.15.2).

Creating (Bloom) bears similarities to rigor (van Hiele) as both are the final tier on the models of understanding. An individual at either of these levels will not be reliant on anyone else in aiding their development (teacher/tutor etc.). This is due to the fact that someone at the creating
(Bloom) level can generate the mathematics they need, and come up with new ideas, to solve problems, as indeed can someone at the level of rigor (van Hiele) as there is no longer any knowledge restrictions. The dominant feature of these two levels however, is the fact that anyone at either of them has an extremely high level of understanding of the content/concepts and is at the top of the understanding hierarchy, with the ability to accurately generate their own mathematics to serve the purpose that they desire.

The author identified the van Hiele model as a more appropriate model to use in this research despite its similarities to Bloom’s taxonomy.

Though both models have been used in an educational setting, a choice had to be made on which of these models was most relevant to the research. Many of the ideas embedded within each model are similar to the ideas in the other, however, an emphasis must be placed on the geometrical nature of the van Hiele model. This research is based on teaching trigonometry and the concepts involved in trigonometry. As the van Hiele model assesses and categorises understanding of geometry there is a direct relationship with the subject matter of this research.

Bloom’s taxonomy is very broad with respect to its potential application to a wide range of educational fields (e.g. English and history (Ferguson, 2002)). A more precise model for understanding in mathematics, specifically geometry was researched heavily and the van Hiele model was an appropriate selection for use in this PhD. research through its high recognition in the field and relevance to the topic. Methods of assessment to place individuals on the levels of the van Hiele model have already been constructed. The application of Bloom’s taxonomy directly to mathematics is problematic. Though it could be possible to determine trigonometric understanding through Bloom’s taxonomy and extend it to teaching, it may not be appropriate. Using Bloom’s taxonomy would give a valid description of what students do and do not understand in trigonometry before teaching commences, however it would be too broad. From the literature it seems a list of concepts that students understand/do not understand could be constructed, but teaching would therefore be unstructured as the teacher would have to jump between concepts in his/her teaching. A conceptual understanding that relates concepts to each other would therefore be ignored in any teaching that would be done. Nevertheless, it can be seen that Bloom’s taxonomy is not too unlike the van Hiele model in terms of structure. Using Bloom’s taxonomy in this research could have proved fruitful, however the
van Hiele model was the more appropriate and convenient choice.

2.17 The SOLO taxonomy vs. the van Hiele Model

Research has been conducted into the possible elaboration of the van Hiele model using the SOLO taxonomy (Olive, 1991; Pegg, 1992; Pegg and Davey, 1998; Pegg, Gutiérrez, and Huerta, 1998; Battista, 2007). What the research shows is that it is a complex topic and so far has not been covered conclusively as many issues and questions still remain unanswered (Battista, 2007). Nevertheless, Pusey (2003) states that both models provide teachers with ways to assess students’ reasoning. She states that both models, in theory, aim to provide assessment guides as well as instructional guides. Pusey (2003) states that one of the main differences is that the SOLO taxonomy is not subject specific. Therefore, a lot of work would be necessary to translate the SOLO taxonomy for each topic in mathematics. Any mapping done between the SOLO taxonomy and the van Hiele model has been more concerned with Piaget’s modes of cognitive development such as:

Table 12: Comparison of van Hiele Model and Piaget’s Modes of Cognitive Development (Pegg and Davey, 1989, p.19)

<table>
<thead>
<tr>
<th>van Hiele level</th>
<th>Mode of cognitive development</th>
<th>Age</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Ikonic</td>
<td>2-6 yrs</td>
</tr>
<tr>
<td>1</td>
<td>Concrete symbolic</td>
<td>7-15 yrs</td>
</tr>
<tr>
<td>2</td>
<td>Concrete symbolic</td>
<td>7-15 yrs</td>
</tr>
<tr>
<td>3</td>
<td>Formal operational 1</td>
<td>16+ yrs</td>
</tr>
<tr>
<td>4</td>
<td>Formal operational 2</td>
<td>None specified</td>
</tr>
</tbody>
</table>

Piaget’s stages are often discussed with reference to the SOLO taxonomy but do not belong to the taxonomy itself. The literature states that the SOLO taxonomy elaborates how children progress through Piaget’s stages (Pegg and Davey, 1989; Pusey, 2003).

No direct comparison between van Hiele and the SOLO taxonomy was drawn in the literature that was reviewed for this research.
The SOLO taxonomy, similar to Bloom’s taxonomy, is not directly related to mathematics (i.e. is not subject specific). This is one reason why the van Hiele model was more appropriate. Not alone is the van Hiele model related to mathematics, but it is based on the understanding of geometry. As this thesis deals with the teaching of trigonometry, a model that relates to geometry is a better choice. If the author would have pursued a model adapted from the SOLO taxonomy, many adaptations would have had to be made to make the SOLO taxonomy specific to mathematics, and then specific to trigonometry. This may have damaged the academic integrity of the model that was developed.

This research aims to develop a purpose-built model for teaching trigonometry. Before the teaching of trigonometry can take place it is vital that a teacher of mathematics knows what previous understanding their students have of various trigonometric concepts. The SOLO taxonomy, similar to Bloom’s taxonomy, may only draw up a list of concepts that students do/do not understand. This would lead to unstructured teaching where a hierarchy of mathematical development cannot be obtained. A teacher would once again have to jump between concepts in his/her teaching. The van Hiele model provides and elaborates upon a hierarchy of levels in geometric development. It also provides assessment methods that pinpoint where students are on this hierarchy of levels. The van Hiele model, with its roots in geometry, and its hierarchial structure was still deemed the most appropriate choice for use in this research.

2.18 Conclusion

A purpose-built trigonometry teaching model was sought in light of the issues in the mathematical education of Irish students and international students in the topic of trigonometry (Chapter 2 Part 1). This teaching model would respond to the gap in the literature with respect to the effective teaching of trigonometry. This chapter also discussed models for understanding mathematical development and/or cognitive development. The main focus of part 2 of this chapter was the van Hiele model. The van Hiele model is a model of how students learn geometry. The literature provides methods of assessing students’ understanding with reference to the levels of the van Hiele model. Though teaching guidelines are provided for the van Hiele model, i.e. how to teach so as to raise student attainment relative to the model, the teaching aspect of the model has been criticised (Pegg, 1997;
Pusey, 2003) and is in need of further elaboration. The literature supports the view that the model is too broad for a specific application to teaching. Despite this, the van Hiele model was still the most appropriate choice as a basis for the purpose-built trigonometry teaching model to be developed. This is primarily due to the fact that the van Hiele model is based on geometry, and that trigonometry is a branch of geometry. Other models that were reviewed such as Bloom’s taxonomy and the SOLO taxonomy, though they could potentially be adapted for a purpose-built model to be developed, were rejected for this research. The author felt that too many adaptations would have been needed as these two models are not specific to mathematics. This would damage the academic integrity of the research. Along with this, numerous types of assessment methods have already been developed for the van Hiele model. The author saw the potential to use the assessment method of Gutiérrez et al. (1991) for this research.

Following the review of literature, the author concluded that a purpose-built model for teaching trigonometry would be developed through an adaptation of the van Hiele model. Using the van Hiele model as a basis, an assessment could be constructed that would place students on the new model. The teaching strategies for the van Hiele model would also be applicable to the purpose-built teaching model due to the geometric focus of the van Hiele model. However, the author would need to address the problem of the broad nature of the van Hiele levels in his work in order to make his model more applicable to teaching. He achieved this through genetic decomposition. This is explained in section 4.2.3.
Chapter 3

Methodology

3.1 Introduction

Research is the “systematic, controlled, empirical and critical investigation of hypothetical propositions about the presumed relations among natural phenomena” (Kerlinger cited in Cohen, Manion and Morrison, 2007, p.6). Similarly, Cohen et al. (2007) explain that research attempts to understand the environment and the phenomena present in that environment. Cohen et al. (2007) state that lay people conduct their own research in very different ways to scientific researchers. Lay people use common sense and base their theories on random events, but scientists carefully construct their theories systematically. Scientists use their theories in specific ways once they are constructed, while lay people use their theories loosely and in an uncritical fashion. This chapter presents the methodological approaches to this research. The kinds of research and paradigms of research considered and implemented throughout the research (Kaplan cited in Cohen et al., 2007) are presented in this chapter as well as the ‘methods’ and ‘methodology’ which analyses these methods.

The chapter begins with the philosophical underpinnings of the research. The epistemological stance taken by the author was that of an interpretivist. An Educational Research Design methodology was selected for the research. This chapter elaborates on the Educational Research Design methodology and provides a comprehensive report on why it was selected. The implementation of the Educational Research Design methodology using proof-of-concept and mixed methods approaches is also discussed. Justification of all methodological decisions must be provided according to Kallet (2004). This

99
is done so that:

- the experiment could be repeated by others to evaluate whether the results are reproducible;
- the audience can judge whether the results and conclusions are valid.

(Kallet, 2004, p.1229)

Cohen, Manion and Morrison (2007) note the idea of ‘fitness for purpose’, i.e. that the purpose of the research dictates the methodological approaches and designs employed. This research follows this idea in that the decisions made with respect to the methods used were done with the purpose of the research being of utmost importance. The chapter describes the methods of the research and how these methods overcame any potential limitations.

### 3.2 Assumptions and Research Paradigms

Hitchcock and Hughes (2002) state that ontological assumptions and epistemological assumptions lead to methodological structures which in turn lead to research instruments and data collection (Figure 9).

Ontological assumptions are those based on the nature of the subject matter being investigated (Hitchcock and Hughes, 2002; Cohen at al., 2007; MacIntosh, 2009). This research is mathematics education research. The nature of mathematics education research means that it is predominantly applicable to classrooms. Schoenfeld (2000, p.641) defines mathematics education research as that which aims to “understand the nature of mathematical thinking, teaching, and learning”. Research into this area should contribute to the field of theory and be useful to the field of mathematics education. The primary ontological assumption is that a person’s view of reality, or in this case mathematics, is shaped by their teacher. The mathematics education literature supports this position (Ball and Bass, 2003; Hill and Ball, 2004; Ball, Thames, and Phelps, 2008; Saritas and Akdemir, 2009).

Epistemological assumptions deal with knowledge, specifically with how knowledge can be acquired and how to communicate knowledge (Hitchcock and Hughes, 2002; Cohen at al., 2007). Following on from the researcher’s ontological assumptions, his assumptions on epistemology are formed from an interpretivist’s viewpoint. There are two main epistemological paradigms used in education research: positivist (or scientific) and interpretivist (or
naturalistic) (Cohen et al., 2007). A paradigm is defined as “a philosophical and theoretical framework of a scientific school or discipline within which theories, laws, and generalizations and the experiments performed in support of them are formulated” (Merriam-Webster, 2014) or in other words any kind of theoretical or philosophical framework. Someone with a positivistic view is someone who believes that knowledge is objective, and their research is based on observations and scientific methods (Cohen et al., 2007). They tend not to be an active participant in the research and mostly play the role of a ‘fly on the wall’. Positivists draw their conclusions from quantitative methods. An interpretivist on the other hand believes that knowledge is constructed personally and subjectively by the individual (Cohen et al., 2007). Conclusions under the interpretive paradigm are primarily drawn from qualitative methods rather than from numerical and statistical analysis. The nature of mathematics education focuses on mathematics as a subject that people construct their own meaning of and gain knowledge from experience and learning. Learning takes place in dynamic environments. Therefore the nature of mathematics education means that the acquisition of knowledge
is viewed as subjective. The interpretive paradigm is duly employed in the majority of mathematics education research and in this doctoral study.

Conducting research from an interpretivistic viewpoint has the following advantages and disadvantages.

- **Advantages:**
  - It facilitates an understanding of the ‘why’s’ and ‘how’s’ of the research;
  - It allows for changes to occur and allows the researcher to react to any changes;
  - It allows for the social aspects of the research to be appreciated;
  - The context in which the research is set is taken into account. Therefore dynamic contexts such as classrooms are part of an interpretivist’s world.

- **Disadvantages:**
  - Collecting data can take a lot of time;
  - Data analysis can be very complicated;
  - Patterns may not emerge in analysis of data and the researcher has to be aware of that;
  - Lay people feel that it is a less credible form of research.

(Raddon, 2010)

An interpretivist views the research process as shown in Figure 10.
Figure 10: Interpretivist View of the Research Process (Raddon, 2010, p.14)
Data analysis paradigms that conform to interpretive research exist. The next section discusses the two data analysis paradigms, qualitative analysis and quantitative analysis, as well as the mixed-methods approach which blends both of these paradigms. Analysing these paradigms provides guidance for the development of research instruments which are developed to gather the researcher’s desired types of data.

3.2.1 Qualitative/Quantitative Paradigms and the Mixed-Methods Approach

Two types of data analysis exist: qualitative data analysis and quantitative data analysis. As discussed in section 3.2, the qualitative paradigm aligns itself more with an interpretivist’s methods of research while the quantitative paradigm falls into a positivist’s methods. This section discusses both types of data analysis, and concludes with a blend of both types, the mixed-methods approach. The author demonstrates the advantages and disadvantages of each type of data analysis and justifies his choice of the mixed-methods paradigm.

3.2.2 Qualitative Paradigm

Qualitative data analysis encompasses the organisation of data and the explanation of data from the participants’ view of the situation (Cohen et al., 2007). The division of data into categories from which patterns and themes emerge is the general approach to qualitative data analysis. There is no single way to analyse qualitative data due to the dynamic nature of the qualitative data itself. This data arises from focus groups, interviews, and any other situation that is non-scientific (i.e. no exact values to be found). Therefore the data that arises cannot be predicted and an analyses of qualitative data has to interpret what is given by participants. This idea of interpreting data relates to an interpretivistic epistemological view. Subjectivity is high in qualitative analysis. Cohen et al. (2007, p.461) state that a researcher using qualitative analysis can aim:

- to describe
- to portray
- to summarise
- to interpret
to discover patterns

to generate themes

to understand individuals and idiographic features

to understand groups and nomothetic features (e.g. frequencies, norms, patterns, ‘laws’)

to raise issues

to prove or demonstrate

to explain and seek causality

to explore

to test

to discover commonalities, differences and similarities

to examine the application and operation of the same issues in different contexts.

Conducting qualitative data analysis has the following advantages and disadvantages (Table 13).

Table 13: Advantages and Disadvantages of Using Qualitative Data Analysis (Langdridge and Hagger-Johnson, 2009, p.15)

<table>
<thead>
<tr>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recognises the subjective experience of participants</td>
<td>Cannot apply traditional notions of validity and reliability on the data</td>
</tr>
<tr>
<td>Often produces unexpected insights about human nature through an open-ended approach to research</td>
<td>It is often not appropriate or even possible to make generalisations or predictions</td>
</tr>
<tr>
<td>Enables an ‘insider’ perspective on different social worlds</td>
<td>Needs justification for it is still not a widely and consistently accepted approach to (psychological) research</td>
</tr>
<tr>
<td>Generally does not impose a particular way of ‘seeing’ on the participants</td>
<td>Lack of replicability</td>
</tr>
</tbody>
</table>

A lack of replicability is the lack of reliability or the lack of the ability to produce similar results if the research was conducted in a different place
and with different participants (Cohen et al., 2007). However, as noted, educational research takes place in dynamic environments. One would not assume that results would be the same across different groups of students. This is due to different groups of students having varying degrees of past mathematical experience. Mathematical ability differences between students themselves can be seen in any typical classroom. Therefore the idea that a piece of educational research would be reliable across every situation, scenario or educational environment is questionable. It can at best be replicable across the ‘typical’ classroom.

3.2.3 Quantitative Paradigm

Quantitative data analysis emanates from the positivistic epistemological view (Hitchcock and Hughes, 2002). It is concerned with numerical analysis. Quantitative data analysis is conducted by gathering numerical data and analysing it through statistical tests. It is a highly objective method to use. Findings or results are usually generalisable to various contexts (Rubin and Babbie, 2010). One key aspect of quantitative research is that the researcher(s) prepares all of their research procedures in advance. He/She knows what statistical tests they will perform before a test/questionnaire is administered (Rubin and Babbie, 2010). This approach does not allow for changes to occur in the research or for dynamic systems, and therefore it is used less often than qualitative research in educational studies. Some advantages and disadvantages of this type of data analysis are given in Table 14.
Table 14: Advantages and Disadvantages of Using Quantitative Data Analysis (University of South Alabama, 2014)

<table>
<thead>
<tr>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tests and validates already constructed theories about why and how phenomena occur</td>
<td>The researcher’s categories that are used may not reflect local constituencies’ understandings</td>
</tr>
<tr>
<td>Tests hypotheses that are constructed before the data are collected</td>
<td>The researchers theories that are used might not reflect local constituencies’ understandings</td>
</tr>
<tr>
<td>Can generalize research findings when the data are based on random samples of sufficient size</td>
<td>The researcher might miss out on phenomena occurring because of the focus on theory or hypothesis testing rather than on theory or hypothesis generation (called the confirmation bias)</td>
</tr>
<tr>
<td>Can generalize a research finding when it has been replicated on many different populations and subpopulations</td>
<td>Knowledge produced might be too abstract and general for direct application to specific local situations, contexts, and individuals</td>
</tr>
</tbody>
</table>

Cohen et al. (2007) note that conducting quantitative analysis has no greater or lesser importance than conducting qualitative analysis. It is all a matter of what type of analysis fits the purpose of the research (Newman and Benz, 1998; Cohen et al., 2007). The differences between both paradigms are given in Table 15.
Table 15: Differences between Qualitative and Quantitative Paradigms (Organisational Heartbeats, 2012)

<table>
<thead>
<tr>
<th>Qualitative</th>
<th>Quantitative</th>
</tr>
</thead>
<tbody>
<tr>
<td>The aim is a complete, detailed description</td>
<td>The aim is an accurate, reliable explanation.</td>
</tr>
<tr>
<td>Used when the researcher has no, or very little idea of what he/she is looking for.</td>
<td>Used when the researcher knows clearly in advance what he/she is looking for.</td>
</tr>
<tr>
<td>Used during earlier phases of research projects.</td>
<td>Used during latter phases of research projects.</td>
</tr>
<tr>
<td>The design starts out quite loose and emerges as the study unfolds.</td>
<td>All aspects of the study are carefully designed before data is collected.</td>
</tr>
<tr>
<td>Researcher is the data gathering instrument.</td>
<td>Researcher uses tools, such as questionnaires to collect data.</td>
</tr>
<tr>
<td>Data is in the form of words, pictures or objects.</td>
<td>Data is in the form of numbers and statistics.</td>
</tr>
<tr>
<td>Subjective - individuals’ interpretation of events is important, e.g., uses observation, in-depth interviews etc.</td>
<td>Objective seeks precise measurement and analysis of target concepts, e.g., uses surveys, questionnaires etc.</td>
</tr>
<tr>
<td>Qualitative data is more ‘rich’, time consuming, and less able to be generalized.</td>
<td>Quantitative data is more efficient, able to test hypotheses, but may miss contextual detail.</td>
</tr>
<tr>
<td>Results may be influenced by the researcher.</td>
<td>Researcher remains objectively separated from the subject matter.</td>
</tr>
</tbody>
</table>

The emphases of both types of data analysis is also quite different (Table 16).
Table 16: Emphasis of Qualitative and Quantitative Paradigms (Rubin and Babbie, 2010, p.36)

<table>
<thead>
<tr>
<th>Attribute</th>
<th>Qualitative</th>
<th>Quantitative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aims</td>
<td>Deeper Understandings; describing contexts; generating hypotheses; discovery</td>
<td>Precision; generalisability; testing hypotheses</td>
</tr>
<tr>
<td>Structure</td>
<td>Flexible procedures; evolve as data are gathered</td>
<td>Research procedures specified in advance</td>
</tr>
<tr>
<td>Setting for data gathering</td>
<td>Natural environment of research participants</td>
<td>Office, agency, or via mail or Internet</td>
</tr>
<tr>
<td>Theoretical approach most commonly employed</td>
<td>Inductive</td>
<td>Deductive</td>
</tr>
<tr>
<td>Sample size likely or preferred</td>
<td>Smaller</td>
<td>Larger</td>
</tr>
<tr>
<td>Most likely timing in investigating phenomena</td>
<td>Early, to gain familiarity with phenomena</td>
<td>Later, after familiarity with phenomena has been established</td>
</tr>
<tr>
<td>Emphasis on objectivity or subjectivity</td>
<td>Subjectivity</td>
<td>Objectivity</td>
</tr>
<tr>
<td>Nature of data emphasised</td>
<td>Words</td>
<td>Numbers</td>
</tr>
<tr>
<td>Depth and generalisability of findings</td>
<td>Deeper, but less generalisable</td>
<td>More superficial, but more generalisable</td>
</tr>
<tr>
<td>Richness of detail and context</td>
<td>Rich descriptions with more contextual detail</td>
<td>Less contextual detail</td>
</tr>
<tr>
<td>Nature of data-gathering methods emphasised</td>
<td>Lengthier and less structured observations and interviews</td>
<td>Various, but highly structured</td>
</tr>
<tr>
<td>Types of designs and methods commonly used</td>
<td>Ethnomography; case studies; life history; focus groups; participatory action research; grounded theory</td>
<td>Experiments; quasi-experiments; single-case designs; surveys</td>
</tr>
<tr>
<td>Data-gathering instruments emphasised</td>
<td>Open-ended items and interviews with probes</td>
<td>Closed-ended items in questionnaires and scales</td>
</tr>
<tr>
<td>Labour intensiveness of data collection for researchers</td>
<td>More time-consuming</td>
<td>Less time-consuming</td>
</tr>
<tr>
<td>-----------------------------------------------------</td>
<td>---------------------</td>
<td>---------------------</td>
</tr>
<tr>
<td>Labour intensiveness of data analysis</td>
<td>More time-consuming</td>
<td>Less time-consuming</td>
</tr>
<tr>
<td>Data analysis process</td>
<td>Search for patterns and meanings in narratives, not numbers</td>
<td>Calculate statistics that describe a population or assess the probability of error in inferences about hypotheses</td>
</tr>
<tr>
<td>Paradigms emphasised in appraising rigor</td>
<td>Contemporary positivist standards might be used, but standards based on interpretivist, social constructivist, critical social science, and feminist paradigms are commonly used</td>
<td>Contemporary positivist standards for minimising bias, maximising objectivity, and statistically controlling for alternative explanations</td>
</tr>
<tr>
<td>Ease of replication by other researchers</td>
<td>Harder</td>
<td>Easier</td>
</tr>
</tbody>
</table>

It is clear that different benefits come with the use of each method. Newman and Benz (1998) state that treating the qualitative and quantitative paradigms as two distinct elements is not in keeping with the philosophy of social science research. Hence comes the argument for a method that blends or combines both paradigms: the mixed-methods approach.

### 3.2.4 The Mixed-Methods Approach

The mixed-methods approach is designed to combine the strengths of both qualitative and quantitative paradigms. This method has been used frequently in recent years (Bryman, 2006). A mixed methods approach was adopted for this research. A mixed, or multi-methods approach, is one where both qualitative and quantitative research approaches are undertaken (Cohen, Manion, and Morrison, 2007; Teddlie and Tashakkori, 2011). Creswell and Plano Clark (2007) note that a piece of research using a mixed-methods approach must employ at least one quantitative method and one qualitative method. A mixed methods approach is endorsed for use in educational research due to its positive effect on triangulation (Cohen, Manion and Morrison, 2007). It therefore is influential in establishing concurrent validity, primarily in qualitative research, which is the “degree to which a measurement instrument produces the same results as another accepted or proven
instrument that measures the same parameters” (Lipsett and Kern, 2009, p.123). The assessment tool which was administered to the research sample on two occasions (pre and post-intervention), was designed to retrieve both types of data, while focus group and journal entries from intervention proceedings were solely qualitative in nature. The strengths and weaknesses of mixed-methods research are shown by Johnson and Onwuegbuzie (2004, p.21) (Table 17).
Table 17: Strengths and Weaknesses of Mixed-Methods Research (Johnson and Onwuegbuzie, 2004, p.21)

<table>
<thead>
<tr>
<th>Strengths</th>
<th>Weaknesses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Words, pictures, and narrative can be used to add meaning to numbers.</td>
<td>Can be difficult for a single researcher to carry out both qualitative and quantitative research, especially if two or more approaches are expected to be used concurrently; it may require a research team.</td>
</tr>
<tr>
<td>Numbers can be used to add precision to words, pictures, and narrative.</td>
<td>Researcher has to learn about multiple methods and approaches and understand how to mix them appropriately.</td>
</tr>
<tr>
<td>Can provide quantitative and qualitative research strengths (sections 3.2.2 and 3.2.3).</td>
<td>Methodological purists contend that one should always work within either a qualitative or a quantitative paradigm.</td>
</tr>
<tr>
<td>Researcher can generate and test a grounded theory.</td>
<td>More expensive.</td>
</tr>
<tr>
<td>Can answer a broader and more complete range of research questions because the researcher is not confined to a single method or approach.</td>
<td>More time consuming.</td>
</tr>
<tr>
<td>In a design with sequential stages, Stage 1 results can be used to develop and inform the purpose and design of the Stage 2 component.</td>
<td>Some of the details of mixed research remain to be worked out fully by research methodologists (e.g., problems of paradigm mixing, how to qualitatively analyze quantitative data, how to interpret conflicting results).</td>
</tr>
<tr>
<td>A researcher can use the strengths of an additional method to overcome the weaknesses in another method by using both in a research study.</td>
<td></td>
</tr>
<tr>
<td>Can provide stronger evidence for a conclusion through convergence and corroboration of findings.</td>
<td></td>
</tr>
<tr>
<td>Can add insights and understanding that might be missed when only a single method is used.</td>
<td></td>
</tr>
<tr>
<td>Can be used to increase the generalisability of the results.</td>
<td></td>
</tr>
<tr>
<td>Qualitative and quantitative research used together produce more complete knowledge necessary to inform theory and practice.</td>
<td></td>
</tr>
</tbody>
</table>
3.3 Research Problem

The initial focus of this research was on improving the teaching of trigonometry. The literature from Ireland indicates that Irish students are struggling with trigonometry at the end of second-level education (State Examinations Commission, 2000; 2001; 2005). Trigonometric weaknesses at the end of second-level have resulted in subsequent difficulties at Higher Education/third-level (O’Donoghue, 2002; Gill, 2006; Faulkner, 2012). Despite these findings, the author was unable to find substantial research into improving the trigonometric understanding of young people (Weber, 2005; Demir, 2012). This doctoral research was therefore narrowed to the idea of creating a purpose-built model for the effective teaching of trigonometry. This model would need to encompass all aspects of trigonometric teaching such as what content should be covered, in what order concepts should be taught, how to teach concepts for a conceptual understanding, as well as allowing for the formative assessment of students. The evidence suggests that these aspects are not all being acknowledged by teachers in the classroom. This research therefore addresses the gap in the literature of effective trigonometry teaching. It provides a foundation on which a teacher can base his/her teaching of the topic.

3.3.1 Aims and Objectives of the Research

The main aim of this research is to develop a purpose-built model for the effective teaching of trigonometry.

The objectives of the research based on the aims are:

- To review current literature in order to gain detailed insights into trigonometry and the teaching and learning of trigonometry, teacher knowledge required and its importance, as well as established models for effective teaching.

- To develop a research-based model for the effective teaching of trigonometry from which methods and materials for improving trigonometric understanding can be derived.

- To create a teaching intervention based on the teaching model.

- To administer this intervention to a sample of pre-service teachers in order to investigate if the model is valid with respect to effectively teaching trigonometry.
• To determine through an analysis of gathered data (both quantitative and qualitative), the impact that the intervention had on the trigonometric understanding of the subjects.

3.3.2 Research Questions

The research questions are aligned with the aims and phases of the research (see sections 3.3.1 and 3.6.1).

Phase 1: Literature Review

• Why is trigonometry included in second-level syllabi?

• How are Irish students and students in other countries performing in the topic of trigonometry?

• If performance is poor, then what is the cause for the poor performance over time?

• How does a teacher affect student learning? What aspects of a teacher’s knowledge base are most important?

• Do teachers currently have this knowledge base in trigonometry?

Phase 2: Developing a Model for the Effective Teaching of Trigonometry

• What models have been formulated in past research for the assessment of knowledge/understanding of a specific topic in mathematics/outside of mathematics?

• How can the model(s) be adapted/applied by the author for the effective teaching of trigonometry?

• Can a diagnostic assessment instrument for the adapted model be created?

Phase 3: Pre-Test of Intervention Sample

• What concepts that are relevant to junior and senior cycle trigonometry are the selected sample struggling to understand?

• Are any other significant findings emerging from the data relating to knowledge of trigonometry in general (e.g. in solving equations/constructing proofs)?
Phase 4: Trigonometry Intervention

- How can the findings from Phase 3 be addressed in the form of a teaching intervention?
- What approach(es) should be used in the trigonometry teaching intervention?

Phase 5: Post-Test & Evaluation of Trigonometry Intervention

- What changes have occurred after the intervention with regards to the samples’ conceptual understanding of trigonometry?
- Have the findings from the post-test highlighted the usefulness/limitations of any of the teaching strategies employed?
- Have the findings validated the model created?
- Can any of the teaching strategies be employed in a secondary school classroom setting? If so, how might they be employed?

The author used a variety of methods to attempt to answer the research questions and address the overall aims and objectives. Before practical field work was conducted the theoretical frameworks that would provide a foundation for the study had to be identified.

3.4 Theoretical Framework

A theoretical framework provides the structure that supports the theory of a research study (Swanson, 2013). A theoretical framework includes existing theories that are used in the particular research study, along with related concepts and definitions (USC, 2012).

3.4.1 Theoretical Framework for this Study

Effective teaching of trigonometry is an under-researched area (Weber, 2005; Demir, 2012). However, findings from recent years in Ireland and in other countries indicate that research into this area is warranted, with second and third-level students failing to grasp basic concepts in the topic (Blackett and Tall, 1991; Chinnappan, Nason, and Lawson, 1996; Kendal and Stacey, 1997; Department of Education and Science, 1999; State Examinations Commission, 2000, 2001, 2003, 2005, 2006a; Orhun, 2001; Weber, 2005; Fi, 2006;
Findings from the research state that students find the topic highly abstract and that poor teaching is one cause of the problems that occur in learning trigonometry (Orhun, 2001; Gür, 2009).

The need for a remedy led the author to construct a model of how to effectively teach trigonometry. The theoretical framework is based on two theories: the van Hiele model and APOS theory. The author identified the van Hiele model (van Hiele, 1984a; 1984b) as one which could provide solid direction for the model to be constructed. The van Hiele model is a level-based hierarchical model primarily focused on how people learn geometry. It also gives teaching phases that one should follow in order to move learners up through the levels of understanding. This model provides the base for the theoretical framework with many properties of the van Hiele model retaining their place in the author’s model. However the author pinpointed certain deficiencies of the van Hiele model that needed to be addressed when developing his purpose-built teaching model:

- The van Hiele model was too general in relation to their levels for an application to trigonometry to be achieved. When consideration is paid to the significantly substandard understanding students have in the topic of trigonometry it is clear that a very direct course of action is required for teachers to implement in their classrooms.

- The teaching phases provided by van Hiele’s therefore would be difficult to implement without a systematic structure in place.

These deficiencies led the author to the second pillar of the theoretical framework, APOS theory (Dubinsky and McDonald, 2002). APOS theory was used by previous researchers to elaborate on what constitutes mathematical thinking (Tall, 2008). APOS theory was therefore identified as a theory which could help with the two issues noted above. APOS theory is concerned with the ‘Actions’, ‘Processes’, ‘Objects’ and their organisation into ‘Schemas’ (see section 4.2.3) that, in tandem, determine the mathematical knowledge that one possesses. The use of this framework allowed the author to transform the van Hiele model from a learning model into a teaching model using ‘genetic decomposition’ on trigonometric schemas. What this means is that the concepts involved in trigonometric schemas were analysed. This in turn permitted a further adaptation of the van Hiele model in order to outline a teaching approach for trigonometry. This provided the direct course of study for trigonometry and the systematic teaching structure
for content that the author desired. In other words, the genetic decomposition outlined what concepts to teach and when to teach them, where each concept would be taught through the teaching phases of the van Hiele model. The following figure (Figure 11) illustrates the use of the theoretical framework in order to achieve the desired teaching model.

Figure 11: Use of Theoretical Framework

A research methodology can be devised that satisfies the overall global ontological and epistemological viewpoints. The methodology must describe the approaches taken in the research as well as an analyses of these approaches. The next section discusses the methodology of this research.
3.5 Research Methodology

A research methodology provides the theory for conducting a piece of research (Kothari, 2004). As mentioned, a methodology should also describe the approaches to the research, as well as the paradigms employed (Kaplan cited in Cohen et al., 2007). The overall methodological approach taken in this research was Educational Design Research. The research also incorporated a proof-of-concept approach as a means to try out the purpose-built model in a typical classroom setting. A mixed-methods approach was taken for the gathering of data (section 3.2.4). This is an approach which combines qualitative and quantitative paradigms. The subsections below elaborate on Educational Design Research and the proof-of-concept approach: what they are and why they were chosen.

3.5.1 Educational Design Research

Research in the field of education receives a lot of criticism due to its supposed weak links to practice (van den Akker, Gravemeijer, McKenney and Nieveen, 2006). This is understandable as much of education research is conducted with an action research approach and generally concludes with findings that are not directly applicable to the general classroom. Van den Akker et al. (2006) also state that educational research is failing to produce any significant advances when compared to advances in the fields of medicine or engineering. Educational Design Research stemmed from the desire to attend to three pressing issues:

1. The relevance of research to policy and practice has to be explicit.
2. Theories have to be constructed through the study of learning processes.
3. Research has to be robust in order to extract more explicit learning.
   
   (van den Akker et al., 2006)

Educational Design Research has been gaining in popularity in the field of educational research (van den Akker et al., 2006). This methodological approach to research contributes to the area of general classroom practice.

The main aim of Educational Design Research is to produce “new theories, artifacts, and practices that account for and potentially impact learning
and teaching in naturalistic settings” (Barab and Squire, 2004, p.2). Another definition is that it is “the systematic study of designing, developing and evaluating educational interventions” (Plomp 2009, p.9). In either case it is a complex approach to research that encompasses many layers of work (McKenney and Reeves, 2012). Educational Design Research is a theoretically oriented approach. Like many forms of research, Educational Design Research uses existing theory as a basis for inquiry. The defining feature of this approach to research is that it uses theory not just as a foundation for inquiry, but as a design for solving issues or problems (McKenney and Reeves, 2012). Educational Design Research is a good fit for this doctoral research as it allows for the development of a new theory or in this case a new model. From the author’s review of literature he wanted to pursue an adaptation of the van Hiele model from a geometry learning model to a trigonometry teaching model. As Education Design Research is a theoretically oriented approach that uses theory, not just solely for an area of inquiry, but for devising solutions, it matches the purpose of the research.

Educational Design Research is conducted in three main phases according to Educause (2012). The phases are Analysis, Design, and Evaluation.

- Analysis - research and learning on the problem at hand and its causes.
- Design - review of theory relevant to topic; various options considered; design created to try out in real scenarios.
- Evaluation - testing and revision of the design created and the ideas on which it was built.

(Educause, 2012)

These phases may be repeated multiple times over the course of a research study. Similarly, Nieveen, McKenney and van den Akker (2006) state that Educational Design Research goes through the phases of Preliminary Research, Prototyping Stage, Summative Evaluation, and Systematic Reflection and Documentation.

- Preliminary Research - The significant problem and the context of the problem are analysed in the literature. A conceptual framework is also developed from the literature.
- Prototyping Stage - The research is designed along with feedback and formative evaluation.
• Summative Evaluation - The transferability of the research and scale of the research are considered, along with a small scale evaluation of how effective the intervention is.

• Systematic Reflection and Documentation - Thorough evaluation of the research is conducted, along with an outcome of design principles and their links to the already devised conceptual framework.

(Nieveen et al., 2006)

These two different views on the phases of Educational Design Research are very similar. This PhD. research was conducted according to the phases provided by Educause (2012). However, it will become apparent in the course of this thesis that the author could also have used the phases of Nieveen et al. (2006) due to the similarities between them.

In either view, one key part of Educational Design Research is that a set of generalisable design principles should be established from the research (Reeves, 2006; van den Akker et al., 2006). Generalisable design principles, in the case of this research, applies to the creation of principles to consider when creating/adapting a teaching model. The design principles that were identified through this research are documented in the final chapter (section 8.5).

Educational Design Research does not come without its critics. The literature suggests that many funding agencies are unwilling to provide funding for studies that use this methodology as they are unfamiliar with it (Educause, 2012). However, it is a methodology that is being used more frequently in recent years and it may not be long before there is increased familiarity (van den Akker et al., 2006). The other criticism of this methodology is that studies using it take a long time (van den Akker et al., 2006; Educause, 2012).

