Recovering Riemannian metrics in monotone families from boundary data

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Abstract. We discuss the inverse problem of determining the anisotropic conductivity of a body described by a compact, orientable, Riemannian manifold \(M\) with boundary \(\partial M\), when measurements of electric voltages and currents are taken on all of \(\partial M\). Specifically we consider a one parameter family of conductivity tensors, extending results obtained in [3] where the simpler Euclidean case is considered. Our problem is equivalent to the geometric one of determining a Riemannian metric in monotone one parameter family of metrics from its Dirichlet to Neumann map on \(\partial M\).

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1. Introduction

In electrical impedance tomography (EIT) one seeks to recover the interior electrical conductivity of an object from measurements of electrostatic potential and current density at the boundary of the object. In an anisotropic medium \(\Omega\), where \(\Omega \subset \mathbb{R}^n\) is a domain, with conductivity tensor the symmetric, positive definite matrix \(\sigma = \sigma(x)\), \(x \in \Omega\), the electrostatic potential \(u\) in the medium satisfies

\[
\text{div}(\sigma \nabla u) = 0, \quad \text{in} \quad \Omega.
\]  

(1)

Complete information about the relationship between applied surface current density and surface voltage is represented by the Dirichlet-to-Neumann map \(\Lambda_\sigma\), associated with \(\sigma\), defined by

\[
\Lambda_\sigma : u|_{\partial \Omega} \mapsto \sigma \nabla u \cdot \nu|_{\partial \Omega},
\]

for any solution \(u \in H^1(\Omega)\) to (1). Here, \(\nu\) denotes the unit outer normal to \(\partial \Omega\). In other words the operator \(\Lambda_\sigma\) maps the Dirichlet data \(u|_{\partial \Omega}\) (the boundary voltage) into the corresponding Neumann data \(\sigma \nabla u \cdot \nu|_{\partial \Omega}\) (the boundary current density).
The inverse problem consists in determining $\sigma$ from the knowledge of $\Lambda_\sigma$.

It is well known that an isotropic (scalar) conductivity is uniquely determined by the boundary data (see [2], [10], [11], [19], [20], [23]), while an anisotropic conductivity tensor is not uniquely determined by the boundary data (see [2], [3], [13], [17], [14], [19], [22]).

In practical applications of electrical impedance tomography and electrical resistivity, imaging anisotropy of conductivity and permittivity is common. In medical EIT (see [1] and references therein), fibrous or layer structures, such as muscles, exhibit electrical anisotropy on a macroscopic scale and red blood cells, aligning with the flow direction, result in a velocity dependent anisotropy in the electrical properties. The anisotropic electrical properties of tissues are also highly frequency dependent. Anisotropic dielectric permittivity is found in crystalline materials as well as liquid crystals. In some materials, such as epoxy resin, an anisotropic permittivity is observed to be linearly related to stress. In geophysics, one finds anisotropy as a result of stratified rocks as well as compaction soil [8].

In many of these applications it is possible to gain some information about the anisotropy. For example, in diffusion tensor MRI, a tensor that is believed to have the same principal directions as the conductivity [24], is measured. In geophysics, seismic measurements might be used to discover the orientation of the layers [21].

In this paper, which generalises the results in [3], we consider the situation where the conductivity (or equivalently the permittivity if the conductivity is negligible and the quasi static approximation is valid) is a known symmetric matrix valued function of one parameter and the parameter is an unknown function of space. This could be a model fitting parameter connecting the conductivity to some other tensor valued material property, or the conductivity could depend on another parameter such as temperature, flow velocity, concentration in a mixture or frequency. As yet such models have not received much attention, in part of course due to the difficulty of measuring anisotropic conductivity in situ. We hope and expect that the theoretical work in this paper will stimulate further physical investigation.

