A Novel Characterization of Patient Arrival Process in Healthcare Facilities

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Abstract: Appropriate modelling for patient arrival is the first step in investigation of factors contributing to waiting time of both patients and the doctor in clinics. In previous practice, scheduled patient’s unpunctuality (i.e., the deviation from the scheduled appointment time) is often modelled either too typical that excludes some fundamental features of clinical observations, or too complex that might bring excessive difficulty to simulate and analyze. From observations of detailed workflow of community clinics, we can find that these models are not generally valid. In this paper, we proposed a modelling technique to characterise patient unpunctuality considering various groups of patient arrival behaviour. The adequacy of this model can be assessed by variation and standard error comparison with normal and Pearson type distributions. A sample simulation is given to exam the effects of various modelling techniques on patient waiting time and the doctor idle time. This proposed modelling technique might provide fundamental basis for improvement of appointment scheme in practical applications.

1 Introduction

Improved patient flow needs to be balanced with ensuring adequate time to complete necessary clinical functions (Waghorn and McKee, 1999; Tai and Williams, 2007). To identify those factors that contribute to unnecessary waiting times of both patients and doctors, the modelling of patient should be investigated first (Kyriacou et al, 1999; Saunders et al, 1989; Hashimoto and Bell, 1996). A review of some published patient arrival patterns indicated certain consistent irregularities in their distribution densities, in particular where some localized modes or near-nodes appear on the distribution tails (Clague et al, 1997; Fontanesi et al, 2000; Gallivan et al, 2002). This raised an interesting question as to how technically to model such aberrations.

Patient arrivals often vary from patient scheduled appointment times. This either contributes to clinician idle time or expands patient waiting lists (Kalai et al, 1992; Kapustiak and Ling, 2000; Asa et al, 1995; Dexter, 1999). In previous practice, scheduled patients’ unpunctuality (i.e., the deviation from the scheduled appointment time) is often modeled in a way that may exclude these persistent underlying features in clinical observations, such as in forcing data onto a normal distribution (Fernando et al, 2007; Alexopoulos et al, 2008; Bagust et al, 1999). Also, the selected distribution may be excessively complex adding to difficulty to simulate and analyze (e.g. the Pearson type and Johnson type distributions) (White and Pike, 1964; Alexopoulos et al, 2008). From re-examination of published observations drawn from detailed workflow analyses of community clinics, we can find that many families of distribution consistently neglect possibly these important features in the original data.

In this paper, a distribution modelling technique is proposed to capture these local peculiarities for scheduled patient arrivals, a method which balances accuracy and complexity by combining some common distributions. These modified forms are presented and a demonstration is made of their use to represent various patient unpunctuality patterns. This proposed modelling technique takes into accounts of the aspects of patient arrival behavior and thus more precise and accurate in modelling clinical
activities, and also it does not increase modelling complexity, because it is created by combining some common distributions and their modified forms, which simplifies parameter extraction under computer programming. The proposed modelling technique might provide useful for improvement of modelling appointment schemes in practical applications.

The adequacy of this model is assessed by variation and standard error comparison with normal and Pearson type distributions using a set of clinical observation record drawn from the literature. Tentative simulation results suggest some significant response to the proposed model. The steps of validation are as follows: firstly, parameters of normal, Pearson and the proposed fitted distributions that are ‘most’ suitable to clinical records are extracted. This process is done using MATLAB programming with MLE (maximum likelihood estimation) experiments; secondly, the standard error (SE) and variation between measure data (clinical observation data) and three sets of modeled data are worked out. Three sets of modeled data is extracted respectively using normal, Pearson and the proposed distribution; finally, a comparison of the variation and SE obtained by using this proposed model with the variation and SE gained from normal and Pearson distributions is made.

The remainder of this paper proceeds as follows: section 2 describes the experimental setup and the methodology used. In section 3, a F3 distribution (3-function distribution) is constructed by combining some common distributions and model verification is accessed using a set of clinical observations of patient arrival in section 4. In section 5, simulation results are given to validate this proposed modelling technique.

2 Comprehensive analysis of modelling distributions

In this section, relevant definitions of description of a clinic are introduced and detailed analysis of clinical observations is given for modelling distributions and estimating parameters.

2.1 Related concepts

Although the clinics surveys represented a variety of specialties and the work done in each was correspondingly varied, the basic definitions of view of the patients’ and doctors’ waiting were similar as follows:

The consultation-time of a patient is the sum of all the times he is claiming the consultant’s attention, or at least preventing him from seeing the next patient’ (White and Pike, 1964). This will include minor interruptions such as answering the telephone or being called to another doctor’s patient, and also any pause after consultation before the next one begins.