An Action Research methodology was also considered. Action Research was rejected because, unlike Educational Design Research, it does not incorporate a ‘proof-of-concept’ approach. Where Action Research deals with constructing theories from the outcomes of educational interventions (Cohen et al., 2007), Educational Design Research allows for a theory to be developed from the literature before the intervention phase. The developed theory is the focus of the research. Educational Design Research was therefore a more appropriate choice as it allowed for the desired adaptation of
the van Hiele model before a trial-run of the model in an intervention or classroom setting to evaluate if the model (or theory) works.

3.5.2 Proof-of-Concept Approach

The field work in this research has been implemented on a relatively small scale for a number of reasons. Though educational research in general desires large-scale field trials, it is not always possible to do so due to the nature of classrooms, students and teachers. A proof-of-concept approach, derived from an engineering perspective, is an approach that tests whether “a design concept will perform as anticipated under certain prespecified conditions” (Dym et al., 2009, p.33). In relation to engineering the proof-of-concept approach would be applied to buildings or aircraft in the sense that it would not be viable for an engineer to build a fleet of aircraft without building one and testing it out. Of course with advancements in computer simulation, a lot of proof-of-concept approaches related to vehicles can now be conducted on computer software through simulations (Dym et al., 2009). In the field of education, a proof-of-concept approach can be considered as an approach which tests a new approach to teaching on a small-scale trial (Simon Fraser University, 2014) and can confirm or discredit the value of the new teaching approach (Bower, Craft, Laurillard and Masterman, 2011), as seen in the work of Treacy (2012).

In the context of this PhD. research a proof-of-concept approach was applicable to the research conducted as the researcher had to know whether the assessment, model, and intervention methods would perform to a high standard with a small sample of pre-service teachers. If this was found to be the case, it would give credence to the idea of these elements of the research working on a larger scale. Similar to the idea of the aircraft above, the author had something to try out, just like an engineer would trial run a plane. The author went into the intervention phase to test his purpose-built model. Therefore the proof-of-concept approach relates to this research as there was something to test out. The proof-of-concept approach was incorporated into the Evaluation phase (Educause, 2012) of the Educational Design Research cycle. Analysing the effectiveness of the teaching model and the reliability of the assessment tool completed the proof-of-concept approach.

The next section discusses the design of the research. The phases of the research are discussed as well as the timeline that they were conducted in. The sample that was selected for use in the research is also discussed.
3.6 Research Design

3.6.1 Phases of the Research

This research was conducted in 6 phases. Figure 12 gives a sequential representation of how the research was conducted and what the general purpose of each phase was.
Began with

Phase 1 Consisted of
A review of literature of general issues related to mathematics education, namely Ireland’s current problems, and international problems. Particular attention given to trigonometry.

An in depth review of some models of teacher knowledge.

A review of research on teachers’ understanding of mathematics.

A detailed analysis of the van Hiele model and some methods of measurement.

Phase 2 Consisted of
Creating a purpose-built model for teaching trigonometry, and an assessment for conceptual understanding of trigonometry.

Phase 3 Consisted of
Assessment & evaluation of participants’ understanding of trigonometry.

Informed & assisted the development of

Phase 4 Consisted of
Intervention development & field testing.

Phase 5 Consisted of
Evaluation of intervention.

Phase 6 Consisted of
Discussion of findings.

Informed

Informed

Informed

Informed

Informed

Resulted in

Figure 12: Research Phases
Phase 1

This phase of the research was the start of the author’s work. It is based solely on the analysis stage of the Educational Design Research methodology. This phase identified trigonometric understanding at second and third-level as a cause for concern in Ireland and abroad. With a significant problem identified, the author investigated reasons for the occurrence of this problem. The literature frequently pointed to ineffective teaching being a major cause for children not understanding mathematics. With this in mind, the author investigated what a teacher needs to know to be effective in the classroom. This is captured in some models of teacher knowledge. Some of these models were mathematics specific while others were not. The models indicate that subject matter knowledge is the foundation of a teacher’s skill set. The literature has shown that teachers have difficulties in various mathematical topics. Finally, phase 1 reviewed some models for understanding mathematical development. The only model reviewed that was specific to mathematics was the van Hiele model of the development of geometric thought. The author saw the potential to adapt the van Hiele to a purpose-built model for the effective teaching of trigonometry because there is a direct relationship between geometry and trigonometry. The literature indicates that this requires a more specific model than the van Hiele model as critics point out that the van Hiele model is too broad for an application to teaching. The research conducted in phase 1 formed the basis for methodological and method decisions to be made for the rest of the research phases.

Phase 2

This phase of the research began the design stage of the Educational Design Research methodology. The author constructed a purpose-built model for the effective teaching of trigonometry in phase 2 based on his research in phase 1. This model was developed by adapting the levels of the van Hiele model for an application to the topic of trigonometry. The author did this by elaborating on the van Hiele levels and deducing their link to trigonometry. An extension of the levels was necessary because the van Hiele model is too broad for an application to teaching. The author employed APOS theory and a genetic decomposition of the levels of his purpose-built model to accomplish this customisation (discussed further in chapter 4). This led to the purpose-built model being a more specific structure which maintained hierarchial and sequential properties, similar to the van Hiele model. As a teacher needs to know what level of understanding a student has before they
begin instruction, the author constructed a Level Finding Assessment assessment to position individuals on his purpose-built model. This assessment was built on the genetic decomposition of the levels. The genetic decomposition provided the concepts that need to be understood at each level of the purpose-built model. The Level Finding Assessment was focused on each of these concepts. Through an appropriate method of analysis of the assessment (adapted from Gutiérrez et al. (1991)) the author was able to pinpoint students’ level on the model in a pilot-test. Two definitions of pilot testing can be found whereby it is referred to as a small scale study, or trial, which precedes and prepares for a major study (Polit, Beck and Hungler, 2001). The second definition is that pilot testing is the ‘trying out’ or pre-testing of a specific research instrument (Baker, 1994). This research follows the latter as the assessment had to be piloted before implementation.

An expert review of the assessment questions was carried out by a selected panel for this purpose. The panel of experts included 1 doctor of mathematics education, a panel of 3 expert mathematicians employed at the University of Limerick, and 6 qualified secondary mathematics teachers. These experts were asked to check the assessment for accuracy in the following particulars:

- Vocabulary;
- Phrasing of questions;
- Mathematical accuracy of assessment;
- Formatting;
- Structure of questions;
- Time duration before completion.

A pilot test was also carried out amongst a class of 19 pre-service secondary mathematics teachers. This pilot test caused various changes to be made to the assessment items and to the evaluation procedures. After this pilot-test involving the pre-service teachers, an inter-rater reliability test was conducted between two doctors of mathematics education and the researcher. The mathematics education experts did not have to be rigorously trained in the evaluation process which highlights the repeatable nature of the assessment (as the evaluation is repeatable by anyone without specific training). The method of evaluation was shown to have substantial/ perfect agreement amongst raters (95% confidence interval of kappa statistic
\[ K \approx [0.6822, 0.8378] \].

This phase therefore allowed the author to make any relevant changes to the Level Finding Assessment before its use in the research. The assessment is discussed further in chapter 4.

Phase 3

This was the pre-test element of the research. A sample of 50 pre-service second-level teachers were assessed (25 third-year students and 25 fourth year students) relative to their level of understanding on the purpose-built model. Their level on the model would dictate where a teaching intervention for each participant would begin. Similar to the pilot-test, the evaluation was carried out in line with the work of Gutiérrez et al. (1991). This phase showed that a student can be assigned to a level of the model through this method of analysis. This phase also showed that the model corresponds to a scale, or is hierarchial in structure. This was shown through Guttman Scalogram Analysis (Torgerson, 1958).

Phase 4

Phase 4 of the research consisted of the development of a teaching intervention based on the findings from the pre-test phase. The purpose-built model was used as the structural foundation of the intervention. In other words, the pre-test demonstrated what level of the model each participant was at, and the model itself outlined what concepts would be covered in each participant’s intervention. The development of the intervention resources and the teaching intervention itself is discussed in chapter 6.

The intervention was administered to the third year group from the pre-test (who were in fourth year at the time of the intervention) in January and February of 2014. 19 students participated in all elements of the intervention and the post-test. The administration of the intervention is also discussed further in chapter 6.

Phase 5

Phase 5 evaluated the intervention. The effectiveness of the intervention with respect to the goals of the intervention were analysed. An evaluation of the intervention under the headings of Shapiro’s (1987) model for evaluating interventions was also conducted. It showed that the intervention
was effective in achieving its goals, appropriate for use, important in the context that it was set, socially acceptable, and socially valid. The evaluation phase also evaluated the intervention with respect to the students themselves. A focus group held with a random selection of the participants provided valuable insights into what the students thought themselves. This focus group was analysed on QSR NVivo (version 10) and categories (or nodes) of what the participants discussed arose in the analysis. Chapter 7 includes an extended discussion on phase 5.

**Phase 6**

Phase 6 outlined the findings of the research. The findings are presented in chapters 7 and 8. The findings relate to the model, intervention, and contributions of this research.

**3.6.2 Chronology of the Research**

**Phase 1: October 2011-June 2012**

Review of up to date literature relevant to topic.

**Phase 2: July-August 2012**

Development of the purpose-built model for the effective teaching of trigonometry. Formulation of the assessment instrument for the research (i.e. the assessment).

**September 2012**

Pilot test the assessment. Write-up of PhD. transfer document.

**October-November 2012**

Transfer.

**December 2012**

Adjustments made to assessment component based on pilot test findings.

**Phase 3: January-March 2013**

Distribution of assessment to participants. Commencement of analysis of pre-test data.

**April 2013**

Analysis of pre-test data from the assessment.
Phase 4: May-December 2013
Design of intervention based on findings from assessment.

January-February 2014
Intervention component carried out.

Phase 5: February 2014

Phase 6: March 2014-January 2015
Findings and write-up of full thesis.

3.6.3 Sampling
A convenience sample was used in this research. Convenience sampling is “the selection of a sample of participants from a population based on how convenient and readily available that group of participants is” (Salkind, 2010, p.254). Though convenience samples are easy to obtain and do not cost a lot of money to access (Salkind, 2010), this sampling method does receive criticism. Salkind (2010) states that studies using convenience samples are not generalisable to other contexts. This has already been discussed in section 3.2.2. The author is aware that this study may not result in the same findings if it was conducted in other contexts. Salkind (2010) does note however that convenience sampling methods are still useful if a model/tool is in the production phase as the study can still provide valuable insights. This is certainly the case in this research.

In the literature that was reviewed it was seen that pre-service teachers are a group that have previously demonstrated deficiencies in their understanding of mathematics. Along with the major finding that Irish students have significant deficiencies in their understanding of trigonometry on leaving secondary school, it is also a case that pre-service teachers at the University of Limerick do not study a specific module in trigonometry. They do engage with it partly in third-level modules, but they do not deeply engage with the topic. Therefore, pre-service teachers at the University of Limerick have a low level of trigonometric experience and understanding upon leaving secondary school and they may not have their understanding improved at third-level. Therefore the sample used in this research is also a purposive
The convenience and purposive sample that was chosen was taken from a second-level pre-service physical education and mathematics teacher training course at the University of Limerick. This course takes four years to complete. The sample was selected from students in their third and fourth years of study. The fourth year group (25 students) participated in the pre-test phase but were unavailable for the intervention phase as they had graduated from the university. The third year group participated in both the pre-test phase (25 students), intervention phase (19 students), and post-test phase (19 students). Six of the third year group chose not to participate in the intervention phase.

3.7 Research Methods

3.7.1 Research Instruments

Research instruments are the tools the researcher uses to gather data (Colton and Covert, 2007). A Level Finding Assessment was used in this research as the main source of data. This is because the model is the primary focus of the research. The assessment was based on concepts found in the genetic decomposition of topics in the purpose-built model. The author kept a research journal during the teaching intervention. These journals ensured that the author had a good record of this aspect of the research and would not forget any comments passed by participants. The journals that were kept were simply a catalogue of events. A focus group was held after the conclusion of the intervention and post-testing phases. The focus group provided qualitative data. This section discusses assessments used in research studies, the use of journal entries in research, and focus groups as a source of data.

3.7.2 Level Finding Assessment

An assessment instrument (Appendix A) was used in this research under a pre-experimental design (or a one-group pre and post-test design) (Sharp, 2012). This assessment in this research aims to position each individual at a level of the purpose-built model, hence it was entitled the ‘Level Finding Assessment’ (hereby referred to solely as the assessment). The author will refer to the test used in this research as an assessment as it is more formative than summative. It is more formative as the assessment provides for the
direction of future instruction under the purpose-built model. Cohen et al. 
(2007) state that a researcher must have eight considerations for designing 
an assessment.

- The purpose of the assessment;
- The assessment specifications (e.g. objectives);
- The contents of the assessment;
- The form of the assessment;
- Writing the assessment;
- The layout of the assessment;
- The time it takes to complete the assessment;
- The scoring of the assessment.

(Cohen at al., 2007)

As mentioned, the purpose of the assessment instrument was to position 
individuals on the purpose-built model. This provides the level that each 
participant’s teaching intervention begins at. The assessment items were 
based on the concepts found in the genetic decomposition of each level of the 
purpose-built model (explained in chapter 4). The assessment instrument 
was a written assessment. Though interviews and video-taping methods 
were used in research related to the van Hiele model, the author was aware 
of a limitation with respect to time. He wished to acquire as large a sample 
as possible from the pre-service teachers at the University of Limerick. It 
would have taken a great deal of time if each participant was required to sit 
an interview where they would have to explain their reasoning. Participants 
may have been put off by this and may not have completed the voluntary 
intervention phase of the research. The written assessment instrument took 
approximately 1 hour to complete. Again, the time it took to complete the 
assessment was limited due to the time-constraints of the participants (see 
section 3.11). The assessment instrument was composed of 16 questions. 
The specific questions are discussed further after the model and genetic 
decomposition have been clarified in chapter 4. The test was evaluated 
using a similar evaluation to that proposed by Gutiérrez et al. (1991) (see 
section 2.15.5). As stated in section 3.6.1, the method of evaluation was 
shown to have substantial/perfect agreement amongst raters (the researcher

130
and two mathematics education specialists) in a pilot-test of the assessment (95% confidence interval of kappa statistic \( K \approx [0.6822, 0.8378] \)). The pilot-test and inter-rater reliability calculations are discussed further in chapter 4.

3.7.3 Focus Groups

A focus group is a form of group interview that relies on interactions between the participants in discussing a particular topic supplied by the researcher (Cohen et al., 2007). They are useful for the acquisition of qualitative data (Cohen at al., 2007). The aim of the focus group was to generate insights into the thoughts and attitudes (O’Meara, 2011) of the participants with respect to learning trigonometry in line with the purpose-built model. The effectiveness of the purpose-built model in teaching trigonometry was evaluated through the assessment but the author wished to gain further qualitative insights from the participants themselves.

The researcher developed a set of focus group questions (Appendix B) that were administered to a group of 4 participants who participated in all phases of the research (pre-test, intervention and post-test) and 1 participant who did not complete the pre-test phase but completed the other two phases. The five people that participated in the focus group were selected from the participants by voluntary means. They were willing to take part when asked by the researcher. The focus group was a semi-structured focus group. This means that the focus group is relatively conversational, though a guide does exist for the focus group proceedings that outlines the subject matter the researcher wishes to cover (Harrell and Bradley, 2009). Deep insights can be gathered using semi-structured focus groups as the participants discuss various topics and share ideas and experiences in a conversational manner. Open-ended questions were used in the focus group proceedings because of this. Open-ended questions invite participants to give personal and honest responses (McKay, 2010). Using this type of questioning added to the validity of any findings from the focus group (Lawson and Philpott, 2008).

3.7.4 Research Journals

Journal entries were maintained by the researcher during the intervention phase of the research (Appendix C). These journals documented events as they happened in the classroom. The researcher wrote the journals based on
his experiences in the classes and the entries may therefore have some level of bias (Ortlipp, 2008). The researcher ultimately decided that the main aim of these journals would be to document events during the intervention process. Nevertheless, the researcher attempted to remain objective in his reflections. Data from these journals is used sparingly in the research findings due to the potential element of bias and the negative effect that this would have on maintaining researcher distance (section 3.10.2).

3.7.5 Instruments and the Mixed-Methods Approach

The instruments of the assessment, focus groups, and journal entries apply to the mixed-methods approach as the assessment instrument gathered quantitative data and qualitative data, the focus group gathered qualitative data, and the journal entries, though seldom used in any findings, provided qualitative data.

Table 18: Coordination of the Research Instruments with the Mixed-Methods Approach

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Qualitative</th>
<th>Quantitative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assessment</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Focus Group</td>
<td>✓</td>
<td>✗</td>
</tr>
<tr>
<td>Journal entries</td>
<td>✓</td>
<td>✗</td>
</tr>
</tbody>
</table>

3.8 Data Collection

Once the instruments for gathering data were designed and any necessary pilot-tests and corrections were conducted, the researcher began phase 3 of the research (section 3.6.1). Pre-test data was collected from third and fourth year pre-service second-level mathematics teachers in March and April of 2013 through the use of the assessment instrument. The assessment was conducted with each group separately during one of their course lectures.

Post-test data was gathered at the end of phase 4 (February 2014) when the intervention was completed. This post-test was only administered to the third-year group from the pre-test phase (who were in fourth year at the time of the intervention). The post-test assessment instrument was identical
to the assessment used for the pre-test. The researcher held the focus group with 5 participants after the post-test was completed.

Journal entries were maintained throughout phase 4, as already mentioned.

3.9 Data Analysis

3.9.1 Assessment Data

The data from the assessment was analysed by the researcher himself. All assessments were assessed by hand using a particular method derived from Gutiérrez et al. (1991). This links to the global ontological assumption that mathematics education research should be applicable to classrooms. A teacher does not need any specialist software knowledge (e.g. SPSS) to evaluate the assessment developed for this research. They can do it themselves by hand using the methods described below.

Gutiérrez et al. (1991) show that quantitative data can be produced from students’ answers to problems in accordance with their work on eight “types” of answers (see section 2.15.5). They show that a weighting can be applied to student responses. The eight types are as follows:

- **Type 0.** No answer provided.
- **Type 1.** Answer provided shows that pupil has not achieved a certain level, but fails to give information about lower levels.
- **Type 2.** Incorrect answers that show some level of reasoning. Incorrect reasoning processes.
- **Type 3.** Correct but insufficiently worked out answers that give a sense of some level of reasoning. Very few explanations provided and incomplete results.
- **Type 4.** Correct/incorrect answers that reflect two van Hiele levels (e.g. levels 2 & 3 or levels 1 & 2). Clear reasoning and sufficient justifications provided.
- **Type 5.** Wrong answers that clearly show a level of reasoning. Answers show reasoning processes that are either complete but incorrect, or correct reasoning processes that do not lead to the correct solution of the task/problem.
• **Type 6.** Correct answers to the task that show a level of reasoning but that are incomplete or insufficiently justified.

• **Type 7.** Complete, correct answers that are fully justified and show a level of reasoning.

(Gutiérrez et al., 1991, p.240)

For the purpose of this research, the types of answers participants gave were judged in line with this work with the only difference being to answer ‘Type 4’ where the answer, being correct or incorrect, reflect levels of the researcher’s purpose-built model for the effective teaching of trigonometry, which is discussed in chapter 4.

Though these ‘types’ of answers are valid, for the purpose of this research an adaptation of these was used. This adaptation is based on a combination of the original ‘types’ of answers (Gutiérrez et al., 1991) and a simplified model outlined in Gür (2009) (Table 19). This contributed to greater inter-rater reliability. The original ‘types’ (Gutiérrez et al., 1991) may be ambiguous for other markers and therefore a more simplistic framework was needed that still achieved a good coverage of content.
Table 19: Simplified ‘Types’ of Answers (Gür, 2009, p.71)

<table>
<thead>
<tr>
<th>Criteria for Scoring</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>I - Correct answer</strong></td>
</tr>
<tr>
<td>Included all components of the validated response, correct answer</td>
</tr>
<tr>
<td><strong>II - Partial Understanding, Misconception or obstacle</strong></td>
</tr>
<tr>
<td><em>At least one of the components of the validated response, but not all the components, just concept or process or mechanical application of a rule, did not involve any justification</em></td>
</tr>
<tr>
<td><em>Included illogical or incorrect information or information different from the correct information</em></td>
</tr>
<tr>
<td><strong>III - Unacceptable</strong></td>
</tr>
<tr>
<td>Irrelevant of unclear responses or not answered or irrelevant answers, repeat information in the question as if it was answer or blank</td>
</tr>
</tbody>
</table>

The author used the following ‘types’ of answers based on an adaptation of Gutiérrez et al. (1991) using the work of Gür (2009):

- **Type 0.** No answer provided/ No workings shown/ Illogical/ Irrelevant/ Incorrect.
- **Type 1.** Incorrect answers that show some level of reasoning (e.g. correct diagram but incorrect rule implemented etc.)
- **Type 2.** Incorrect answer but pupil demonstrates good levels of understanding (e.g. correct diagram, correct rule used, implementation of the rule incorrect)
- **Type 3.** Correct/ Incorrect answer due to incomplete/ unjustified answer or pupil makes error in their calculations.
- **Type 4.** Correct answer that is fully justified.

The researcher indicates that in a manner similar to the van Hiele model, that the levels in his purpose-built model are divided into different degrees of acquisition. These degrees of acquisition were proposed by Gutiérrez et al. (1991) and are logical in the sense that each level is not viewed as
an “all or nothing” acquisition, or in other words that it is not viewed that an individual has either a complete grasp of the level, or no knowledge at all.

Figure 13: Degrees of Acquisition of a van Hiele Level (Gutiérrez et al., 1991, p.238)

Based on the answers provided by participants to the first section of the assessment, a weighting was applied to their responses. This weighting can be related to Figure 13 in the sense that if a participant receives a score of 25 for level 5 in the purpose-built model, then he or she is in the low acquisition stage of that level.

Table 20: Weighting of Different Types of Answers (Gutiérrez et al., 1991, p.241)

<table>
<thead>
<tr>
<th>Type</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>0</td>
<td>0</td>
<td>20</td>
<td>25</td>
<td>50</td>
<td>75</td>
<td>80</td>
<td>100</td>
</tr>
</tbody>
</table>

However, due to the change to the ‘types’ of answers, the weighting to be applied was as follows:

Table 21: Weighting of Different ‘Types’ of Answers to be Used in this Research

<table>
<thead>
<tr>
<th>Type</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>0</td>
<td>20</td>
<td>75</td>
<td>90</td>
<td>100</td>
</tr>
</tbody>
</table>

These weightings were acquired from each level by cross referencing the ‘types’ of answers used in this research with those of Gutiérrez et al. (1991). The types of answers in this research have the following characteristics with respect to degrees of acquisition:
• Type 0. Indicates no acquisition of a level.

• Type 1. Beginning of acquisition of a level. Vague traces or flashes of a level of reasoning. Very incomplete. Usually short answers.

• Type 2. Advanced phase in the transition between 2 levels. Participant uses reasoning methods imperfectly and may resort to using thinking methods from lower levels.

• Type 3. This answer was a type formulated in between types 2 and 4. It is designed to account for calculation errors only (i.e. not conceptual misunderstandings). Therefore a very high weighting of 90 is assigned to answers of this type as the concepts are understood but the answer is still not correct.

• Type 4. Fully acquired a concept within a level.

A marking scheme was developed where each type of answer for each question on the Level Finding Assessment was elaborated upon (Appendix L). This gave the researcher a particular marking scheme to follow allowed for the pinpointing of answer types.

After the participants had completed the assessment their answers were assigned vectors \((l, t)\), where \(l\) denotes the level of the question and \(t\) is the type of answer provided. The weighting of each type of answer for each level \((l, w)\) was then applied and an arithmetical average was calculated to uncover the degree of acquisition each participant had of each level.
Example from the pilot test:

Table 22: Vector \((l, t)\) Example from Pilot Test (Pilot Participant 1 (Appendix D))

<table>
<thead>
<tr>
<th>Item</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level ((l))</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Type ((t))</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Item</th>
<th>10</th>
<th>11(i)</th>
<th>11(ii)</th>
<th>11(iii)</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level ((l))</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Type ((t))</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 23: Weighting \((l, w)\) Example from Pilot Test (Pilot Participant 1)

<table>
<thead>
<tr>
<th>Item</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level ((l))</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Weight ((w))</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>20</td>
<td>75</td>
<td>75</td>
<td>75</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Item</th>
<th>10</th>
<th>11(i)</th>
<th>11(ii)</th>
<th>11(iii)</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level ((l))</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Weight ((w))</td>
<td>0</td>
<td>100</td>
<td>100</td>
<td>90</td>
<td>75</td>
<td>20</td>
<td>20</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 24: Weighting Average Example from Pilot Test (Pilot Participant 1)

<table>
<thead>
<tr>
<th>Level</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight average</td>
<td>100</td>
<td>100</td>
<td>60</td>
<td>56.25</td>
<td>73</td>
<td>20</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 25: Degrees of Acquisition Example from Pilot Test (Pilot Participant 1)

<table>
<thead>
<tr>
<th>Level</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree of acquisition</td>
<td>Comp.</td>
<td>Comp.</td>
<td>Inter.</td>
<td>Inter.</td>
<td>High</td>
<td>Low</td>
<td>None</td>
</tr>
</tbody>
</table>

3.9.2 Focus Group Data

The focus group was transcribed by the researcher before an analysis was conducted. QSR NVivo 10 was used to analyse the qualitative data from the transcription. The researcher used Thematic Analysis in the analysis of the transcriptions. Thematic Analysis classifies issues that participants in the research present in the qualitative data they provide (e.g. through interviews, focus groups etc.) (Fereday and Muir-Cochrane, 2006; Alhojailan, 2012). It then identifies themes or patterns relating to the data. This form of analysis is widely considered to be the most appropriate method for research conducted under the interpretive epistemology as it enhances meaning and determines relationships between data (Alhojailan, 2012).

Thematic Analysis allows a researcher to apply coding and categorisation to data. It goes beyond analysis that deals with identification. For example, Thematic Analysis would not dwell on the fact that a participant supplied an incorrect word when explaining a mathematical concept. Alhojailan (2012) states that Thematic Analysis provides the themes that present themselves in the data. This allows for summary methods of interpreting the data richly, deeply, and in a complex way.

3.9.3 Journal Entry Data

The author kept research journals to document events so that he would not forget what happened in the intervention classes. No in-depth analysis of the journal entries was undertaken. Anecdotal evidence from the journal entries is used occasionally to illustrate or emphasise a point. The journals only serve to give a better picture of what occurred during the research.
3.10 Research Issues

The researcher had to ensure that some conditions were considered/put in place before each phase of the research was conducted in order to ensure that the research conducted was valid and reliable. Without these two important prerequisites, the research would most likely be unacceptable to the mathematics education community, or any scientific community.

3.10.1 Ethics

The application for ethical approval (Appendix E) was sent to the relevant parties on the 9th of November 2012. This application followed the ethical practices outlined by the University of Limerick’s Research Ethics Committee (ULREC). The application stated that:

- All data would be confidential and no reports the author makes will mention anything about who supplied the information.
- All data will be stored on the supervisor’s password protected computer.
- Any participant in the research would take part voluntarily.
- The research would only be conducted amongst mathematics education students.
- Consent forms (Appendix F) would be provided to anyone that was taking part in the research.

Ethical approval was granted for this research on the 2nd of January 2013 by ULREC.

3.10.2 Researcher Distance

The researcher maintained objectivity by using assessment evaluation methods already proposed in the literature (Gutiérrez et al., 1991). This ensured that the evaluation of assessments was objective after the data was provided by participants in the research. The focus group proceedings and thematic analysis of the focus group transcriptions was also objective as the researcher only analysed the data provided by the participants. As noted, research journals are used sparingly as a source of data and only serve to corroborate or emphasise findings. They are not used as a source of new findings. Hence, the findings from this research can be viewed as valid, and are as reflective of the data as possible.
3.10.3 Validity

Cohen et al. (2007) state that validity is a necessity for qualitative and quantitative research and go so far as to say that without validity then the research is not worth anything to anyone. They summarise by stating that validity is “the touchstone of all types of educational research” (p.134). The traditional form of validity is that the research instrument(s) measure(s) what it/they intend(s) to measure, however, Cohen et al. (2007) note that nowadays, many different forms of validity exist such as content validity, criterion-related validity, and internal validity (full list available in Cohen et al., 2007, p.133). The most important forms of validity for this research are descriptive validity, theoretical validity, internal validity and content validity.

Descriptive validity is defined as being “the notion of ‘truth’ in research - what actually happened (objectively factual)” (Cohen et al., 2007, p.135). As already stated in the previous section (section 3.10.2) the researcher maintained objectivity at all times in any analysis of data and results reported. This research reflects the data as the assessment evaluation method and method of Thematic Analysis applied to qualitative data were both objective.

Theoretical validity is defined as where “theory (here) is regarded as explanation” (Cohen et al., 2007, p.135). This form of validity is relevant to the purpose-built teaching model developed for the research. The van Hiele model (chapter 2), through its adaption, explained why the model is valid. The measurement techniques for the van Hiele model of Gutiérrez et al. (1991) (section 2.15.5) explain why the measurement strategies are valid.

Internal validity refers to the explanation of a particular finding in the research being sustained by the data (Cohen et al., 2007). In other words that any finding in the research can be corroborated by the data. This aspect of validity is strongly maintained across the research. Any findings from the pre-test, intervention and post-test that are reported can be significantly corroborated in the raw data gathered. This will be clear in any relevant section where findings or results are reported.

Content validity is demonstrated when the research instrument shows “that it fairly and comprehensively covers the domain or items that it purports to cover” (Cohen et al., 2007, p.137). The instruments in this research,
such as the assessment tool or the resource pack, are all grounded in the theory that accumulated in the development of the purpose-built teaching model. Every instrument used was either informed by literature, or more so, by the purpose-built teaching model developed (which itself was informed by literature).

3.10.4 Reliability

Reliability is how consistent a piece of research is. In other words, reliability refers to the research producing similar results with a different sample or with different evaluators (e.g., in the case of evaluating an assessment). In terms of research, it is stated to be a “synonym for dependability, consistency and replicability over time, over instruments and over groups of respondents” (Cohen et al., 2007, p.146). This research ensures reliability of the purpose-built teaching model developed and the assessment procedure through Guttman Scalamagram Analysis (section 5.9). This analysis showed the high consistency of results between the researcher and two other assessors, who were relatively untrained with respect to the model and the assessment.

The reliability of the research is high for multiple raters. In other words, if a number of people evaluate the same assessment using the evaluation proposed for this research, they will get similar results. Reliability across groups may not be high. Different groups that could take part in a similar piece of research would come from various different backgrounds and would all have different levels of mathematical abilities. For example, this PhD. research was conducted with a sample of pre-service second-level mathematics teachers. If it were conducted with first year second-level students (approx 13 years old) different findings may arise.

3.10.5 Triangulation

Triangulation is defined as a “method used by qualitative researchers to check and establish validity in their studies by analyzing a research question from multiple perspectives” (Guion et al., 2002, p.1). Multiple types of triangulation are presented in the literature such as data triangulation, investigator triangulation, theory triangulation, methodological triangulation, environmental triangulation, and time triangulation (Guion et al., 2002; Cohen et al., 2007). The two methods of triangulation used in this research were data triangulation and methodological triangulation.
Data triangulation is the use of different sources of information in order to increase accuracy of intended measurement (i.e. validity) (Guion et al., 2002; Cohen et al., 2007). In this research data triangulation was achieved through the collection of data using assessments, focus groups, and journal entries based on empirical evidence. The findings from each of these will be discussed where appropriate throughout the thesis.

Methodological triangulation is the use of multiple sources of information that are compared to see if similar findings are coming to the fore (Guion et al., 2002; Cohen et al., 2007). Methodological triangulation was achieved in this research by corroborating evidence between each of the sets of data noted above under the topic of data triangulation. Again, this corroboration will become clear in the final chapters.

3.11 Limitations of the Research

There were a number of limitations to the research conducted:

- **Time constraints (assessment)**

  The assessment in phase 3 of the research (pre-intervention) was bound by time constraints. The assessment could not last longer than 50 minutes as participants had to attend other classes.

- **Time constraints (intervention)**

  The intervention could only take place from January 2014. The author did not want to stretch the intervention beyond week 4 of a 12-week semester in order to be mindful of the workload required from the students in their own degree work. The author also wanted to keep as many participants taking part as possible.

- **Response rate**

  A group of 20 people took part in all intervention classes (19 of which had completed a pre-test and 1 had not). Five students who had completed the pre-test did not take part in the intervention classes, however three of this five did complete a post-test. One further students completed the pre-test, half of the required intervention classes, and the post-test.
3.12 Conclusion

This chapter demonstrated the researcher’s philosophical stance and how this affected his approach to the research. The Educational Design Research methodology that was used with a proof-of-concept approach, as well as the research designs and research methods that developed under this methodology were described. Each phase of the research has been discussed. Various issues related to the research such as validity, reliability, limitations, and ethics were also dealt with and curtailed in this chapter. With the research methodology and all that it entails clear, the next chapter looks at the guiding tool of the research: the purpose-built model for the effective teaching of trigonometry.
Chapter 4

A Purpose-Built Model for the Effective Teaching of Trigonometry

4.1 Introduction

Chapter 2 Part 1 described the difficulties that Irish students and students abroad have with learning trigonometry. The literature indicates that trigonometry is one of the most difficult areas for students to learn (Cockcroft, 1982; Blackett and Tall, 1991; Gür, 2009) despite the topic being included in every mainstream post primary mathematics syllabus worldwide (Delice, 2002). This part of chapter 2 also demonstrated that trigonometry is an under-researched area in mathematics education (Weber, 2005; Demir, 2012).

Chapter 2 Part 2 showed that despite these findings, a model for how to teach trigonometry effectively was not found. As poor teaching was found to be one of the largest contributors to students’ deficiencies in trigonometry (Orhun, 2001; Gür, 2009), a model for teaching trigonometry was needed. Models for the effective teaching of mathematics can have a positive effect on problem solving approaches (Ernest, 1989), cooperative learning (Rogoff, Matusov and White, 1996), information processing of students (Joyce, Showers and Rolheiser-Bennett, 1987), and overall increased student learning (Joyce, Showers and Rolheiser-Bennett, 1987). Likewise, similar outcomes can be expected from an effective model for teaching trigonometry. Some general educational models of understanding were found in the literature
such as Bloom’s Taxonomy (section 2.13; Bloom et al., 1956) and the SOLO taxonomy (section 2.14; Biggs and Collis, 1982). However, these models are not mathematics specific. Subsequently the van Hiele model (section 2.15; van Hiele 1984a; 1984b) was identified by the author as a model that could be transformed into a purpose-built model for the effective teaching of trigonometry. This was initially hypothesised because the van Hiele model has a geometric focus and trigonometry is an extension of geometry.

The author transformed the van Hiele model in two stages to develop his purpose-built teaching model. The first transformation was the extension of the van Hiele model from a geometry learning model to a trigonometry learning model. The second transformation was the adaptation of the trigonometry learning model from a model of how people learn, to a model of how to teach. This led the author to the use of APOS theory. These adaptations and extensions are discussed in detail in this chapter. The chapter also discusses the properties and teaching strategies of the purpose-built teaching model. The links between the purpose-built teaching model and the Irish syllabus are also demonstrated. The chapter concludes with a discussion on the construction of an assessment instrument to position individuals on the purpose-built model before teaching can commence.

4.2 A Purpose-Built Model for the Effective Teaching of Trigonometry

The author’s purpose-built teaching model is a stage based hierarchial model comprising several levels, some of which are further subdivided. The author at this point gives the features of the model that are discussed in the remainder of this chapter (Table 26).
Table 26: Features of the Purpose-Built Teaching Model

<table>
<thead>
<tr>
<th>Feature</th>
<th>Discussion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model for learning trigonometry</td>
<td>Extension of the van Hiele model from geometry to trigonometry.</td>
</tr>
<tr>
<td>Levels of the model for the purpose of teaching</td>
<td>Identification of trigonometric concepts for teaching and the sequence of teaching (achieved with the use of APOS theory and genetic decomposition).</td>
</tr>
<tr>
<td>Teaching strategies for the purpose-built teaching model</td>
<td>Teaching phases used for teaching trigonometry under the purpose-built teaching model.</td>
</tr>
<tr>
<td>Properties of the purpose-built teaching model</td>
<td>Properties of the levels and teaching strategies of the purpose-built teaching model.</td>
</tr>
<tr>
<td>Syllabus mapping</td>
<td>Mapping of the purpose-built teaching model and the Irish second-level mathematics syllabus.</td>
</tr>
<tr>
<td>Valid assessment instrument</td>
<td>The development of a valid assessment instrument to position individuals on the purpose-built teaching model before instruction begins.</td>
</tr>
</tbody>
</table>

The purpose-built model for the effective teaching of trigonometry was developed by initially transforming the van Hiele model of geometric thought (Usiskin, 1982; Mayberry, 1983; van Hiele, 1984a; van Hiele, 1984b; Burger and Shaughnessy, 1986; Crowley, 1987; Gutierrez, Jaime, and Fortuny, 1991) (section 2.15). The first step in developing a trigonometry teaching model was to customise the van Hiele model for the treatment of trigonometry. This led to a learning model for trigonometry. The development of this learning model is discussed in the next section.