The physical problem of recovering the conductivity of a body by measurements of electric voltage and current density on its surface is closely related to the geometric problem of determining a Riemannian metric from its Dirichlet-to-Neumann map for harmonic functions (see [6], [13], [17], ). For an orientable manifold $M$ of dimension $n > 2$, the electric field is the 1-form $du \in \Omega^1(M)$ while the current density corresponds to an $(n - 1)$-form and the conductivity tensor $\sigma \in \Omega^1(M) \otimes (\Omega^{n-1}(M))^*$, which can be viewed as a linear map taking electric field to current density (Ohm’s law). The electrical power dissipation is then $du \wedge \sigma du \in \Omega^n(M)$ and must be a non-vanishing $n$-form that is symmetric: $\alpha \wedge \sigma \beta = \beta \wedge \sigma \alpha$ for all $\alpha, \beta \in \Omega^1(M)$. In dimension $n > 2$, the conductivity $\sigma$ uniquely determines a Riemannian metric $g$ such that

$$\sigma = *_g,$$

(2)
where \( *_g \) is the \textit{Hodge star} operator mapping 1-forms on \( M \) into \((n - 1)\)-forms (see [6], [13], [17]). The Dirichlet-to-Neumann map associated to \( g \) is therefore defined as the operator \( \Lambda_g \) mapping functions \( u|_{\partial M} \in H^{1/2}(\partial M) \) into \((n - 1)\)-forms \( \Lambda_g(u|_{\partial M}) \in H^{-1/2}(\Omega^{n-1}(\partial M)) \)

\[
\Lambda_g(u|_{\partial M}) = i^*(*_g du),
\]

for any \( u \), solution to

\[
\Delta_g u = 0, \quad \text{in} \quad M,
\]

where \( i : \partial M \to M \) is the inclusion mapping, \( i^* \) is the pull-back of \( i \) and \( \Delta_g = -*_g d*_g d \) is the \textit{Laplace Beltrami operator} on functions. (4) in coordinates becomes

\[
\sum_{i,j=1}^n (\det g)^{-\frac{1}{2}} \frac{\partial}{\partial x^i} \left\{ (\det g)^{\frac{1}{2}} g^{ij} \frac{\partial u}{\partial x^j} \right\} = 0, \quad \text{in} \quad M.
\]

For the case \( n = 2 \) the situation is different as the two-dimensional conductivity determines a conformal structure of metrics under scalar field, i.e. there exists a metric \( g \) such that \( \sigma = \gamma*_g \), for a positive function \( \gamma \) (see [6], [13], [17]).

In the case of a non-orientable manifold the current density \( -\sigma du \) must be considered as a twisted \((n - 1)\)-form, that is it takes its values in the (non-trivial) orientation line bundle. We omit the non-orientable case from this paper for the sake of clarity.

The problem of recovering the Riemannian metric by boundary data in the inverse conductivity problem has been studied in the past and in recent years. Kurylev gave a fruitful insight on the study of inverse problems on Riemannian manifolds in [12], where the problem of reconstructing the coefficients of an elliptic operator from its boundary spectral data is presented. We also refer to [9], where the authors investigated whether the so-called boundary distance representation of a Riemannian manifold determines the Riemannian manifold. See also [15]. Lassas and Uhlmann [14] recovered a connected compact real-analytic Riemannian manifold \((M, g)\) with boundary by making use of the Green’s function of \( \Delta_g \). See also [16].

In [3] the Euclidean case where the anisotropic conductivity tensor \( \sigma \) is \textit{a priori} known to be of type \( \sigma(x) = \sigma(x, a(x)) \), with \( x \in \Omega \) (\( \Omega \) is a domain in \( \mathbb{R}^n \)), is considered, where the one parameter matrix valued functions \( t \to \sigma(x, t) \) is \textit{a priori} known to satisfies the so-called monotonicity assumption

\[
D_t \sigma(\cdot, t) \geq \text{Const}.I > 0.
\]

