Exact solution for single-scale Gaussian random transport

James P. Gleeson*
Department of Applied Mathematics, University College, Cork, Ireland
(Received 18 October 2001; published 15 February 2002)

A quadrature expression is derived for the probability density function of passive tracers advected from a point by a one-dimensional, single-scale, Gaussian velocity field. The effect of trapping on the tracer moments and the Lagrangian velocity variance is explicitly demonstrated.

DOI: 10.1103/PhysRevE.65.037103 PACS number(s): 05.60.−k, 02.50.Ey, 02.50.Fz, 05.40.−a

The transport of tracers by a random velocity field is a problem of interest in a variety of areas, including fluid turbulence [1,2], wave turbulence [3], random media [4], and stochastic population growth models [5]. The basic stochastic differential equation is of the form

$$\frac{dx}{dt}=u(x,t),$$

where u is a random field, i.e., u varies in space and time such that only its statistical (average) properties are known. Generally the field may be homogeneous in space and stationary in time, but with finite correlation length and time scales. The quantities of interest are the statistical properties of the (non-Markovian) process $x(t)$; in particular its density function $\Theta(x,t)$ from which the moments $\langle x^n\rangle$ may be calculated. For example, the transport of passive tracers by a turbulent velocity field is described by such an equation, with $x(t)$ denoting the position of a tracer at time t. A turbulent velocity field $u(x,t)$ is found by solving the Navier-Stokes equations, but even the simpler case of a Gaussian random velocity field remains an open problem [1,2,6]. Many approximation methods or closure schemes have been proposed but for the validation of these, and also of computational approaches, exact solutions for special cases are desirable but regrettably rare [1]. This paper describes an exact solution for the density function of tracers transported in one spatial dimension by the separable Gaussian field $u(x,t)=v(x)w(t)$. The field varies on a single spatial scale, and deterministically in time. This is a gross simplification of the physical problems of interest, but given the scarcity of exact solutions in this area we believe that this result, and the methods employed to derive it, merit attention.

We consider motion in a random field defined by

$$\frac{dx}{dt}=v(x)w(t),$$

$$x(0)=0, \tag{1}$$

where $v(x)$ is a homogeneous Gaussian random process of mean zero, and $w(t)$ is a deterministic function of the time. This equation describes the position $x(t)$ of a tracer released at $x=0$ at time $t=0$, and the distribution of such tracers at time $t>0$ over an ensemble of realizations defines a concentration (probability density) function $\Theta(x,t)$. Our goal is the exact quadrature expression (19) for Θ in the special case of a single-scale field $v(x)$.

The Gaussian random field $v(x)$ is described by its correlation function

$$\langle v(x)v(x')\rangle=\alpha^2R(x-x'), \tag{2}$$

where the angle brackets denote ensemble averaging and the dependence of R upon $x-x'$ follows from the homogeneity of the field. Taking $R(0)=1$ defines the variance of the field to be $\langle v^2\rangle=\alpha^2$. An alternative description of R is by its Fourier transform, written as

$$R(\xi)=\int_{-\infty}^{\infty}e^{-ik\xi}E(k)dk. \tag{3}$$

The non-negative function $E(k)$ is called the energy spectrum of the field, in analogy with transport problems in fluid turbulence [2]. A general homogeneous Gaussian random process of mean zero $v(x)$ may be approximated in each realization by

$$v_N(x)=\frac{1}{\sqrt{N}}\sum_{n=1}^{N}A_n\cos(k_nx)+B_n\sin(k_nx), \tag{4}$$

where A_n and B_n are chosen from independent zero-mean Gaussian distributions of variance α^2, and the k_n are chosen from a distribution shaped so as to yield the desired energy spectrum [4]. The Gaussian nature of the field results from passing to the $N\to\infty$ limit. In a single-scale field, the energy spectrum has the special form

$$E(k)=\frac{1}{2}\delta(k-k_0)+\frac{1}{2}\delta(k+k_0) \tag{5}$$

for some characteristic wave number k_0. This implies that the correlation function R is given by Eq. (3) as

$$R(x-x')=\cos[k_0(x-x')]. \tag{6}$$

Moreover, the spectrum (5) is obtained from the expansion (4) of the random field if each k_n is chosen to be k_0 or $-k_0$ with equal probability. The result is that Eq. (4) may be rewritten as