The unpunctuality of a patient indicates the difference between his time of appointment and time of arrival, whether the patient is early or late. The mean unpunctuality of a group of patients is defined as the average result of their individual unpunctualities, and it can be zero even if the patients are individually unpunctual. Generally in simulation, a positive and negative unpunctuality values indicate a late arrival and an early one, respectively.

The doctor’s waiting-time or idle-time means the sum of the times, between the doctor’s arrival at the clinic and the last appointment time, in which the doctor is not consulting because there are no patients waiting to be seen. It is possible that, after the last appointment time, the doctor has no patients waiting for attention, but all the patients called have not arrived. It seems reasonable (and was what happened in most clinics) for the doctor to leave the clinic at this stage, possibly returning at some convenient time later to attend to any late arrivals.
A patient's waiting-time means the period between his arrival at the clinic and his first being seen by the doctor; it includes waiting by patients who arrive early.

### 2.2 Analysis of clinic observations of patient unpunctuality and consultation times

A set of data for patient unpunctuality is applied for model extraction and simulation from the Pike and White’s (1964) work, which includes a recorded unpunctuality of 1352 patients at the out-patient department of a Scottish hospital.

![Figure 1: Frequency distribution of patient unpunctuality](image)

Fig 1 presents the frequency distribution function of patient unpunctuality. Reviewing the data of patient arrival from clinics, some features are summarised from fig 1. For scheduled patient arrivals (not necessarily punctual), firstly, patients did not, on average, arrive late, but they were far from punctual in the sense that many individuals arrived well before or well after their particular appointment-times. Secondly, patients mean unpunctuality possibly be a little earlier than the appointment time, but not zero. Finally, frequency distribution of patient unpunctuality shows unsymmetrical; there are three modes before, near and after the appointment time as we point out with three circles in fig 1.

As addressed, in previous studies, selected distributions for modelling either patient unpunctuality or consultation times are not suitable mainly for two reasons. One is selected distributions may be simple and explicit but fail to depict some important features of clinical data, typically normal distributions; the other is, selected distributions may be accurate in modelling, but too complex in form which makes the model quite different for simulations under alternative strategies and further applications, typically Pearson type distributions and Johnson type distributions.

In light of analysis above, the observed frequency distributions have several local modes, in which case none of the standard distributions will provide an adequate representation. Therefore, we constructed a distribution incorporating these features with some common distributions combined. In statistics, if the observations can be separated into two cases, with \( p_j \) being the proportion of observations for case \( j \), then a density \( f_j(x) \) is fit to the class \( j \) observations, the overall density function of observations is given by,

\[
f(x) = p_1 f_1(x) + p_2 f_2(x)
\]  

(1)
This density function can be extended to present distributions with finite modes,

\[ f(x) = \sum_{j=1}^{\infty} p_j f_j(x) \quad \left( \sum_{j=1}^{\infty} p_j = 1, \ p_j \geq 0 \right) \]  

(2)

According to the findings we gained from the clinical data, we noticed three local modes in fig 1 and thus we model patient unpunctuality using a F3 (three functions) distribution \( f(x) \) that is combined by three distributions in light of analysis above. Detailed modelling technique is presented in section 3.

2.3 Parameter estimation

After selecting proper probability distributions for patient arrivals, it turns out that one needs to estimate the unknown parameters. Different distributions have different numbers of parameters to determine, for instance, the exponential distribution requires the estimation of a single rate parameter, the normal requires estimation of the mean and variance, and the three-parameter Weibull requires estimates for its location, shape, and scale parameters.

Maximum likelihood estimation (MLE) test is widely used to identify if the observation data are independent (or random) sample from some underlying distribution (Law and Kelten, 2000; Ross, 1989; Wegman, 1982) and also we apply MLE to estimate the parameters in our model. Given that we observed a set of independent identically distributed data as patient unpunctuality \( X_1, X_2, \ldots X_N \). Let \( p_\theta(x) \) denote the probability mass function for this distribution and we define the likelihood function \( L(\theta) \) as follows:

\[ L(\theta) = p_\theta(X_1) p_\theta(X_2) \cdots p_\theta(X_n) \]  

(3)

The value of \( \theta \), that maximise the joint probability mass function \( L(\theta) \); that is, \( L(\theta_j) \geq L(\theta) \), for all possible values of \( \theta \), ‘best explains’ the data have collected. Differential calculus can be used to get the optimal \( \theta_L \), which satisfies,

\[ \frac{\partial L(\theta)}{\partial \theta} = 0, \quad \frac{\partial^2 L(\theta)}{\partial^2 \theta} < 0 \]  

(4)

Besides maximum likelihood test, Chi-Square and Kolmogorov-Smirnov tests are also widely used tools for Goodness-of-Fits tests as well. We applied MLE to extract parameters extraction in our model with MATLAB® programming.