The aim of this paper is to consider the more general case of a Riemannian manifold \((M, g_0)\) of dimension \( n \geq 3 \), where a one parameter family of metrics of type

\[
t \to g_t(x) := g(x, t),
\]
is prescribed on $M$, for any $t \in [\lambda^{-1}, \lambda]$, with $\lambda > 0$ constant and such that $g(x, 0) = g_0(x)$. Denoting by $*_t$ the Hodge star operator associated to the metric $g_t$, we assume that the following monotonicity condition is satisfied
\[
*_{0}((D_t*_{t}) \theta \wedge \theta) \geq \text{Const.} *_{0}(\theta \wedge \theta), \quad \text{for any } \theta \in \Omega^1(M).
\] (6)

The results obtained in [3] are given in terms of the Euclidean metric $(g_0)_{ij} = \delta_{ij}$, here we allow $g_0$ to be a general Riemannian metric and condition (6) is given in terms of it. The case of a manifold with a flat metric $g_0$ will be still more general than the one treated in [3].

Results of stability and uniqueness at the boundary and then global uniqueness in the interior are proven in the present paper.

The paper is organized as follows. Section 2 contains the statements of the main results (Theorems 2.3, 2.4, 2.5 and Corollary 2.6). In Section 3 we prove results of the existence of singular solutions on a Riemannian manifold. In Section 4 we give the proofs of the main results. For sake of brevity we only give the proof of Theorem 2.3, 2.4 as proofs of Theorem 2.5, and Corollary 2.6 follow the same line of proof of Theorems 2.3, 2.4 and the arguments used in [2], [3].

2. Main results

Let $(N, g_0)$ be a $C^\infty$ open, bounded Riemannian manifold of dimension $d \geq 3$.

**DEFINITION 2.1** For any $x^0 \in N$, $v \in T_{x^0}N$, we denote by $\rho_{v,x^0}(s)$ the geodesic of length $s$, starting at $x^0$ with direction $v$.

**DEFINITION 2.2** For any $x^0 \in N$, we denote by $B_{N,r}(x^0)$ the geodesic ball
\[
B_{N,r}(x^0) = \{ x \in N | d(x, x^0) < r \},
\]
where $d(\cdot, \cdot)$ is the geodesic distance on $N$ induced by $g_0$.

Let $M \subset N$ be a compact submanifold of $N$, of dimension $3 \leq n \leq d$, with Lipschitz boundary $\partial M$; the definition of Lipschitz boundary we will be using is the one formulated below.

**DEFINITION 2.3** Given positive numbers $L, r, h$ satisfying $h \geq Lr$, we say that a compact manifold $M \subset N$ has Lipschitz boundary if, for every $x^0 \in \partial M$, there exists a chart $(U, \{x_i\}_{i=1}^n)$ around $x^0$ in $N$ and an $(n-1)$-dimensional submanifold $\mathcal{M} \subset U$, with $x_n = 0$, such that $x^0 \in \mathcal{M}$ and such that $\partial M \cap C_{r,h}$ is the graph of a Lipschitz function $f : \mathcal{M} \longrightarrow \mathbb{R}$ which satisfies
\[
|f(x') - f(y')| \leq L d(x', y'),
\]
for any $x', y' \in \mathcal{M} \cap C_{r,h}$, where $\nu = -\frac{\partial}{\partial x_n}$ on $\partial M \cap U$ and
\[
C_{r,h} = \{ x = \rho_{v,y}(s) | y \in B_{\mathcal{M},r}(x^0), -h < s < h \}
\]
is the geodesic cylinder in $N$ of base $B_{\mathcal{M},r}(x^0)$ and height $h$. Moreover
\[
M \cap C_{r,h} = \{ x \in C_{r,h} | y \in B_{\mathcal{M},r}(x^0), -h < s < 0 \}.
\]
Let us denote by $\mu_{g_0}$ the volume form associated to the metric $g_0$ and by $\nabla$ the Levi-Civita connection on $(N, g_0)$; the class $\mathcal{H}$ of metrics $g_t(x) := g(x, t)$ admissible for our problem is given by the following definition. In the sequel we will make use of both notations $g_t(x)$ and $g(x, t)$, depending on the contest.