4.2.1 1st Adaptation: Geometry to Trigonometry

The author took the level indicators of the van Hiele model and extended them to the context of trigonometry for the first step in developing his purpose-built teaching model.
Level 0 (visualisation) of the van Hiele model is concerned with learning the visual appearance of shapes. In relation to trigonometry, this means that students would recognise a triangle by appearance.

Level 1 (analysis) of the van Hiele model concerns the analysis of the components of shapes. In relation to building a conceptual framework of trigonometry, students analyse the components of various types of triangles and analyse elementary trigonometric concepts.

Level 2 (abstraction) of the van Hiele model encompasses the stage of learning where students notice relationships between figures. An extension to trigonometry would hold a similar principle. Learners of trigonometry at this stage explore relationships between trigonometric concepts. For example, the relationships that exist between right-triangle concepts and the unit circle.

Level 3 (deduction) of the van Hiele model is when the learner can construct proofs based on their conceptual understanding of geometry. Similar to the principle of a student constructing geometric proofs, an extension to trigonometry holds that a student can construct trigonometric proofs. A student could prove for example that \( \cos^2 \theta + \sin^2 \theta = 1 \).

Level 4 (rigor) of the van Hiele model is when the student views geometry in the abstract. Students are no longer confined to using trigonometry in familiar trigonometric settings. They can apply their understanding of trigonometry to any setting where it may be beneficial to use trigonometric concepts.

This extension shows that the van Hiele model can be applied to trigonometry as a model of learning. However, it cannot yet be deemed to be a teaching model as it is too broad. A learning model is a model that seeks to identify how people learn mathematics, whereas a teaching model is a model which focuses on how to promote learning (Rogoff et al., 1996).

The levels so far give general principles of trigonometric learning but no specific teaching structure. This reflects the criticisms of the van Hiele model (section 2.15.8) that state that it is too broad for an application to teaching and cannot be specified to the teaching of geometric concepts in schools. A more solid teaching structure was needed for each level of the proposed model. The author needed to transfer the model from a learning model
to a teaching model. After a list of trigonometric concepts to teach were identified from the Project Maths syllabus the author needed to identify what teaching is required to develop an understanding of these concepts. This led to the use of APOS theory. These factors that helped the author to develop a teaching model from the learning model are discussed in the next section.

4.2.2 2nd Adaptation: Learning to Teaching

The first issue in transferring from a learning to a teaching model was to identify the trigonometric concepts that one needs to teach. Trigonometric content on Irish primary (Government of Ireland, 1999) and secondary (Junior Cycle: NCCA, 2010; Senior Cycle: NCCA, 2011) mathematics syllabi were analysed and connections were made between this content and the levels outlined in section 4.2.1 (Table 27).

Table 27: Irish Trigonometric Content and Their Link to the Levels of the Trigonometry Learning Model

<table>
<thead>
<tr>
<th>School level</th>
<th>Content</th>
<th>Learning</th>
<th>Level of model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary</td>
<td>Angles, compare shapes, construction of triangles when given side lengths and angles, classify types of triangles, sum of angles</td>
<td>Visual recognition and properties</td>
<td>Visualisation (level 0) and analysis (level 1).</td>
</tr>
<tr>
<td>Secondary junior cycle</td>
<td>Right-triangles, trigonometric ratios (concept and application), Pythagoras’ theorem, surd form (Higher level only)</td>
<td>Right-triangles</td>
<td>Analysis (level 1).</td>
</tr>
<tr>
<td>Secondary senior cycle</td>
<td><em>Ordinary level:</em> Pythagoras’ theorem, areas of triangles, sine and cosine rules, trigonometric ratios, surd form. <em>Higher level:</em> Use trigonometric in 3D problems, graph basic trigonometric functions, graph trigonometric functions of the form $f(\theta) = a + b\sin c\theta$ and $g(\theta) = a + b\cos c\theta$, solve trigonometric equations for all solutions, radian measure, derive trigonometric formulae, apply trigonometric formulae, construct trigonometric proofs</td>
<td>Unit circle, trigonometric functions, proofs</td>
<td><em>Ordinary level:</em> Analysis (level 1) and abstraction (level 2). <em>Higher level:</em> Abstraction (level 2) and deduction (level 3).</td>
</tr>
</tbody>
</table>
Content covered at primary level is focused on the visual appearance of triangles and the identification of various types of triangles based on their appearance and basic properties. Primary students should also be able to construct triangles based on basic properties (side lengths/angles) and acknowledge that the sum of all interior angles is $180^\circ$ as well as use this to solve problems. Primary level trigonometry therefore coincides with levels 0 and 1 of the trigonometry learning model (section 4.2.1) as the main aims of learning are the visual appearance of triangles and the basic properties of triangles.

At junior cycle in second-level the assumption is that students have acquired the concepts from primary level (above). The main focus of learning at junior cycle is on right-triangle concepts. As the concepts are more advanced than visual recognition and deal with the analysis of right-triangle concepts, the learning that occurs at junior cycle relates to level 1 of the trigonometry leaning model. The aims for junior cycle trigonometry are for the learner to:

- recall basic trigonometric facts;
- construct basic triangles and identify their properties;
- solve problems;
- interpret diagrams;
- analyse information presented in unfamiliar contexts (real-life scenarios);
- select appropriate formulae to solve problems

(NCCA, 2010)

Exploration of relationships is not stressed in the content that is studied. This is due to the content being primarily confined to right-triangle concepts. Therefore level 2 (abstraction) is not a focus of learning at junior cycle.

At senior cycle of second-level it is again assumed that students have obtained the required trigonometric learning at junior cycle, respective to the students’ level of study. In other words if a student studies Higher level in senior cycle, it is assumed that they have gained the necessary learning for
Higher level at junior cycle. Ordinary level at senior cycle focuses on levels 1 and 2 of trigonometry learning model. A lot of focus is on the analysis of individual trigonometric concepts (e.g. Pythagoras’ theorem and trigonometric ratios). Connections are made between right-triangle concepts and the sine and cosine rules. The senior cycle curriculum urges that students make connections between concepts in this strand (Strand 2: Geometry and Trigonometry) and to concepts in other strands (see section 2.5) (NCCA, 2011). Therefore level 2 of the trigonometry learning model is also engaged with at Ordinary level.

Students studying Higher level mathematics at senior cycle engage with numerous concepts which have connections to previously acquired learning. Graphing trigonometric functions and solving trigonometric equations for example can be linked to right-triangle concepts and the unit-circle (which is not specified as a concept of study in the syllabus but is used as a topic of study on the Project Maths website (Project Maths, 2010)). Connections between concepts is a strong area of learning at Higher level and therefore Higher level corresponds with level 2 of the trigonometry learning model. As students at Higher level also have to derive trigonometric formulae and construct logical and formal trigonometric proofs, this level of study also corresponds with level 3 of the trigonometry learning model.

Though the author had identified some concepts to teach from Table 27, a systematic teaching model had not yet been developed. Table 27 only provides global topics to teach where connections between concepts are not always recognised. However, an initial design of the levels of the purpose-built teaching model was developed at this stage.

The purpose-built teaching model developed in this research has 7 levels. Due to the depth of trigonometric content at the extended levels of analysis and abstraction in the trigonometry learning model, the researcher divided each of these levels into two levels for the development of the purpose-built teaching model. Level 1 (analysis) of the trigonometry learning model was divided into two levels as two distinct conceptual areas were identified from the Irish syllabi (Table 27). The first was the analysis of various components in different types of triangles. The second was the in-depth analysis of right angled triangles (Pythagoras’ theorem, trigonometric ratios etc.). Level 2 (abstraction) of the trigonometry learning model was divided into two levels due to the same idea. As this level deals with relationships the unit circle was the first conceptual area that was identified. From the outset, learners can
begin to relate the unit circle to right triangle concepts already learned. The second area identified was trigonometric functions (graphing and analysis). Students relate the unit circle to the graphing of trigonometric functions. Because of their learning of the unit circle before trigonometric functions, students can also relate how right triangle concepts apply to the graphs of trigonometric functions.

Figure 14: Comparison of the Proposed Levels of a Purpose-Built Model and the van Hiele Model/Trigonometry Learning Model

The author required a framework that allowed for a systematic structure for teaching where connections between concepts are stressed and the levels of the purpose-built teaching model can be obtained. The author was led to APOS theory.
4.2.3 APOS Theory and the Genetic Decomposition of the Purpose-Built Model

APOS theory is concerned with the ‘Actions’, ‘Processes’, ‘Objects’ and their organisation into ‘Schemas’ (Dubinsky and McDonald, 2002) that, in tandem, determine the mathematical knowledge that one possesses. APOS theory was chosen as it was fit for the purpose of this research. As it is a constructivist theory that describes “that an individual’s understanding of a mathematical topic develops through reflecting on problems and their solutions” (Dubinsky, 2014, p.8) it harmonised with the van Hiele model, and hence the trigonometry learning model, in its constructivist stance and also allowed for a teaching structure to be established through genetic decomposition. Genetic decomposition is the detailed analysis of mathematical content in a structure for learning (Dubinsky and McDonald, 2002). In this research APOS theory and its accompanying genetic decomposition transferred the trigonometry learning model to a teaching model as it identified what trigonometric concepts to teach and in what order to teach them. This is discussed in detail in this section.

Activities refer to the transformation of objects which an individual regards as external by the use of external cues (Asiala et al., 1997; Dubinsky and McDonald, 2002). For example in relation to the concept of the use of Pythagoras’ theorem, if an individual is asked to prove if a triangle is a right triangle when given the length of the three sides, they would only be able to complete the ‘activity’ when given the external cue to use Pythagoras’ theorem. This is due to the nature of the question as being external, i.e. they can apply the theorem but not in this context in which the problem is set.

Processes is the term used to represent the repetition of actions that lead to mental constructions called processes (Asiala et al., 1996; Dubinsky and McDonald, 2002). Using the previous example the process would be that once the action is carried out and repeated, then the individual can perform this action without the external cue to use Pythagoras’ theorem. The individual relates this question to the concept of Pythagoras’ theorem and the action can be carried out.

Objects refer to the individual’s ability to view the processes as a “totality” (Dubinsky and McDonald, 2002) and acknowledgment that transformations can act on it. Using the Pythagoras’ theorem example an individual
who views this concept as an object realises that the theorem can be used in different contexts, for example proving a triangle is a right triangle, calculating distances, finding the measurement of an unknown side of a right triangle etc.

*Schemas* refer to the frameworks within the mind of an individual that connect the actions, processes and objects (Dubinsky and McDonald, 2002). These schemas can be used to solve problems that involve the concept that they have built the framework on. Chinnappan (1998) provides an example of a schema being formed on the concept of a right triangle (Figure 15).

![Figure 15: Right Triangle Schema Example 1 (Chinnappan, 1998)](image)

A similar example is provided by Chinnappan (2008) as a result of the findings of his work (Figure 16).

![Figure 16: Right Triangle Schema Example 2 (Chinnappan, 2008)](image)
Chinnappan (1998) also states that this schema would become more complex as a student’s knowledge of geometry increases. For example, an individual would be aware that right triangles could be incorporated into a rectangle etc. However, when the schemas provided by Chinnappan are analysed it can be seen that they only provide elements of learning for right-angled triangles. They do not provide a teaching structure that allows for the building of a wider conceptual understanding.

With reference to the example of Pythagoras’ theorem the schema that is created would include the necessary existence of a right angle, if true then there is an implied existence of a right angle, the labels of the sides opposite, adjacent and hypotenuse, the formula $a^2 + b^2 = c^2$, and so on. This schema would grow as the individual’s exposure to the concept grows (i.e. application questions would add the applicability of the theorem to areas such as engineering, the use of the theorem in complex numbers, vectors, proofs, amongst others). However, following Chinnappan’s (1998) work this schema could be elaborated further. For example the formula $a^2 + b^2 = c^2$ could be its own schema of what it means to solve this equation. Elements of this schema would include multiplication of numbers, the meaning of an index and equations. Again each of these could be elaborated upon and fine tuned. From this it can be seen that highly complex schemas could be drawn and mapped and far more detail could be produced than in Figures 15 and 16.

In Dubinsky’s work the detailed analysis of schemas is referred to as ‘genetic decomposition’ (Asiala et al., 1996; Dubinsky and McDonald, 2002). This means that the concepts involved in a schema are analysed and a model is produced as to what it means to understand the concept based on the parts used to form the concept (Asiala et al., 1996). Asiala et al. (1996, p.7) refer to genetic decomposition as a “structured set of mental constructs which might describe how the concept can develop in the mind of an individual”. In terms of teaching, genetic decomposition provides a systematic structure where an instructor teaches one concept identified in the genetic decomposition and then teaches the next concept once the preceding one has been acquired. Connections can therefore be made between concepts as the lessons progress through the genetic decomposition. An important note on genetic decomposition highlighted by many researchers is that the initial analysis of the concept is greatly influenced by the researcher’s own understandings of the concept and this impacts on the schemas produced for the purpose of research (Asiala et al., 1996; Dubinsky and McDonald, 2002).
An example of a model of concepts in trigonometry is provided by Orton (2004).

![Concept Map of Trigonometry (Orton, 2004, p.16)](image)

Figure 17: Concept Map of Trigonometry (Orton, 2004, p.16)

It can be seen that the schema in this model are growing structures. For
example, one must know about the concept of right angles before working on the concepts of sine, cosine and tangent. These concepts must be understood before working on the concepts of the sine and cosine rules.

Each of the 7 levels in this research were focused on individually in order to construct the genetic decomposition of the purpose-built model for the effective teaching of trigonometry. The concepts that relate to each level were extracted in order to know what concepts an individual at each level understands. The concepts that are outlined in Table 27 are synonymous with the ‘objects’ of Dubinsky et al.’s (2002) work and that the ‘actions’ and ‘processes’ that accompany these objects are implied in the understanding of the concept. Following on from this after each level is attained an individual should understand where that whole level fits into the overall picture of trigonometry. For example, if an individual has just mastered level 1 of the model (Property Relations), then he/she should be aware of how the properties of triangles impact on the appearance of triangles (which he/she will be aware of from level 0). This schema will keep building as the individual progresses through the levels and an interconnected framework of concepts should be developed by the student.

Note that the ‘Actions’ for each level are the same (Tables 28 and 29). This is due to the fact that processes and actions deal with the same concepts. The difference is that the understanding of a concept as an action requires the teacher to cue students when they need to use the concept(s). Therefore the concepts at the stage of actions are the same as those at the stage of processes in Tables 28 and 29. The difference is that a teacher is expected to provide cues at the actions stage but they do not need to provide cues if the student views a concept as a process.
Table 28: Genetic Decomposition of the Purpose-Built Teaching Model (Levels 0-3)

<table>
<thead>
<tr>
<th>Level</th>
<th>Schema</th>
<th>Object(s)</th>
<th>Process(es)</th>
<th>Actions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0</td>
<td>Appearance Recognition</td>
<td>Recognition of the visual appearance of a triangle.</td>
<td>Differentiating between triangles and other shapes.</td>
<td>Actions lead to processes through the use of ‘cues’.</td>
</tr>
<tr>
<td>Level 1</td>
<td>Property Relations</td>
<td>Isosceles triangles, scalene triangles, equilateral triangles, and angle summation.</td>
<td>Recognise the type of triangles by name. Measuring angles. 180° in a triangle. Calculating the measurement of a missing angle.</td>
<td>Actions lead to processes through the use of ‘cues’.</td>
</tr>
<tr>
<td>Level 2</td>
<td>Right-triangles &amp; Pythagoras</td>
<td>Right angles. Hypotenuse, adjacent and opposite sides. Pythagoras' theorem. Sine, cosine and tangent ratios.</td>
<td>Measure and recognise right angles. Label the sides of a right triangle. Utilise Pythagoras' theorem in various problems and acknowledge its use for right triangles only. Use each trigonometric ratio in problems.</td>
<td>Actions lead to processes through the use of ‘cues’.</td>
</tr>
<tr>
<td>Level 3</td>
<td>Unit Circle Concepts</td>
<td>Radians. Quadrants. Unit Circle. Sine and cosine rules.</td>
<td>Conversion of degrees to radians and vice versa. Labelling and sketching the unit circle with respect to sine and cosine. Explaining CAST notation. Evaluating trigonometric expressions. Solving for angles in simple trigonometric expressions (e.g. ( \cos \theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}, \frac{5\pi}{4} )). Using sine and cosine rules to find unknown sides and angles.</td>
<td>Actions lead to processes through the use of ‘cues’.</td>
</tr>
</tbody>
</table>
Most of the objects and processes in Tables 28 and 29 are on second-level syllabi worldwide (Delice, 2002). Though level 6 of the model is not covered at second-level, the author suggests that a teacher may find it beneficial for use with very gifted students. The genetic decomposition gives a systematic teaching structure based on students achieving an understanding of the overall level. It therefore alleviates the criticism of the broad nature of other models, such as the van Hiele model. The actions lead to processes. Processes lead to objects. Objects lead to schemas. If students grasp the

<table>
<thead>
<tr>
<th>Level</th>
<th>Schema</th>
<th>Object(s)</th>
<th>Process(es)</th>
<th>Actions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 4</td>
<td>Further Rules &amp; Trigonometric Functions Understood</td>
<td>Solving more complex trigonometric equations. Sinusoidal functions. Tangent function. Inverse trigonometric functions.</td>
<td>Manipulation of trigonometric identities. Solutions to more complicated trigonometric equations can be found (e.g. ( \cos 2A + 3 \sin A - 2 = 0, \ 0 \leq A \leq 360^\circ )). Principal values. Graphing and analysis. Limits.</td>
<td>Actions lead to processes through the use of ‘cues’.</td>
</tr>
<tr>
<td>Level 6</td>
<td>Trigonometry Seen in the Abstract</td>
<td>Oscillations (waves, music, springs). Complex numbers. Spherical trigonometry. Lesser met functions (secant, cosecant, cotangent).</td>
<td>Application of more abstract content to situations, problems etc. (e.g. spherical trigonometry to navigation). Understanding of the connection of lesser met functions to the unit circle.</td>
<td>Actions lead to processes through the use of ‘cues’.</td>
</tr>
</tbody>
</table>
concepts as interconnected schemas they will have achieved an understanding of each level, and this allows for future conceptual learning of the topic.

It must be noted here that similar to the elaboration on Chinnappan’s (1998) work, each of the processes highlighted in the genetic decomposition of the purpose-built model for the effective teaching of trigonometry can be elaborated upon. For example the process within level 4 of graphing and analysis can be extended to co-ordinate geometry (plotting points, labelling axes etc.), functions (substituting values, drawing function tables, domain and range), and so on.
4.2.4 The Purpose-Built Model for the Effective Teaching of Trigonometry

The levels of the purpose-built model for the effective teaching of trigonometry are synthesised as follows:

- **Level 0: Appearance Recognition**

  This is the foundation set for the study of trigonometry. At this level individuals see triangles as a three sided shape and nothing more. They can distinguish a triangle from other shapes but are unaware of the properties of triangles, for example that the angles in a triangle all add up to $180^\circ$.

- **Level 1: Property Relations**

  Individuals at this level are aware of the different types of triangles (e.g. isosceles, equilateral, scalene). Measurement of angles in a triangle is also understood and can be carried out. It is understood that the angles all add up to $180^\circ$ however the presence of right angles in some triangles is still not acknowledged.

- **Level 2: Right-triangles & Pythagoras**

  At this level students recognise the existence of right angles and acknowledge that if a right angle exists in a triangle then that is the largest angle that the triangle has. The labels of hypotenuse, opposite, and adjacent can be applied to right-angled triangles. Pythagoras’ theorem is the main focus of learning and understanding here and it encompasses a whole level of the model as it is crucial to so much of trigonometry. The trigonometric ratios of sine, cosine, and tangent are also understood once the foundation of Pythagoras’ theorem has been laid down. Basic constructions (orthocentre, circumcentre etc.) can be carried out.

- **Level 3: Unit Circle Concepts**

  A prerequisite for this level is that radian measure is understood and that students have the ability to transfer from degrees to radians and vice-versa. A student at this level understands the unit circle and how it applies to trigonometric ratios. They also understand the CAST notation, and can apply their understanding in order to find, for example that the $\cos 210^\circ = -\frac{\sqrt{3}}{2}$. Students at this level
understand how to use sine and cosine rules in their calculations and recognise when to use each one of them.

- **Level 4: Further Rules & Trigonometric Functions Understood**

  Students are able to graph trigonometric functions and inverse trigonometric functions and understand the usefulness of this. Solutions to trigonometric equations can also be found. The capability to produce proofs is still not apparent.

- **Level 5: Derivation & Proof**

  At this level students are capable of producing reasoned proofs from familiar or unfamiliar starting points, using what they have acquired in their trigonometric training so far. For example, they can prove that \( \cos^2 \theta + \sin^2 \theta = 1 \) from their understanding of the unit circle, trigonometric ratios, and Pythagoras’ theorem. Reasoned definitions for concepts that they have met already can be produced by the means of thinking and explaining (almost like a teacher would do).

- **Level 6: Trigonometry Seen in the Abstract**

  Students can use their understanding of all aforementioned trigonometric concepts to help them to solve problems in various fields. Students are not limited to using these concepts in familiar situations and can apply them to problems where they may prove to be beneficial (e.g. music/complex numbers/springs etc.).

The genetic decomposition of the purpose-built teaching model still holds true for each of these levels.

The adaptation of the van Hiele model is evident throughout this model. Level 0 of the purpose-built teaching model is directly linked to the visualisation level of van Hiele’s work as it is solely concerned with recognition of the shape of triangles. Level 1 of the van Hiele model (analysis) is linked to level 1 of this model as some properties of triangles are beginning to be noticed and understood. Level 2 of the purpose-built teaching model is a further building upon level 1 in the van Hiele model. This level is included due to the critical nature of Pythagoras’ work on the understanding of trigonometry, and therefore it gets an individual section. Levels 3 and 4 of the purpose-built teaching model relate to level 2 of the van Hiele model.
(abstraction) as higher levels of understanding of concepts, and the interrelationships between concepts, begin to be formulated in level 3 and are built upon further in level 4. Level 3 of the van Hiele model (deduction) links to level 5 of the purpose-built teaching model as individuals do not need to rely on memorisation for constructing proofs as they have a sound and accurate grasp of the concepts. The final levels of each of the models are linked by the abstract nature in which the topic is viewed.

It is also important to note that from level 1 up to and including level 6 individuals should be able to solve problems, be they real-life problems or otherwise, on the levels they have achieved. The word ‘levels’ is used here instead of level because if an individual is at level 4 in terms of their ability then he/she should be able to solve problems from level 0 up to and including level 4. A support-like structure can be arranged in the purpose-built teaching model where each level forms a base of support for the next level, such as in Figure 18.

![Figure 18: Purpose-Built Model for the Effective Teaching of Trigonometry](image-url)
4.2.5 Teaching Strategies for the Purpose-Built Teaching Model

Upon reflection of the research conducted into the van Hiele model it was concluded that the most appropriate teaching strategies used in order to promote understanding and movement up through the levels were the phases proposed by van Hiele (1984b) (section 2.15.4). This is due to the strong ties between the purpose-built teaching model and the van Hiele model. The teaching phases for the van Hiele model may prove to be effective for the purpose-built teaching model as the primary criticism of the van Hiele model, its broad nature, has been alleviated through genetic decomposition.

- Phase 1: Inquiry/Information (van Hiele, 1984b) - Present topic/concept of study.
- Phase 2: Directed/Guided Orientation (van Hiele, 1984b) - Exploration of area of study.
- Phase 3: Explicitation (van Hiele, 1984b) - Terminology and symbols introduced.
- Phase 4: Free Orientation (van Hiele, 1984b) - Higher complexity tasks.
- Phase 5: Integration (van Hiele, 1984b) - Summary of learning and link to previously acquired learning.

These phases were implemented as they would be implemented in a geometry class, as depicted by Abdullah and Zakaria (2013, p.255).

![Figure 19: Phases of Learning Geometry (Abdullah and Zakaria, 2013, p.255)](image-url)
The genetic decomposition of the purpose-built teaching model align with the teaching phases to allow a teacher to build from the base of the model (Figure 18). The teacher can begin with a concept, teach it through the five phases, and then proceed to the next concept in the genetic decomposition. This is only done if the students have demonstrated an understanding of the preceding concept.

4.2.6 Properties of the Purpose-Built Teaching Model

The properties of the purpose-built teaching model cannot all be said to be the same as those of the van Hiele model (section 2.15.3). Some of the following van Hiele properties have been examined however, with respect to the purpose-built teaching model:

- Sequential - move through levels in the designated order;
- Advancement - depends more on teaching and instruction provided than on the age of the student;
- Intrinsic/Extrinsic Objects - objects intrinsic at one level but extrinsic at the next;
- Linguistics - teacher should not use language above the level that the student is working at;
- Mismatch - instruction that is at a different level to the level a pupil may be at.

(van Hiele, 1984b)

The sequential property of the van Hiele model does apply to the purpose-built teaching model. Guttman Scalogram Analysis was performed after the pre-test phase. This analysis is discussed in detail in section 5.9. The \( R_{ep} \) value obtained in this analysis was above the standard figure of 0.9 for each set of levels, with a \( R_{ep} \) interval of \([0.9055,1]\) being found. This means that the model significantly corresponds to a ‘perfect scale’, meaning it is hierarchical in nature. This infers that the levels of the model are sequential.

The properties of advancement through the model depending more on instruction than on the age of students, and intrinsic/extrinsic objects, were outside the area of interest in this research.
The final two van Hiele properties of linguistics and mismatch apply to the purpose-built teaching model. These are inferred from the hierarchial nature of the model, which was proven by Guttman Scalogram Analysis as stated. If a teacher uses language or instruction that is at a higher level of the purpose-built teaching model than the students are able for it is clear that confusion will occur. Students would not be mentally capable of handling the higher order subject matter.

The development of the purpose-built teaching model has been described. Though the researcher had developed this model he had to investigate if it was applicable to teaching at second-level. The next section briefly discusses the links between the purpose-built teaching model and teaching trigonometry at second-level in Ireland.

4.3 The Relationship Between the Purpose-Built Teaching Model and the Irish Syllabus

Each object outlined in the genetic decomposition of the purpose-built teaching model is relevant to teaching trigonometry at second-level in Ireland (section 2.5). The objects at level 0 and level 1 are applicable to the syllabus as it is important for a teacher and students to understand what a triangle is and to be able to identify different types of triangles. The objects at these levels will form the foundation for the teaching and learning that will occur in higher levels. The proceeding objects at various levels of the purpose-built teaching model and their links to Junior and Leaving Certificate syllabi are listed below.

- The objects at level 2 all correspond to the Junior Certificate syllabus at all levels (Foundation, Ordinary and Higher) (NCCA, 2010, p.20).
- The objects at level 3 correspond to the Leaving Certificate Ordinary (sine and cosine rules) and Higher (sine and cosine rules, radians, quadrants and the unit circle) levels (NCCA, 2011, p.23).
- Level 4 objects all correspond to the Leaving Certificate syllabus at Higher level only (NCCA, 2011, p.23).
- Level 5 objects are relevant to the element of ‘synthesis and problem solving skills’ highlighted in the Project Maths syllabus. The objects here require students/teachers to apply their knowledge and skills to
solve problems in familiar and unfamiliar contexts, analyse information, communicate mathematics verbally and in written form, and devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions. These are all aspects of understanding which are required according to the Project Maths syllabus at Leaving Certificate level. The main focus of level 5 of the purpose-built teaching model is the aspects of derivation and proof. The Leaving Certificate syllabus at Higher level states that students should be able to derive certain trigonometric formulae (NCCA, 2013).

- The objects outlined in level 6 are not included in Leaving Certificate Mathematics syllabi. However this level is important for teachers in order to adapt for exceptional pupils that they may have in their classes (NCCA, 2007). When we consider that exceptionally able pupils have the potential to:

  grasp the formal structure of a problem: can generate ideas for action;
  recognise pattern: can specialise and make conjectures;
  reason logically: can verify, justify and prove;
  think flexibly, adopting problem solving approaches;
  may leap stages in logical reasoning and think in abbreviated mathematical forms;
  are able to generalise from examples;
  are able to generalise approaches to problem-solving;
  use mathematical symbols as part of the thinking process;
  may work backwards and forwards when solving a problem;
  remember mathematical relationships, problem types, ways of approaching problems and patterns of reasoning;

(NCCA, 2007, p.24)

it is important that a teacher can challenge these individuals, aid in their development, progress their learning, and maintain their enthusiasm and interest in mathematics.

The model also adheres to syllabi in other places such as New York (USA) (Common Core, 2013), England (Department for Education, 2014)
With the purpose-built teaching model in mind it is important that a teacher knows the current level of his/her students before they begin instruction in order to prevent mismatch. An assessment was designed that would position students on the purpose-built teaching model. This assessment was evaluated in line with Gutiérrez et al. (1991). The exact evaluation methods for the assessment were described in section 3.9.1. The next section outlines the design of the assessment, the necessary alterations that were made after expert reviews and pilot-tests, the individual assessment questions, and the inter-rater reliability of the evaluation methods analysed from the pilot-test.

4.4 Constructing a Valid Assessment to Position Individuals on the Purpose-Built Teaching Model

The assessment instrument was finalised in December 2012 after an initial design was developed, a review by an expert panel was conducted, and a pilot-test was carried out. The finalised assessment was based on the genetic decomposition of the purpose-built teaching model. Each level of the purpose-built teaching model was assessed through one or more questions that were based on concepts found through genetic decomposition. However, time was a limitation of the assessment as stated in section 3.11. The author had to be aware that the assessment was limited to a one-hour time period as participants had to be able to complete it in a one-hour class. Therefore the researcher had to be selective about what concepts to assess at each level, and what types of questions to include. Sixteen individual assessment items were included in the final questionnaire (Appendix A). Table 30 demonstrates the concepts assessed in each question and what level of the purpose-built teaching model each relates to.
### Table 30: Concepts Assessed in Finalised Assessment Items

<table>
<thead>
<tr>
<th>Question</th>
<th>Concept</th>
<th>Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Visual appearance of a triangle</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>Properties of types of triangles</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>Angle summation</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>Trigonometric ratios</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>Pythagoras’ theorem</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>Evaluation trigonometric expressions</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>Quadrants</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>Cosine rule</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>Radians</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>Solving trigonometric equations</td>
<td>4</td>
</tr>
<tr>
<td>11</td>
<td>Trigonometric functions</td>
<td>4</td>
</tr>
<tr>
<td>12</td>
<td>Limits</td>
<td>4</td>
</tr>
<tr>
<td>13</td>
<td>Proof</td>
<td>5</td>
</tr>
<tr>
<td>14</td>
<td>Proof</td>
<td>5</td>
</tr>
<tr>
<td>15</td>
<td>Lesser met functions (cot ( \theta ), sec ( \theta ), cosec ( \theta ))</td>
<td>6</td>
</tr>
<tr>
<td>16</td>
<td>Vectors</td>
<td>6</td>
</tr>
</tbody>
</table>

Some of the questions on the assessment are based in a context/scenario. This links back to the model with respect to the aspect of problem solving that individuals should be able to achieve, whilst using the concepts from each of the levels which were found through genetic decomposition (section 4.2.3). The scenarios model real-life situations as these types of questions have been highlighted through the years as being critical to mathematics education. One of the researchers which placed major importance on the use of real-life scenarios was Freudenthal (1968; 1973). Freudenthal’s work, which was partly based on the van Hiele model (section 2.15.7), shaped the formation of the realistic mathematics education (RME) in his views of how mathematics should be relevant to society, be relevant to life, and be meaningful. This has also been echoed with respect to assessment.
Alonso-Tapia (2002) states that, in order for assessment tasks to be effective, they should be open-ended tasks that require application and use of knowledge in solving transfer tasks (i.e., tasks that require the use of understanding to solve problems in unfamiliar contexts). PISA uses Freudenthal’s idea of mathematising (Shiel et al., 2007) in their assessment of secondary pupils’ understanding of mathematics. Mathematising is the process that takes place when solving a real-life problem (or word problem) by using mathematics (Grigoras and Hoede, 2008).

However, this was not the initial design of the assessment. The development of the assessment over time (July 2012 through December 2012) which eventually led to the final assessment instrument is described in this section. The initial design, review by expert panel, pilot-test, and individual assessment items for the final research instrument are all discussed.

4.4.1 Initial Design of Assessment

The overall design and idea behind the initial assessment was that the questions must assess the participants’ conceptual understanding of trigonometry. The stages involved in producing the assessment was similar to the stages that led to the ARTIST assessment (ARTIST, 2012) which was an assessment of conceptual understanding for statistics. In line with the ARTIST assessment, the researcher first compiled a bank of questions and then removed those which were procedural in nature. The assessment was initially designed in two sections (Appendix G). The first section was initially a multiple choice question (MCQ) type of assessment, with each question consisting of five possible answers from which to choose. The second section was an open-ended type of assessment. Open-ended problems encourage pupils to come up with their own problem solving strategies for the problem they are working on and allow for numerous possible solutions (Wee and Looi, 2009). In this second section participants would have been required to provide explanations and diagrams in order to explain concepts such as ‘What is a radian?’.

The assessment was initially designed in this way in order to receive both qualitative data and quantitative data. The quantitative data was planned to arise from participants’ answers and responses to the first MCQ section and the qualitative data to stem from the second open-ended assessment.
After the initial design of the assessment it was reviewed by an expert panel in order to make any necessary changes before a pilot-test could be conducted.

4.4.2 Development of Assessment Through Review by Expert Panel

For this research reviews of the assessment questions were carried out by an expert panel consisting of 1 doctor of mathematics education, 3 expert mathematicians employed at the University of Limerick, and 2 qualified secondary mathematics teachers. These experts were asked to check the assessment for accuracy in the following particulars:

- Vocabulary;
- Phrasing of questions;
- Mathematical accuracy of assessment;
- Formatting;
- Structure of questions;
- Time duration before completion.

The feedback was used to adapt the original untested research instrument (i.e. before pilot tests were carried out) (Appendix G) for the production of the final assessment instrument (Appendix A). The following table (Table 31) outlines the changes that occurred to the assessment for Appendices G - K as a result of the expert reviews.
Table 31: Reviews by Experts

<table>
<thead>
<tr>
<th>Feedback for Appendix _</th>
<th>Expert(s)</th>
<th>Changes made</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>G</td>
<td>Mathematics education specialist</td>
</tr>
<tr>
<td></td>
<td>H</td>
<td>1 mathematician</td>
</tr>
<tr>
<td></td>
<td>I</td>
<td>2 secondary teachers</td>
</tr>
<tr>
<td></td>
<td>J</td>
<td>Feedback from 2 mathematicians</td>
</tr>
</tbody>
</table>

After changes had been made to the assessment following the reviews by the expert panel, the author had to pilot-test the assessment instrument (Appendix K). A small scale pilot-test was conducted first and this was followed by a larger scale pilot-test. The small scale pilot-test was conducted to investigate if the proposed evaluation methods (see section 4.4.3) would actually work in-situ. The larger scale pilot-test was done with the purpose of fine-tuning any remaining issues that needed to be adjusted.
4.4.3 Pilot-Tests

Two definitions of pilot testing can be found whereby it is referred to as a small scale form, or trial, which precedes and prepares for a major study (Polit, Beck and Hungler, 2001). The second definition is that pilot testing is the ‘trying out’ or pre-testing of a specific research instrument (Baker, 1994). Both are valid definitions for the pilot-test in this research as the assessment had to be piloted before implementation. A small scale pilot-test was conducted with two second-level mathematics teachers and then a larger pilot-test was done with 19 pre-service teachers in order to see how the assessment and marking scheme worked in practice.