*Email address: j.gleeson@ucc.ie

$$v_N(x)=\cos(k_0x)\left[\frac{1}{\sqrt{N}}\sum_{n=1}^{N}A_n\right]+\sin(k_0x)\left[\frac{1}{\sqrt{N}}\sum_{n=1}^{N}B_n\right]. \tag{7}$$

But the sum of N independent, identically distributed Gaussian variables is $\sqrt{N}$ times a similar Gaussian variable, i.e., we may write Eq. (7) as

$$v(x)=A\cos(k_0x)+B\sin(k_0x), \tag{8}$$

with A and B chosen from independent, zero-mean Gaussian distributions of variance α^2. Thus we have shown that for the special spectrum (5), a Gaussian random field can always be simplified to Eq. (8).

Next, we solve the equation

$$\frac{dx}{dt}=[A\cos(k_0x)+B\sin(k_0x)]w(t) \tag{9}$$

with initial condition $x(0)=0$ in each realization, and then consider the distribution $\Theta(x,t)$, which results from averaging over the random A and B. Integrating Eq. (9) to the form

$$\int_0^x\frac{d\xi}{A\cos(k_0\xi)+B\sin(k_0\xi)}=\int_0^t w(\tau)d\tau, \tag{10}$$

yields the implicit solution

$$\tanh\left[\frac{1}{2}k_0rT\right]\cos\theta=\sin\left[\frac{1}{2}k_0x\right], \tag{11}$$

where we have introduced polar coordinates for the (A,B) plane,

$$A=r\cos\phi,$$

$$B=r\sin\phi, \tag{12}$$

with $\theta=\phi-k_0x/2$ for convenience. The integrated time variable is

$$T=\int_0^t w(\tau)d\tau.$$

For given values of x, t, and k_0, Eq. (11) defines a curve in the (A,B) plane. Each point on the curve corresponds to a pair of random numbers (A,B) whose use in Eq. (9) allows the point x to be reached at time t, see Fig. 1. The total probability density $\Theta(x,t)$ is then found by calculating the total probability of the (r,θ) values along the curve. Note that Eq. (11) may be solved for the angle θ,

$$\theta(r;x,t)=\cos^{-1}\left[\frac{\sin(\frac{1}{2}k_0x)}{\tanh(\frac{1}{2}k_0rT)}\right]. \tag{13}$$

Also, the minimum value of r along the curve corresponds to $\theta=0$ $(\phi=k_0x/2)$, with corresponding minimum radius

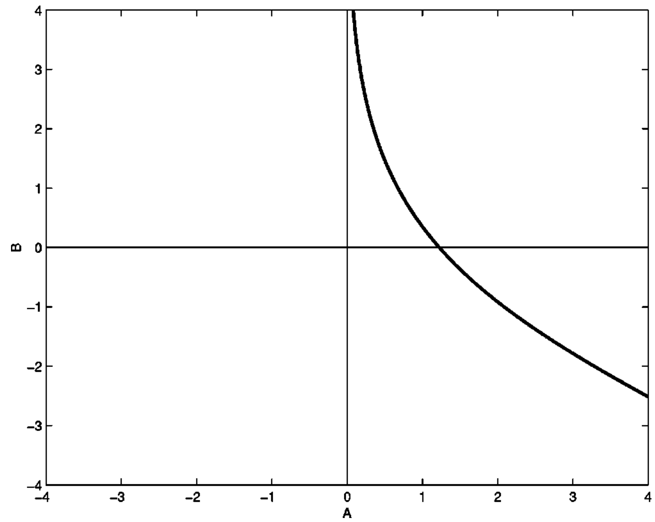

FIG. 1. An example of curve (11) in the (A,B) plane. Values of the random numbers (A,B) that lie on this curve allow the point $x=1$ to be reached when $T=1$. Here the parameter k_0 has been taken to be 1.

$$r_{\min}=\left|\frac{2}{k_0T}\tanh^{-1}\left[\sin\left(\frac{1}{2}k_0x\right)\right]\right|. \tag{14}$$