**Definition 2.4** Given $p > n$, $\lambda$, $E > 0$, and denoting by $T^2_0(M)$ the bundle of covariant tensors of type $(2, 0)$ on $M$, we say that the metric $g_t(\cdot, \cdot) \in \mathcal{H}$ if it satisfies the following conditions

\[ g_t \in W^{1,p}(M \times [\lambda^{-1}, \lambda], T^2_0(M)); \quad (7) \]

\[ D_t g_t \in W^{1,p}(M \times [\lambda^{-1}, \lambda], T^2_0(M)); \quad (8) \]

\[ \operatorname{Ess} \sup_{t \in [\lambda^{-1}, \lambda]} \left( \| g_t(\cdot) \|_{L^p(M, \mu_{g_0})} + \| \nabla X g_t(\cdot) \|_{L^p(M, \mu_{g_0})} + \| D_t g_t(\cdot) \|_{L^p(M, \mu_{g_0})} \right) \leq E, \]

\[ \text{for any smooth vector field } X \in C^\infty(TM), \text{ with } \| X \|_{L^\infty(M, \mu_{g_0})} = 1. \quad (9) \]

\[ \lambda^{-1} |\xi|^2 \leq g^{ij}(x) \xi_i \xi_j \leq \lambda |\xi|^2, \text{ for almost every } x \in \Omega, \]

\[ \text{for every } t \in [\lambda^{-1}, \lambda], \xi \in \mathbb{R}^n. \quad (10) \]

\[ \ast_0 \left( (D_t \ast g(x, t)) \theta \wedge \theta \right) \geq E^{-1} \ast_0 (\ast_0 \theta \wedge \theta), \text{ for almost every } x \in \Omega, \]

\[ \text{for every } t \in [\lambda^{-1}, \lambda], \xi \in \mathbb{R}^n. \quad (11) \]

(10) and (11) are a condition of uniform ellipticity and a condition of monotonicity with respect to the variable $t$ (see [3]).

**Remark 2.1** The volume form associated to the metric $g_0$ is specified in (9), but, since $M$ is compact, all the $L^p$-norms related to different volume forms are equivalent, therefore a different choice of the volume form will maintain $\operatorname{Ess} \sup$ appearing in (9) bounded, although constant $E$ will depend on the volume form. For sake of brevity we will denote any $L^p$ norm by omitting to specify the volume form $\mu_{g_0}$ for now on, by meaning that these norms are calculated in terms of $\mu_{g_0}$.

**Remark 2.2** Conditions (7)-(9), combined together with the Sobolev imbedding theorems for $p > n$ on manifolds with Lipschitz boundary (see [7, chapter 7, p. 158]), lead to

\[ g_t^{-1} \in W^{1,p}(M \times [\lambda^{-1}, \lambda], T^2_0(M)); \quad (12) \]

\[ D_t g_t^{-1} \in W^{1,p}(M \times [\lambda^{-1}, \lambda], T^2_0(M)). \quad (13) \]
Furthermore

\[ \text{Ess sup}_{t \in [\lambda^{-1}, \lambda]} \left( \| g_t^{-1}(\cdot) \|_{L^p(M)} + \| \nabla_x g_t^{-1}(\cdot) \|_{L^p(M)} + \| D_t g_t^{-1}(\cdot) \|_{L^p(M)} \right) \]

\[ + \| D_t \nabla_x g_t^{-1}(\cdot) \|_{L^p(M)} \leq F(\mathcal{E}, n), \]

for any smooth vector field \( X \subseteq C^\infty(TM) \), with \( \| X \|_{L^\infty(M)} = 1 \).  \( (14) \)

where \( F(\mathcal{E}, n) > 0 \) is a constant depending on \( \mathcal{E}, n \) only. Moreover, if we define

\[ G_t(x) := \| g_t(x) \|^{\frac{1}{2}} g_t^{-1}(x), \]

then

\[ G_t \in W^{1,p}(M \times [\lambda^{-1}, \lambda], T_0^2(M)); \]  \( (15) \)

\[ D_t G_t \in W^{1,p}(M \times [\lambda^{-1}, \lambda], T_0^2(M)); \]  \( (16) \)