A small scale pilot-test was carried out amongst 2 secondary school teachers (at Appendix I) in order to view the assessment in practice before a larger scale pilot-test. An evaluation was carried out on the 2 assessments. The first section (open-ended) of the assessment provided valuable insights and a ‘type’ of answer (see section 3.9.1) could be applied to each response provided by the pilot participants. However, the second section (MCQ’s) failed to provide adequate information. The multiple choice questions were assessed using:

\[ K = R - \frac{Q - R}{A - 1} \]

(Scharf and Baldwin, 2007)

where \( K \) is the number of questions for which the participant knows the right answer, \( R \) is the number of right answers given, \( Q \) is the total number of questions attempted (of which selections of answer (e) I don’t know were excluded), and \( A \) is the number of possible answers for each question. This provides an approximation of how many questions participants actually know, rather than correct answers obtained from guessing. The problem lay in the fact that though a score could be obtained, it did not allow for the assignment of a level on the purpose-built teaching model for the participants. Along with this, conceptual misunderstandings/lack of understanding could not be pinpointed due to the multiple choice nature of this section. Therefore, the researcher decided to reject the idea of MCQ’s for this research and developed an assessment focused solely on open-ended questions. This led to the assessment shown in Appendix K.

A further pilot-test was carried out amongst a class of 19 first year pre-service secondary mathematics teachers after the open-ended assessment
was developed (Appendix K). This pilot-test made various changes to the assessment items and to the evaluation procedures/marking scheme.

As explained in Section 3.9.1, the open-ended problems were assessed using a marking scheme which relied on the assignment of one of five types of answers (type 0 - type 4) to each response provided by the participants. Each question has its own unique structure for grading that cross-references the types of answers in 3.9.1 with the specific context of the question (Appendix L). For example, before the pilot-test with the 19 pre-service teachers, question 6 had the following types of answers:

- **Type 0**: No answer/ totally incorrect attempt;
- **Type 1**: Correct diagram but clearly no reasoning from there;
- **Type 2**: Correct diagram, attempt at solution, but incorrect;
- **Type 3**: Correct diagram, error in answer only (i.e. they seem to know the answer but took it down wrong);
- **Type 4**: Correct answer.

The pilot-test with 19 pre-service teachers aided with developing the types of answers such as these further, and making them more representative of the weightings that would be applied after the type of answer was identified.

The evaluation of question 6, as stated above, originally had a type 2 answer of ‘Correct diagram, attempt at solution, but incorrect’. However, the answers provided by many of the participants in this pilot study reflected this type of answer and were not synonymous with someone who has a high acquisition of the content (i.e. a weighting of 75). This answer was altered to be a type 1 answer and therefore the answers provided by the participants reflected a low acquisition, which in all cases was deemed to be the case. The type 2 answer was formulated from some of the responses provided by the pre-service teachers, and reads ‘Correct diagram, labelled correctly, noted that \( \cos(-240^\circ) = \cos 120^\circ \), error from there’.

Question 7 had an original type 2 response of ‘Correct answer of quadrants 1 & 3 but no justification’. From the responses provided by the participants, it was observed that many of them could provide the quadrants but could not articulate why they had chosen them. As this is the concept
assessed their answers again did not reflect a high acquisition of the content. This answer was changed to a type 1 response. The type 3 response was also downgraded to become a type 2. The new type 3 answer was constructed by building upon the original type 3 response. Type 3 now reads as ‘Correct diagram, correct answer for 2 quadrants and 1 explanation correct, error in the other explanation’ to reflect the weighting attached to the type of answer.

Question 12 also had to have alterations to the types of answers participants provided. The original type 2 answer read ‘Wrong answer (e.g. \( \frac{\sin(7(0))}{0} = \frac{0}{0} = \text{undefined} \)’. However, from the responses provided it was clear that participants did not have a high acquisition of this content and their responses did not warrant a score of 75. This answer was changed to be a type 1 response. The new type 2 answer was developed from the viewing of one particular participant who had attempted the question correctly but did not provide an answer. Their answer was clearly more deserving of a type 2 weighting than those who stated \( \frac{\sin(7(0))}{0} = \frac{0}{0} = \text{undefined} \). The new answer reads ‘Correct attempt at working out the question but no final answer given’.

One question had to be changed from the findings of the pilot-test. The original question 11 (below) was found to have received surprising responses from the participants. Many of the participants scored higher on level 4 questions than they had on lower levels. This had to be investigated.
**Question 11:** The height \( h(t) \) metres of the tide above mean sea level on January 24th in Cape Town is modelled approximately by

\[
h(t) = 3 \sin 30t
\]

where \( t \) is the number of hours after midnight.

(i) Graph \( y = h(t) \) for \( 0 \leq t \leq 24 \).

(ii) When is high tide and what is the maximum height?

(iii) If a ship can cross the harbour provided the tide is at least 2 metres above mean sea level, when is crossing possible on January 24th? (approximate from graph)

Upon asking a lecturer who taught this group in a module held previous to the pilot-test, the researcher was informed that this form of question was a certainty on the participants’ examination for that module (which was recent with respect to the pilot test). Many of the responses seemed to be rote-learned as one or two provided excessive information which was irrelevant to the question they were asked. The lecturer stated that these excessive responses contained questions that would have been asked of the pupils in the module. From the fact that the responses appeared rote-learned in nature the question was removed and replaced.

The question that took its place was designed from the point of view of:

1. **Fitness for purpose**

   The question had to fit with the schema and objects for level 4 of the purpose-built teaching model. It had to assess for the conceptual understanding of trigonometric functions.

2. **Time constraints**

   The question should take less time than the original question 11. This way the participants would even have more time to complete the assessment as the original question 11 had taken a lot of time to complete.
The question that was implemented in place of the original question 11 was:

**Question 11:** The mean monthly maximum temperatures for Cape Town in South Africa are:

<table>
<thead>
<tr>
<th>Month</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp (°C)</td>
<td>28</td>
<td>27</td>
<td>25</td>
<td>22</td>
<td>18</td>
<td>16</td>
<td>15</td>
<td>16</td>
<td>18</td>
<td>21</td>
<td>24</td>
<td>26</td>
</tr>
</tbody>
</table>

The graph over the period of a year is shown:

Find a cosine model for this data (i.e. find a model in the form $T = A \cos Bt + C$).

This question requires participants to find a cosine model for some data. It requires them to understand:

1. Cosine functions in the form $y = A \cos Bx + C$;
2. Periods;
3. Amplitudes;
4. Principal axes (these should also be familiar to the participants from any previous work on any type of functions).

Cosine functions of the form $y = A \cos Bx$, and hence periods and amplitudes, are a required to be understood at Leaving Certificate Higher level (NCCA, 2011). Therefore the pre-service teachers should have a good conceptual understanding of them. The $+C$ aspect of a function of the form
After these changes were made to the assessment and assessment rubric (‘types’ of answers (Appendix L)) the researcher had a final assessment instrument to use in his research. The next section describes each item on the final assessment and how it applies to the purpose-built teaching model.

4.4.4 Individual Assessment Items

The genetic decomposition of the purpose-built teaching model informed the questions that formed the assessment, or research instrument used. In this section the design of each individual item included in the assessment will be described. Each question will also be related to a level in the purpose-built model for the effective teaching of trigonometry.

**Level 0 (Appearance Recognition): Question 1:**

Which of these are triangles? *(Please circle correct answer(s))*

This question assesses the participants’ abilities to simply identify triangles by shape when given a selection of shapes. The question assesses if the participants understand the ‘object’ of the visual appearance of a triangle.
Level 1 (Property Relations): Question 2:
Write under each of the triangles below as to which is equilateral, isosceles, and scalene.

Question 2 assesses the participants’ knowledge of the objects of isosceles, equilateral and scalene triangles. They should be able to link the terms to each of the triangles given in the question.

Level 1 (Property Relations): Question 3:
Calculate the missing angle in the triangle below.

Question 3 assesses the understanding of the object of angle summation. At this stage of the model, however, it is more a process than an object as it is the calculation that is important. The concept of 180° in a triangle still has to be acknowledged.
Level 2 (Right-Triangles & Pythagoras): Question 4:
The string of a kite is 120 metres long and makes an angle of 60° with the horizontal. What is the height of the kite if the bottom of the string touches the ground?

This question examines the level of understanding participants have of trigonometric ratios through their use in an application scenario. It is done in this way as the participants should be able to problem solve using the concepts they know, as shown in the diagram of the model (Figure 18). The question assesses the processes of labelling the hypotenuse, adjacent, and opposite sides, as well as the calculations that follow.

Level 2 (Right-Triangles & Pythagoras): Question 5:
Two cyclists start from the same location. One travels due north and the other due east, at the same speed. Find the speed of each in miles per hour if after 2 hours they are $17\sqrt{2}$ miles apart.

This question tests the participants’ abilities to utilise Pythagoras’ theorem (or any other applicable mathematical idea) in a context that requires them to apply it to a real-life scenario. In order to do this they must visualise the problem and problem solve using their understanding of applying the correct concept.

Level 3 (Unit Circle Concepts): Question 6:
What is the value of $\sin(-240°)$? Explain your work.

This question assesses the participants’ understanding of trigonometric expressions, specifically evaluating them. The concept of quadrants and angles are all included in this assessment questions as participants will have to place the angle in the relevant quadrant, and then evaluate the expression correctly.

Level 3 (Unit Circle Concepts): Question 7:
The tangent of an angle ($\tan \theta$) is positive in quadrants ___ and ___. Explain your answer.

This question further illustrates the participants’ understanding of the quadrant concept. In order to demonstrate accurate understanding of the concept participants must explain that sine and cosine are both positive in quadrant 1 and by using $\tan \theta = \frac{\sin \theta}{\cos \theta}$ that $\tan \theta$ is also positive in quadrant
1. This similar concept also applies to quadrant 3 where cosine and sine are both negative.

**Level 3 (Unit Circle Concepts): Question 8:**
A geologist wants to measure the diameter of a circular crater. From her camp inside the crater it is 4 miles to the northern-most point and 2 miles to the southern-most point. If the angle between the two lines of sight is $120^\circ$, what is the diameter of the crater?

This question further investigates the concepts within level 3. It assesses participants’ abilities at visualising the problem and then using their understanding of the concept of the cosine rule to solve it. Participants must understand the application of the cosine rule to solve this problem. If they do not know when to apply the rule in solving such problems the correct solution will not be found.

**Level 3 (Unit Circle Concepts): Question 9:**
Question 9 of the assessment asks participants ‘What is a radian? (A visual aid in your explanation may help)’. This question stems from the Leaving Certificate Higher level syllabus requiring pupils to “use the radian measure of angles” (NCCA, 2011, p.23). The participants should therefore understand the concept of a radian and be able to explain what it is as the syllabus requires pupils also to be able to communicate mathematics verbally and in written form (NCCA, 2011). This question will reflect a concept met at level 3 in the purpose-built teaching model. A visual explanation is requested as it is important that the participants have a mental picture of what a radian is, and that they understand what it is visually.

**Level 4 (Further Rules & Trigonometric Functions Understood): Question 10:**
This question (Solve $\cos 2\theta + 3 \sin \theta - 2 = 0$ for all values $0^\circ \leq \theta \leq 360^\circ$ and explain your work where possible) stems from findings in the Chief Examiner’s Reports that stated that pupils had insufficient understanding of how to find all solutions to problems (State Examinations Commission, 2000). This type of question is required to be understood at Leaving Certificate Higher level (NCCA, 2011). This question assesses acquisition of level 4 of the purpose-built teaching model.
**Level 4 (Further Rules & Trigonometric Functions Understood):**

**Question 11:**
The mean monthly maximum temperatures for Cape Town in South Africa are:

<table>
<thead>
<tr>
<th>Month</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp (°C)</td>
<td>28</td>
<td>27</td>
<td>25.5</td>
<td>22</td>
<td>18.5</td>
<td>16</td>
<td>15</td>
<td>16</td>
<td>18</td>
<td>21.5</td>
<td>24</td>
<td>26</td>
</tr>
</tbody>
</table>

The graph over the period of a year is shown:

Find a cosine model for this data (i.e. find a model in the form $T = A \cos Bt + C$).

This question reflects the concept of ‘graphing and analysis’ of trigonometric functions found in the genetic decomposition of the model. The inclusion of this question has already been discussed in section 4.4.3. Graphing and analysis of trigonometric functions was found at level 4 of the purpose-built teaching model.

**Level 4 (Further Rules & Trigonometric Functions Understood):**

**Question 12:**
Limits is the focus of assessment in this question (as was found to be poorly understood by the Chief Examiner’s Reports (2005, 2001, 2000)). This question tests the participants’ conceptual understanding in that it uses the fact that $\lim_{x \to 0} \frac{\sin 7x}{7x} = 1$. Therefore participants should understand that they must multiply above and below by 7 to obtain $7 \lim_{x \to 0} \frac{\sin 7x}{7x}$ which evaluates to 7(1) which equals 7. This question also analyses the concept of manipulation of content which is crucial to trigonometry, as found in the genetic decomposition of level 4.

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Level 5 (Derivation & Proof): Question 13:
Prove: \( \tan x - \frac{1}{2} \sin 2x = \tan x \sin^2 x. \)

This question analyses the participants’ concept of proof. Though this concept is applicable to various areas in mathematics the nature of this question is based solely on trigonometry. It assesses the three concepts of ‘choosing what to manipulate using trigonometric identities for simplification’, ‘manipulation of content in question’ and ‘the concept of proof’ found in the genetic decomposition of level 5 of the model.

Level 5 (Derivation & Proof): Question 14:
What error can be found in the following “proof” of Pythagoras’ theorem that ultimately may not constitute it as a proof?
Proof: In a right-angled triangle let \( a \) and \( b \) be its legs, \( c \) its hypotenuse, and \( \beta \) the angle opposite leg \( b \). Then, by the definition of the trigonometric functions:

\[
\sin \beta = \frac{b}{c} \\
\cos \beta = \frac{a}{c}
\]

As is provided in the mathematical tables \( \cos^2 \beta + \sin^2 \beta = 1 \). Therefore:

\[
\left( \frac{a}{c} \right)^2 + \left( \frac{b}{c} \right)^2 = 1
\]

This implies that \( a^2 + b^2 = c^2 \).

This question assesses the participants’ abilities to analyse information and apply their understanding to solve this problem which may be unfamiliar: two skills highlighted in the Leaving Certificate syllabus (NCCA, 2011). This “proof” of Pythagoras’ theorem was noticed to be false by retired Harvard professor Eng. María L. Mean (Bogomolny, 2012). It should be clear to participants that \( \cos^2 \beta + \sin^2 \beta = 1 \) is derived from Pythagoras’ theorem and should not be used in a proof for Pythagoras’ theorem. Participants are assessed on this question on their acquisition of level 5 of the purpose-built teaching model. With respect to the genetic decomposition of the model it assesses the concept of proof as well as the concept of Pythagorean identities.
**Level 6 (Trigonometry Seen in the Abstract): Question 15:**

State which lines in the unit circle below correspond to $\sin \theta, \cos \theta, \tan \theta, \sec \theta, \cot \theta,$ and $\cosec \theta$ and explain your answer.

Note: $\cot \theta = \frac{1}{\tan \theta}, \sec \theta = \frac{1}{\cos \theta}, \cosec \theta = \frac{1}{\sin \theta}$

Question 15 assesses participants’ knowledge of where to find $\sin \theta, \cos \theta,$ and $\tan \theta$ on a diagram. It also assesses their conceptual understanding of similar triangles in order to produce $\sec \theta, \cot \theta,$ and $\cosec \theta$. Similar triangles is relevant to the Junior Certificate Ordinary level syllabus (NCCA, 2011, p.80). This will assess participants’ acquisition of level 6 of the purpose-built teaching model as it is abstract in nature due to the fact that the individuals will not have encountered $\sec \theta, \cot \theta,$ or $\cosec \theta$ on a regular basis, if at all.
**Level 6 (Trigonometry Seen in the Abstract): Question 16:**
A 50 pound speaker is suspended from the ceiling by two support braces. If one of them makes an angle of 60° with the ceiling, and the other makes an angle of 30° with the ceiling, what are the tensions ($\mathbf{T}_1$ and $\mathbf{T}_2$) on each of the supports? (Note: The speaker is to remain stationary.)

This question again places trigonometry in a more abstract field, leaning towards physics. Application of trigonometric concepts (as in vectors (see Appendix M)) is required for the solution to be achieved.

The researcher did not use the second pilot-test for the sole purpose of adjusting the final research instrument. He also used it to serve the purpose of ensuring that the assessment evaluation method/marking scheme outlined in section 3.9.1 could be used by any teacher, not just the researcher himself. The next section demonstrates that there was a substantial agreement between the researcher and two mathematics educators who had not used the evaluation methods before.

**4.4.5 Inter-Rater Reliability from the Pilot Test**
After the pilot-testing phase involving the 19 first year pre-service secondary mathematics teachers, an inter-rater reliability test was conducted between two doctors of mathematics education and the researcher. The mathematics education experts did not have to be rigorously trained in the evaluation process/marking scheme which highlights the repeatable nature of the assessment (as the evaluation is repeatable by anyone without specific training).
The pilot inter-rater reliability test illustrated the reliability of the assessment tool, as well as the ‘types’ of answer participants could provide. Reliability is important as one needs to be sure that the test would produce the same results irrespective of who is correcting the assessment. Graham, Milanowski and Miller (2012) state that inter-rater reliability is of the utmost importance when it informs assessment results which are the basis for professional development planning. This rationale can be mirrored directly in this research as the assessment forms the basis of the intervention (i.e. development) phase. The two mathematics education specialists independently evaluated a random sample of 10 assessments from the overall 19, resulting in a total of 180 questions for each of them to assess (18 total questions before the final assessment was developed - see Appendix K). The researcher had corrected all 19 assessments but the same 10 were used to compare with the results provided by each of the mathematics education specialists.

The corrections by the mathematics education specialists were conducted in two stages. After the first of the specialists had marked the 10 assessments, both sets of data up to then were compared (the researcher and the first specialist). From this analysis, certain ‘types’ of answers were adjusted due to the accuracy and advice of the mathematics education specialist. The adjusted marking scheme (or adjusted ‘types’ of answers) were provided for the second mathematics education specialist for their correction of the assessments.

The kappa statistic is a numerical value which relates to the chance of agreement occurring between two or more independent assessors who are evaluating the same thing (in this case all three individuals are evaluating the same assessment) (Viera and Garrett, 2005). The kappa statistic was calculated twice. The first calculation was for the correlation between the researcher and the first mathematics education specialist, and the second calculation was for the correlation between the researcher and the second mathematics education specialist. Tables from Viera and Garrett (2005) were referred to for the interpretation of the kappa statistic (Tables 32 and 33).
Table 32: Interpretation of the Kappa Statistic (Viera and Garrett, 2005, p.362)

<table>
<thead>
<tr>
<th>Interpretation of Kappa</th>
<th>Poor</th>
<th>Slight</th>
<th>Fair</th>
<th>Moderate</th>
<th>Substantial</th>
<th>Almost Perfect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kappa</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 33: Interpretation of the Kappa Statistic (Viera and Garrett, 2005, p.362)

<table>
<thead>
<tr>
<th>Kappa</th>
<th>Agreement</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 0</td>
<td>Less than chance agreement</td>
</tr>
<tr>
<td>0.01 – 0.20</td>
<td>Slight agreement</td>
</tr>
<tr>
<td>0.21 – 0.40</td>
<td>Fair agreement</td>
</tr>
<tr>
<td>0.41 – 0.60</td>
<td>Moderate agreement</td>
</tr>
<tr>
<td>0.61 – 0.80</td>
<td>Substantial agreement</td>
</tr>
<tr>
<td>0.81 – 0.99</td>
<td>Almost perfect agreement</td>
</tr>
</tbody>
</table>

In order to calculate the kappa statistic, a table was constructed which outlined the number of times a ‘type’ of answer was agreed upon and disagreed on by the two assessors in question. For example in Table 34, the number 61 at the intersection point of both ‘type 4’ answers means that on 61 occasions both assessors agreed that a response was of ‘type 4’. A similar concept is employed in the table for disagreement. For example, at the intersection of ‘type 2’ provided by the researcher and ‘type 3’ provided by the first mathematics education specialist, the number 2 means that the researcher deemed a response to be of ‘type 2’ and the specialist deemed the response to the same question to be of ‘type 3’ on 2 occasions.
Table 34: Agreement of Answer Types Between the Researcher and Mathematics Education Specialist 1

<table>
<thead>
<tr>
<th>'Type' of answer</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Researcher</td>
<td>61</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>67</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0</td>
<td>7</td>
<td>0</td>
<td>1</td>
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<td>5</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>16</td>
<td>51</td>
<td>68</td>
</tr>
<tr>
<td>Total</td>
<td>61</td>
<td>15</td>
<td>7</td>
<td>45</td>
<td>52</td>
<td>180</td>
</tr>
</tbody>
</table>

Using this table, the kappa statistic could be calculated. Two values $p_o$ and $p_e$ had to be calculated. The observed agreement ($p_o$) is the proportion of ‘types’ of answers which the assessors agree on. In this case the assessors agree on 142 (61 + 7 + 5 + 18 + 51) of the total 180 questions. Therefore:

$$p_o = \frac{142}{180} \approx 0.79$$

The second value that had to be calculated is the expected agreement ($p_e$). This was calculated as follows:

$$p_e = \left[ \frac{61}{180} \times \frac{67}{180} \right] + \left[ \frac{15}{180} \times \frac{8}{180} \right] + \left[ \frac{7}{180} \times \frac{16}{180} \right] + \left[ \frac{45}{180} \times \frac{21}{180} \right] + \left[ \frac{52}{180} \times \frac{68}{180} \right]$$

$$p_e \approx 0.2716$$

The kappa statistic $K$ is calculated by the use of the formula:

$$K = \frac{p_o - p_e}{1 - p_e}$$

$$K = \frac{0.79 - 0.2716}{1 - 0.2716}$$

$$K \approx 0.712$$

According to the interpretation values provided by Viera and Garrett (2005) (Tables 32 and 33) this value corresponds to a substantial agreement.
The standard error \( SE(K) \) was calculated using the formula provided in Bland (2008).

\[
SE(K) = \sqrt{\frac{p(1-p)}{n(1-p_e)^2}}
\]

\[
SE(K) = \sqrt{\frac{0.79(1-0.79)}{180(1-0.2716)^2}} \approx 0.0417
\]

This leads to a 95% confidence interval for \( K \) (as \( K \) is normally distributed) of:

\[
K \pm 1.96(SE(K)) = 0.712 \pm 1.96(0.0417) \approx [0.63, 0.7937]
\]

of which the entire interval corresponds to substantial agreement.

The second assessor corrected the assessments after a more precise version of the marking scheme was developed (i.e. after the alteration of certain ‘types’ of answers). The kappa statistic was again calculated for the agreement between this assessor and the researcher.

Table 35: Agreement of Answer Types Between the Researcher and Mathematics Education Specialist 2

<table>
<thead>
<tr>
<th>'Type' of answer</th>
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<th>3</th>
<th>2</th>
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<th>0</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Researcher</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>66</td>
</tr>
<tr>
<td>4</td>
<td>65</td>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>66</td>
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<td>24</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>11</td>
<td>54</td>
<td>67</td>
</tr>
<tr>
<td>Total</td>
<td>68</td>
<td>12</td>
<td>5</td>
<td>35</td>
<td>60</td>
<td>180</td>
</tr>
</tbody>
</table>

The kappa statistic calculation produced the following values:

\[
p_o = \frac{149}{180} \approx 0.83
\]

\[
p_e = \left[ \frac{68}{180} \times \frac{66}{180} \right] + \left[ \frac{12}{180} \times \frac{11}{180} \right] + \left[ \frac{7}{180} \times \frac{12}{180} \right] + \left[ \frac{35}{180} \times \frac{24}{180} \right] + \left[ \frac{60}{180} \times \frac{67}{180} \right]
\]

189
\[ p_e \approx 0.2944 \]

\[ K = \frac{p_o - p_e}{1 - p_e} \]

\[ K = \frac{0.83 - 0.2944}{1 - 0.2944} \]

\[ K \approx 0.76 \]

\[ SE(K) \approx 0.0397 \]

\[ 95\% CI \approx [0.6822, 0.8378] \]

This \( K \) value is a higher value and tends more to the higher end of the interval of values deemed to demonstrate a substantial agreement and some correspond with an almost perfect agreement (values \( > 0.81 \)). In the context of this research this significant substantial agreement indicates the strength of agreement between the researcher and the other raters who are relatively untrained with respect to this model and the assessment. This highlights the fact that this research would be repeatable by others and that the results are valid.

The mathematics education specialists also reported an anomaly in the findings of the original question 11 and questioned the reason as to how the participants were achieving higher in this question compared to questions on lower levels. This further supports the decision from the pilot test phase to amend the original question.

4.5 Conclusion

This chapter described the development of the purpose-built model for the effective teaching of trigonometry. The development of the model through its adaptation from the van Hiele model was discussed. Each level of the model was outlined and concepts within each level were illustrated through genetic decomposition. The author also demonstrated the properties of the teaching model and how the model would be implemented in a classroom through the five teaching phases of the van Hiele model. The chapter also
showed that the assessment evaluation methods have a substantial level of agreement between different evaluators. This was relative to two researchers who were untrained with respect the evaluation methods, giving credence to the idea of teachers in schools being able to use these methods.

The model, as well as its assessment (section 4.4) and evaluation of the assessment methods (section 3.9.1) were implemented amongst a sample of pre-service mathematics teachers at the University of Limerick. This was done in an attempt to position these pre-service teachers at a level on the purpose-built teaching model. Using these results a teaching intervention could be constructed in order to improve the trigonometric conceptual understanding of this group. The next chapter analyses the results that arose from this assessment.
Chapter 5

Discussion of Findings from the Pre-Intervention Test

5.1 Introduction

The purpose-built model for the effective teaching of trigonometry (chapter 4) and its accompanying assessment instrument (section 4.4) have been previously discussed. This chapter discusses the findings that arose from the analysis of the pre-test phase of the research. Results from the assessment evaluation as well as exploratory findings are discussed. These findings provided the foundation on which the teaching intervention phase was built.

The pre-intervention test (or pre-test) was administered to two groups; third year and final year pre-service mathematics teachers (the academic year 2012/2013). The majority of the third year group who completed the pre-test made up the intervention group in the following year. The intervention was based on findings that arose from the pre-test.

The pre-tests were administered to the sample in March and April of 2013. Each group was assessed in an assigned one-hour lecture. The evaluation of all of the assessments was conducted by the researcher in accordance with the assessment rubric which outlined the types of answers for each question (Appendix L). Each participant was assigned a degree of acquisition for each level of the purpose-built teaching model in line with the evaluation methods. This chapter discusses the findings that arose from the evaluation of the pre-test assessments.
5.2 Understanding With Respect to the Levels of the Purpose-Built Model

The first element to inspect in the pre-test was the conceptual understanding of pre-service teachers with respect to the purpose-built teaching model that was developed. This section analyses the degrees of acquisition the participants had of each level, as found in the pre-test phase. Figures 20, 21 and 22 outline the percentages of third year pre-service participants, fourth year (final year) pre-service participants, and all pre-service participants, respectively, who were found to have a complete acquisition of each individual level. This section is an overview of the degrees of acquisition of the participants. A more detailed analysis of misunderstandings and misconceptions will be discussed in section 5.6. The findings discussed in the following sections have been obtained from the evaluation of the pre-test assessment according to the ‘types’ of answers provided by the participants. For the complete raw data see Appendices N and O. Note that coding for individuals in the raw data is assigned as 3-P[x] or 4-P[x]. For example the seventh third year participant is coded as 3-P7.

![Diagram of Level Acquisition]

Figure 20: % and Number of Third Year Participants With a Complete Acquisition of Each Individual Level
Figure 21: % and Number of Fourth Year Participants With a Complete Acquisition of Each Individual Level

Figure 22: % and Number of All Pre-Service Participants With a Complete Acquisition of Each Individual Level
At a glance, the pre-service participants all demonstrated complete acquisition of level 0 and level 1 (with one exception in each level which will be discussed in section 5.4). This demonstrates the participants’ understanding of the very basic concepts of trigonometry (see Table 36). The participants can differentiate a triangle from other figures and can give the basic properties of a triangle (summation of angles etc.).

Table 36: % of Pre-Service Teachers in Each Degree of Acquisition of Levels 0 and 1

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1</td>
<td>98%</td>
<td>2%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Level 0</td>
<td>98%</td>
<td>2%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

There is a significant decrease in the understanding of concepts at the next level, level 2. 76% of participants were found to have a complete acquisition of this level. However, 4% of participants were found to have a high acquisition, 14% an intermediate acquisition, and 6% were found to have no acquisition of this level. These findings indicate that concepts that relate to right triangle ideas, Pythagoras’ theorem, and trigonometric ratios caused problems for a selection of the sample of pre-service participants.

Table 37: % of Pre-Service Teachers in Each Degree of Acquisition of Level 2

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 2</td>
<td>76%</td>
<td>4%</td>
<td>14%</td>
<td>0%</td>
<td>6%</td>
</tr>
</tbody>
</table>

Level 3 was only completely acquired by 4% of the sample. 8% of the sample were found to have a high acquisition of the level, 22% an intermediate acquisition, 52% a low acquisition, and 14% were found to have no acquisition of this level. This highlights that there were misunderstandings amongst 96% of the sample in relation to concepts linked to the unit circle, as well as the sine and cosine rules. It is therefore the case that connections between concepts as outlined in the extension of the van Hiele model from geometry to trigonometry (section 4.2.1) were not understood.
Table 38: % of Pre-Service Teachers in Each Degree of Acquisition of Level 3

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 3</td>
<td>4%</td>
<td>8%</td>
<td>22%</td>
<td>52%</td>
<td>14%</td>
</tr>
</tbody>
</table>

At level 4 of the model 0% of participants were found to have a complete acquisition. One of the questions failed to return any answer of at least ‘Type 1’ by the third year participants. Only 4% were found to have a high acquisition of this level, 2% had an intermediate acquisition, and 94% had either a low or no acquisition (48% and 46% respectively).

Table 39: % of Pre-Service Teachers in Each Degree of Acquisition of Level 4

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 4</td>
<td>0%</td>
<td>4%</td>
<td>2%</td>
<td>48%</td>
<td>46%</td>
</tr>
</tbody>
</table>

The trend continued at level 5 of the model with 60% of the participants demonstrating no acquisition of the level (an increase of 14% on level 4). 4% had a complete acquisition, 0% a high acquisition, 22% an intermediate acquisition, and 14% a low acquisition.

Table 40: % of Pre-Service Teachers in Each Degree of Acquisition of Level 5

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 5</td>
<td>4%</td>
<td>0%</td>
<td>22%</td>
<td>14%</td>
<td>60%</td>
</tr>
</tbody>
</table>

At the final level, level 6, 2% of participants demonstrated a complete acquisition, 2% a high acquisition, 2% an intermediate acquisition, 6% a low acquisition, and 88% showed no acquisition (again an increase of 28% on level 5).
Table 41: % of Pre-Service Teachers in Each Degree of Acquisition of Level 6

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 6</td>
<td>2%</td>
<td>2%</td>
<td>2%</td>
<td>6%</td>
<td>88%</td>
</tr>
</tbody>
</table>

The summary of the third year, fourth year, and total participants’ degrees of acquisition of each level are depicted in Tables 42, 43 and 44 respectively.

Table 42: % of Third Year Participants in Each Degree of Acquisition of Each Level

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 6</td>
<td>0%</td>
<td>4%</td>
<td>4%</td>
<td>4%</td>
<td>88%</td>
</tr>
<tr>
<td>Level 5</td>
<td>0%</td>
<td>0%</td>
<td>20%</td>
<td>16%</td>
<td>64%</td>
</tr>
<tr>
<td>Level 4</td>
<td>0%</td>
<td>4%</td>
<td>4%</td>
<td>52%</td>
<td>40%</td>
</tr>
<tr>
<td>Level 3</td>
<td>4%</td>
<td>12%</td>
<td>24%</td>
<td>52%</td>
<td>8%</td>
</tr>
<tr>
<td>Level 2</td>
<td>80%</td>
<td>0%</td>
<td>16%</td>
<td>0%</td>
<td>4%</td>
</tr>
<tr>
<td>Level 1</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Level 0</td>
<td>96%</td>
<td>4%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

Table 43: % of Fourth Year Participants in Each Degree of Acquisition of Each Level

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 6</td>
<td>4%</td>
<td>0%</td>
<td>0%</td>
<td>8%</td>
<td>88%</td>
</tr>
<tr>
<td>Level 5</td>
<td>8%</td>
<td>0%</td>
<td>24%</td>
<td>12%</td>
<td>56%</td>
</tr>
<tr>
<td>Level 4</td>
<td>0%</td>
<td>4%</td>
<td>0%</td>
<td>44%</td>
<td>52%</td>
</tr>
<tr>
<td>Level 3</td>
<td>4%</td>
<td>4%</td>
<td>20%</td>
<td>52%</td>
<td>20%</td>
</tr>
<tr>
<td>Level 2</td>
<td>72%</td>
<td>8%</td>
<td>12%</td>
<td>0%</td>
<td>8%</td>
</tr>
<tr>
<td>Level 1</td>
<td>96%</td>
<td>4%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Level 0</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

197
Table 44: % of Pre-Service Participants in Each Degree of Acquisition of Each Level

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 6</td>
<td>2%</td>
<td>2%</td>
<td>2%</td>
<td>6%</td>
<td>88%</td>
</tr>
<tr>
<td>Level 5</td>
<td>4%</td>
<td>0%</td>
<td>22%</td>
<td>14%</td>
<td>60%</td>
</tr>
<tr>
<td>Level 4</td>
<td>0%</td>
<td>4%</td>
<td>2%</td>
<td>48%</td>
<td>46%</td>
</tr>
<tr>
<td>Level 3</td>
<td>4%</td>
<td>8%</td>
<td>22%</td>
<td>52%</td>
<td>14%</td>
</tr>
<tr>
<td>Level 2</td>
<td>76%</td>
<td>4%</td>
<td>14%</td>
<td>0%</td>
<td>6%</td>
</tr>
<tr>
<td>Level 1</td>
<td>98%</td>
<td>2%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Level 0</td>
<td>98%</td>
<td>2%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

It can be seen in Tables 43 and 44 that some anomalies arose in the degrees of acquisition. For example, there were a higher percentage of participants with a complete understanding of level 5 than level 4 of the purpose-built model. As the model had been found to be hierarchial through Guttman Scalogram Analysis (as will be explained in section 5.9) this was an unexpected occurrence. The next section investigates the inconsistent degrees of acquisition with reference to research conducted into the van Hiele model, which is also hierarchial in structure (Mayberry, 1983).

5.3 Inconsistency of the Model

It can be seen in the previous section that the model is inconsistent at certain points. For example, 0% of the participants were found to have a complete acquisition of level 4 of the model, while 4% were found to have a complete acquisition of level 5. Though the model is structured in a sequential manner (i.e. that the concepts within a level must be understood before the next level of understanding commences), it is not fixed as such. Various studies (Usiskin, 1982; Mayberry, 1983; Gutiérrez and Jaime, 1987; Gutiérrez, Jaime and Fortuny, 1991) have found similar unusual patterns in their research.
In the above figure (Figure 23) it can be seen that in the 4-level van Hiele model research, conducted by Gutiérrez et al. (1991), the results of the unit within the research on the left showed that this individual had a higher degree of acquisition of level 3 than level 2. The unit within the research on the right had a higher degree of acquisition on both levels 3 and 4 than level 2.

Gutiérrez, Jaime and Fortuny (1991) state that this could be due to the teaching methodologies implemented with the students. Though this is one possible reason in relation to the sample in this doctoral research, it is also important to consider the context in which this research is set. The sample in this research have all followed the traditional Irish secondary mathematics syllabus, i.e. before Project Maths. Within this syllabus, certain proofs are required to be learned and were given in distinct sections in Leaving Certificate textbooks (Humphrey, 2003). This still occurs in Project Maths textbooks (Keating, Mulvany, Murphy and O’Loughlin, 2011). These proofs have been rote learned in the past by students (NCCA, 2005). Nevertheless, the students in this sample had exposure to the concept of mathematical proof since second-level. Along with the exposure at secondary level, the sample in this research all have had elements of proof involved in their studies at third level. They have all studied Technological Mathematics 1 and 2, Algebra 1 and 2, Analysis, Linear Algebra 1, History of Mathematics, and Mathematics Laboratory all of which contain some element of the concept of proof. They also have completed teaching practice (third years had done 6 weeks of teaching practice, final years had done 16 weeks) where they may
have taught some aspect of proof, though this is unlikely. The fourth year participants have additionally had 9 weeks of Group Theory and 9 weeks of Differential Equations modules (up to the time the assessment was conducted). Other than those two modules the mathematical backgrounds of the third and fourth year participants are very similar.

These aspects may help to explain why some of the participants performed better on level 5 of the model, due to its focus on the concept of proof. The participants have previously studied the concept of proof in many areas (e.g. at third-level in Algebra 1 they were required, for example, to prove that $\sqrt{2}$ is irrational), and over many years. However, the concepts within level 4 of the model were studied significantly less. The participants in this research would all have studied trigonometric functions at secondary level and would have studied them again as part of the syllabus for Technological Maths at third level. However, it is quite clear from the results of the assessment for this research that the participants have never fully engaged with the concepts within level 4 with 92% of participants having a low or null acquisition of the level.