3 Novel modelling techniques

In this section, a detailed modelling technique for clinic studies is proposed in light of analysis above for patient unpunctuality.

As analyzed in previous section, three local modes are found in frequency distribution of patient unpunctuality and thus a F3 distribution \( f(x) \) can be created which combined three sub-distributions to represent different patient groups. A normal distribution is applied as the main distribution for its explicitly and generality to represent patient group which is regarded as punctual (i.e., arrivals within 20 minutes earlier or later than scheduled appointment time), along with two modified lognormal distributions as the supplementary distributions in each side. The two supplementary distributions are applied to describe the patient groups that arrive earlier or later than 20 minutes. Here we choose normal and lognormal distributions as sub-distributions because they are simple, common and also accurate enough when combined together. The probability density functions (PDFs) of three sub-distributions are listed below.
\[ f_1(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(\ln(x-\delta_1)-\mu_1)^2}{2\sigma_1^2}} & (x < \delta_1) \\ 0 & (\text{otherwise}) \end{cases} \] (5)

\[ f_2(x) = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}} \] (6)

\[ f_3(x) = \begin{cases} \frac{1}{x\sqrt{2\pi\sigma_3^2}} e^{-\frac{(\ln(x-\delta_3)-\mu_3)^2}{2\sigma_3^2}} & (x > \delta_3) \\ 0 & (\text{otherwise}) \end{cases} \] (7)

\( f_2(x) \) is for normal distribution; \( f_1(x) \) and \( f_3(x) \) are both slightly modified from lognormal distributions to fit the clinical data of patient unpunctuality. \( f_1(x) \) is established by introducing translation and rotation into a standard lognormal distribution, and \( f_3(x) \) is established by introducing translation only into a standard lognormal distribution. The sum of proportions of three sub distributions \( p_1 \), \( p_2 \) and \( p_3 \) should be 1 considering the normalisation of probability distribution function. Thus, this proposed F3 distribution \( f(x) \) can be expressed as,

\[ f(x) = p_1 f_1(x) + p_2 f_2(x) + p_3 f_3(x) \] (8)

\[ p_1 + p_2 + p_3 = 1 \] (9)

**Figure 2: Fitted frequency distribution of patient unpunctuality**

Fig 2 presents fitted frequency distributions of patient unpunctuality, using this proposed modelling technique. The parameters of \( f(x) \) for F3 distribution are obtained under MLE (maximum likelihood estimation) test using MATLAB. The blue dash line with markers is the constructed F3 distribution \( f(x) \), which is combined by three dash lines respectively corresponding to \( f_1(x) \), \( f_2(x) \) and \( f_3(x) \).
In the next section, validation of this proposed modelling technique is accessed by fitness comparisons between F3 distribution and normal and Pearson distributions.

4 Fitness validation of novel modelling techniques

The proposed modelling technique is to create a F3 distribution combining three sub-distributions in consideration for fundamental attributes of patient arrival behavior. In this section, the adequacy of proposed distribution is examined by comparing with normal and Pearson type distributions.

As addressed, previous studies mainly focus on two kinds of distributions in modelling clinical activities. One is selected distributions may be explicit and easy to model but fail to characterize some obvious attributes of either consultation time or patient arrival, typically normal distributions; the other is, selected distributions may depict some characterization correctly, but too complex in its form (contains too many parameters or complex factorial operations etc) which results in huge computation in modelling and simulation, typically Pearson type and Johnson type distributions. The proposed distribution is compared with normal and Pearson type distributions to examine the fitness of data.