\[ \text{Ess sup}_{t \in [\lambda^{-1}, \lambda]} \left( \| G_t(\cdot) \|_{L^p(M)} + \| \nabla_x G_t(\cdot) \|_{L^p(M, \mu_0)} + \| D_t G_t(\cdot) \|_{L^p(M)} \right) \]

\[ + \| D_t \nabla_x G_t(\cdot) \|_{L^p(M)} \leq C(\mathcal{E}, \mathcal{F}), \]

for any smooth vector field \( X \subseteq C^\infty(TM) \), with \( \| X \|_{L^\infty(M)} = 1 \).  \( (17) \)

where \( C(\mathcal{E}, \mathcal{F}) > 0 \) is a constant depending on \( \mathcal{E}, \mathcal{F} \) only.

Denoting by \( \langle \cdot, \cdot \rangle \) the \( L^2(\partial \Omega) \)-pairing between \( H^{1/2}(\partial M) \) and \( H^{-1/2}(\Omega^{n-1}(\partial M)) \), the Dirichlet-to-Neumann map

\[ \Lambda_{g(x,a)} : H^{1/2}(\partial M) \rightarrow H^{-1/2}(\Omega^{n-1}(\partial M)) \]

can be defined by its weak formulation

\[ \langle \Lambda_{g(x,a)} u, v \rangle = (-1)^{n-1} \int_M \ast_g(x,a) du \wedge d\phi, \]  \( (18) \)

for any \( \phi \in H^1(M) \) and any \( u \in H^1(M) \) weak solution to

\[ \Delta_{g(x,a)} u = 0, \quad \text{in} \quad M. \]  \( (19) \)

We shall denote by \( \| \cdot \|_* \) the norm of bounded linear operators between \( H^{1/2}(\partial M) \) and \( H^{-1/2}(\Omega^{n-1}(\partial M)) \).

The first result is a stability result of the metrics at the boundary.

THEOREM 2.3 (Lipschitz stability at the boundary). Let \((N, g_0)\) be a \( C^\infty \) open, bounded \( n \)-dimensional Riemannian manifold. Given \( p > n \), let \( M \subset N \) be a compact submanifold of \( N \) of dimension \( n \geq 3 \), with Lipschitz boundary \( \partial M \). Suppose \( a \) and \( b \) are two functions on \( M \) satisfying

\[ \lambda^{-1} \leq a(x), b(x) \leq \lambda, \quad \text{for each} \quad x \in M, \]  \( (20) \)

\[ \| a \|_{W^{1, p(M)}} \| b \|_{W^{1, p(M)}} \leq E \]  \( (21) \)

and \( g(x, t) \in \mathcal{H} \). Then we obtain

\[ \| g(x, a(x)) - g(x, b(x)) \|_{L^\infty(\partial M)} \leq C \| \Lambda_{g(x,a)} - \Lambda_{g(x,b)} \|_*, \]  \( (22) \)

where \( C \) is a positive constant depending only on \( n, p, L, r, h, \text{diam}(M), \lambda, E \) and \( \mathcal{E} \).
THEOREM 2.4 (Hölder stability of derivatives at the boundary). Given $p$, $n$, $M$, $(N, g_0)$ as in Theorem 2.3, let $a, b$ satisfy (20), (21) and $g \in \mathcal{H}$. Suppose there exist a point $y \in \partial M$ and a neighborhood $U$ of $y$ in $M$, a positive integer $k$ and some $\alpha, 0 < \alpha < 1$ such that

$$g(x, t) \in C^{k, \alpha}(\bar{U} \times [\lambda^{-1}, \lambda], T^{2,0}(M)),$$

$$a - b \in C^{k, \alpha}(U),$$

with

$$\|g\|_{C^{k, \alpha}([\lambda^{-1}, \lambda], T^{2,0}(M))} \leq E_k,$$

$$\|a - b\|_{C^{k, \alpha}(U)} \leq E_k. \tag{25}$$

Then, for any neighborhood $W$ of $y$ in $M$ such that $W \subset U$ and any smooth vector field $Z \in C^\infty(TM)$, we have