As mentioned in section 5.2, one participant failed to demonstrate a complete acquisition of level 0, and another participant failed to demonstrate a complete acquisition of level 1. The next section discusses these two participants and explains the issues that their pre-tests presented.
5.4 Results Caused by Mistakes and not from Lack of Understanding

The participant without a complete acquisition of level 0 (Participant 3-P3; Appendix N) had a complete acquisition of levels 1 and 2 and a high acquisition of levels 3, 4 and 6. This fact raises questions about how the participant faltered at level 0. The answer the participant provided for level 0 was:

1. Which of these are triangles? (Please circle correct answer(s))

![Figure 24: Participant 3-P3 Response to Question 1](image)

It is logical, from the other responses provided by this individual to further questions on trigonometry that they do understand what a triangle is visually. Therefore it is fair to assume their response to this question was a mistake not caused by a misunderstanding/misconception.

The individual who failed to show a complete understanding of level 1 (Participant 4-P10; Appendix O), again demonstrated a complete acquisition of level 2 and therefore, at this level of understanding it can be assumed that their response to a question posed at level 1 held a genuine error and not one formulated by a misunderstanding/misconception. Upon observing the response it is likely that the response incorporated a mental calculation error.
3. Calculate the missing angle in the triangle below.

![Triangle Image]

Figure 25: Participant 4-P10 Response to Question 3

One of the main concerns from the pre-test findings was the low acquisition of level 4 of the purpose-built model. It was clear that no participant from either group had a complete acquisition of this level. The next section discusses the difficulties at this level of understanding for the sample.

5.5 Difficulties at Level 4 of the Model

Level 4 of the model was the only level where 0% of the sample demonstrated a complete acquisition of the level. There may be multiple reasons for this, however, the author will illustrate a few reasons from the participants’ past mathematical experiences of the concepts within this level.

Questions 10, 11 and 12 in the assessment tool related to level 4 of the model (Appendix A). The types of answers provided by the 50 participants for question 10 were distributed as follows:

<table>
<thead>
<tr>
<th>Question 10 Answer Type</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 4</td>
<td>6%</td>
</tr>
<tr>
<td>Type 3</td>
<td>30%</td>
</tr>
<tr>
<td>Type 2</td>
<td>14%</td>
</tr>
<tr>
<td>Type 1</td>
<td>34%</td>
</tr>
<tr>
<td>Type 0</td>
<td>16%</td>
</tr>
</tbody>
</table>
Question 10, which is based on solving trigonometric equations, was answered with responses of types 3 or 4 in 36% of the assessments (a weighting of 90 or 100). Just over \(\frac{1}{3}\) of the participants had a complete acquisition of the concepts in this question. This is not ideal, however investigation into further concepts within this level of understanding proved to have more alarming results.

The concepts within question 11 were understood less by the participants. In Table 46 it can be seen that 0% of participants responded with an answer of type 4. Though the concepts within this question would have been met by these individuals before (see section 5.3), they clearly have not engaged with them in a relational manner (Skemp, 1976) and therefore have not acquired these concepts as ‘objects’ (see section 4.2.3). It is clear that these concepts have, at best, only been acquired as ‘processes’ (section 4.2.3). This is confirmed as this particular question is based in a context and once the concepts have been applied outside of a procedural situation the participants are unable to apply the concept.

<table>
<thead>
<tr>
<th>Question 11 Answer Type</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 4</td>
<td>0%</td>
</tr>
<tr>
<td>Type 3</td>
<td>2%</td>
</tr>
<tr>
<td>Type 2</td>
<td>0%</td>
</tr>
<tr>
<td>Type 1</td>
<td>4%</td>
</tr>
<tr>
<td>Type 0</td>
<td>94%</td>
</tr>
</tbody>
</table>

Table 47: Answer Types for Question 12

<table>
<thead>
<tr>
<th>Question 12 Answer Type</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 4</td>
<td>6%</td>
</tr>
<tr>
<td>Type 3</td>
<td>0%</td>
</tr>
<tr>
<td>Type 2</td>
<td>0%</td>
</tr>
<tr>
<td>Type 1</td>
<td>12%</td>
</tr>
<tr>
<td>Type 0</td>
<td>82%</td>
</tr>
</tbody>
</table>
Question 12 also clarified that the participants have a poor understanding of the concepts with this level. Limits have been shown to be understood poorly at Leaving Certificate level (section 2.6.2) and this seems to continue at third level, as is to be expected according to Weber (2005). Even though question 12 was not based in a context, the participants were unable to explain and illustrate their work, with only 6% achieving the correct calculation and communication of the concept.

Though the problems that occurred at level 4 upwards of the model have been documented, there must be underlying elements that contribute to this. Certain concepts in preceding levels must be misunderstood in order to produce the lack of understanding of level 4. Level 0 and level 1 can be seen to be acquired to a very high standard (Table 36), however, elements of misunderstanding begin to come to the fore in levels 2 and 3. In the next section answers of type 0 or type 1 will be classified as inadequate as they are answer types which comprise of low levels of understanding. Their respective weighings reflect no or low acquisition.

5.6 Dominant Misconceptions and Misunderstandings

As the higher levels of the model (levels 4-6) include many connections with cosine, sine and tangent functions, it is important that these basic concepts are understood. Question 4 in the assessment tool assessed a basic application of trigonometric ratios. Of the 50 participants sampled, 3 of them supplied inadequate answers of type 0 or type 1. This is a predictor that higher order problems incorporating trigonometric ratios/expressions/functions will cause further problems in understanding.

Question 5 of the assessment tool could be attempted using Pythagoras’ theorem/trigonometric ratios/sine rule. 12 inadequate answers (24% of answers) were provided to this question. An examination of the assessment scripts supports that the predominant reason for these answers was the inability to picture/visualise the problem correctly. Visualisation is a skill that is certainly required in trigonometry, and elsewhere in geometry. Many of the participants did not make it to the stage where they could relate their visual portrayal of the scenario to a mathematical situation/solution.
The decline in acquisition becomes increasingly apparent at level 3 where the complete acquisition drops from 76% (at level 2) to 4% (at level 3) (Table 44). Questions 6 through 9 assess this level of the model.

30 inadequate answers (60% of answers) were provided to the problem posed in question 6. Visualisation of the problem proved to once again be an issue for the participants. An angle of $-240^\circ$ was not understood by many individuals. Many of them provided a diagram similar to that in Figure 26.

![Figure 26: Common Diagram in Responses to Question 6](image)

Many of the responses therefore showed that the participants do not fully understand what an angle means in relation to the unit circle and hence, what quadrant a certain angle should be in.

The responses to question 7 were inadequate in 43 (86%) cases (12 type 0’s and 31 type 1’s). The dominant response to this question involved the use of the CAST notation. This response if explained would have been adequate, however every participant, with only one exception, failed to explain what CAST means. It was provided as a rote-learned response. The vast majority of the 31 answers of type 1 were similar to the following:
The tangent of an angle ($\tan \theta$) is positive in quadrants 1 and 3. This is because in quadrant 1 all are positive and in quadrant 3 $\tan$ is positive.

This answer does not demonstrate any mathematical understanding of the concept, it is solely an image with letters to represent trigonometric functions in particular quadrants. The CAST notation was frequently used in traditional Leaving Certificate textbooks (e.g. Humphrey, 2003, p.111) and is still used in Project Maths textbooks without any further explanation to develop a mathematical understanding of the rule (e.g. Keating et al., 2011, p.245). It further highlights and verifies that the participants do not understand the concept of quadrants to a high degree.

23 inadequate responses (46%) were provided to question 8 which assessed the cosine rule concept. 17 of these responses were of type 0. Again, visualisation of the problem appears to be an issue for these 17 participants. The 6 answers of type 1 included the correct visualisation of the problem, however the participants were unable to apply the correct concept from this point onwards. Even after correctly visualising the problem, the application of mathematical methods was not always achieved. For example:
Participant 3-P7 visualised the problem well despite the diagram not being accurate (as the distance of 2 miles looks longer in their diagram that the distance of 4 miles). This participant applied labels of A, O, and H, which stand for adjacent, opposite and hypotenuse sides and proceeded to use trigonometric ratios in an attempt to solve the problem. Of course this could not be done due to the non-existence of a right angle. (Note: This answer was of type 1).

The vast majority (82%) of the participants gave inadequate responses to question 9 which asked them to explain the concept of a radian. The 22 responses of type 1 all mentioned the term ‘angle’. This implies that the participants know that a radian is linked to angles, however, they do not fully understand the concept.

The intervention phase for each third year participant was planned with consideration of the findings from the pre-test evaluation. The next section briefly discusses the percentage of each group that would start their intervention at each of the levels.
5.7 Intervention Initiation

It is proposed for this research that the intervention phase be carried out beginning at the first level where the participants have not demonstrated a complete acquisition in the pre-test. This is due to the hierarchial nature of the model. It is proposed that the concepts within a level must be completely acquired before the next level can be acquired. Though some research notes that this does not necessarily have to be the case, it is the chosen way to approach the intervention in this research.

For the fourth year participants, the percentage of interventions that would start at each individual level is as follows:

![Diagram showing percentage of fourth year participants whose intervention would start at each level](image)

*Figure 28: % of Fourth Year Participants Whose Intervention Would Start at Each Level (n = 25)*
For the third year participants it is:

Figure 29: % of Third Year Participants Whose Intervention Would Start at Each Level (n = 25)

From the overall sample therefore, the following percentages of participants would begin their intervention at the following levels:
As noted, it is the third year group that participated in the intervention. Therefore Figure 29 represents the findings related to the sampling frame for the teaching intervention that took place. Six individuals from the third year group chose not to participate in the intervention (as it was voluntary) and therefore Figure 29 had to be adjusted when comparing pre and post-test findings. This is described in more detail in chapter 7.

Participants demonstrated low levels of acquisition of the purpose-built model. The researcher now discusses the level of the purpose-built model that second-level teachers should be on. The next section demonstrates the levels of the purpose-built model that each of the participants should be at in order to teach each of the levels at second-level for a relational understanding.
5.8 Level of Expectation

A strong subject matter knowledge base is required by mathematics teachers (Shulman, 1986; 1987; Ernest, 1989; Fennema and Franke, 1992; Rowland, 2007; Ball et al., 2008; O’Meara, 2011). The importance of subject matter knowledge has been identified in multiple pieces of literature. The author now indicates where a teacher should be on the model in order to teach effectively.

When we compare the purpose-built model to the mathematics syllabus at Leaving Certificate Higher level it is fair to conclude that levels 0 to 5 (inclusive) of the model are all relevant to senior cycle mathematics. In order to teach to achieve the Project Maths objective of developing a relational understanding of trigonometry (NCCA, 2011) a teacher should understand all of the concepts of level 0 through 5. This was not reflected in the results from the pre-test phase.

Level 6 is not deemed critical for effective teaching at secondary school level. This level is only applicable to a teacher’s teaching if there is an exceptional pupil in their class who needs challenges of a higher order (NCCA guidelines, 2007; section 4.3). This level may not be a “necessity” to teach the second-level syllabus for a relational understanding, however it may be deemed important for adapting one’s teaching for all pupils, depending on the abilities of students in the class.

The pre-test assessments also provided the researcher with the opportunity to investigate if the purpose-built teaching model was hierarchial in structure. The van Hiele model was shown to be hierarchial (section 2.15.3) and the hypothesis was that the purpose-built model would be too. This is due to the strong links between both models. The hierarchial structure of the purpose-built model was shown through Guttman Scalogram Analysis as described in the next section.
Hierarchial Structure of the Model and Assessment Tool Using Guttman Scalogram Analysis

5.9.1 Introduction

Guttman Scalogram Analysis (Torgerson, 1958) uses a coefficient of reproducibility (\(Rep\)) to determine if a model corresponds to a “perfect scale” (Torgerson, 1958, p.318). In the case of this research, this analysis was used to investigate if the purpose-built model for the effective teaching of trigonometry is hierarchial in nature and hence if the assessment tool relates to this in terms of a perfect scale.

Reproducibility is the key idea to a Guttman Scale. Guttman used this idea when forming the definition of error. An error in the usage of Guttman Scalogram Analysis is viewed as an error in reproducibility. In this case, an error is viewed as when a participant in the research has obtained a larger degree of acquisition in a higher level on the model than in a preceding one (e.g. a participant has a complete acquisition of level 4 and only a high acquisition of level 3).

This analysis was conducted in accordance with the conditions of Guttman Scalogram Analysis. Torgerson (1958) outlines five points that should be adhered to in order to ensure that the \(Rep\) obtained is representative of the data, and is not falsely high.

1. **Number of answer categories**: Torgerson (1958) states that for multi-category items (such as the levels of the model in this research), that a small number of questions would suffice, however, they state that as the number of questions increases so does the accuracy of the results. The small number they mention is meant to be less than 10. In this research 16 questions were asked so this adheres to the requirement.

2. **Range of marginal frequencies (n/a)**: Not applicable to this research as each level is as important to the model as the others.

3. **The pattern of errors**: Torgerson (1958) states that error should be “random” (p.324). This means that a large number of participants should not all be shown to have the same pattern of errors. This is clearly adhered to in the participants’ degrees of acquisition as the...
degrees of acquisition are unique for each individual who took part. There was no deterministic elements to the assessment tool.

4. **Item reproducibility:** It is stated that Reps should all be 0.85 or more, however, the calculations in this research have surpassed this value and the more recently acknowledged value of 0.9. More researchers now base 0.9 as the benchmark for reproducibility.

5. **Improvement:** Torgerson (1958) states that every category (participant results) should have more non-error than error. In this research, non-error is an agreement between markers, and error is disagreement. For example in table 49 it is clear there are 5 cases of non-error and 1 case of error. This is the case for all of the participants.

The coefficient of reproducibility (Rep) is calculated by the following formula:

$$Rep = 1 - \frac{\text{total number of errors}}{\text{total number of responses}}$$

or

$$Rep = 1 - \frac{\text{total number of errors}}{\text{number of items} \times \text{number of subjects}}$$

In this research the formula would appear as

$$Rep = 1 - \frac{\text{total number of errors}}{\text{number of levels} \times \text{number of participants}}$$

According to Torgerson (1958) a Rep of 0.9 or more indicates scalability. A Rep was calculated in this research for all levels of the model (0-6), and also for levels 0-5, levels 0-4, levels 0-3 and finally levels 0-2. The final Rep (levels 0-2) produced a Rep of 1 and therefore all preceding levels would also carry a Rep of 1.
5.9.2 Method of Calculation

The first step in calculating the Rep’s was to organise the results from the pre-test phase so they could be coded for analysis. As the main parameter of interest was to determine if the model and assessment were scalable (hierarchical), the data categorising the degrees of acquisition was used (Appendix N and Appendix O). The coding for each of these were as follows:

- 4 - Complete acquisition
- 3 - High acquisition
- 2 - Intermediate acquisition
- 1 - Low acquisition
- 0 - No acquisition

This was performed so that an error could be identified and numbered. For example participant 3-P7 (Appendix N) had degrees of acquisition:

<table>
<thead>
<tr>
<th>Participant</th>
<th>Level 0</th>
<th>Level 1</th>
<th>Level 2</th>
<th>Level 3</th>
<th>Level 4</th>
<th>Level 5</th>
<th>Level 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-P7</td>
<td>Complete</td>
<td>Complete</td>
<td>Complete</td>
<td>Low</td>
<td>Low</td>
<td>Inter</td>
<td>None</td>
</tr>
</tbody>
</table>

The coding for Rep calculation was therefore:

<table>
<thead>
<tr>
<th>Participant</th>
<th>Level 0</th>
<th>Level 1</th>
<th>Level 2</th>
<th>Level 3</th>
<th>Level 4</th>
<th>Level 5</th>
<th>Level 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-P7</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

From this we can see that there is one error for this participant as his degree of acquisition of level 5 is higher than that of level 4. This highlights one error in the hypothesised hierarchical structure.

This method of analysis was conducted for the sample of 50 pre-service secondary mathematics teachers (Tables 50 and 51) and the total number of errors with respect to the model at the various level intervals (0-6 etc.) were noted (Tables 52, 53, 54, 55 and 56). For example, in Table 52 for participant 4-P16 there were zero errors with respect to the model, but for
participant 3-P16 there were 2 errors. This was a total of 2 errors for these two participants. This analysis was conducted for all of the fourth year participants (4-P[x]) and all of the third year participants (3-P[x]) for all relevant level intervals (0-6 (Table 52), 0-5 (Table 53), 0-4 (Table 54), 0-3 (Table 55), 0-2 (Table 56)) in the model.

Note: An asterisk (*) indicates that this error has been nullified for the purpose of this research due to the rationale noted in section 5.4.
<table>
<thead>
<tr>
<th>Participant</th>
<th>Level 0</th>
<th>Level 1</th>
<th>Level 2</th>
<th>Level 3</th>
<th>Level 4</th>
<th>Level 5</th>
<th>Level 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-P1</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3-P2</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3-P3</td>
<td>3*</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3-P4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3-P5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3-P6</td>
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<td>4</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3-P7</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3-P8</td>
<td>4</td>
<td>4</td>
<td>4</td>
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<td>1</td>
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<td>0</td>
</tr>
<tr>
<td>3-P9</td>
<td>4</td>
<td>4</td>
<td>4</td>
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</tr>
<tr>
<td>3-P10</td>
<td>4</td>
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<td>2</td>
<td>1</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3-P11</td>
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<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
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<td>3-P12</td>
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<td>4</td>
<td>4</td>
<td>2</td>
<td>1</td>
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<td>0</td>
</tr>
<tr>
<td>3-P13</td>
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<td>4</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3-P14</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3-P15</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3-P16</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
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<td>3-P17</td>
<td>4</td>
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<td>4</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3-P18</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3-P19</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3-P20</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
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</tr>
<tr>
<td>3-P21</td>
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</tr>
<tr>
<td>3-P22</td>
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</tr>
<tr>
<td>3-P23</td>
<td>4</td>
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<td>2</td>
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<tr>
<td>3-P24</td>
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</tr>
<tr>
<td>3-P25</td>
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<td>4</td>
<td>2</td>
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</tr>
</tbody>
</table>
Table 51: Fourth Year Coding for Guttman Scalogram Analysis

<table>
<thead>
<tr>
<th>Participant</th>
<th>Level 0</th>
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<th>Level 2</th>
<th>Level 3</th>
<th>Level 4</th>
<th>Level 5</th>
<th>Level 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-P1</td>
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<td>4</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4-P2</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4-P3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4-P4</td>
<td>4</td>
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<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4-P5</td>
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<td>2</td>
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<td>1</td>
</tr>
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<td>0</td>
</tr>
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<td>2</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
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<td>4-P8</td>
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</tr>
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</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>4-P20</td>
<td>4</td>
<td>4</td>
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<td>2</td>
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<td>2</td>
<td>3</td>
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<td>0</td>
</tr>
<tr>
<td>4-P24</td>
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<td>0</td>
</tr>
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<td>0</td>
</tr>
</tbody>
</table>
Table 52: Guttman Scalogram Analysis: Errors of Levels 0-6

<table>
<thead>
<tr>
<th>Participant</th>
<th>4-P[x] Errors (0-6)</th>
<th>3-P[x] Errors (0-6)</th>
<th>Total Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>x=1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>x=2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>x=3</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>x=4</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>x=5</td>
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<td>0</td>
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</tr>
<tr>
<td>x=6</td>
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<td>0</td>
<td>2</td>
</tr>
<tr>
<td>x=7</td>
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<td>1</td>
</tr>
<tr>
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<td>0</td>
<td>1</td>
</tr>
<tr>
<td>x=9</td>
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<td>1</td>
</tr>
<tr>
<td>x=10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>x=11</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>x=12</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>x=13</td>
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<td>28</td>
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Table 53: Guttman Scalogram Analysis: Errors of Levels 0-5

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<th>3-P[x] Errors (0-5)</th>
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<td>Total</td>
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<td>21</td>
</tr>
</tbody>
</table>
Table 54: Guttman Scalogram Analysis: Errors of Levels 0-4

| Participant | 4-P|x| Errors (0-4) | 3-P|x| Errors (0-4) | Total Errors |
|-------------|---------------------------------|---------------------------------|-------------|
| x=1         | 0                               | 0                               | 0           |
| x=2         | 0                               | 0                               | 0           |
| x=3         | 0                               | 0                               | 0           |
| x=4         | 1                               | 0                               | 1           |
| x=5         | 0                               | 0                               | 0           |
| x=6         | 1                               | 0                               | 1           |
| x=7         | 0                               | 0                               | 0           |
| x=8         | 0                               | 0                               | 0           |
| x=9         | 0                               | 0                               | 0           |
| x=10        | 0                               | 0                               | 0           |
| x=11        | 0                               | 0                               | 0           |
| x=12        | 0                               | 0                               | 0           |
| x=13        | 1                               | 0                               | 1           |
| x=14        | 0                               | 0                               | 0           |
| x=15        | 0                               | 0                               | 0           |
| x=16        | 0                               | 1                               | 1           |
| x=17        | 0                               | 0                               | 0           |
| x=18        | 1                               | 0                               | 1           |
| x=19        | 0                               | 0                               | 0           |
| x=20        | 0                               | 0                               | 0           |
| x=21        | 1                               | 0                               | 1           |
| x=22        | 0                               | 0                               | 0           |
| x=23        | 1                               | 0                               | 1           |
| x=24        | 0                               | 0                               | 0           |
| x=25        | 0                               | 0                               | 0           |
| Total       |                                  |                                  | 7           |
Table 55: Guttman Scalogram Analysis: Errors of Levels 0-3

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</table>
Table 56: Guttman Scalogram Analysis: Errors of Levels 0-2

| Participant | 4-P|x| Errors (0-2) | 3-P|x| Errors (0-2) | Total Errors |
|-------------|-----------------|-----------------|--------------|
| x=1         | 0               | 0               | 0            |
| x=2         | 0               | 0               | 0            |
| x=3         | 0               | 0               | 0            |
| x=4         | 0               | 0               | 0            |
| x=5         | 0               | 0               | 0            |
| x=6         | 0               | 0               | 0            |
| x=7         | 0               | 0               | 0            |
| x=8         | 0               | 0               | 0            |
| x=9         | 0               | 0               | 0            |
| x=10        | 0               | 0               | 0            |
| x=11        | 0               | 0               | 0            |
| x=12        | 0               | 0               | 0            |
| x=13        | 0               | 0               | 0            |
| x=14        | 0               | 0               | 0            |
| x=15        | 0               | 0               | 0            |
| x=16        | 0               | 0               | 0            |
| x=17        | 0               | 0               | 0            |
| x=18        | 0               | 0               | 0            |
| x=19        | 0               | 0               | 0            |
| x=20        | 0               | 0               | 0            |
| x=21        | 0               | 0               | 0            |
| x=22        | 0               | 0               | 0            |
| x=23        | 0               | 0               | 0            |
| x=24        | 0               | 0               | 0            |
| x=25        | 0               | 0               | 0            |
| Total       | 0               | 0               | 0            |
5.9.3 Results

The $Rep$ calculation for errors occurring for levels 0-6, 0-5 etc. was

$$Rep = 1 - \frac{\text{total number of errors}}{\text{number of levels} \times \text{number of participants}}$$

- $Rep_{(0-6)} = 1 - \frac{28}{7 \times 50} = 0.92$
- $Rep_{(0-5)} = 1 - \frac{21}{6 \times 50} = 0.93$
- $Rep_{(0-4)} = 1 - \frac{7}{5 \times 50} = 0.972$
- $Rep_{(0-3)} = 1 - \frac{3}{4 \times 50} = 0.985$
- $Rep_{(0-2)} = 1 - \frac{0}{3 \times 50} = 1.00$

The findings demonstrate that the $Rep$’s for each set of levels are in excess of the 0.9 standard figure. This infers that the model is hierarchical in nature and hence, the assessment for each level reflects this structure.

Green (1956) provides a formula for the approximation of the standard error of the coefficient of reproducibility:

$$SE = \sqrt{\frac{CR(1 - CR)}{NK}}$$

where $CR$ is the $Rep$ obtained, $N$ is the number of individuals, and $K$ is the number of items (in this case the number of levels).

- $SE_{0-6} = \sqrt{\frac{0.92(1 - 0.92)}{50(7)}} \approx 0.0145$
- $SE_{0-5} = \sqrt{\frac{0.93(1 - 0.93)}{50(6)}} \approx 0.0147$
- $SE_{0-4} = \sqrt{\frac{0.972(1 - 0.972)}{50(5)}} \approx 0.0104$
- $SE_{0-3} = \sqrt{\frac{0.985(1 - 0.985)}{50(4)}} \approx 0.0086$

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Therefore the Rep’s for each set of levels are:

\[ Rep_{(0-6)} = 0.92 \pm 0.0145 \]
\[ Rep_{(0-5)} = 0.93 \pm 0.0147 \]
\[ Rep_{(0-4)} = 0.972 \pm 0.0104 \]
\[ Rep_{(0-3)} = 0.985 \pm 0.0086 \]
\[ Rep_{(0-2)} = 1.00 \]

Therefore the very least possible value for the Rep’s found was 0.9055 which is above the 0.9 standard figure, indicating a significant level of reproducibility.

5.10 Conclusion

This chapter has demonstrated the findings from the pre-test for those that completed it (third and fourth year pre-service teachers). The analysis outlined the percentage of the groups who were at each level of the purpose-built model for the effective teaching of trigonometry. It also provided an extended analysis on the percentage of participants with various degrees of acquisition of each level. It was shown that the sample of pre-service teachers at the University of Limerick was not at a level required for the teaching of trigonometry at second-level, with a major lack of understanding being demonstrated at levels 3 and 4 of the purpose-built model. The chapter also demonstrated the hierarchial structure of the purpose-built model through the use of Guttman Scalogram Analysis.

The findings and results that arose from the pre-test formed the basis for the teaching intervention that was administered. The next chapter discusses the development of the teaching intervention, the design of resources for the intervention, and the administration of the intervention.
Chapter 6

Intervention Development and Administration

6.1 Introduction

Begeny et al. (2012, p.215; cited in Johnson and Street, 2013) state that teaching interventions are “intensive, supplemental curricula that teach the struggling learner what a good core curriculum fails to teach him or her”. Johnson and Street (2013) expand on this statement by noting that just as a curriculum must provide evidence that it covers the entire scope of an area of study, so must a teaching intervention. This is achieved by the provision of evidence of successful and timely remedies for deficits in knowledge.

The sampling frame for the teaching intervention was composed of the third year pre-service teachers from the pre-test phase (who at the time of the intervention were fourth (final) year students). A sample of individuals at levels 2 and 3 were selected for the intervention (as these comprised the majority (96%) of this group). The aim was to raise the level of those at level 2 to a level of 3 or 4, and to raise those at level 3 to a complete acquisition of level 4.

In order to achieve this, an intervention first took place for the individuals who did not demonstrate a complete acquisition of level 2. These individuals were given the necessary instruction to move them up to a complete acquisition of level 2 as a means to begin learning at level 3.

The next intervention took place a number of weeks later with the sample
of individuals who did not demonstrate a complete acquisition of level 3 and who the researcher wished to transition to a complete acquisition of level 4. The previous group (level 2 participants) were also invited to attend this intervention if they wished to do so. This was done in order to attempt moving them from their initial level of 2 up to a complete acquisition of level 4.

6.2 Aims of the Intervention

It was beneficial before designing the intervention to outline the main aims of what the intervention was to achieve. This provided valuable direction for the intervention and kept the research focused, thus eliminating the possibility of straying from the objective. As already mentioned, the primary aim of the intervention was to raise the conceptual understanding of the participants in the research by moving them up through the levels of the model of understanding of trigonometry. However, there were other aims for the intervention, developed with the audience (pre-service teachers) in mind. Research states that it is always important to keep your audience in mind when designing a teaching plan (Seldin, Miller and Seldin, 2010; Stanford University, 2013). Many of these auxiliary aims were developed with consideration of the models of teacher knowledge (section 2.7). For example, O’Meara (2011) stated that a teacher should have a knowledge of real-life applications and also a knowledge of connections as part of their subject matter knowledge.

The aims of this intervention were:

- To raise the levels of conceptual understanding of the participants with respect to the purpose-built teaching model.
- To develop the visualisation skills of the participants.
- To expose the pre-service participants to the five teaching phases of the purpose-built teaching model that they are experiencing.
- To assist the participants in linking the content to real-life scenarios.
- To expand the participants’ knowledge at the mathematical horizon (Ball et al., 2008) through the use of anecdotes and historical points.
- To provide the participants with resources that may be beneficial to their own future teaching.
To further analyse the purpose-built teaching model developed and the teaching intervention employed.

### 6.3 Background to the Intervention

Consideration was paid to the aims of the teaching intervention, which were based on research and the results from the pre-test phase, when devising a strategy for the intervention. Research from many disciplines states that answers are needed to the questions of why an instructor should teach certain content, what content he/she should teach, and how he/she should teach it (Goodyear and Allchin, 1998; Salemi, 2005; National Council of Teachers of English (NCTE), 2007).

With this sample of pre-service teachers the question of why teach the content was answered through research conducted into elements of teacher knowledge (section 2.7). Through this research it was found that teacher subject matter knowledge is a critical element of a teacher’s knowledge base. In the Irish context it was found that secondary pupils, upon leaving secondary school, do not have a good understanding of trigonometry (State Examinations Commission, 2000; 2001; 2005). Drawing upon the work of Furinghetti (2000) and Weber (2005) this may be the case at third level as the misunderstandings may not be rectified. The pre-test phase confirmed this to be the case (section 5.2), though it must be noted that the third level education of the participants had not yet ended. However, they were not going to study any further trigonometry before graduation.

The question of what to teach was answered by consulting the results from the pre-test and by reflecting on the genetic decomposition of the model (Tables 28 and 29). The concepts for each individual level of the model (0-6) were outlined as the ‘objects’ in the genetic decomposition. In order to gain an understanding of these objects, individuals will be led through the ‘processes’ in the genetic decomposition.

For example an individual at level 3 of the model (which is the intervention level used for the majority of the intervention sample (76%)) will have to learn and understand the concepts of radians, quadrants, the unit circle, and the sine and cosine rules. These will be achieved through the acquisition of the processes listed in Table 28.
The question of how to teach the concepts and content would dictate how the teaching intervention would look and be carried out. Due to the strength of focus on the van Hiele model the author needed to use a framework that was valid with respect to this model. Upon reflection of the research conducted into the van Hiele model it was concluded that the most valid teaching framework used in order to promote understanding, and movement up through the levels was that proposed by van Hiele (1984b) (section 2.15.4):

- Phase 1: Inquiry (van Hiele, 1984b)
- Phase 2: Directed Orientation (van Hiele, 1984b)
- Phase 3: Explicitation (van Hiele, 1984b)
- Phase 4: Free Orientation (van Hiele, 1984b)
- Phase 5: Integration (van Hiele, 1984b)

These five phases were used in the intervention classes with the goal of improving the intervention sample’s understanding of trigonometric concepts at their respective levels and, ultimately, raise the level of each individual with respect to the model.

6.4 Type of Intervention

The complex nature of this research, including the level structure of the model, as well as the assessment of individuals, was all under consideration when selecting the type of intervention that would be implemented. Certain types of instructional interventions were considered, for example behavioural interventions and process-based interventions (Sandoval, 1993; O’Meara, 2011). The author decided, due to the nature of the research up to this point, to plan the intervention using a model of behaviour change that most suited the research, namely a stage-based model of behaviour change (Bridle et al., 2005), and to deliver the intervention using the phases of learning provided by van Hiele (1984b) whilst bearing in mind general pedagogical principles (section 6.6.1).

The model that was selected in planning the intervention was the stage-based model of behaviour change (Bridle et al., 2005). Though this type of intervention is used mainly in health research, the links that exist between the characteristics of stage-based interventions and the intervention that
was to be conducted in this research were strong. Bridle et al. (2005, p.284) state that stage-based interventions are “most effective when they are tailored to an individual’s current stage”, similar to the placing of individuals on the levels of the purpose-built teaching model in this research. It is also noted that stage-based interventions are more effective than those that are delivered to large groups of people under a ‘one size fits all’ mentality (Lichtenstein and Glasgow, 1992; cited in Bridle et al., 2005). Taking this stage-based model as a basis for planning the intervention conformed to the research undertaken to this point. The intervention used the results from the pre-test so it could tailor the intervention to those at different levels, which according to the ideas of stage-based models should increase effectiveness. Doing this also meant that not every participant would be attending every class, eliminating the possibility of adopting a one size fits all mentality in the design of the intervention.

6.5 The Intervention

The intervention phase of this research took place over the course of five one-hour classes at the beginning of 2014 and was taught by the researcher himself who, as noted, is a fully qualified second-level mathematics teacher. Each class took place in the University of Limerick. A resource pack (section 6.7; Appendix Q) was given to each participant at the start of the intervention. These classes were followed by a post-test which was the same assessment tool from the pre-test, as well as a focus group. The classes provided as part of this intervention are detailed in the slides used in each class (Appendix P). The five phases of learning (section 6.3) were followed in each class. The researcher also paid close attention to the pedagogical considerations outlined in section 6.6.1 during the course of the classes. The tasks selected were taken from the resource pack and were in line with points made in section 6.6.2.

6.6 Development of Intervention

Though the stages of learning have been considered and the type of intervention selected, pedagogical principles which are conducive to learning concepts must also be implemented in the intervention classes. The next section outlines the pedagogical considerations that were used for the intervention.
Along with this, one of the most important contributors to learning is the types of tasks participants will be participating in. The guiding principles for the development of these tasks are outlined in section 6.6.3.

### 6.6.1 Pedagogical Considerations

It must first be said that the researcher is a fully trained second-level mathematics teacher. He has a sound knowledge of effective pedagogy and experience teaching at second-level and higher education.

Many interventions have taken place in research regarding the van Hiele model. Some of these interventions were undertaken with school pupils (Parsons, Stack and Breen, 1998; Mistretta, 2000; Clements, Battista, and Sarama, 2001) and others with teachers (Fuys, Geddes and Tischler, 1988; Swafford, Jones and Thornton, 1997; Sharp, 2001). The interventions with students were all based on improving the van Hiele level of pupils through various teaching methodologies (ICT, logo activities etc.). The interventions that have taken place with teachers have predominantly been concerned with how to evaluate pupils’ van Hiele levels such as the studies of Fuys et al. (1988) and Sharp (2001).

The intervention for this research was designed to incorporate elements that would aid in developing visualisation skills and problem-solving skills (as these were found to be problematic in the pre-test phase). The study of Swafford et al. (1997) concentrated on teachers’ content knowledge of geometry and also on teachers’ understanding of the van Hiele model. This study ultimately resulted in a higher level of understanding of geometry by the teachers, as well as changes in instructional practices to reflect the van Hiele model. Swafford et al. (1997) used an instructional approach of “teaching via problem solving” for their content knowledge intervention with a sample of teachers. As this study achieved goals that mirror the aims of the intervention for this research, an adaptation of the methods used by Swafford et al. (1997) was selected. Taplin (2013) provides seven specific characteristics of a problem-solving teaching approach:

- Student/student and student/teacher interactions (Van Zoest et al., 1994);
- Mathematical dialogue between students (Van Zoest et al., 1994);
- Initiation of problem by teacher (background information etc.) and
development of strategies, solutions etc. by students (Cobb et al., 1991);

- Teachers accepting both right and wrong answers in a non-evaluative way (Cobb et al., 1991);

- Teachers play the role of facilitator (guiding, probing etc.) (Lester et al., 1994);

- Teachers step in and intervene when necessary (Lester et al., 1994);

- Students encouraged to generalise about concepts and rules (Evan and Lappin, 1994).

These principles guided pedagogy within the intervention. The structure of lessons, as mentioned in section 6.3 (inquiry, directed orientation etc.) dictates the progression of the lessons and tasks, while these principles of teaching via problem-solving encompass the interactions, dialogue and facilitation of learning that take place in the lessons.

The structure and pedagogical principles have been outlined, however, another important consideration was the types of tasks that the intervention participants would be taking part in.

6.6.2 Types of Tasks

The types of tasks that are used in an educational setting are critical to the learning that takes place. Tasks provide meaning on what mathematics is about and what the learner needs to know and be able to do in order to do mathematics (NCTM, 1991). Leading on from this, Henningsen and Stein (1997, p.525) note that “the nature of tasks can potentially influence and structure the way students think and can serve to limit or to broaden their views of the subject matter with which they are engaged”. Hence, a poorly designed task may lead to a lack of mathematical structure within the mind of the individual who is learning the content. For a teacher, or in the case of this research a pre-service teacher, it is very important that they have a strong knowledge of mathematical content, as seen in the models of teacher knowledge. The tasks developed for this intervention facilitated a sound conceptual structure of the content that connected concepts with other areas of mathematics.
The following three factors proved most powerful in maintaining a high-level of engagement with tasks (Henningsen and Stein, 1997):

- Tasks must build on pupils’ prior knowledge;
- Appropriate amount of time given for tasks;
- Sustained pressure on pupils to provide explanation and meaning to their work.