PDF of Normal distribution is expressed in equation (6) already and we do not repeat it again. Pearson type distributions have several different PDFs forms according to its types. We choose Pearson type VII to model unpunctuality as White and Pike (1964) did. The PDF expression of Pearson type VII is,

$$f(x) = \frac{1}{\alpha B(m - \frac{1}{2}, \frac{1}{2})} \left[ 1 + \left( \frac{x - \lambda}{\alpha} \right)^2 \right]^{-m}$$

where $B$ is Beta function that can be expressed as,

$$B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} \cdot dt$$

The steps of validation are as follows: firstly, parameters of normal, Pearson and the proposed F3 distributions that are ‘most’ suitable to clinical observations are extracted. This process is done by MATLAB programming with MLE (maximum likelihood estimation) test; secondly, the standard error ($SE$) and variation between measure data (clinical observation data) and three sets of modeled data are worked out. Three sets of modeled data is extracted respectively using normal, Pearson and the F3 distribution; finally, comparison of the variation and $SE$ gained from proposed model with the variation and $SE$ gained from normal and Pearson distributions is made for model validation. Variation and $SE$ are expressed as,

$$Var = \frac{\sum_{i=1}^{N} (Model(X_i) - Meas(X_i))^2}{N}$$

$$SE = \frac{\sum_{i=1}^{N} |Model(X_i) - Meas(X_i)|}{N}$$

Where $Meas(X_i)$ represents the clinical observations of $X_i$ (i=1, 2, ..., N) and $Model(X_i)$ means the modeled values using computer programming and $N$ is the sampling number.
Fig 3 is the comparison of unpunctuality frequency distributions modeled respectively from normal, Pearson and the proposed F3 distributions for patient recorded unpunctuality.

Table 1 presents the summary of validation results extracted directly from fig 3 above, which shows the comparison of variations and SEs of three modelled frequency distributions of patient unpunctuality. From table 1, it may be seen that both variation and standard error decreased significantly with proposed F3 distributions compared with normal and Pearson distributions. This proposed modelling technique is created by combining common distributions to represent various local modes, which represents various arrival types in clinical observation and thus improved accuracy, also the proposed modelling technique does not increase model complexity because it is created with common distributions and easy for parameter extraction under computer programming.

<table>
<thead>
<tr>
<th></th>
<th>Normal distribution</th>
<th>Pearson distribution</th>
<th>F3 distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variation</td>
<td>28</td>
<td>29</td>
<td>14</td>
</tr>
<tr>
<td>SE</td>
<td>3.7</td>
<td>3.4</td>
<td>2.5</td>
</tr>
</tbody>
</table>

Table 1: Comparison of variations/SEs of three distributions

In the next section, simulation experiments under alternative modelling techniques of patient arrival are given to exam the validation of this proposed modelling technique.

5 Simulation validation of novel modelling technique

The objective of investigation for clinical activities, either patient arrival and or consultation times, is to find out the factors contributing to waiting time and then propose better appointment schemes which both decrease patient waiting time and the doctor idle time (Callahan and Redmon, 1987; Tai and Williams, 2008; Williams, 2009).
In this section, simulations under alternative modelling technique of patient arrival are given to exam the effects on patient waiting time and the doctor idle time.

Patients arrived by appointment but not necessarily punctually, queued to be seen in turn once by a single doctor, and then left (Pike, 1963). Simulation assumptions are given in light of this: firstly, double and multiple consultations can be ignored because either the proportion of patients is very small, or the duration of consultation is very short. Second, the effect of diagnostic tests or treatment prior to the consultation can be ignored because the patients' appointment-times can be adjusted to allow for the time it takes to have the prior test or treatment and finally, a clinic carried on by two or more doctors usually comprises two or more examples of the model we have described, each doctor having his own list.

In light of discussions, a mathematical model was constructed of an out-patient clinic staffed by a single doctor. Patient unpunctuality is modelled respectively using normal, Pearson type and the proposed F3 distributions to generate 50 patient arrivals. Patient consultation time is randomly generated following a normal distribution with mean of 10 minutes and standard deviation of 3 minutes. Patient inter-arrival appointment time is scheduled as 10 minute which is the same with average consultation time. Ten times of independent simulation duplications are done to gain average patient waiting time and the doctor idle time.