$$\| \nabla^k_Z(g(x, a) - g(x, b)) \|_{L^\infty(\partial M \cap \bar{W})} \leq C \| \Lambda g(x, a) - \Lambda g(x, b) \|_{\star}^{\delta_k}, \tag{27}$$

where $\nabla^k_Z$ is the $k$th covariant derivative with respect to the vector field $Z$ and $\delta_k = \prod_{j=0}^{k} \frac{\alpha}{\alpha + j}$. Here $C > 0$ is a constant which depends only on $n$, $p$, $L$, $r$, $h$, $\text{diam}(M)$, $\text{dist}(W \cap \partial M, M \setminus U)$, $\lambda$, $E$, $\mathcal{E}$, $\alpha$, $k$, $E_k$ and $Z$.

The following uniqueness result can be obtained under a slightly weaker assumption.

THEOREM 2.5 (Uniqueness at the boundary). Let $p$, $n$, $M$, $(N, g_0)$, $a$, $b$, $g$ as in Theorem 2.3. Suppose there exist a point $y \in \partial M$, a neighborhood $U$ of $y$ in $M$ and a positive integer $k$ such that

$$a - b \in C^k(\bar{U}). \tag{28}$$

If

$$\Lambda g(x, a) = \Lambda g(x, b),$$

then

$$\nabla^j(a - b) = 0 \quad \text{on} \quad \partial M \cap \bar{U}, \quad \text{for any} \quad j \leq k, \tag{29}$$

where $\nabla$ denotes the gradient in a general coordinate system. If in addition

$$g(x, t) \in C^{k}(U \times [\lambda^{-1}, \lambda], T^{2,0}(M)),$$

then, for any neighborhood $W$ such that $W \subset U$ and any smooth vector field $Z \in C^\infty(TM)$, we have

$$\nabla^j_Z(g(x, a)) = \nabla^j_Z(g(x, b)), \quad \text{on} \quad \partial M \cap \bar{U}, \quad \text{for any} \quad j \leq k. \tag{31}$$
The following corollary is a well-known consequence of the previous theorem in the Euclidean case (see [2], [3]) and can be easily adapted to the case of a Riemannian manifold.

**COROLLARY 2.6** (Uniqueness in the interior). Let $n, M, (N, g_0)$ be as in Theorem 2.3. Let $a, b$ be two functions satisfying (20) and (21) with $p = \infty$. Let $g(x, t) \in \mathcal{H}$ and in addition $g \in W^{1, \infty}(M \times [\lambda^{-1}, \lambda, T_0^2(M)])$. Suppose that $M$ can be partitioned into a finite number of Lipschitz submanifolds, $\{A_j\}_{j=1}^n$, such that $a - b$ is analytic on $\bar{A}_j$, for any $j = 1, \ldots, n$. If $\Lambda g(x, a) = \Lambda g(x, b)$, then

$$g(x, a) = g(x, b) \quad \text{on} \quad M. \quad (32)$$

### 3. Singular solutions

Let $(N, g_0)$ be the $C^\infty$ orientable Riemannian manifold of dimension $n \geq 3$, introduced in Section 2 and let $g$ be a metric on $N$ satisfying

$$\| g^{ij} \|_{W^{1, p}(N)} \leq E, \quad i, j = 1, \ldots, n, \quad (33)$$

where $p > n$ and $E$ is a positive constant. Let us consider the Laplace Beltrami operator on functions, associated to $g$, $\Delta_g = -\star_g d \star_g d$, which in coordinates is

$$\Delta_g = -\sum_{i, j = 1}^n |g|^{-\frac{1}{2}} \frac{\partial}{\partial x^i} \left\{ |g|^{\frac{1}{2}} g^{ij} \frac{\partial}{\partial x^j} \right\}, \quad \text{on} \quad N, \quad (34)$$