Drawing from this, consideration was placed on each of these three factors when designing tasks. The researcher ensured that all tasks built on prior knowledge and were given the appropriate amount of time. Throughout the tasks, explanations from the participants were required for everything they did (e.g. “why have you found the length of that side?” “what is the purpose of finding that angle?”). This ensured that the researcher, as tutor, was aware of the conceptual understanding/misunderstanding that was occurring in the class, and could adapt the intervention accordingly.

For the design of the tasks themselves, the researcher wished to have them grounded in real-life contexts. This means that each task was applicable to a real-life scenario. This draws from the works of Ball et al. (1998) and O’Meara (2011). As the audience was pre-service teachers, it was important that they understand and can apply their knowledge to real-life situations. Exposure to this aspect of learning was necessary. Therefore, though the intervention was contributing to the participants’ subject matter knowledge and conceptual understanding, it also contributed to the participants’ ability to relate mathematical material to real-life concepts, as advocated in the models of teacher knowledge. A model for creating tasks that contributes to such aims is provided by Lesh et al. (2000) and the tasks are known as Model-Eliciting Activities (MEAs).

### 6.6.3 Designing Model-Eliciting Activities (MEAs)

MEAs are defined to be thought revealing activities that focus on the development of constructs (Lesh et al., 2000). Lesh et al. (2000) state that these activities encourage and lead to deeper and higher levels of understanding of mathematical content. As we will see the activities are also situated in a real-life context which was one of the auxiliary aims of this PhD. intervention. Moore and Diefes-Dux (2004) in their research on using MEAs with undergraduate engineering students also state that these tasks aid with the
organisation of mathematical understanding and problem solving.

There are six principles for designing an MEA (Lesh et al., 2000; Lesh et al., 2003; Moore and Diefes-Dux, 2004). The six principles are:

Table 57: Principles for Guiding MEA Development (Moore and Diefes-Dux, 2004, p.10)

<table>
<thead>
<tr>
<th>Principle</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model Construction</td>
<td>Ensures the activity requires the construction of an explicit description, explanation, or procedure for a mathematically significant situation.</td>
</tr>
<tr>
<td>Reality</td>
<td>Requires the activity to be posed in a realistic engineering context and to be designed so that the students can interpret the activity meaningfully from their different levels of mathematical ability and general knowledge.</td>
</tr>
<tr>
<td>Self-Assessment</td>
<td>Ensures that the activity contains criteria the students can identify and use to test and revise their current ways of thinking.</td>
</tr>
<tr>
<td>Model-Documentation</td>
<td>Ensures that the students are required to create some form of documentation that will reveal explicitly how they are thinking about the problem situation.</td>
</tr>
<tr>
<td>Construct Share-Ability and Re-Usability</td>
<td>Requires students to produce solutions that are shareable with others and modifiable for other engineering situations.</td>
</tr>
<tr>
<td>Effective Prototype</td>
<td>Ensures that the model produced will be as simple as possible yet still mathematically significant for engineering purposes.</td>
</tr>
</tbody>
</table>

Here the word engineering is used as in this study of Moore and Diefes-Dux (2004) the focus was using MEAs with engineering students.

Every task that was included in the resource pack developed for the intervention was included with consideration of MEA’s. Ensuring that tasks were not procedural was key in the development and selection of them. For example:

233
3.1 A Week on Earth

A line from the Sun to the Earth sweeps out an angle of how many radians in 1 week? Assume the Earth’s orbit is circular and there are 52 weeks in a year. Express the answer in terms of $\pi$ and as a decimal correct to two decimal places.

Figure 31: Example of a Task for Concept at Level 3 (p.174 of resource pack)

This task adheres to the principles of a MEA as discussed in Table 58.

Table 58: MEA Example - A Week on Earth

<table>
<thead>
<tr>
<th>Principle</th>
<th>Problem Characteristic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model Construction</td>
<td>Requires students to construct relationship between weeks in a year and radians swept by a line from the Earth to the Sun.</td>
</tr>
<tr>
<td>Reality</td>
<td>Realistic rotation of Earth around the Sun. Appropriate for different abilities.</td>
</tr>
<tr>
<td>Self-Assessment</td>
<td>Requires the use of radians and hence self-assessment of the radian concept.</td>
</tr>
<tr>
<td>Model-Documentation</td>
<td>Students need to document their thinking in attempting to answer problem. Creation of relationship between weeks and radians is not created in one-line.</td>
</tr>
<tr>
<td>Construct Share-Ability and Re-Usability</td>
<td>Solution can be shared. Meaningful discussion is available.</td>
</tr>
<tr>
<td>Effective Prototype</td>
<td>Mathematically significant and simple solution that can be understood.</td>
</tr>
</tbody>
</table>
6.7 Resource Pack for Participants

A resource pack (Appendix Q) was designed and constructed in the Summer, Autumn and Winter of 2013. The resource pack consists of 4 books:

- **Book 1: A Purpose-Built Model for the Effective Teaching of Trigonometry** - This booklet is composed of certain extracts from the PhD. thesis of the author with the aim of providing information on the purpose-built teaching model for trigonometry. The booklet briefly explains the van Hiele model of geometric understanding and how it was adapted to be specific to the understanding of trigonometry. The levels within the model are elaborated upon and explained. (e.g. Figure 32).

- **Book 2: FAQs in Trigonometry: Concepts Decoded** - This booklet aims to alleviate some of the issues that may occur in the teaching and understanding of trigonometric concepts. Each of the concepts explained in this booklet correspond with the level of understanding they are listed under in the purpose-built teaching model. (e.g. Figures 33 and 34).

- **Book 3: Real-Life Examples of Trigonometry** - This provides various real-life examples which can be used in the teaching of trigonometry. These examples should act as a stimulus and reference for the creation of further examples. (e.g. Figure 35).

- **Book 4: Trigonometry Workbook** - This booklet is composed of various real-life problems relevant to each of the levels in the purpose-built teaching model. (e.g. Figure 36).

Each of the books includes content from the levels of interest for the interventions (levels 2 to 4) as well as content from preceding levels (levels 0 and 1). These are included due to the hierarchical nature of the model. Material and resources for levels 5 and 6 of the model have been excluded as these levels were outside the area of research for the intervention.

Using this layout the researcher ensured that the aims of the intervention were acknowledged in their entirety. The following table (Table 59) illustrates how each booklet relates to each individual intervention aim:
Table 59: How the Resource Booklets Adhere to the Aims of the Intervention

<table>
<thead>
<tr>
<th></th>
<th>Raising Conceptual Under-</th>
<th>Visualisation / Problem Solving</th>
<th>Teaching Stages</th>
<th>Linking to Real-Life</th>
<th>Mathematical Horizon</th>
<th>Future Teaching</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Book 1: Model</strong></td>
<td></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td><strong>Book 2: FAQs</strong></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td><strong>Book 3: Real-Life Examples</strong></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td><strong>Book 4: Workbook</strong></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

Book 1 provided the participants with the teaching model that they were learning from. It therefore does not play any part in the raising of conceptual understanding of the participants, development of visualisation and problem-solving skills, linking content to real-life scenarios, or knowledge at the mathematical horizon. Its sole purpose is to introduce the participants to the levels within the model in order to provide clarity and eliminate any confusion that could exist in the layout of the proceeding booklets that state the intended level of concepts, problems etc. It is envisaged that, through knowledge of the model before the experience of learning through it, that this booklet will contribute to the participants’ future teaching.
they have met already can be produced by the means of thinking and
explaining (almost like a teacher would do).

- Expert in Trigonometry

Level 6: Trigonometry Seen in the Abstract

Trigonometry can be applied, for example, to exponential functions and
complex numbers (for example in Euler's formula \( e^{i\theta} = \cos \theta + i\sin \theta \)).

The adaptation of the van Hiele model is evident throughout this model (Figure 2.1). Level 0 of the model of the progression of trigonometric thought is directly linked to the visualisation level of van Hiele's work as it is solely concerned with recognition of the shape of triangles. Level 1 of the van Hiele model (analysis) is linked to level 1 of this model as some properties of triangles are beginning to be noticed and understood. Level 2 in the model of progression of trigonometric thought is a further building upon level 1 in the van Hiele model. This level is included due to the critical nature of Pythagoras' work on the understanding of trigonometry, and therefore it gets an individual section.

Levels 3 and 4 of this model relate to level 2 of the van Hiele model (abstraction) as higher levels of understanding of concepts, and the interrelationships between concepts, begin to be formulated in level 3 and are built upon further in level 4. Level 3 of the van Hiele model (abstraction) links to level 5 of this model as individuals do not need to rely on memorisation for constructing proofs as they have a sound and accurate grasp of the concepts. The final levels of each of the models are linked by the abstract nature in which the topic is viewed in.

---

Figure 32: Excerpt from Book 1 (page 18 of resource pack)
Book 2 illustrates many of the concepts that relate to Junior and Senior Certificate syllabi. This booklet does not link to real-life scenarios as the author found it important that students gain a thorough understanding of the concepts before attempting to view them in real-life scenarios. This links to the idea of mismatch as noted by Crowley (1987) in relation to van Hiele instruction (section 2.15.3). Through a strong conceptual understanding, which includes visualisation of the concepts, the participants will understand the links with the real-life examples provided in book 3 (i.e. learning is generative (Carpenter and Lehrer, 1999)).
3.3 CAST Explained

The following diagram of CAST notation is used in many secondary school textbooks as an aid to remembering what trigonometric functions (sine, cosine and tangent) are positive in each quadrant.

![CAST notation diagram]

Figure 3.10: CAST notation

However, this is not a mathematical explanation!!! It is just a way of remembering a few facts (but some people take it as a justification). But what is the mathematical way to explain these facts? Let's start first with cosine!

![Cosine graph]

Figure 3.33: Excerpt from Book 2 (Level 3 - page 56 of resource pack)
4.1.2 Cosine function

The cosine function can be graphed again using the information from the unit circle. Some of the facts that we know are:

\[
\begin{align*}
\cos 0 &= 1 \\
\cos \frac{\pi}{2} &= 0 \\
\cos \pi &= -1 \\
\cos \frac{3\pi}{2} &= 0 \\
\cos 2\pi &= 1
\end{align*}
\]

ENSURE that you know why each of these is true!!!!

We can plot each of the points \((0, 1), (\frac{\pi}{2}, 0), (\pi, -1), (\frac{3\pi}{2}, 0),\) and \((2\pi, 0)\) and simply join them up with a curve.

Of course this still is not technically correct, but only true for the interval from 0 to 2\(\pi\). Cosine is again periodic (which simply means recurring!)

EXPLAIN why cosine repeats after 2\(\pi\), 4\(\pi\), 6\(\pi\) etc.
Book 3 builds upon the concepts explained in book 2 by linking those concepts to concrete real-life examples and natural phenomena. The author does note at this point that the example of music provided in this booklet may appear to be quite contrived, however, it is included in the booklet to show that trigonometric functions can be applied to music, which most likely is something of interest to all of the participants.
4.1 Introduction

This level is about the understanding of trigonometric functions (primarily graphing and analysis) and solving trigonometric equations. The author in this section proposes various topics, such as music, as a means to understand trigonometric functions and also solving trigonometric equations. The content of this section can be followed if desired, or indeed adapted.

4.2 Trigonometric functions

In the booklet 'FAQs in Trigonometry: Concepts Decoded' we see how to graph basic sine, cosine and tangent functions. However, though the concepts and graphs may seem a little abstract they do have real world connections. Consider the following scenario:

Throughout the day, the depth of water at the end of a clock in Bar Harbor, Maine (USA) varies with the tides. The table shows the depths (in feet) at various times during the morning. (Source: Nautical Software, Inc.)

<table>
<thead>
<tr>
<th>Time, t</th>
<th>Depth, y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Midnight(0 A.M.)</td>
<td>2.91</td>
</tr>
<tr>
<td>2 A.M.</td>
<td>8.51</td>
</tr>
<tr>
<td>4 A.M.</td>
<td>11.3</td>
</tr>
<tr>
<td>6 A.M.</td>
<td>8.49</td>
</tr>
<tr>
<td>8 A.M.</td>
<td>2.88</td>
</tr>
<tr>
<td>10 A.M</td>
<td>0.1</td>
</tr>
<tr>
<td>Noon(12 P.M.)</td>
<td>2.91</td>
</tr>
</tbody>
</table>

A meteorologist hires you to create a graph for the data and also a trigonometric function (in the form $y = a \cos(bt - c) + d$, or $y = a \sin(bt - c) + d$) to model the data so that he can predict the depth of water at any time.

First you can graph the data from the table.

We can find where our principal axis ($d$) is quite easily. With a basic cosine function, the principal axis is the $x$-axis and the graph oscillates above and below this axis the same amount. With this graph the principal axis must be halfway between the highest and lowest depths.

$$d = \frac{1}{2} \left( \text{max depth} + \text{min depth} \right)$$

$$d = \frac{1}{2} (11.3 + 0.1)$$

$$d = 5.7$$
Book 4 allows participants to engage with the concepts the were explained in the preceding booklets and solve contextualised problems which are relevant to these concepts.
3.5 The Royal Gorge

The highest bridge in the world is the bridge over the Royal Gorge of the Arkansas River in Colorado. Sightings to the same point at water level under the bridge are taken from each side of the bridge which is 268m long. How high (h) is the bridge (correct to 2 decimal places)?

3.6 Mount Everest

The highest mountain peak in the world is Mt. Everest, located in the Himalayas. The height of this enormous mountain was determined in 1856 by surveyors using trigonometry long before the mountain was first climbed in 1953. This difficult measurement had to be done from a great distance. At an altitude of 4433m on a different mountain, the straight line distance to the peak of Mt. Everest is 43452m and its angle of elevation is 5.52 degrees. Approximate the height, in metres, of Mt. Everest (correct to 1 decimal place).

3.7 Golf Accuracy

A golfer hits a drive 237.74m on a hole that is 365.76m long. The shot is 15° off target.

(a) What is the distance x from the golfer’s ball to the hole (correct to 2 decimal places)?
(b) Assume the golfer is able to hit the ball precisely the distance found in part (a). What is the maximum angle θ by which the ball can be off target in order to land no more than 10m from the hole (correct to 2 decimal places)?

Figure 36: Excerpt from Book 4 (Level 3 - page 175 of resource pack)
6.8 Conclusion

This chapter discussed and analysed the intervention phase of the research. Each element in the design of the intervention was demonstrated in detail as well as the implementation of the intervention itself. Though certain notes were made in this chapter on the results from the intervention, nothing substantial was discussed or analysed. The next chapter focuses on the findings from the post-test through the analysis that was conducted. After the findings have been demonstrated and analysed the contributions of this thesis will be synthesised and areas for possible future work and research can be identified in the final chapter.
Chapter 7

Discussion of Results from the Post-Intervention Test

7.1 Introduction

The previous chapter discussed the intervention that the participants in the research took part in. Each element of the intervention (design of intervention, resources developed, and implementation) was elaborated upon and clarity was provided on each element that was designed. This chapter reports on the findings that arose from the intervention that was developed and administered. Pre and post-test findings are compared and an extended discussion of exploratory findings is made. This chapter also reports on the opinions of those that participated in the intervention and highlights what the participants found beneficial and effective. The chapter concludes with an evaluation of the intervention in accordance with Shapiro’s framework. The proof-of-concept is demonstrated as the purpose-built teaching model performed to a high standard in-situ. The proof-of-concept approach is concluded upon in the final chapter.

Table 60 demonstrates the results of the intervention group from the pre-test (third year group at the time of pre-test, fourth year group at the time of the intervention). For a more comprehensive analysis and discussion of the results from the pre-test phase refer to chapter 5.
Table 60: % of Third Year Participants in Each Degree of Acquisition of Each Level (Pre-test)

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 6</td>
<td>0%</td>
<td>4%</td>
<td>4%</td>
<td>4%</td>
<td>88%</td>
</tr>
<tr>
<td>Level 5</td>
<td>0%</td>
<td>0%</td>
<td>20%</td>
<td>16%</td>
<td>64%</td>
</tr>
<tr>
<td>Level 4</td>
<td>0%</td>
<td>4%</td>
<td>4%</td>
<td>52%</td>
<td>40%</td>
</tr>
<tr>
<td>Level 3</td>
<td>4%</td>
<td>12%</td>
<td>24%</td>
<td>52%</td>
<td>8%</td>
</tr>
<tr>
<td>Level 2</td>
<td>80%</td>
<td>0%</td>
<td>16%</td>
<td>0%</td>
<td>4%</td>
</tr>
<tr>
<td>Level 1</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Level 0</td>
<td>96%</td>
<td>4%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

The data that is reported on in this section is based solely on the results of the 19 participants who completed all phases of the research (pre-test, intervention, and post-test). The remaining 6 individuals who completed the pre-test either had no intervention (3 of the 6; 3-P20, 3-P24 & 3-P25), no intervention and post test (2 of the 6; 3-P5 & 3-P16), or had only half of an intervention (1 of the 6; 3-P1). Along with these 6 individuals, one other individual completed the intervention and post-test, but did not complete the pre-test (this person is coded as ‘D’ in post-test data - Appendix R). The individual with only half an intervention, as well as the final individual who had no pre-test, were not included. The percentage of the 19 third year participants who demonstrated different degrees of acquisition of each level in the pre-test phase, and who completed all phases of the intervention, is shown in Table 61.
Table 61: % of Intervention Participants in Each Degree of Acquisition of Each Level (Pre-test)

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 6</td>
<td>0%</td>
<td>5.3%</td>
<td>5.3%</td>
<td>0%</td>
<td>89.5%</td>
</tr>
<tr>
<td>Level 5</td>
<td>0%</td>
<td>0%</td>
<td>15.8%</td>
<td>10.5%</td>
<td>73.7%</td>
</tr>
<tr>
<td>Level 4</td>
<td>0%</td>
<td>5.3%</td>
<td>0%</td>
<td>42.1%</td>
<td>52.6%</td>
</tr>
<tr>
<td>Level 3</td>
<td>5.3%</td>
<td>10.5%</td>
<td>21.1%</td>
<td>52.6%</td>
<td>10.5%</td>
</tr>
<tr>
<td>Level 2</td>
<td>78.9%</td>
<td>0%</td>
<td>21.1%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Level 1</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Level 0</td>
<td>94.7%</td>
<td>5.3%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

Summarising the results of the pre-test, the concepts within levels 3 and 4 were not understood well by the group. Understanding of concepts at levels 5 and 6 of the model was almost non-existent as the majority of the group had no acquisition of either level. Concepts at levels 2, 3 and 4 in the genetic decomposition were the concepts of interest to the intervention (for the majority of the participants it was levels 3 and 4 only).

This chapter will present and analyse the results of the post-test, whilst comparing them to the results of the pre-test.
7.2 Results of Post-Test

This analysis will begin, similar to the pre-test analysis (chapter 5), by examining the percentage of the intervention participants (third year pre-test group) who had a complete acquisition of each level after the intervention took place. Figure 37 illustrates these percentages.

![Diagram of percentage of intervention participants with complete acquisition of each level](Image)

Figure 37: % of Intervention Participants With a Complete Acquisition of Each Individual Level

It can be seen from this figure that every intervention participant had a complete acquisition of levels 0 and 1 of the model. However, a decrease in acquisition can be seen at level 2.

7.2.1 Level 2

89.5% of intervention participants demonstrated a complete acquisition of level 2, with the remaining 10.5% demonstrating an intermediate acquisition. These values are better than those in the pre-test (see Table 63) (See Appendix S for comparison of pre and post-test degrees of acquisition).
Table 62: Post-Test Degrees of Acquisition of Level 2

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 2</td>
<td>89.5%</td>
<td>0%</td>
<td>10.5%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

One of the participants (coded as 3-P4 in pre and post-test data (Appendices N and R)) who demonstrated an intermediate acquisition of level 2 in the post-test had been shown to have previously had a complete acquisition in the pre-test phase. This is the only person that regressed in degree of acquisition at this level. The other participant that demonstrated an intermediate acquisition (3-P23) was one to have been previously tested in the pre-test phase to have an intermediate acquisition. This participant missed one of her intervention classes, which was the level 2 intervention class, and hence did not increase in degree of acquisition of this level. As she attended all other classes she was still included in the data set. She is the only participant included in the data set that missed some part of the intervention.

For each level of the model, the improvement in degrees of acquisition can also be identified (Note that there is a limit of 4 increases in degrees of acquisition due to the degrees being on an interval of none-complete, i.e. the largest possible increase is from no acquisition to complete, an increase of 4 degrees of acquisition). Three intervention participants increased by two degrees of acquisition. All three increased from an intermediate acquisition to a complete acquisition. These three participants were all part of the intervention class for level 2. This gives credence to the statement that the intervention class at level 2 and its use of the van Hiele teaching sequence aids in conceptual understanding with respect to the purpose-built model developed for this research.

The average weighting of responses demonstrates that improvement has occurred amongst the group. The average weighting for the 19 intervention participants responses to level 2 problems in the pre-test was 86.84 (median weighting of 95), while the post-test average was an increased 94.08 (median of 100). This is an increase of approximately 8.3% on the pre-test average weightings. Statistical significance tests were not conducted because the assignment of weightings to the types of answers provided by participants was relatively subjective.
To conclude on performance at level 2, a table can be provided which gives a comprehensive account of the degrees of acquisition that can be attributed to the group after both the pre-test and post-test phases.

Table 63: Pre and Post-Test Degrees of Acquisition of Level 2

<table>
<thead>
<tr>
<th>Test</th>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre</td>
<td>Level 2</td>
<td>78.9%</td>
<td>0%</td>
<td>21.1%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Post</td>
<td>Level 2</td>
<td>89.5%</td>
<td>0%</td>
<td>10.5%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

Clearly, an increase in understanding has occurred amongst the intervention group at level 2. Level 3 of the model is now focused on and the changes, if any, that occurred with respect to understanding at this level are examined.

7.2.2 Level 3

Performance at level 3 by the intervention group culminated in 52.6% demonstrating a complete acquisition of the level and 42.1% a high acquisition. The remaining 5.3% were shown to have an intermediate acquisition.

Table 64: Post-Test Degrees of Acquisition of Level 3

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 3</td>
<td>52.6%</td>
<td>42.1%</td>
<td>5.3%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

The vast majority of the group therefore demonstrated in the post-test to have a high level of understanding as 94.7% were shown to have a complete or high acquisition.

This is the level of the model which had the greatest level of improvement after the intervention. Only 5.3% of this group were shown to have a complete acquisition of this level in the pre-test. This increased to a figure of 52.6%. The higher end of the acquisition spectrum (complete and high) now accounts for 94.7% of the intervention participants, an increase from the initial 15.8% from the pre-test results. Dealing with the bottom two facets of the acquisition spectrum (low and none) it can be seen that 63.1% lay in these two degrees in the pre-test and that this has dramatically been reduced to 0%.
The improvement in degrees of acquisition of individual participants can again be shown (Appendix S). Only 2 of the intervention participants (3-P9 and 3-P21) remained at the same degree of acquisition of level 3 after the intervention. One of these (3-P21) had been pre-tested to have a complete acquisition and therefore could not improve further (because of the limit noted in section 7.2.1). 10.5% of the intervention group increased by one degree of acquisition, with one of these participants (3-P3) being limited to a one degree increase (high acquisition in the pre-test). 47.4% increased by two degrees of acquisition (33.3% of this 47.4% were limited to two degree increases). 31.6% increased by three degrees of acquisition (with 83.3% of this group being limited to three degree increases). Therefore the dominant increase in degrees of acquisition was a two degree increase. The following table outlines the increases in the particular degrees of acquisition that took place:

Table 65: Increases in Degrees of Acquisition at Level 3

<table>
<thead>
<tr>
<th>Degree of acquisition increase</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>None - Complete</td>
<td>0</td>
</tr>
<tr>
<td>Low - Complete</td>
<td>5</td>
</tr>
<tr>
<td>Intermediate - Complete</td>
<td>3</td>
</tr>
<tr>
<td>High - Complete</td>
<td>1</td>
</tr>
<tr>
<td>None - High</td>
<td>1</td>
</tr>
<tr>
<td>Low - High</td>
<td>5</td>
</tr>
<tr>
<td>Intermediate - High</td>
<td>1</td>
</tr>
<tr>
<td>None - Intermediate</td>
<td>1</td>
</tr>
<tr>
<td>Low - Intermediate</td>
<td>0</td>
</tr>
<tr>
<td>None - Low</td>
<td>0</td>
</tr>
</tbody>
</table>

Engagement with the content at level 3 was successfully completed by the participants in the intervention classes (Journal Entries 2 and 3: Appendix C). This appears to have improved their visualisation and understanding of concepts at this level.

The average weighting of the group at level 3 in the pre-test was 39.14 (median of 28.75), while the post-test average was 81.91 (median of 81.25). This is an increase of 109% on the pre-test weighting average.
The summary of degrees of acquisition of level 3 from both pre and post-tests is provided in table 66.

Table 66: Pre and Post-Test Degrees of Acquisition of Level 2

<table>
<thead>
<tr>
<th>Test</th>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre</td>
<td>Level 3</td>
<td>5.3%</td>
<td>10.5%</td>
<td>21.1%</td>
<td>52.6%</td>
<td>10.5%</td>
</tr>
<tr>
<td>Post</td>
<td>Level 3</td>
<td>52.6%</td>
<td>42.1%</td>
<td>5.3%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

The next level that is focused on is level 4 of the purpose-built teaching model.

7.2.3 Level 4

Level 4 was another level of interest in the intervention classes. The post-test results showed that one person had a complete acquisition (this individual (3-P3) had previously been found to have a high acquisition and so the final transition must only have been necessary). 10.5% of intervention participants demonstrated a high acquisition of level 4, 31.6% an intermediate acquisition, 52.6% a low acquisition, and 0% having no acquisition.

Table 67: Post-Test Degrees of Acquisition of Level 4

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 4</td>
<td>5.3%</td>
<td>10.5%</td>
<td>31.6%</td>
<td>52.6%</td>
<td>0%</td>
</tr>
</tbody>
</table>

The improvement at this level was not as significant as the increase at level 3. The findings indicate that the level 4 intervention has enabled one participant (3-P3) to transition from a high to complete acquisition and extinguished the existence of no acquisition amongst the intervention participants. More participants are now located in high and intermediate degrees of acquisition compared to at the time of the pre-test.

Though the results do not appear to be strongly different, some changes have certainly occurred to participants’ understand of concepts. Only one participant (3-P15) remained in the same degree of acquisition (low) as in the pre-test. 78.9% of intervention participants increased by one degree of acquisition and 15.8% increased by two degrees. The specific increases that occurred are again outlined in the following table (Table 68):
Table 68: Increases in Degrees of Acquisition at Level 4

<table>
<thead>
<tr>
<th>Degree of acquisition increase</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>None - Complete</td>
<td>0</td>
</tr>
<tr>
<td>Low - Complete</td>
<td>0</td>
</tr>
<tr>
<td>Intermediate - Complete</td>
<td>0</td>
</tr>
<tr>
<td>High - Complete</td>
<td>1</td>
</tr>
<tr>
<td>None - High</td>
<td>0</td>
</tr>
<tr>
<td>Low - High</td>
<td>2</td>
</tr>
<tr>
<td>Intermediate - High</td>
<td>0</td>
</tr>
<tr>
<td>None - Intermediate</td>
<td>1</td>
</tr>
<tr>
<td>Low - Intermediate</td>
<td>5</td>
</tr>
<tr>
<td>None - Low</td>
<td>9</td>
</tr>
</tbody>
</table>

The majority of increases occurred in either the transition from no acquisition to low acquisition, or from low acquisition to intermediate acquisition (both accounting for approximately 77.8\% of increases). The persistent difficulties at level 4 of the model are discussed in detail in section 7.4.

Average weightings increased at level 4 from 19.04 (median of 10) to 44.3 (median of 10), an increase of 132.7\%. However, the average weighting is still low.

The summary of the degrees of acquisition of participants from pre and post-tests at level 4 is:

Table 69: Pre and Post-Test Degrees of Acquisition of Level 2

<table>
<thead>
<tr>
<th>Test</th>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre</td>
<td>Level 4</td>
<td>0%</td>
<td>5.3%</td>
<td>0%</td>
<td>42.1%</td>
<td>52.6%</td>
</tr>
<tr>
<td>Post</td>
<td>Level 4</td>
<td>5.3%</td>
<td>10.5%</td>
<td>31.6%</td>
<td>52.6%</td>
<td>0%</td>
</tr>
</tbody>
</table>

Though levels 5 and 6 were not dealt with through the teaching intervention, a brief analysis of findings from these two levels is presented in the next two sections.
7.2.4 Level 5

Though level 5 of the model was not a level of interest to the teaching intervention, it can be shown that some changes (though very minor) have occurred to the intervention participants’ degrees of acquisition. As expected, no participant was found to have a complete acquisition or high acquisition of the level after the intervention, due to the concept of proof in trigonometry not being explored in the intervention. However, some improvements in the lower degrees of acquisition did take place. In the pre-test it was found that 73.7% of the intervention group had no acquisition of this level. The post-test has found this number to have been reduced.

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 5</td>
<td>0%</td>
<td>0%</td>
<td>26.3%</td>
<td>21.1%</td>
<td>52.6%</td>
</tr>
</tbody>
</table>

More of the group are now located in the intermediate degree of acquisition with 26.3% now in this degree compared to a pre-test value of 15.8%. 6 intervention participants have improved their degree of acquisition (5 improved by one degree, 1 improved by two degrees), 1 intervention participant (3-P7) has regressed one degree (from intermediate to low), and 12 remain unchanged. The following table outlines the increases in degrees of acquisition that have taken place:
We can see from this table that any increases have occurred at the lower three degrees of acquisition, however again it must be noted that certain constraints are in place due to the results of the pre-test. For example, no participant can be found in the interval of high-complete improvement as no one was found to be in the high degree of degree of acquisition in the pre-test. The improvements are not strong enough to warrant an in depth analysis. It is a fact of the research that this level was not a focus of the intervention and therefore improvements were not strong. Average weightings for the 19 intervention participants increased by 49.6% as it increased from approximately 16.184 (median of 10) to approximately 24.21 (median of 10). However, the post-test results are still low.

### Table 71: Increases in Degrees of Acquisition at Level 5

<table>
<thead>
<tr>
<th>Degree of acquisition increase</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>None - Complete</td>
<td>0</td>
</tr>
<tr>
<td>Low - Complete</td>
<td>0</td>
</tr>
<tr>
<td>Intermediate - Complete</td>
<td>0</td>
</tr>
<tr>
<td>High - Complete</td>
<td>0</td>
</tr>
<tr>
<td>None - High</td>
<td>0</td>
</tr>
<tr>
<td>Low - High</td>
<td>0</td>
</tr>
<tr>
<td>Intermediate - High</td>
<td>0</td>
</tr>
<tr>
<td>None - Intermediate</td>
<td>1</td>
</tr>
<tr>
<td>Low - Intermediate</td>
<td>2</td>
</tr>
<tr>
<td>None - Low</td>
<td>3</td>
</tr>
</tbody>
</table>

### Table 72: Pre and Post-Test Degrees of Acquisition of Level 5

<table>
<thead>
<tr>
<th>Test</th>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre</td>
<td>Level 5</td>
<td>0%</td>
<td>0%</td>
<td>15.8%</td>
<td>10.5%</td>
<td>73.7%</td>
</tr>
<tr>
<td>Post</td>
<td>Level 5</td>
<td>0%</td>
<td>0%</td>
<td>26.3%</td>
<td>21.1%</td>
<td>52.6%</td>
</tr>
</tbody>
</table>

#### 7.2.5 Level 6

The improvements at level 6 are not significant enough to be analysed as only three of the intervention participants increased in degrees of acquisition
(2 single degree increases, and 1 two degree increase) and one intervention participant (3-P22) regressed by one degree (intermediate to low). Average weightings increased from approximately 8.29 (median of 0) to approximately 13.68 (median of 0), an increase of 65.1%. However, this is still a low average weighting. Again, increasing conceptual understanding at level 6 of the model was not an objective of the teaching intervention and an increase amongst the intervention participants was not anticipated.

7.2.6 Summary of Post-Test Findings

The research intervention improved degrees of acquisition across the three levels of interest in the model. At level 2, though pre-tested to a high degree of acquisition, it was necessary to transition four participants from an intermediate acquisition to a complete acquisition. Three of these successfully accomplished this after participating in the intervention. The other participant did not attend the level 2 intervention as previously discussed (section 7.2.1).

Level 3 necessitated improvement from more participants than level 2 as pre-tests showed that it was acquired to lower degrees. There was a strong increase at this level of understanding with 94.7% of post-tests demonstrating a high or complete acquisition, in comparison to the 15.8% in the pre-test. The average weighting of the group suggests that the group as a whole has moved from a low-intermediate acquisition of this level (39.14) to a high acquisition (81.91).

Level 4 was the final level dealt with in the intervention. The increase in degrees of acquisition at this level was not as strong as the increase at level 3 with 15.8% of participants demonstrating a high or complete acquisition of the level after the intervention. However, this is compared to 5.3% of participants being at the same degrees of acquisition before the intervention. Average weightings suggest that the group as a whole has moved from a low acquisition (19.04) to an intermediate acquisition (44.3) of this level and so further work would need to be undertaken. The persistent difficulties at this level are discussed further in section 7.4.

Levels 5 and 6 of the model were not dealt with in the intervention. These levels were shown to have experienced some improvement but average weightings suggest that participants still have little to no understanding of these levels as a group. Therefore a further analysis of these levels did
not take place.

A synopsis of pre and post-test results in terms of degrees of acquisition is given in Table 73.

Table 73: Comparison of Degrees of Acquisition in Each Level from Pre & Post-Tests

<table>
<thead>
<tr>
<th>Test</th>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre</td>
<td>Level 6</td>
<td>0%</td>
<td>5.3%</td>
<td>5.3%</td>
<td>0%</td>
<td>89.5%</td>
</tr>
<tr>
<td>Post</td>
<td>Level 6</td>
<td>5.3%</td>
<td>0%</td>
<td>5.3%</td>
<td>10.5%</td>
<td>79%</td>
</tr>
<tr>
<td>Pre</td>
<td>Level 5</td>
<td>0%</td>
<td>0%</td>
<td>15.8%</td>
<td>10.5%</td>
<td>73.7%</td>
</tr>
<tr>
<td>Post</td>
<td>Level 5</td>
<td>0%</td>
<td>0%</td>
<td>26.3%</td>
<td>21.1%</td>
<td>52.6%</td>
</tr>
<tr>
<td>Pre</td>
<td>Level 4</td>
<td>0%</td>
<td>5.3%</td>
<td>0%</td>
<td>42.1%</td>
<td>52.6%</td>
</tr>
<tr>
<td>Post</td>
<td>Level 4</td>
<td>5.3%</td>
<td>10.5%</td>
<td>31.6%</td>
<td>52.6%</td>
<td>0%</td>
</tr>
<tr>
<td>Pre</td>
<td>Level 3</td>
<td>5.3%</td>
<td>10.5%</td>
<td>21.1%</td>
<td>52.6%</td>
<td>10.5%</td>
</tr>
<tr>
<td>Post</td>
<td>Level 3</td>
<td>52.6%</td>
<td>42.1%</td>
<td>5.3%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Pre</td>
<td>Level 2</td>
<td>78.9%</td>
<td>0%</td>
<td>21.1%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Post</td>
<td>Level 2</td>
<td>89.5%</td>
<td>0%</td>
<td>10.5%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Pre</td>
<td>Level 1</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Post</td>
<td>Level 1</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Pre</td>
<td>Level 0</td>
<td>94.7%</td>
<td>5.3%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Post</td>
<td>Level 0</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

Some of the group (3 individuals) from the pre-test did not take part in the intervention but did complete a post-test. Though an in-depth analysis of these cannot be undertaken as the number of people is so small, an exploratory account can be provided on how they performed in both assessments.
7.3 Results of Participants With No Intervention

Participants 3-P20, 3-P24 and 3-P25 all completed both pre and post-tests. However, these individuals did not take part in any intervention class (as the intervention was voluntary for the group).

The results from the post-tests completed by these three individuals show that no increase in degrees of acquisition occurred at any level of the model. Again, similar to the last section, some limits exist in the findings (e.g. 3-P20 could not increase in degree of acquisition of level 2 as they were pre-tested to have a complete acquisition). However, some regression was noted by two of the three individuals.