In order to calculate the probability density $\Theta(x,t)$ we first consider the cumulative probability that $x(t)$ has a value greater than x,

$$C(x,t)=\frac{1}{2\pi\alpha^2}\int\int_{\text{right of curve}}e^{-(r^2/2\alpha^2)}r\,d\theta\,dr, \tag{15}$$

with the integration region being to the right of the curve defined by Eq. (11), see Fig. 1. It follows that the probability that $x(t)$ lies between x and $x+dx$ is

$$\Theta(x,t)dx=C(x,t)-C(x+dx,t), \tag{16}$$

and so the probability density function is given by

$$\Theta(x,t)=-\frac{d}{dx}C(x,t). \tag{17}$$

The area integral giving $C(x,t)$ may be written as

$$C(x,t)=\int_{r_{\min}}^{\infty}dr\int_{-\theta(r;x,t)}^{\theta(r;x,t)}d\theta\frac{r}{2\pi\alpha^2}e^{-r^2/2\alpha^2}, \tag{18}$$

with $\theta(r;x,t)$ given by Eq. (13) and $r_{\min}$ by Eq. (14). The angle integral is evaluated immediately to yield $2\theta(r;x,t)$ and following differentiation with respect to x we obtain our result

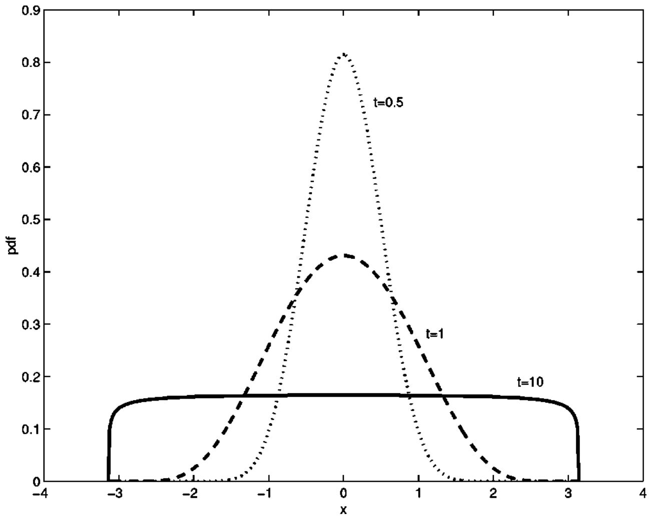

FIG. 2. Probability density function Θ at times $t=0.5$ (dotted), $t=1$ (dashed), and $t=10$ (solid). Parameters are $k_0=1$, $\alpha=1$, and $w(t)\equiv 1$.

$$\Theta(x,t)=\frac{k_0}{2\pi\alpha^2}\cos\left(\frac{1}{2}k_0x\right)\times\int_{r_{\min}}^{\infty}\frac{re^{-r^2/2\alpha^2}}{\sqrt{\tanh^2(\frac{1}{2}k_0rT)-\sin^2(\frac{1}{2}k_0x)}}dr. \tag{19}$$

The probability density function may now be evaluated using numerical quadrature, and we plot in Fig. 2 its shape at various times for the case $w(t)\equiv 1$. The parameters k_0 and α are chosen to be 1, and note the general case may always be reduced to this by transforming to nondimensional variables,

$$\tilde{x}=k_0x,$$

$$\tilde{t}=\alpha k_0 t.$$

The formula (19) also permits the analysis of several limiting cases, some of which we now examine briefly.

The limit $k_0\to 0$ of Eq. (19) is readily shown to yield

$$\Theta(x,t)|_{k_0=0}=\frac{1}{\pi\alpha^2}\int_{x/T}^{\infty}\frac{re^{-r^2/2\alpha^2}}{\sqrt{r^2T^2-x^2}}dr, \tag{20}$$

and by changing the variable of integration to $y=r^2-x^2/T^2$, this evaluates to

$$\Theta(x,t)|_{k_0=0}=\frac{1}{\sqrt{2\pi}\alpha T}e^{-x^2/2\alpha^2T^2}. \tag{21}$$

This is the expected Gaussian distribution of tracers transported by the simple time-dependent velocity field $v(x)=Aw(t)$ resulting from the $k_0\to 0$ limit in Eq. (9).