<table>
<thead>
<tr>
<th>Measure</th>
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<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>S. Mean</th>
<th>S. Var</th>
</tr>
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<tr>
<td>Normal Distribution</td>
<td>16.9</td>
<td>19.4</td>
<td>14.4</td>
<td>26.2</td>
<td>26.5</td>
<td>13.4</td>
<td>12.2</td>
<td>26.8</td>
<td>13.1</td>
<td>12.7</td>
<td>15.6</td>
<td>37.7</td>
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<tr>
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<td>18.9</td>
<td>12.3</td>
<td>14.7</td>
<td>25.5</td>
<td>18.8</td>
<td>18.0</td>
<td>21.9</td>
<td>19.9</td>
<td>17.5</td>
<td>17.6</td>
<td>23.2</td>
</tr>
<tr>
<td>Fitted Distribution</td>
<td>14.9</td>
<td>16.3</td>
<td>20.7</td>
<td>13.1</td>
<td>21.3</td>
<td>16.3</td>
<td>15.8</td>
<td>15.0</td>
<td>12.1</td>
<td>11.3</td>
<td>15.7</td>
<td>10.8</td>
</tr>
</tbody>
</table>

Table 2: Average waiting time of patients under three distribution settings of patient unpunctuality (min Note: results are obtained from ten independent simulation replications. S.Mean: Sample Mean; S.Var: Sample Variance)

<table>
<thead>
<tr>
<th>Measure</th>
<th>1</th>
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<th>7</th>
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<th>9</th>
<th>10</th>
<th>S. Mean</th>
<th>S. Var</th>
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<tbody>
<tr>
<td>Normal Distribution</td>
<td>56.9</td>
<td>78.5</td>
<td>77.5</td>
<td>63.9</td>
<td>86.6</td>
<td>47.9</td>
<td>36.9</td>
<td>69.6</td>
<td>43.2</td>
<td>68.6</td>
<td>63.0</td>
<td>269.1</td>
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<td>Pearson Distribution</td>
<td>98.2</td>
<td>41.3</td>
<td>81.1</td>
<td>53.4</td>
<td>57.2</td>
<td>45.8</td>
<td>95.7</td>
<td>65.9</td>
<td>52.3</td>
<td>73.0</td>
<td>66.4</td>
<td>404.0</td>
</tr>
<tr>
<td>Fitted Distribution</td>
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<td>57.3</td>
<td>42.3</td>
<td>31.0</td>
<td>42.0</td>
<td>64.5</td>
<td>72.3</td>
<td>64.5</td>
<td>56.1</td>
<td>63.7</td>
<td>58.3</td>
<td>275.6</td>
</tr>
</tbody>
</table>

Table 3: Idle time of the doctor under three distribution settings of patient unpunctuality (min Note: results are obtained from ten independent simulation replications. S.Mean: Sample Mean; S.Var: Sample Variance)
Table 2 and 3 are respectively simulation outputs of average patient waiting time and the doctor idle time under different modelling techniques of patient unpunctuality. Results are obtained from ten simulation replications and the sample mean and variance are given in the last two rows in both tables. The data in second column of both tables are the simulation results, whose simulation input (patient unpunctuality) is randomly generated from frequency distribution of clinical observations in figure 1. Comparing results of three distributions with ‘measure data’ in table 2 and 3, it is found that, the F3 distribution is closest to the ‘measure’ result and also the variation of the F3 distribution is the least among the four, which means it is less fluctuate with stochastic events and thus more reliable in simulation. For average patient waiting time, Pearson distribution is better than normal distribution since the S. Mean (17.6) gained from Pearson distribution is closer to measure S. Mean (15.6) than normal distribution (18.2) and also the S. Var of Pearson is less than S. Var of normal which represents results of Pearson is more reliable. For the doctor idle time in table 3, it seems that normal distribution is preferable than Pearson distribution in S. Mean, but worse in S. Var, meaning the result is closer to practical (measure) result in average level, but quite fluctuate for specific stochastic simulation replications.

6 Conclusion
In this paper, patient unpunctuality is modeled by constructed distributions, which are combined by some common distributions and their modified forms, considering various patient unpunctuality patterns. The adequacy of this model is accessed by comparison of variation and SE using a set of clinic patient unpunctuality observations. Simulations under various modelling techniques of patient unpunctuality are given to analyze the effects of various modelling techniques on patient waiting time and the doctor idle time. Compared with previous modelling studies, the proposed modelling technique could depict the attributes of clinic observations more precisely without increasing modelling complexity, because it is created by combining quite common distributions and their modified forms and the parameter extraction is easy under computer programming.

The objective of modelling clinical activities is to propose better appointment schemes to improve patient flow and regarding this more future works need to be done. Simulation under alternative strategies with various clinical sizes, different scheduled appointment intervals should be investigated to exam the effects on clinical efficiency and the doctor idle time should also be considered as a clinical performance measure in the future work.

References