3-P20 was found to have remained at the same degree of acquisition at levels 0 through 2 of the model (all complete). A one degree regression was found at level 3 where this individual decreased from a high acquisition to an intermediate acquisition. Level 4 remained the same (low acquisition). At level 5 this individual regressed again, this time by two degrees of acquisition from an intermediate acquisition to no acquisition. Results showed that this individual regressed from an overall weighting of 62.68 to an overall weighting of 56.79 between pre and post-tests (note that overall weighting averages were taken as the average of the total scores for each level). A comparison of acquisition degrees across pre and post-tests for 3-P20 is as follows:

<table>
<thead>
<tr>
<th>Level</th>
<th>Pre</th>
<th>Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Complete</td>
<td>Complete</td>
</tr>
<tr>
<td>1</td>
<td>Complete</td>
<td>Complete</td>
</tr>
<tr>
<td>2</td>
<td>Complete</td>
<td>Complete</td>
</tr>
<tr>
<td>3</td>
<td>High</td>
<td>Intermediate</td>
</tr>
<tr>
<td>4</td>
<td>Low</td>
<td>Low</td>
</tr>
<tr>
<td>5</td>
<td>Intermediate</td>
<td>None</td>
</tr>
<tr>
<td>6</td>
<td>None</td>
<td>None</td>
</tr>
</tbody>
</table>

3-P24 remained at the same degree of acquisition of each level. No increase or regression took place at any level. In terms of this individual’s
average weighting it can be seen that there was a negligible difference between a pre-test weighting of 53.81 and a post-test weighting of 53.63.

Table 75: 3-P24 Pre and Post-Tests

<table>
<thead>
<tr>
<th>Level</th>
<th>Pre</th>
<th>Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Complete</td>
<td>Complete</td>
</tr>
<tr>
<td>1</td>
<td>Complete</td>
<td>Complete</td>
</tr>
<tr>
<td>2</td>
<td>Complete</td>
<td>Complete</td>
</tr>
<tr>
<td>3</td>
<td>Low</td>
<td>Low</td>
</tr>
<tr>
<td>4</td>
<td>Low</td>
<td>Low</td>
</tr>
<tr>
<td>5</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>6</td>
<td>None</td>
<td>None</td>
</tr>
</tbody>
</table>

3-P25, as stated, did not show any increase in degrees of acquisition for any level in the model. This individual remained at the same degree of acquisition of levels 0 through 3 of the model. One degree regressions were identified at levels 4 (intermediate to low) and 5 (low to none) of the model. The participant could not regress at level 6, having previously tested as having no acquisition.

Table 76: 3-P25 Pre and Post-Tests

<table>
<thead>
<tr>
<th>Level</th>
<th>Pre</th>
<th>Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Complete</td>
<td>Complete</td>
</tr>
<tr>
<td>1</td>
<td>Complete</td>
<td>Complete</td>
</tr>
<tr>
<td>2</td>
<td>Complete</td>
<td>Complete</td>
</tr>
<tr>
<td>3</td>
<td>Intermediate</td>
<td>Intermediate</td>
</tr>
<tr>
<td>4</td>
<td>Intermediate</td>
<td>Low</td>
</tr>
<tr>
<td>5</td>
<td>Low</td>
<td>None</td>
</tr>
<tr>
<td>6</td>
<td>None</td>
<td>None</td>
</tr>
</tbody>
</table>

Average weightings for 3-P25 display a decrease from 63.57 in the pre-test, to 57.86 in the post-test.

For each of these individuals, a decrease in understanding, however minuscule, has taken place. As every person that took part in the intervention
demonstrated an increase in understanding (as average weightings increased for all 19 intervention participants) and these three individuals have each displayed some regression in terms of understanding, it gives some credence to the effectiveness of the intervention compared to a control group. In this research a control group of adequate size could not be compiled due to the small sample to choose from.

7.4 Possible Reasons for Persistent Difficulties at Level 4

Even though an increase in understanding was shown to have occurred amongst the intervention group at level 4 of the model, the post-test results indicate that difficulties still persist (see Section 7.2.3 / Table 69). Some potential reasons for this will be outlined in this section.

One reason for this persistent lack of understanding is the amount of time devoted to the intervention at level 4. The pre-test indicated that level 4 had a more substantial lack of understanding than level 3, and that level 3 had a larger lack of understanding than level 2. While level 2 was given one intervention class and level 3 was given two intervention classes, it might have been deemed necessary to give level 4 more than two classes (because of the substantial lack of understanding at this level). As stated in section 3.11, time constraints were a limitation of this research. However, if this type of intervention were to be done again, with a level(s) having been tested as poorly acquired, more time would be devoted to such levels.

Another potential reason for the discrepancies amongst the effectiveness of interventions at levels 3 and 4 was the past experiences of the participants with the concepts at these levels. The concepts at level 3 would certainly have been studied by the participants at some time in their educational lives (most likely at Leaving Certificate level). However, the concepts at level 4 may not. It is a fact that these participants did engage with trigonometric functions, in terms of graphing and analysis, in the university module ‘technological mathematics’. However, it was found in the focus group (which is discussed in section 7.7.3), that technological mathematics was a module where they did not attend lectures and studied for the examination on the night before it took place. True engagement with the concepts in level 4 may therefore not have been accomplished by the participants prior to the intervention. This, in contrast to the concepts at level 3 being somewhat
familiar to participants could be a source of the continued difficulties after the intervention classes. Their experiences with level 4 concepts has been too brief.

Though level 4 did not accomplish as much as was hoped for, the intervention was still deemed a success and this contributes to the proof-of-concept approach. The next section demonstrates why the intervention can be construed as a success.

7.5 Was the Intervention a Success?

To evaluate the success of the intervention it is necessary to review the aims of the intervention to analyse if they were achieved. The aims for this intervention (as with section 6.2) were:

- To raise the levels of conceptual understanding of the participants with respect to the purpose-built teaching model.
- To develop the visualisation skills of the participants.
- To expose the pre-service participants to the five teaching phases of the purpose-built teaching model that they are experiencing.
- To assist the participants in linking the content to real-life scenarios.
- To expand the participants’ knowledge at the mathematical horizon (Ball et al., 2008) through the use of anecdotes and historical points.
- To provide the participants with resources that may be beneficial to their own future teaching.
- To further analyse the purpose-built teaching model developed and the teaching intervention employed.

Each of these aims are examined individually.

7.5.1 Raising Conceptual Understanding

It was shown at the beginning of this chapter (section 7.2) that an improvement in the level of conceptual understanding of the intervention group has been achieved. The majority of this improvement was found in the levels of the model which were of interest to the intervention (levels 2, 3, and 4). Though this can be seen through the qualitative data relating to the levels
that the participants were at pre and post-intervention, it can also be analysed through the quantitative weightings applied to responses. An effect size can be calculated.

Many research journals are abandoning the use of statistical significance values and favouring the use of effect size (Cohen et al., 2007). Effect size quantifies the difference between two groups of people (Cohen et al., 2007). The concept of effect size would apply to this research in quantifying the difference between the pre-test group and the post-test group. As the intervention dealt with levels 2 through 4 of the model, only weightings from these three levels will be used in the calculation of effect size. Note that the weightings, though somewhat subjective, have been found to be coherent with results (see Appendix T). As weightings are the same across both pre and post-tests, and as both assessments are identical, the effect size could be calculated. The effect size is the most applicable method of quantifying the overall difference between pre and post-test groups, relative to the research conducted.

Cohen et al. (2007, p.521) cite Glass et al. (1981) in the provision of a formula for calculating effect size:

\[
\text{effect size} = \frac{\text{mean of experimental group} - \text{mean of control group}}{\text{standard deviation of the control group}}
\]

The pre-test mean for levels 2-4 of the purpose-built model (only relative to those who completed all phases of the research) was 48.34105 and standard deviation was 35.41. The post-test group had a mean of 73.4289.

\[
\Rightarrow \text{effect size} = \frac{73.4289 - 48.34105}{35.41} = 0.7085
\]

This corresponds to a moderate effect according to Cohen et al. (2007, p.521). The effect size for each individual level can also be calculated.

Level 2:
\[
effect size = \frac{94.07895 - 86.8421}{20.06659} = 0.3606
\]

Level 3:
\[
effect size = \frac{81.90789 - 39.14474}{25.55989} = 1.6731
\]

Level 4:
\[
effect size = \frac{44.3 - 19.03632}{16.87166} = 1.4974
\]
The effect size at level 2 is 0.3606 and what is referred to as a modest effect (Cohen et al., 2007). However, the majority of those at this level in the pre-test demonstrated a complete acquisition and were limited for improvement. Therefore this value may not be representative of the true increase in understanding, and hence, effect size. The other two effect sizes are greater than 1 and therefore correspond to a strong effect (Cohen et al., 2007). The majority of participants were not bound by constraints at levels 3 and 4.

Therefore the intervention can be deemed a success due to the moderate-strong effect it had on increasing conceptual understanding. The proof-of-concept is also demonstrated by these results as the purpose-built model performed to a high standard in a small scale trial.

7.5.2 Development of Visualisation Skills

Participants state that the intervention helped them to get an image of the concepts and that this made it easier to understand the concepts in question (see section 7.7.1). However, in relation to the assessment the main fact that shows improved visualisation of problems and hence a contribution to overall problem-solving skills is the distinction between answers of type 0 and type 1 in the pre and post-tests.

Answers of type 0 and type 1 for the purpose of this research were defined as (section 3.9.1):

- **Type 0.** No answer provided/ No workings shown/ Illogical/ Irrelevant/ Incorrect.
- **Type 1.** Incorrect answers that show some level of reasoning (e.g. correct diagram but incorrect rule implemented etc.)

The distinction can be seen here in visualisation. An answer of type 0 means that no visualisation has taken place, while an answer of type 1 means that an appropriate diagram has been constructed. Bear in mind that, as noted in previous work, an answer of type 1 does not mean the problem was solved correctly. However, as one of the findings from the pre-test showed flawed visualisation of problems, that is the aspect of interest at this point.

The pre and post-tests displayed the following number of answer types 0 and 1 for questions 4 through 9 (as these were found to be poorly visualised
in pre-test and correspond to levels 2 and 3 of the purpose-built teaching model). Note that values from the pre-test are only with respect to the 19 participants that completed all phases of the research.

Table 77: Answer Types 0 and 1 for Questions 4 Through 9

<table>
<thead>
<tr>
<th>Question</th>
<th>Test</th>
<th>No. of type 0</th>
<th>No. of type 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Pre</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Post</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>Pre</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Post</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>Pre</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>Post</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>Pre</td>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>Post</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>Pre</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Post</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>Pre</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Post</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Answers of type 0 decreased by approximately 83% after the intervention was conducted. As this type of answer is the only one categorised by no visualisation of a problem then it can be concluded that visualising has improved. The number of type 1 answers are not necessarily relevant in the table above but have been displayed due to the distinction drawn between answers of type 0 and 1 at the beginning of this section.

7.5.3 Exposition to Teaching Phases

This was achieved through the intervention classes (Appendix P) and can be corroborated through the response to the intervention by the participants (section 7.7). Participants, without stating explicitly the inquiry phase or the others, appeared to use these stages in their own way, as shown in the focus group (see section 7.7.4). They came up with an idea of taking a problem and teaching it over a number of classes in order to teach numerous concepts (section 7.7.4). The idea resembled the five phase teaching sequence
they were exposed to during the intervention. Other validations of this aim being reached are found in the understanding participants had of the intervention (Intervention Acceptability) (section 7.8.3) and the use of the intervention teaching methods in their own teaching (section 7.7.4).

7.5.4 Linking of Content to Real-Life Scenarios

It must be stated that it is unknown if participants could create their own real-life examples for use in their own teaching. In fact, it is doubtful that this would be the case at level 4 due to the persistent lack of understanding and comments such as that passed about all functions in Intervention Class 5 (Appendix C).

However some other aspects of linking content to real-life were identified. The participants understand the difference between a real-life problem and a contrived one (see section 7.7.1). This is an important factor to consider. It is furthermore important that participants find the use of contrived questions irritating, but have used them nonetheless (e.g. Sean’s Farmers Field in section 7.7.1). The recognition of real-life problems as “easier” highlights the increased visualisation of participants. The fact that they can see the image of a concept makes understanding the concept more attainable and they foresee the use of this in their own future teaching (see section 7.7.1).

Linking of content to real-life, the importance of it, and the use in the participants’ own teaching can be seen throughout section 7.7 and the focus group (Appendix B).

7.5.5 Expansion of Knowledge at the Mathematical Horizon

The use of anecdotes (such as the story of Pythagoras and the Pythagoreans) was a feature of the intervention classes through the researcher’s/tutor’s own knowledge of historical points and points raised in the resource pack (booklet entitled ‘FAQs in Trigonometry: Concepts Decoded’). However, the main expansion of knowledge at the mathematical horizon was through the contextual applications that were used in the intervention classes. The researcher points to his own past experience of trigonometry where any contextualised problems were based on finding a distance to travel, or finding some distance applicable to a construction of a building. The applications used in the intervention such as music, the earth, and the old farmer’s almanac give a very different perspective and show how trigonometric concepts
can apply to various other phenomena which may have been unknown to the participants. This is confirmed through the comments passed on the music application (Appendix B) and comments passed on the old farmer’s almanac problem (Appendix B). These comments stated that the problems used in the intervention were “more realistic” (3-P18) that other problems the participants had previously engaged with in their studies (Appendix B). This knowledge at the mathematical horizon is vital to teaching (section 2.7) because it provides contexts for learning various concepts and how people use those concepts in real-life. This also strengthens the linking of content to real-life scenarios (section 7.5.4) in the fact that pupils are less limited as to what they can apply concepts to.

7.5.6 Provision of Resources

Resources were successfully provided to participants through Dropbox. These resources will be a valuable commodity to their future teaching of trigonometry not just in terms of the problems that it provides, but in terms of the preparation which these teachers wish to put into their lessons before teaching concepts (Appendix B).

7.6 Comparison of Intervention Group to Previous Final Year Pre-Service Teachers

To compare results from the intervention group at the time of fourth year (final year in university in academic year 2013/2014) and the final year group from the pre-test (final year students in academic year 2012/2013), we can first examine those with a complete acquisition of each level.
Figure 38: % of 19 Intervention Participants (Final Year Students 2013/2014) With a Complete Acquisition of Each Individual Level

Figure 39: % of 25 Pre-Test Final Year Participants (2012/2013) With a Complete Acquisition of Each Individual Level
There is a clear difference between the intervention group and last year’s final year group. Major differences can be seen at levels 2 and 3 of the model where 72% and 4% respectively of last year’s group had a complete acquisition. Approximately 90% and 53% of the intervention group have a complete understanding of levels 2 and 3 respectively. The persistent difficulties at level 4 have already been discussed in section 7.4, and as noted, levels 5 and 6 were not of interest to the intervention. However these findings are only concerned with complete acquisitions of levels. Further differences can be identified when the full spectrum of degrees of acquisition are considered.

The following tables (Tables 78 and 79) demonstrate the differences between the intervention group, and the previous year’s final year pre-service teachers in terms of degrees of acquisition of each individual level of the model.

Table 78: % of Intervention Participants (Final Year Students 2013/2014) in Each Degree of Acquisition of Each Level

<table>
<thead>
<tr>
<th>Level</th>
<th>Complete</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
<th>None</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 6</td>
<td>5.3%</td>
<td>0%</td>
<td>5.3%</td>
<td>10.5%</td>
<td>79%</td>
</tr>
<tr>
<td>Level 5</td>
<td>0%</td>
<td>0%</td>
<td>26.3%</td>
<td>21.1%</td>
<td>52.6%</td>
</tr>
<tr>
<td>Level 4</td>
<td>5.3%</td>
<td>10.5%</td>
<td>31.6%</td>
<td>52.6%</td>
<td>0%</td>
</tr>
<tr>
<td>Level 3</td>
<td>52.6%</td>
<td>42.1%</td>
<td>5.3%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Level 2</td>
<td>89.5%</td>
<td>0%</td>
<td>10.5%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Level 1</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Level 0</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

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Other than the increase in complete acquisition it can be seen that there is a strong decrease in the ‘no acquisition’ degree. This shows that less of this year’s group (intervention group) have no understanding of concepts.

At level 3 of the model, where 92% of last year’s fourth years lay in the ‘none-intermediate’ interval (inclusive), a greater percentage of the intervention group (94.7%) lie in the ‘high-complete’ interval. These findings are in direct contrast to each other and in a very positive way. It again strengthens the necessity and effectiveness of the intervention. At level 4, 96% of last year’s fourth year group lay in the ‘none-low’ interval and this has been reduced to 52.6% for the intervention group. 4% of last year’s fourth years lay in the ‘high-complete’ interval and this has been increased to 15.8% for the intervention group. Again it must be stated that, though level 4 has not experienced as strong an increase as level 3, it is an increase nonetheless.

The intervention participants are further along the purpose-built teaching model in comparison to last year’s group. When we take into account the literature that was discussed in section 2.7, this can only be said to improve the intervention group’s future teaching.

With the findings in terms of increased understanding of trigonometry demonstrated, the focus is now turned to the intervention participants themselves. The next section reports on the findings from a focus group which was conducted with a sample of the participants after the intervention. The focus group provided strong evidence that the intervention was also a success from the learner’s point of view.
7.7 Response to Intervention by Participants

A focus group (Appendix B) conducted after the intervention, was completed in order to gain insights directly from the participants themselves. Participants that took part were coded as they had been in pre and post-test analyses. Participants 3-P6, 3-P14, 3-P17, 3-P18, and ‘D’ (who had an intervention and post-test only, i.e. no pre-test) took part in focus group proceedings. These focus group participants were selected randomly from the intervention group. The focus group is used for the majority of the analysis as the research journals that were kept were based on observation by the researcher and may not be viewed as entirely objective (Cohen et al., 2007). All of the qualitative data from the focus group was coded on QSR NVivo (version 10) for analysis (Appendix U). The focus group was transcribed by the researcher before data analysis could be conducted. The data was provided by participants who experienced the intervention from a learner’s perspective and was relatively objective, with the only source of potential bias being that the researcher conducted the focus group. The nodes that arose through thematic analysis (section 3.9.2) of the data were:

- Effectiveness of the intervention;
- Necessity of the intervention;
- University course;
- Use in their own teaching;
- Examination focus.

Note that in the analysis (on QSR NVivo) some comments from participants were coded multiple times under different nodes.
Child nodes were identified for some of these parent nodes. Figure 41 shows the child nodes that were identified. These child nodes will be elaborated upon in the coming sections.

![Diagram of child nodes from thematic analysis](image)

**Figure 41: Child Nodes from Thematic Analysis**

### 7.7.1 Effectiveness of the Intervention

The node ‘effectiveness of intervention’ was created through thematic analysis. Child nodes of ‘good’ (29 references) and ‘bad’ (8 references) were created as extensions of this parent node due to these themes arising from the data provided by the participants in the focus group.
The ‘good’ points raised by participants accounted for 7.61% of the focus group discourse. The main focus of the good points was on heightened knowledge that the participants felt that they had gained from the intervention. Increased conceptual knowledge/understanding was noted on 19 occasions. Participants mentioned that the intervention “removed a couple of misconceptions” (3-P18) and made many concepts “a whole lot simpler” (3-P14) than they thought. Participants also noted that they felt more confident in trigonometry because of the increased knowledge they had gained from the classes.

Along with this, the understanding of concepts on a visual level was evident from the participants’ comments with statements such as “periods and things like that I wouldn’t have associated with... visually as much” (3-P17) and that the classes helped them “kind of graphically represent(ed) it rather than just making it this untangible (intangible) concept” (3-P18). One student noted that he found trigonometry more interesting after the intervention because it made more sense to him (Participant D). More findings on applying ideas from the intervention to their teaching will be discussed in section 7.7.4, however one pupil linked increased visualisation of concepts to his teaching as he felt the methods of “being able to actually see it like rather than just seeing numbers and words and formulas, being able to actually see what it means and why, and the relevance of it” (3-P6) would be understood much better by pupils.

Another ‘good’ point that participants noted was the applications that they encountered throughout the classes. The music activity that was used in one of the classes was highlighted as something that was interesting (3-P14). This is corroborated by comments noted in Journal Entry 4 (Appendix C). Participants in the focus group also noted the “easier” aspect of problems based on application. The contextualised problems made the participants “see it, and like see how it applies” (Participant D). Participants noticed that the problems they encountered were “realistic” (3-P18) compared to contrived questions that they had encountered in the past. Two participants engaged in discussion about “Sean’s farmers field” (Participant D) which was a real-life example they had used in previous teaching (on teaching practice) which they stated “never applied to anything” (3-P18). They noted that the application of concepts removed elements of abstractness and put concepts in more context.
Though positive points were noted, there were still some negative issues raised about the intervention experience. Participants still had concerns over some of the content that they covered stating “I still don’t understand a lot of it though” (3-P17). The researcher had provided information on the teaching stages the participants were experiencing during the classes, as well as the levels of understanding that were being covered. This may have left some fear in the participants about the amount of content in trigonometry. The researcher had stated that the intervention would only be covering up to level 4 on the purpose-built teaching model and the participants seemed apprehensive about this. Points were raised that the “intervention improved my knowledge of trigonometry, but it also highlighted that it’s not where it should be” (3-P18) and that one participant, though pleased with his increased knowledge, felt “less confident because I didn’t realise there is so much to know but now there is” (3-P17).

It is clear from the focus group that the strengths of the intervention under the purpose-built teaching model outweigh its weaknesses. Another node that was coded on QSR NVivo was the necessity of the intervention for the participants that they themselves were forthcoming in admitting to.

7.7.2 Necessity of the Intervention

The node ‘necessity of the intervention’ for the participants accounted for 8.02% of the focus group dialogue. Child nodes of ‘content knowledge’, ‘learning’ and ‘worries for teaching Leaving Certificate’ were identified under this node.

Participants stated that the intervention “Highlighted how needed an improvement of maths is for me” (3-P14). The researcher infers that this participant was referring to trigonometry, and not mathematics in general. Other participants noted that they were “way behind” (3-P17) where they felt they should be in terms of content knowledge and felt that they would have to make up for it with a lot of preparation before teaching trigonometry. Though these comments are somewhat negative, the participants did note that certain concepts such as the unit circle were “simpler” (3-P14) than what they thought. Concepts like the unit circle that are crucial to trigonometric understanding at higher levels need to be understood, and the participants viewing this concept in a new light is a good point to take from the intervention. They noted that when they learned the unit circle it was taught with no context or relevance (Participant D). One might argue
here that because of this it comes as no surprise that concepts at level 3 of the purpose-built model and hence, trigonometric functions at level 4 of the model were not understood in the pre-test phase (section 5.5).

The child node of ‘learning’ also arose in the analysis. The ‘learning’ node refers to visual learning, relational learning, and university learning. The visual learning that was necessary is similar to that noted in section 7.7.1. On the point of relational learning it is noteworthy that even these students at this stage of education, being pupils who have performed well in the education system, still find concepts when taught in an abstract manner difficult to understand:

“sine theta and all the tan theta, 2 theta and all this stuff, but they didn’t have any context associated with them... That’s where I’d be... It doesn’t mean anything to me at all... Whatever.”

(3-P17)

The contextualised problems helped this group of participants to learn the relations between concepts and understand each concept better (also noted in section 7.7.1). The group noted various points about their university course and what they felt was a necessity for the improvement of the third-level course they had almost completed. One participant stated

“I think our course is missing so much.. of things like that. Huge amounts. I think there should be something like that for each strand (of the Project Maths syllabus).”

(3-P17)

The group concurred that their course placed too much emphasis on mathematics modules where content is taught in an abstract manner and in some cases had an attendance of up to 300 students. More on their ideas about university learning will be discussed in section 7.7.3. The amount of discussion that took place on this matter lead to it being given its own node in the analysis.

As stated the amount of discussion that took place on the university course that the participants were enrolled in resulted in the course itself being given its own node. The next section analyses the qualitative data under this node.
7.7.3 University Course

The university course that the participants were all enrolled in (Physical Education and Mathematics) was a topic of discussion amongst the participants that accounted for 13.79% of the focus group discourse. Some elements of the focus group were coded under the node of university learning which has already been discussed under previous nodes (section 7.7.2). In terms of content knowledge, or lack of, this was also coded under university as the researcher wished to point out the failings of third-level to rectify misunderstandings and lack of understanding amongst this group of pre-service mathematics teachers. This corroborates points made by Furinigetti (2000), Selter (2001) and Weber (2005) (section 6.3). Statements demonstrating that the group did not learn enough mathematical content at third-level which is directly applicable to their teaching at second-level highlights this (e.g. “I don’t know what I’ve picked up in the four years” (3-P14)). The fact that mathematics modules were few in number in the course studied was mentioned by the group and that many of the mathematics modules they did do were highly abstract and did not benefit their learning or future teaching.

The participants provided ideas drawing from their experience of the intervention. They felt that the intervention methods could be adapted into pedagogy modules where they would learn content as well as pedagogy. A philosophy for these pedagogy modules was put forward by the group where a module would encompass learning “what you teach, and this is a really good way of teaching it, and these are really good ideas” (3-P14) as opposed to the abstract mathematics modules that pre-service mathematics teachers currently undertake.

That being said, it is notable that these ideas could not all be implemented, even if the students feel it to be more beneficial, due to teaching council requirements (section 2.10). However, these are some considerations for future planning of mathematics pedagogy modules. Pre-service teachers currently do two mathematics pedagogy modules. More will be discussed on this in chapter 8.

7.7.4 Use in Their Own Teaching

6.94% of the focus group was concerned with the applicability of the intervention to the participants’ own teaching. Some topics already discussed
such as increased content knowledge are directly applicable to their future teaching as increased knowledge directly relates to more effective teaching (chapter 2). Other child nodes that arose were ‘preparation’, ‘use of intervention resources’, ‘pedagogy’, and ‘combination of resources and pedagogy’.

Preparation for teaching was highlighted strongly by the participants at the start of the focus group, with all of them conceding that they would have to put in a lot more preparation for teaching trigonometry, in terms of their own learning. They admitted that they would have to ensure that they would understand more trigonometric concepts. This links back to the fear aspect mentioned in section 7.7.1. It seems that the participants did not realise how much content is in trigonometry, which is understandable when we consider that the vast majority were at levels 2 and 3 on the purpose-built teaching model. Participants mentioned that the intervention showed them how much work was necessary before they could teach the topic. Increased investment of time into trigonometry was a consensus that was reached by all participants in the focus group.

The use of intervention resources was another child node that arose. The real-life examples provided in the intervention classes were excerpts from the intervention resource pack. Participants were happy to have been given these. They said that they planned to ‘steal’ (Participant D) some of the real-life examples for their own teaching as a means to put the concepts in context, which they stated help them to understand the content in the intervention classes. Other uses of the resources were noted in conjunction with pedagogical practices.

The applicability of the intervention to the participants in terms of pedagogy, as noted, was another child node that arose. The main points made here was that visualising concepts and teaching concepts through contextualised problems would help the participants’ future students to understand concepts. One participant said:

“Just to be able to visualise it as well like, to actually, rather than just learning off formulas and seeing sine, do you know what it actually looks like and get them to see that like cause they’ll remember that I think an awful lot more than just the formula.”

(3-P6)
Another participant stated that visualising the concept alleviated the issue of “making it this untangible (intangible) concept” (3-P18). Though the participants have been instructed on the benefits of contextualised problems and visualising concepts in university pedagogy modules, they appeared to have limited exposition to these on a first-hand basis. This could be due to the mathematics modules they have experienced in university being taught in an abstract manner (section 7.7.1) and in many cases failing to adhere to pedagogical practices recommended in their pedagogy modules. The mismatch between pedagogy modules and mathematics modules appears to be a source of frustration for this group.

The final child node under the parent node of ‘use in their own teaching’ was ‘combination of resources and pedagogy’. Participants linked the resources they were supplied with and how they could influence pedagogy. The group had an idea to take one contextualised problem and teach concepts through that problem. They hypothesised that you could take a problem at the start of a week (or any time) and teach many concepts through that problem. One idea put forward was (Note: This quote is worded by the author to ensure that it can be understood):

In an ideal world you could take a problem on a Monday in your class that none of the students would know how to do. You could break it down over the course of a week, or however long it takes.

(3-P18)

The scenario was hypothesised where their future pupils would be given a problem that they would have no idea how to solve, and by the end of using this problem in teaching, the pupils would have experienced many concepts and understood each of those concepts. Relational understanding (Skemp, 1976) was a topic that was discussed again and again, though the participants did not explicitly state that phrase. One thing that was noted was that this idea for use in pedagogy would restrict the teaching of concepts in an isolated manner and it would aid in connecting all of the concepts together.

‘Examination focus’ was the final node that arose in the analysis of the data. Though the intervention was not based on examinations (except perhaps in the post-test), the participants referred to examinations throughout the focus group.
7.7.5 Examination Focus

This section links back to section 2.4 in corroborating the evidence that university students (namely pre-service teachers) still see examinations as the most important element of teaching and learning.

The group noted Leaving Certificate examinations on a few occasions (2.23% coverage). The main aspect that they discussed in relation to the Leaving Certificate was the worry that they would not do better on a Leaving Certificate paper than they did four years ago (when they completed their own Leaving Certificate examinations). They discussed how they understood that the content covered at university level should give them a higher level of knowledge than that at Leaving Certificate level, however, the reality of this was questioned. One participant stated that he “really thought he (I) would just walk out of this (university course) and someone handing (him) a Leaving Cert paper and he would (I’d) get an A in it every single time. Without a shadow of a doubt and which he (I) wouldn’t now” (3-P18) as he had to prepare a lot before doing grinds with students at Leaving Certificate level. They noted that any teacher who is teaching at Leaving Certificate level should be able to do the post-test that they received. However, they were pessimistic about the actuality of that statement, namely amongst their own class of pre-service teachers stating that they would be “surprised” (3-P17) if anyone in their class had fared well in the assessment.

Another discussion took place on two particular university modules that the group felt were a waste of time. Though these modules were since removed from the course of study of pre-service mathematics teachers, some other relevant information exists. The group felt that the modules Technological Maths 1 and Technological Maths 2 were a waste of time for them as they “knew that stuff” (3-P18). They stated that they “didn’t go to the lectures” (3-P18) and they “all got B’s and A’s in it. By studying the night before” (3-P18). As the content on these modules was largely based on content that would be covered at Leaving Certificate Higher level the group felt at the time that they could “learn your Leaving Cert stuff there quickly the night before and you got a B or an A. And no one really cared either way” (3-P18). This shows the focus the group had during their university studies in taking procedural approaches to examinations as they stated that they rote learned material on the day before examinations. It also raises questions which the researcher failed to probe on during the focus group (as he
did not anticipate this discourse). Though the group feel that more Leaving Certificate material should be covered, they also state that they did not attend lectures when the material was based on Leaving Certificate content. These were contrasting and confusing elements of the focus group. However, attention must be paid to the note made by this group that they would only find it relevant if the content was taught as a means to promote effective pedagogy. This will be discussed further in the final chapter (chapter 8).

At this point, and taking this data analysis into consideration, we can evaluate the intervention using the framework of Shapiro (1987) that outlines four attributes of an intervention that must be considered in order to evaluate it: Treatment Effectiveness, Treatment Integrity, Treatment Acceptability, and Social Validity.

### 7.8 Intervention Evaluation

Shapiro’s (1987) intervention evaluation framework was not originally based on educational interventions, but on medical research. In recent years, however, Shapiro’s framework has been adopted for use in educational research, primarily as a means to evaluate educational interventions (Hourigan and O’Donoghue, 2009; O’Meara, 2011; Prendergast, 2011; Faulkner, 2012; Treacy, 2012). All of this research demonstrates the use of Shapiro’s framework as a means to evaluate educational interventions. The research states that four criteria must be considered in order to evaluate it correctly:

- Treatment Effectiveness;
- Treatment Integrity;
- Treatment Acceptability;
- Social Validity.

These will be discussed individually in the evaluation of the intervention in sections 7.8.1 through 7.8.4. Note that the titles will be changed by replacing the word ‘Treatment’ with ‘Intervention’ due to the adaptation of Shapiro from a medical treatment context to an educational intervention context.
7.8.1 Intervention Effectiveness

Intervention effectiveness is defined as “the amount of change or improvement evident among the participant group, ideally in comparison with the control group who have not experienced the intervention” (Hourigan and O’Donoghue, 2009, p.136). The change or improvement that is evident from this teaching intervention is in relation to the change or improvement in the participants’ conceptual understanding of trigonometry relative to the purpose-built teaching model.

The intervention was effective in its purpose at levels 2 and 3 of the model as it raised conceptual understanding with respect to the model (post-test results show - 100% complete acquisition at level 2; 94.7% complete/high acquisition of level 3). The intervention at level 4 of the model was not as effective. Data suggests that an increase of conceptual understanding did occur (15.8% complete/high acquisition of level 4) but to a lesser degree than that at levels 2 and 3. As discussed, more time would need to be devoted to level 4 (section 7.4).

Nevertheless the intervention was effective in its purpose (section 7.5.1) as it did increase conceptual understanding of trigonometry amongst the participating group.

7.8.2 Intervention Integrity

Shapiro (as cited in O’Meara, 2011) defines treatment (or intervention) integrity as that which complies with three conditions:

- The appropriateness of the intervention;
- The importance of the outcome;
- The significance of its goals.

Much of the research conducted so far made the process of ensuring intervention integrity easier.

The intervention can be deemed appropriate by what was seen in chapters 2 and 5. The extensive review of literature (chapter 2) provided the author with a solid foundation on which to base this research. The fact that the research was conducted with pre-service teachers also complies with the research on the difficulties that this sample of the population experience in
relation to subject matter knowledge. The results of the pre-test (chapter 5) also corroborate that which was found in the literature. A distinct lack of conceptual understanding of trigonometry was found to exist amongst the sample of pre-service teachers in the University of Limerick and this strengthens the appropriateness of the intervention.

The importance of the intervention can be extended from the appropriateness of the intervention. As the intervention was deemed appropriate through the review of literature and the results of the pre-test, it can also be said to be important in order to combat these issues. Along with this, the participants themselves recognised and understood the importance of such an intervention for their understanding of trigonometric concepts for their future teaching, as was seen in section 7.7. As the participating group also recognised the importance of the intervention, the fact that the importance was explicit (understood by all) rather than implicit (understood by researcher only) can only strengthen the importance factor.

The significance of the intervention can again be viewed from the perspective of the appropriateness and importance of the intervention. It is significant as the problem exists. The intervention is also significant when we view the objectives of the research (section 6.2), namely:

- To raise the levels of conceptual understanding of the participants with respect to the purpose-built teaching model.
- To further analyse the purpose-built teaching model developed and the teaching intervention employed.

From the point of view of the research, the attempt to raise the conceptual understanding of the participants could be attempted through an educational intervention. The reanalysis of the purpose-built teaching model also required information from a teaching intervention, in order to see if the model and teaching sequence were valid for use in teaching trigonometry.

7.8.3 Intervention Acceptability

The author was diligent in ensuring the intervention was acceptable. As cited by O’Meara (2011) (from Shapiro (1987)), in order to ensure acceptability, one must ensure that the intervention considers the:

- time and cost of the intervention;
• method of delivery;
• effectiveness and integrity of the intervention;
• possible side effects that occur;
• understanding of the intervention by those involved;
• reproducibility and transportability of the intervention.

(adapted from O'Meara, 2011, p.270)

The cost of the intervention was not applicable to this research as the participants were not charged for it and the intervention did not require them to spend money on travel as it took place in the University of Limerick where they would be studying regardless. The intervention resources were transmitted via Dropbox and did not cost the participants anything. Also, the researcher was the only necessary instructor and therefore other tutors/facilitators were not needed and hence payment of tutors was not necessary. The only cost was in the printing of one resource pack for the tutor to refer to in the classes. The time aspect of the intervention was considered carefully by the researcher. The time aspect of the intervention was discussed in section 3.11. Though the time that the researcher wanted to devote to the intervention was given, some of the classes did have to be grouped into double-classes (i.e. two classes were completed in one joint session). Other than the intervention classes themselves, the intervention resources were developed over a period of five months which included from their initial design, to the proof reading of said resources by an expert mathematician employed at the University of Limerick.

The method of delivery of the intervention was in keeping with the teaching sequence advocated by the van Hiele’s (section 2.15.4) and good pedagogical practices found through research (section 6.6.1) as a means of progressing through the levels of the purpose-built teaching model. The method of delivery was also approved by the participants who took part in the intervention classes in terms of time, content and instruction (section 7.7) (Appendix B).

The effectiveness and integrity of the intervention were discussed in sections 7.5 and 7.7 and hence are deemed to be satisfactory.
There were no possible side effects associated with the intervention other than the main desired effects of increased subject matter knowledge of trigonometry.