The limit of Eq. (19) for large values of T is straightforward, and yields the piecewise-constant distribution

$$\Theta(x)|_{T\to\infty}=\begin{cases}\dfrac{k_0}{2\pi}, & |x|<\dfrac{\pi}{k_0},\\ 0, & \text{otherwise.}\end{cases} \tag{22}$$

Note that this limit corresponds to large times $t\to\infty$ if $w(t)$ is a constant or an increasing function of the time. Of particular interest is the case $w(t)\equiv 1$ corresponding to a time-independent (''frozen'') velocity field, and we use this example in the figures. The zero probability of particle transport beyond $x=\pm\pi/k_0$ can be understood by noting that the velocity (8) must have a zero within this interval, and particles are trapped at zeros of $v(x)$ for all time.

The moments of the distribution are denoted by $\langle x^n\rangle$, for example, the tracer variance is given by

$$\langle x^2(t)\rangle=\int_{-\infty}^{\infty}x^2\Theta(x,t)dx. \tag{23}$$

The moments can be calculated from Eq. (19) by numerical integration (see Fig. 3), but their values for long times can be

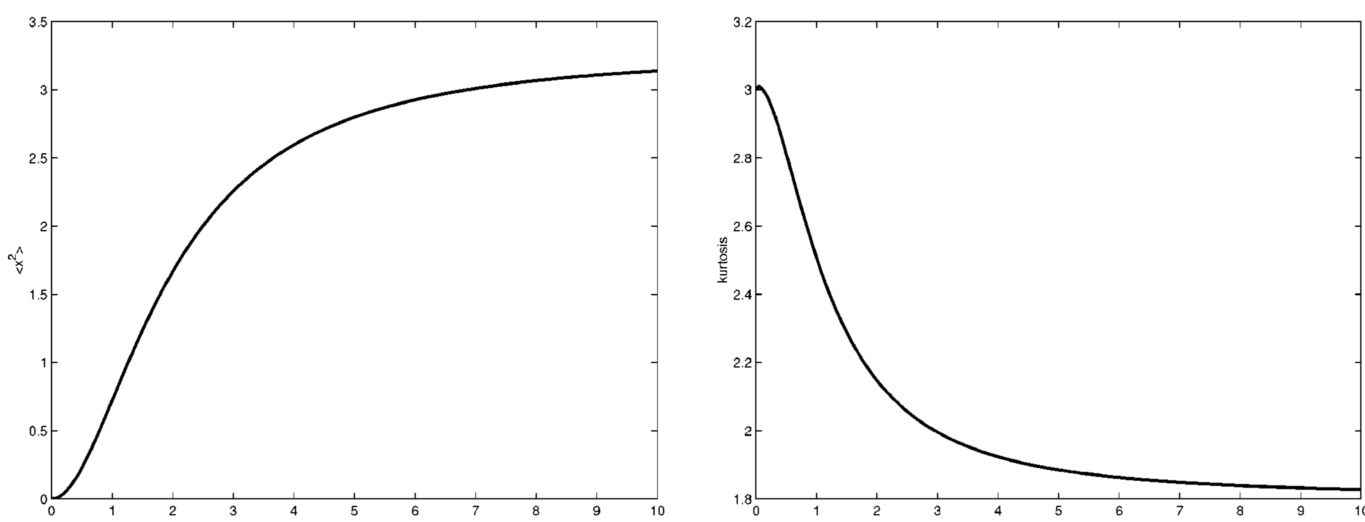

FIG. 3. Tracer variance and kurtosis as functions of the time t. Parameters are $k_0=1$, $\alpha=1$, and $w(t)\equiv 1$.

easily found using Eq. (22). In particular, we note the variance has the $T \to \infty$ value

$$\langle x^2 \rangle |_{T\to\infty} = \frac{\pi^2}{3k_0^2}, \tag{24}$$

and the kurtosis or flatness is given by

$$\left. \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2} \right|_{T\to\infty} = \frac{9}{5} \tag{25}$$

and is independent of k_0.