Understanding of the intervention was necessary for all of those involved, not just the researcher. The participants had to understand the intervention in order to gain the increased understanding that was desired. From the data put forward in the focus group (Appendix B), it is clear that participants understood the purpose and content within the intervention classes at the time of the classes. As stated in section 7.7, the understanding of the intervention was found to be explicit rather than implicit.

Reproducibility and transportability of the intervention is one of the strongest points of this intervention. The content that is covered in the classes only needs a regular classroom with a computer projector. Though some activities that were conducted in the intervention classes for this research may have needed some knowledge of golf or playing a musical instrument (Appendix P) there are many more activities that can be put in the place of these from the resource pack. This intervention could easily be carried out in other teacher training institutions or schools. Of course if this was to happen more instructors/facilitators would be needed. Those instructors would need the necessary understanding of trigonometry before teaching the classes. One possible consideration would be the assessment of participants in an intervention like this elsewhere. One would have to conduct a pre-test with their students and would therefore need to invest time into reading and understanding the prescribed assessment procedures and evaluation protocol.

The author is confident that each of the considerations one must take into account in making an acceptable intervention have been addressed and attended to adequately.

7.8.4 Social Validity

Shapiro (as cited in O’Meara, 2011) notes four elements to address in ensuring that an intervention is socially valid:

- The immediacy and degree of change;
- The effort in implementation;
The theoretical orientation;

- The intervention facilitator.

(adapted from O’Meara, 2011, p.274)

The element of the *immediacy and degree of change* has already been discussed in section 7.5.1. This chapter has analysed and further discussed the change that took place after the intervention was conducted. It was shown that the change was significant enough for the intervention to have satisfied the element in question.

*The effort in implementation* of the intervention was not overtly strenuous. The author will state that the pre-test phase and all that it entailed (such as the design of the assessment and evaluation of assessment) was tedious and required a lot of effort, as did the planning of the intervention (resource pack, selection of material for intervention classes, finalising times for classes). However, the actual implementation of the classes was not as tedious. The researcher had a number of years tutoring and lecturing at the University of Limerick as well being a fully qualified second-level mathematics teacher and therefore felt comfortable in the classroom. This could be said to have alleviated some of the the effort involved compared to someone who perhaps may not feel comfortable speaking and teaching in front of groups.

The intervention was strong in its *theoretical orientation*. Every part of the intervention, from the rationale of the intervention, through to the analysis of data from the intervention was grounded in literature. The goals and aims of the intervention were formulated from research undertaken previously such as from the pre-test phase (which was all grounded in literature) and from the literature itself. The intervention classes were based on the van Hiele teaching sequence that was shown to aid in the transition from level-to-level in the van Hiele model. As the purpose-built teaching model formulated for this research was adapted from the van Hiele model, this teaching sequence was appropriate to use as a theoretical background to the intervention classes. Therefore the intervention had a theoretical orientation that was informed by literature and data gathered previously to ensure that the intervention was not only valid in its own right, but valid theoretically.

The *intervention facilitator* was the researcher himself. As the researcher, as tutor, had a knowledge of each individual participant with respect to their
level of understanding on the purpose-built teaching model, as well as the teaching sequence in place for the classes, he was the most appropriate facilitator for the intervention. The researcher is a fully qualified secondary mathematics teacher and also has had experience teaching at third-level. This strengthens his credentials as facilitator for the intervention.

Taking everything discussed in this chapter into consideration, the researcher developed some recommendations for any future teaching/interventions conducted under the purpose-built teaching model.

7.9 Recommendations for the Conduction of Future Interventions

The researcher can make two recommendations for any future interventions that would take place using the model constructed for this research and all that it entails (assessment methods etc.).

1. *Ensure that time dedicated to the intervention is with respect to the pre-test.*

The intervention that was conducted for this research took into account the level that every possible participant was at in the model. However, too much emphasis was placed on this and the research failed to infer from pre-test analysis that more time would be necessary for an intervention at level 4 compared to level 3. It is a strong recommendation that someone would allocate more time to levels demonstrating a low acquisition from a group. Of course a pre-test would still be in place before the conduction of such research and therefore levels which have a low acquisition amongst a sample would become apparent through analysis, similar to the analysis conducted for this research.

2. *Persist with intervention with pre-service teachers up to and including level 4 and then re-evaluate and plan for further interventions at levels 5 and 6.*

This is a similar idea to the cycles in Educational Design Research. The researcher advises that one would conduct an intervention of levels up to level 4 with pre-service teachers before commencing an intervention at level 5. It was shown in this research that,
though level 3 was completely acquired by the majority, level 4 was still a source of difficulty. This would need to be improved upon before level 5 could be intervened upon. The researcher urges that any future intervention takes into account the findings from this research in this respect.
7.10 Conclusion

This chapter provided a description and analysis of the findings from the research intervention conducted. It was shown that improvement occurred at the levels of the model which were of interest to the intervention (levels 2, 3 and 4). The largest increase in acquisition was at level 3. Level 2 was limited in its increase due to the acquisition degrees of none-complete.

Persistent difficulties were found to still exist at level 4 of the model and potential reasons for this were noted. This led to the recommendation that more time be allocated for future interventions on levels that were pre-tested to a low standard (none/low acquisition).

However, intervention methods designed for the model of understanding constructed were found to increase understanding with respect to the model and allow for progression through the levels with moderate-strong effect sizes. The comparison of the intervention group to a small number of individuals who did not take part in the intervention (but completed both pre and post-tests) indicates that only regression in acquisition will occur amongst pre-service teachers if certain interventions for subject matter knowledge are not put in place. However, the note was made that, due to limitations of the research, the group with no intervention was small in number and therefore this finding is just an indication, not a certainty.

The proof-of-concept has been validated in this chapter. The findings show that teaching trigonometry in accordance with the purpose-built teaching model has increased the sample’s understanding of trigonometry. The findings also indicate that teaching under the purpose-built model is an accepted and approved method according to those who learned through it.

The final chapter summarises the research that was conducted. The conclusions that have been drawn are discussed and the research questions that were posed at the beginning of the research are answered. The contributions of this research to mathematics education are identified and the researcher outlines his potential future work.
Chapter 8

Thesis Contributions and Future Work

8.1 Introduction

The conclusions made by the researcher as well as their implications for mathematics education research are discussed in this chapter. The research questions that were posed at the beginning of this work are answered with reference to the literature and the analysis of gathered data throughout this study. The proof-of-concept approach has been concluded and accepted with reference to the performance of the purpose-built teaching model in-situ. Design principles to consider when creating/adapting a teaching model which were developed through this research are also discussed. Recommendations for the conduct of future teaching and research are discussed based on the findings from this work. The chapter concludes with a discussion of the contribution of this doctoral research to mathematics education and possible avenues for future research.

8.2 Summary

The main focus of this thesis was to develop a model for the effective teaching of trigonometry. This was identified as the main aim due to findings from the literature in Ireland and abroad.

Literature from Ireland indicates that trigonometry is a difficult topic for the nation’s second-level students to learn. Consistently low results at Junior (Department of Education and Science, 1999; State Examinations Commis-
sion, 2003; 2006a) and Leaving Certificate (State Examinations Commission, 2000; 2001; 2005) level have been observed by the Chief Examiner. The failure to understand this topic at second-level has led to a continued lack of understanding at third-level (O’Donoghue, 2002; Gill, 2006). Similar findings to Ireland have been identified at second-level in Turkey (Gür, 2009; Orhun, 2001) and with pre-service and in-service teachers in Turkey (Topçu et al., 2006; Tuna, 2013). Second (Kendal and Stacey, 1997 in Weber, 2005) and third-level (Weber, 2005; Fi, 2006) students in the United States have also demonstrated difficulties with the topic, as well as second-level students in England (Blackett and Tall 1991; Ofsted, 2012) and third-level students in Australia (Chinnappan et al., 1996). Poor teaching was found to be one of the most important contributors to students’ weak understanding of mathematical content (Cockcroft, 1982; Fennema and Franke, 1992; Nye, Konstantopoulos and Hedges, 2004; Center for Public Education, 2005; Rivkin, Hanushek and Kain, 2005). Based on these findings, the author aimed to develop a purpose-built model for the effective teaching of trigonometry.

The van Hiele model was identified as an appropriate model for the purpose of this research. This was because the van Hiele model has a geometric focus, and trigonometry is a branch of geometry. The author developed his purpose-built teaching model by transforming the van Hiele model from a model that describes how people learn geometry to a model of how to teach trigonometry. He did this by extending the five level van Hiele learning model from geometry to a seven level learning model for trigonometry. The researcher then conducted a genetic decomposition of the seven levels to produce the purpose-built teaching model from this learning model. The purpose-built teaching model and its accompanying teaching strategies were applied with a group of 19 final year pre-service second-level mathematics teachers at the University of Limerick in order to examine if the model would work in practice, and hence function as a proof-of-concept.

The teaching intervention showed that the purpose-built model was effective for the levels dealt with in the intervention (levels 2, 3 and 4). The moderate-strong effect sizes observed after the intervention in a comparison of pre and post-test results led to the acceptance of the proof-of-concept. The model worked in practice in a small scale trial and this provides evidence that it may work on a larger scale. This research was centered on this proof-of concept approach as a means to test the purpose-built teaching model before applying it on a larger scale. This will be discussed as an avenue for future research in section 8.8.
The author responds to the research questions that were posed at the beginning of this research based on his findings throughout the study.

8.3 Research Questions Revisited

The research questions devised during the process of this research will be answered in this section. Though only a brief synopsis of the answer to each research question is provided, the sections of this thesis that go into further detail will be noted for a reference point to find further information.

8.3.1 Phase 1

Table 80: Phase 1 Research Questions

<table>
<thead>
<tr>
<th>Phase</th>
<th>Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase 1</td>
<td>Why is trigonometry included in second-level syllabi?</td>
</tr>
<tr>
<td></td>
<td>How are Irish students and students in other countries performing in the topic of trigonometry?</td>
</tr>
<tr>
<td></td>
<td>If performance is poor, then what is the cause for the poor performance over time?</td>
</tr>
<tr>
<td></td>
<td>How does a teacher affect student learning? What aspects of a teacher’s knowledge base are most important?</td>
</tr>
<tr>
<td></td>
<td>Do teachers currently have this knowledge base in trigonometry?</td>
</tr>
</tbody>
</table>

1. Why is trigonometry included in second-level syllabi?

Delice (2002) states that trigonometry is a topic of study in every mathematics syllabus worldwide. It is a topic that highlights connections of concepts within the topic itself, and connections to different areas of mathematics. It is also a topic that highlights other important areas of mathematics such as the importance of notation, analysis skills, characteristics and properties of figures, visualisation, working to precise definitions, development of mathematical language, fluency in mathematics, and the concept of mathematical proof (MSEB, 1990; Leitzel, 1991; Joint Mathematical Council of the United Kingdom, 1997; NCTM, 2014).
2. **How are Irish students and students in other countries performing in the topic of trigonometry?**

Chief Examiner reports based on Irish Junior (Department of Education and Science, 1999; State Examinations Commission, 2003; 2006) and Leaving Certificate (State Examinations Commission, 2000; 2001; 2005) examinations indicate that Irish second-level students find trigonometry a very difficult topic to learn. The basic concepts of trigonometry were found not to be understood at both Junior and Leaving Certificate levels. The weak understanding exhibited at second-level in Ireland has been shown to effect transition to, and success in, mathematics in higher education (O’Donoghue, 2002; Gill, 2006).

Reports from Turkey (Orhun, 2001; Topçu et al., 2006; Gür, 2009; Tuna, 2013), the United States (Weber, 2005; Fi, 2006), Australia (Chinnappan et al., 1996), and England (Blackett and Tall, 1991; Ofsted, 2012) indicate that similar findings exist in other countries at both second and third-levels.

3. **If performance is poor, then what is the cause for the poor performance over time?**

The literature indicates that poor teaching is one of the primary causes of the difficulties encountered by students when trying to learn trigonometry (Orhun, 2001; Gür, 2009) and trying to learn mathematics in general (Nye, Konstantopoulos and Hedges, 2004; Center for Public Education, 2005). Poor teaching contributes to students finding the topic highly abstract and difficult to understand. The research states that trigonometry should be taught through the use of visual aids and that this is not happening in classrooms (Weber, 2005). In general, mathematics instruction that relies heavily on memorisation and abstract ideologies was found to be a cause of poor understanding amongst students (Cockcroft, 1982; Sherman et al., 2010).

4. **How does a teacher affect student learning? What aspects of a teacher’s knowledge base are most important?**

It was found that teachers need to be proficient in pedagogical skills, knowledge of the curriculum and knowledge of the psychology of learning, amongst others (Shulman, 1986; 1987; Ernest, 1989; Fennema and Franke, 1992; Rowland, 2007; Ball et al., 2008; O’Meara, 2011). However, the most important element of a teacher’s knowledge base was found to be subject matter knowledge as it provides the foundation upon which every other skill is built. This finding had an impact on the final research question at this phase.

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5. *Do teachers currently have this knowledge base in trigonometry?*

Findings from Ireland indicate that many teachers of mathematics do not have the necessary qualifications to teach mathematics at second-level (section 2.10). No study that investigated Irish teachers’ (pre or in-service) understanding of trigonometry was found in the literature that was reviewed for this research. However, the findings from research conducted in other countries indicates that the answer to this research question is no.

Topçu et al. (2006) showed that both pre and in-service second-level teachers in Turkey did not understand the concept of radians to a high standard. This finding is corroborated by Tuna (2013). Fi (2006) found that pre-service secondary teachers in the United States did not understand trigonometric functions. Chinnappan et al. (1996) found that pre-service secondary teachers did not understand the relationships between trigonometric concepts.

Each of these studies took place in countries where students find trigonometry difficult and demonstrated low levels of understanding in the topic (Orhun, 2001; Weber, 2005; Chinnappan et al., 2006).

### 8.3.2 Phase 2

<table>
<thead>
<tr>
<th>Phase</th>
<th>Questions</th>
</tr>
</thead>
</table>
| Phase 2 | What models have been formulated in past research for the assessment of knowledge/understanding of a specific topic in mathematics/outside of mathematics?  
How can the model(s) be adapted/applied by the author for the effective teaching of trigonometry?  
Can a diagnostic assessment instrument for the adapted model be created? |

293
1. What models have been formulated in past research for the assessment of knowledge/understanding of a specific topic in mathematics/outside of mathematics?
Models have been formulated in past research for analysing general areas of mathematical knowledge (e.g. Ball et al.’s (2008) model for general areas of mathematics teacher knowledge (section 2.7)), however, models for assessing/categorising knowledge of a particular topic in mathematics were few in number. General learning taxonomies do exist (e.g. Bloom et al. (1956) and the SOLO taxonomy (Biggs and Collis, 1982) (sections 2.13 and 2.14)), however these are not mathematics specific. The van Hiele model (chapter 2) was a model that was researched which categorised the learning of geometry. Other models similar to this that were mathematics specific were not found in the literature that was reviewed.

2. How can the model(s) researched be adapted/applied by the author for the effective teaching of trigonometry?
The author adapted the van Hiele model through an analysis of the levels of the van Hiele model (section 2.15). Figure 14 outlines the comparison of the van Hiele model and the purpose-built model for the effective teaching of trigonometry through the comparison of the levels that each is composed of. It was this ‘level’ approach that would go on to form a large part of the purpose-built teaching model. The seven levels that were identified from the extension of the van Hiele model of geometric learning to the topic of trigonometry were elaborated on and specialised to teaching with the use of APOS theory, specifically with the conduction of a genetic decomposition of the seven levels (Dubinsky and McDonald, 2002) (section 4.2.3).

3. Can a diagnostic assessment instrument for the adapted model be created?
The author identified that a diagnostic assessment could be developed based on the objects found in the genetic decomposition of the purpose-built model. The objects were assessed through the assessment items (Section 4.4.4) and therefore each assessment item linked back to the purpose-built model in the fact that each object lay in one particular level (Section 4.2.3). This gave evidence relating to what each participant understood at each level of the purpose-built model after the evaluation was carried out (Section 3.9.1).
8.3.3 Phase 3

<table>
<thead>
<tr>
<th>Phase</th>
<th>Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase 3</td>
<td>What concepts that are relevant to junior and senior cycle trigonometry are the selected sample struggling to understand? Are any other significant findings emerging from the data relating to knowledge of trigonometry in general (e.g. in solving equations/constructing proofs)?</td>
</tr>
</tbody>
</table>

1. **What concepts that are relevant to junior and senior cycle trigonometry are the selected sample struggling to understand?**
   
   This question was answered through the pre-test phase of the research and through the use of the purpose-built teaching model (chapter 5). It was found that a small selection of the sample did not have a complete conceptual understanding of Pythagoras’ theorem or trigonometric ratios. However, the dominant lack of understanding lay in the concepts at levels 3 and 4 of the model. From the results of the pre-test it was shown that the sample of pre-service teachers did not understand quadrants or radians. This affected the acquisition of level 4 and it was conclusive that very little, if any, of the sample understood trigonometric functions in terms of graphing and analysis. As was noted in section 2.5 trigonometric functions (graphing and analysis) are forming a large part of the Project Maths Syllabus at Leaving Certificate level and therefore the pressure is on teachers (both pre and in-service) to understand them before they teach them.

2. **Are any other significant findings emerging from the data relating to knowledge of trigonometry in general (e.g. in solving equations/constructing proofs)?**
   
   The concept of a negative angle was shown to have been misunderstood by many participants in the pre-test. The CAST idea was also relied on by the majority of participants assessed and this of course is not deemed a mathematical explanation. The participants seemed to think that certain rote learned ideas like CAST were a valid reason for performing certain actions. One other significant finding from the pre-test phase was that visualisation of problems was not something that the sample excelled at. Section 5.6 discussed this in more detail. Another finding is that participants
were not able to apply the correct mathematical concept to a problem even if visualisation of the problem was achieved. Participants therefore showed that they did not understand concepts at levels 2 (small selection of sample), 3, and 4 and hence 5 and 6.

8.3.4 Phase 4

Table 83: Phase 4 Research Questions

<table>
<thead>
<tr>
<th>Phase 4</th>
<th>How can the findings from Phase 3 be addressed in the form of a teaching intervention?</th>
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<tbody>
<tr>
<td></td>
<td>What approach(es) should be used in the trigonometry teaching intervention?</td>
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</table>

1. **How can the findings from Phase 3 be addressed in the form of a teaching intervention?**

It was proposed that a stage based intervention would be the most appropriate approach where each individual’s intervention would begin at the first level that they demonstrated an incomplete acquisition of. This would eliminate a ‘one size fits all’ approach to the intervention and tailor the intervention as much as possible to the abilities of each participant. It would not have been possible to conduct an intervention with each individual separately. The approach that was used catered for the abilities of the participants as much as possible in this intervention.

2. **What approach(es) should be used in the trigonometry teaching intervention?**

The researcher implemented a five phase teaching sequence (inquiry, directed orientation, explicitation, free orientation, integration) that was advocated by the van Hiele’s (1984b) as a means to progress through levels of the van Hiele model. This was implement due to the strength of connection between the purpose-built teaching model and the van Hiele model. It was therefore envisaged that the use of this teaching sequence would allow for progression through the model that was constructed. In the intervention classes themselves, general pedagogical principles (section 6.6.1) were also adhered to. The problems that were used in the intervention classes were all taken from the resource pack which was constructed for the purpose of the intervention.
These problems were designed using the theoretical underpinnings of MEAs (Model Eliciting Activities) (section 6.6.3). MEAs were chosen as they serve the purpose of increasing deeper understanding of content and establishing an understanding of real-life problem solving strategies.

Once the teaching strategies and content were decided upon and devised, the question of how to implement the approaches in the form of a teaching intervention was answered. The implementation also had to consider the results of the pre-test before implementation. An intervention was first conducted with participants who were found to have an incomplete acquisition of level 2 of the model from the pre-test. Participants who were found to have an incomplete acquisition of level 3 from the pre-test joined in after the level 2 intervention was concluded. As noted, more time should have given to the level 4 intervention classes (section 7.4).

8.3.5 Phase 5

<table>
<thead>
<tr>
<th>Phase</th>
<th>Questions</th>
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</thead>
<tbody>
<tr>
<td>Phase 5</td>
<td>What changes have occurred after the intervention with regards to the samples’ conceptual understanding of trigonometry?</td>
</tr>
<tr>
<td></td>
<td>Have the findings from the post-test highlighted the usefulness/limitations of any of the teaching strategies employed?</td>
</tr>
<tr>
<td></td>
<td>Have the findings validated the model created?</td>
</tr>
<tr>
<td></td>
<td>Can any of the teaching strategies be employed in a secondary school classroom setting? If so, how might they be employed?</td>
</tr>
</tbody>
</table>

1. What changes have occurred after the intervention with regards to the samples’ conceptual understanding of trigonometry?

Chapter 7 analysed the results from the post-tests (i.e. after the intervention was carried out). A clear improvement occurred amongst the intervention group, with the strongest improvement being seen at level 3 of the model. The improvement at level 4 was not as significant but some improvement was shown to have taken place. The conclusion that was made on this was that
time spent on level interventions should be related to the degrees of acquisition that were shown to exist amongst a group in the pre-test (section 7.9).

Participants have moved with respect their acquisition of the levels of the model. According to that which decided what level an individual’s intervention would begin at (section 5.7), it is a fact that 57.9% of the intervention group would begin their next intervention at a different level to which their intervention started at on this occasion. One participant (3-P3) even demonstrated an understanding that would lead to his next intervention beginning at level 5 of the model.

2. Have the findings from the post-test highlighted the usefulness/limitations of any of the teaching strategies employed?
The findings indicate that the teaching strategies led to increased understanding of trigonometry amongst the participating group. Limitations were not identified, however, an issue was found with respect to the time needed to increase understanding based on the results of the pre-test. It was found that improvements at level 4 of the purpose-built model were not as significant as the improvements at preceding levels. The results from the pre-test indicated that level 4 had lower degrees of acquisition than the preceding levels. Therefore the researcher infers that more time should be devoted to teaching at levels that have been pre-tested to a have a low acquisition.

3. Have the findings validated the model created?
The purpose-built teaching model has been validated through the proof-of-concept approach which is discussed further in section 8.4. The effect size that was generated by comparing pre and post-test data was shown to have validated the model and its incorporated teaching strategies. The assessment instrument and hence, the purpose-built model was also shown to be hierarchial through Guttman Scalogram Analysis.

The purpose-built teaching model was also shown to be valid from the learner’s perspective. Focus group data demonstrated that the participants who learned in accordance with the purpose-built found it effective and stated that it developed their understanding and interest in the topic.

4. Can any of the teaching strategies be employed in a secondary school classroom setting? If so, how might they be employed?
As the concepts to be taught under the purpose-built teaching model were
initially identified from the Project Maths syllabus (section 4.2.2), there is no reason to doubt that the model could be applied in any secondary school classroom. These concepts also exist in syllabi in other countries and so could be applied globally (section 4.3). However, one possible restriction to doing this in an effective manner is that teachers may not understand trigonometry to a high enough level to implement the purpose-built model in a way that develops connections and conceptual understand of the topic. Many Irish second-level teachers of mathematics, as shown in section 2.10, do not possess the qualifications required to teach the subject. Research from Turkey, the United States, Australia and England also shows that second-level mathematics teachers (both pre and in-service) have deficiencies in their understanding of trigonometry (section 2.6.1).

In order to implement the model appropriately, a teacher should have a level of understanding of at least level 5 of the purpose-built model as stated in section 5.8.

8.4 Proof-of-Concept

The proof-of-concept approach, as discussed in section 3.5.2, is an approach that tests whether “a design concept will perform as anticipated under certain prespecified conditions” (Dym et al., 2009, p.33). The question which the research wanted to investigate was whether the purpose-built teaching model would perform to an adequate standard in a classroom setting in order to deem it an effective teaching model.

The purpose-built teaching model and its accompanying teaching strategies and assessment instruments were shown to have performed to a high standard with the sample in this research. The assessment instrument developed from the genetic decomposition of the levels of the purpose-built teaching model was shown to be reliable across markers (section 4.4.5) and corresponded to a perfect scale (as it was above the standard Guttman Scalogram Analysis figure of 0.9) (section 5.9). It was therefore inferred that the purpose-built teaching model itself corresponds to a perfect scale as the assessment was taken from the genetic decomposition of its levels.

The teaching intervention conducted in accordance with the purpose-built teaching model was shown to have a moderate-strong effect on the conceptual understanding of the participants in the intervention. Every par-
participant increased in acquisition of the levels dealt with in the intervention. A small number of individuals who did not take part in the intervention but did complete pre and post-tests were found to have all regressed in acquisition. This therefore gives an indication that only regression will occur without proper interventions being put in place.

The proof-of-concept was accepted in this small scale trial, giving credence to the idea of the purpose-built teaching model being effective in developing understanding of trigonometry on a larger scale.

8.5 Design Principles

The researcher developed a set of design principles in keeping with Educational Design Research based on his work in developing the purpose-built teaching model. Generalisable design principles, in the case of this research, applies to the creation of principles that one needs to consider when developing a teaching model (section 3.5.1). These design principles, as stated should be generalisable, and are therefore not specific to the topic of trigonometry. The set of principles that author has developed are described below.

1. Identify what the student already knows;

   The guiding tool of the teaching intervention in this research was the identification of where each individual was on the purpose-built teaching model. The author identifies this as the first design principle for designing teaching models. This design principle relates to van Hiele’s (1984b) idea of mismatch and helps in minimising the chance of mismatch occurring in the classroom. A teacher should know what the students already know and tailor their instruction appropriately to eliminate a ‘one size fits all’ approach to teaching (Lichtenstein and Glasgow, 1992; cited in Bridle et al., 2005). If a teacher knows what concepts their students understand then it is easier to make connections between these concepts and the new concepts they wish their students to learn. Whether this is done through an assessment, or interviews, or any other form of diagnostic test is up to the researcher. However, it is advised that the method of data collection is relevant to the teaching model they are developing. In this research, for example, the assessment instrument linked to the purpose-built teaching model
and aligned for instruction to begin at a particular level.

2. Build knowledge from what the student already knows;

A teacher can begin instruction once a diagnostic instrument/method has been designed, implemented and analysed. It is advised that the teacher links new mathematical concepts to that which is already known. This alleviates some of the ‘abstract’ issues noted by the subjects that took part in this research. The participants in the focus group stated that they were taught trigonometric concepts in a way that meant nothing to them (section 7.7.2). If instruction is based on concepts that students understand already, whilst introducing new language and images gradually, the potential of making the content too abstract is lessened.

3. Build students’ mathematical knowledge in a way that connects mathematical concepts.

This also links to the findings from a teacher’s diagnostic instrument/method. Once a teacher knows what concepts a student already knows then they can teach in a way that introduces new concepts with reference to previously acquired ones. Doing this highlights connections between concepts and builds schemas (Dubinsky and McDonald, 2002; Dubinsky, 2014), which in turn, lead to higher levels of mathematical understanding. This point is different from point 2 (above), though they may appear similar. Point 2 is in relation to keeping the content as comprehensive as possible to eliminate mismatch (van Hiele, 1984b). Point 3 is to build and reinforce that mathematical concepts are interrelated and develops students’ conceptual understanding of the subject.

A teaching model for any topic in mathematics (similar to this research being based on trigonometry), can be developed with these design principles in mind. No matter what topic in mathematics that one is teaching it is vital that the teacher knows what the students already understand, alleviates mismatch and confusion by building new knowledge on previously acquired knowledge, and also builds new knowledge in a way that connects concepts together which promotes conceptual understanding of the topic.
8.6 Recommendations

The researcher makes certain recommendations based on the findings of the research. These recommendations are categorised in terms of general mathematics pedagogy, future research on trigonometry, and future considerations for planning content to be covered in teacher training institutes with secondary mathematics teachers.

General mathematics pedagogy:

- It is important that teachers of mathematics at any level pay attention when using problems based on a context or problems with any element of visualisation. With Project Maths now in effect these types of problems will become much more prominent in Irish classrooms. The researcher recommends that a teacher should not just *formatively assess mathematical content*, but also *assess the sketching and visualisation of problems*. The possibility that a teacher would allocate time to teach visualisation of problems exists, however due to the limitations of this research, this is an avenue that was not pursued.

- The researcher *recommends the use of visual aids when teaching concepts for understanding*. It was shown in this research that critical concepts such as the unit circle were previously taught to the participants of this research study in a way that did not facilitate the development of conceptual understanding. With concepts like this underpinning so much more in the topic of trigonometry, it is unsurprising to see Irish students performing poorly in State examinations. Unless the didactic and rote methods traditionally employed in schools change, the success of Project Maths is questionable.

- The use of ‘traditional’ real-life examples should be updated. Throughout this research the author has encountered many real-life problems that deal with distances away from certain things, or heights of objects. In a world dominated by technology, and where more interesting things can be found with the click of a mouse, the author urges that educators *be creative when developing tasks* for their classes. It is recommended that they look for interesting and unexpected examples where trigonometry, and mathematics in general, can be applied.

Future research on trigonometry:

- The researcher, in conjunction with the findings of this thesis and the literature review, urges that *more research is conducted into content*
knowledge of trigonometry. This does not necessarily mean solely with pre-service teachers but at all levels of education. Various avenues are still available, for example, more work into teaching methods that increase understanding in trigonometry could be conducted, and other methods of evaluating the model produced in this research could be devised. The amount of literature available on trigonometry is extremely sparse and this is not the first time this has been noted (Weber, 2005). The fact that performance in trigonometry is not of a high standard (one of lowest achievement areas in the Irish mathematics syllabus) highlights that research into trigonometry and trigonometric understanding is necessary.

- For future research conducted using the purpose-built teaching model the researcher advises the continued use of the pre-test, intervention, and post-test phases. The recommendations for the use of the model all involve increased allocation of time. As time was a limitation of this research the researcher was bound by it. However, if time was not an issue the researcher would allow more time for participants when completing both pre and post-tests (possibly with assessments having more problems that cover and validate findings further). The researcher would advise the continuation of conducting the intervention up to level 4 and then follow with a post-test before initialising an intervention at level 5. It is also advised that the time allocated to teaching interventions is respective of the pre-test results (i.e. if the majority have shown a very low acquisition of level 4 in the pre-test, a lot of time is needed to raise degrees of acquisition, while if they have pre-test to high degrees of acquisition, not much time is needed).

Future considerations for content covered with pre-service teachers:

- It is recommended from the findings of this research that teacher training institutions need to improve the levels of understanding that their pre-service mathematics teachers have in mathematical topics. The cycles of Furinghetti (2000) and Weber (2005) must be stopped. One recommendation from participants, in conjunction with the intervention methods, is provided in the next point.

- The conduct of mathematics modules that blend pedagogy and content knowledge. The recommendation from participants was that this would be done for each strand of the Project Maths syllabus. The mantra for these modules was put forward, being that the module
would aid in ‘learning the content area and learning how to teach that content area at the same time’. This would be an avenue for future research as will be outlined in section 8.8.

8.7 Contributions

The author’s work throughout this doctoral research study has led to various contributions to the field of mathematics education research and contributions to the resource set of the mathematics teacher. These contributions can be considered in any future research conducted into the teaching and learning of trigonometry or research into models for teaching.

The purpose-built model for the effective teaching of trigonometry

This is the primary contribution of this doctoral work. This model was found to be needed from the findings demonstrated in the review of literature. The review of literature indicated that even though trigonometry is not understood well at second or third-level, a model for teaching it effectively was not found. The seven level hierarchial purpose-built model developed in this research outlines what trigonometric concepts to teach and when to teach them. According to the findings of this doctoral research, the purpose-built model is the only one of its kind. The goal of the purpose-built model is to help teachers to develop students’ conceptual understanding of trigonometry. The transformation of the van Hiele model is evident in the purpose-built teaching model. The work involved in developing the purpose-built teaching model included extending the five level van Hiele model of learning geometry to a seven level trigonometry teaching model. The researcher had to further elaborate upon each of these levels with the use of APOS theory. APOS theory allowed for the development of a systematic teaching model to be achieved.

Further analysis of the model has shown it to be effective in teaching trigonometry, effective from the point of view of the learner, as well as hierarchial in structure. The model is therefore useful to mathematics educators because it is a teaching model that provides a systematic teaching structure which facilitates conceptual learning of trigonometry, a topic that students find difficult to learn. The model is also beneficial to mathematics education researchers. The ethos of the overall research is one that can be taken on board by the mathematics education research community - models that
explicitly demonstrate how to teach topics of mathematics in a way that
develops understanding and connections between concepts have to be made
available. Though this model deals specifically with trigonometry, the au-
thor is confident that the general methods and methodologies in this research
could be used to develop teaching models for other areas of mathematics.

Assessment instrument and accompanying evaluation method
to position individuals on the purpose-built teaching model

This contribution relates to the purpose-built teaching model discussed
above as well as the recommendation that a teacher should begin instruc-
tion from the point of current student understanding. The assessment in-
strument that was developed and its accompanying evaluation method was
shown to be an appropriate instrument for positioning individuals on the
purpose-built teaching model. This allows for effective and timely teaching
or teaching interventions to be implemented under the purpose-built
teaching model.

A set of generalised design principles to consider when de-
veloping a teaching model

The design principles that were identified through the research are gener-
alised for the development of further teaching models which do not have to
be specific to trigonometry. Any researcher/teacher that is developing their
own teaching model is advised to consider these design principles when plan-
ning and conducting their research.

An approach for converting a learning model to a teaching
model

The stages that were followed in the transformation of the van Hiele model
from a model of how people learn geometry to a model for how to teach
trigonometry is, as far as this research could show, a novel approach to do-
ing this. The learning model is first adapted/extended to a general learning
model for the topic of study, which was trigonometry in the case of this
research. After this is accomplished a researcher should identify what con-
cepts need to be taught under this general learning model. An extension of
these concepts follows that allow for connections to be made and a concep-
tual understanding to be developed. This final transition was accomplished
with the use of APOS theory in this research.
Identified trigonometry as an area that requires more research

Though this is not a contribution which the author can take full credit for as it was identified in past research by Weber (2005), this research shows that further research into trigonometric teaching and learning is required in Ireland and abroad. Though this research showed that trigonometry is a topic that is not understood well by Irish second-level pre-service mathematics teachers at the University of Limerick, the literature indicates that it is also not understood by second-level students in Ireland. Second-level students, third-level students, pre-service teachers, and in-service teachers in other countries have demonstrated similar trigonometry abilities to Ireland.

8.8 Future Work

Based on the research conducted, purpose-built teaching model produced, and the findings of this study the researcher wishes to continue research into pre-service teachers’ subject matter knowledge of trigonometry. The future work that is envisaged is that the researcher will:

- Further analyse and refine the purpose-built teaching model that was developed so that it can be used on a larger scale than a sample of pre-service teachers in the University of Limerick.

- Conduct another test with the intervention group next year to see what conceptual understanding has remained after a long period of time. This will be conducted through sending the assessment to the group wherever they may be. It must be said that this may cause issues due to lack of response rate and the possibility of them not following the rules of the assessment.

- Construct a module for pre-service teachers that blends pedagogy and content knowledge of trigonometry. Before this module the pre-test will be administered and after the module the post-test will be administered.

- Evaluate the response of second-level pupils to the intervention methods (teaching strategies and resources). Their interest in the resources, as well as the increase in understanding that occurs through these methods, in comparison to traditional methods, will be evaluated.

- Construct other interventions for levels 5 and 6 of the model so a “finished article” of intervention methods can be accessed.
• Potentially adapt the model of understanding constructed to produce further models that may aid in increasing conceptual understanding of other topics in mathematics.

8.9 Final Comment

Trigonometry is an essential topic in mathematics. Despite this fact, this doctoral thesis has shown that young people at second and third-level struggle to learn it. This thesis reported on the development of a model that helps to teach this important topic of mathematics at second and third-levels. It was shown that the model and its accompanying teaching strategies helped in producing more knowledgeable teachers than the year before in terms of trigonometric content knowledge. However, the researcher recognises that this is not the end of his work. This model must now be developed and refined further so that it can be applied by any teacher in any secondary school in Ireland, and abroad. Though this research has given credence to the idea of the model working through the proof-of-concept approach, it still needs to be tested in a second-level setting after appropriate developments for second-level are made.

As a final comment the author would like to stress the importance that more research be carried out on trigonometry. If the research is based on models for understanding, models for teaching, different methods to improve knowledge of the topic, or on the subject matter knowledge of teachers on the topic, it does not matter. It is all contributing to more knowledge in the world of mathematics education, and only then can improvements and progress be made. Through this research and through the researcher’s personal experience with teaching trigonometry, he has seen the issues that students have with trying to learn the topic. These issues extend to the most basic concepts, even with high achieving students. He hopes that the purpose-built model he has developed is beneficial to instructors and hence, that it helps future students to understand trigonometry better.
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