The intensity of the velocity experienced by each tracer as it moves according to Eq. (1) is a random function of the time, and is referred to as the Lagrangian velocity $u(t)$. We now show that the variance of the Lagrangian velocity is a decreasing function of the time. Note that this implies, in particular, that the Lagrangian velocity correlation is nonstationary. We begin by noting that the velocity of the tracer, which is at position x at time t, is given by Eqs. (9) and (12),

$$u = r\cos(\phi - k_0 x) w(t), \tag{26}$$

and the position x is related to r and ϕ through Eq. (11),

$$x = \frac{2}{k_0}\sin^{-1}\left[\frac{\tanh(k_0 r T/2)\cos\phi}{\sqrt{1+\tanh^2(k_0 r T/2) - 2\tanh(k_0 r T/2)\sin\phi}} \right]. \tag{27}$$

Substituting Eq. (27) into Eq. (26) gives the Lagrangian velocity at time t,

$$u(t) = \frac{r\cos\phi\,\mathrm{sech}^2(k_0 r T/2) w(t)}{1+\tanh^2(k_0 r T/2) - 2\tanh(k_0 r T/2)\sin\phi}. \tag{28}$$

The variance is found by averaging u^2 over the ensemble of possible values of r and ϕ, i.e.,

$$\langle u^2 \rangle = \int_0^\infty dr \frac{r e^{-r^2/2\alpha^2}}{2\pi\alpha^2} \int_0^{2\pi} d\phi\; u(t)^2, \tag{29}$$

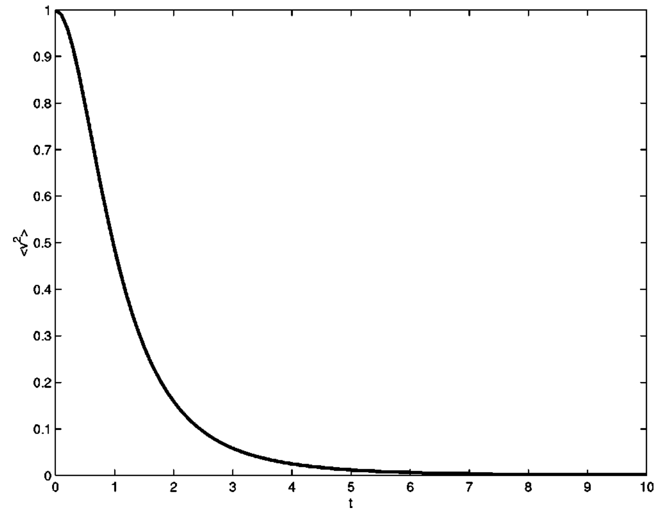

FIG. 4. Lagrangian noise variance $\langle u^2 \rangle$ as a function of the time t. Parameters are $k_0 = 1$, $\alpha = 1$, and $w(t) \equiv 1$.

with $u(t)$ given by Eq. (28). The angle integral may be calculated exactly to give the quadrature formula

$$\langle u^2 \rangle = \frac{w^2(t)}{2\alpha^2} \int_0^\infty dr\, r^3 \mathrm{sech}^2(k_0 r T/2) e^{-r^2/2\alpha^2}. \tag{30}$$

This decays to zero as $T \to \infty$ due to the inevitable trapping of tracers at zeroes of $v(x)$; in Fig. 4 we plot the results of numerical integration of Eq. (30) with $k_0 = \alpha = 1$ and $w(t) \equiv 1$.

In summary, we have derived an exact quadrature expression (19) for the probability density function of tracers advected by the single-scale velocity field (9), and the corresponding Lagrangian velocity variance (30). Long-time limits for transport in time-independent fields are given by Eqs. (22), (24), and (25).

Support from the Institute of Nonlinear Science, University College Cork and helpful discussions with Dr. Supratik Roy are gratefully acknowledged.

[1] A.J. Majda and P.R. Kramer, Phys. Rep. **314**, 237 (1999).
[2] J.P. Gleeson, Phys. Fluids **12**, 1472 (2000).
[3] A.M. Balk, Phys. Lett. A **279**, 370 (2001).
[4] J. M. Ziman, *Models of Disorder* (Cambridge University Press, Cambridge, 1979), Chap. 3.
[5] J. P. Gleeson and H. V. Kojouharov (unpublished).
[6] D.C. Swailes and K.F.F. Darbyshire, Physica A **262**, 307 (1999).