where $|g|$ denotes the determinant of $g_{ij}$. Clearly, for any chart on $N$, there exists a positive constant $\lambda$ such that $\Delta_g$ satisfies the ellipticity condition

$$\lambda^{-1} |\xi|^2 \leq g^{ij}(x)\xi_i\xi_j \leq \lambda |\xi|^2, \quad (35)$$

for all $x$ in the domain of the chart and all $\xi \in \mathbb{R}^n$. Here we denote the Euclidean norm on $\mathbb{R}^n$ simply by $| \cdot |$. Let us also consider the geodesic ball

$$B_{N,r}(\bar{x}) = \{ x \in N \mid d(x, \bar{x}) < r \},$$

where $d$ is the geodesic distance induced by $g_0$ and $\bar{x} \in N$. We will simply denote $B_{N,r}$ by $B_r$ when it will be clear from the context what is the manifold $N$ we are referring to. Let us denote $G = |g|^{\frac{1}{2}} g^{-1}$, where $g$ is the matrix $\{g_{ij}\}_{i,j=1}^n$ and $g^{-1}$ is its inverse $\{g^{ij}\}_{i,j=1}^n$. The following theorem provides the construction of singular solutions obtained in [2], [3], on a geodesic ball of a Riemannian manifold.

**THEOREM 3.1** (Singular solutions on manifolds). If $\Delta_g$ is the Laplace Beltrami operator satisfying (33)-(35), for any $m = 0, 1, 2, \ldots$ there exists $u \in W^{2, p}_{loc}(B_r \setminus \{\bar{x}\}) \cap W^{1, 2}(N)$ solution to

$$\Delta_g u = 0, \quad \text{in} \quad N, \quad (36)$$
such that there exist coordinates $(x^i)_{i=1}^n$ on $N$ with
\begin{equation}
    u(x) = |J(x - \bar{x})|^{2-n-m} S_m \left( \frac{J(x - \bar{x})}{|x - \bar{x}|} \right) + w(x), \quad \text{in} \quad B_r \setminus \{\bar{x}\},
\end{equation}
where $S_m$ is a spherical harmonic of degree $m$, $J = \sqrt{G^{-1}(\bar{x})}$ and $w$ satisfies
\begin{equation}
    |w(x)| + |x| |Dw(x)| \leq C |x|^{2-n-m+\alpha}, \quad \text{in} \quad B_r \setminus \{\bar{x}\},
\end{equation}
\begin{equation}
    \left( \int_{s<|x|<2s} |D^2w|^p \right)^{\frac{1}{p}} \leq C s^{-m-n+\alpha + \frac{n}{p}}, \quad \text{for every} \quad s, \ 0 < s < r/2.
\end{equation}
Here $\alpha$ is any number such that $0 < \alpha < 1 - \frac{n}{p}$, and $C$ is a constant depending only on $\alpha, n, p, r, \lambda$, and $E$. Furthermore
\begin{equation}
    \|du\|_{g_0} \leq C d(x, \bar{x})^{1-n-m}, \quad \text{for every} \quad x \in B_r(\bar{x}) \setminus \{\bar{x}\}
\end{equation}
\begin{equation}
    \|du\|_{g_0} > \frac{1}{2} d(x, \bar{x})^{1-n-m}, \quad \text{for every} \quad x \in B_{r_0}(\bar{x}) \setminus \{\bar{x}\},
\end{equation}
where $r_0$ is a positive constant which depends only on $\lambda, E, p, m$ and the diameter of $N$, $\text{diam}(N)$.

\textbf{Proof of Theorem 3.1} By [3, Theorem 3.4] and by choosing normal coordinates on $B_r(\bar{x})$ we can construct $u_m$ solution to
\begin{equation}
    \Delta_g u_m = 0, \quad \text{in} \quad x \in B_r(\bar{x}) \setminus \{\bar{x}\}
\end{equation}
and $u_m$ satisfies (37)-(39). By expressing $g_0$ in normal coordinates we obtain
\begin{equation}
    (g_0)_{ij}(x) = \delta_{ij} + O(d(x, \bar{x})^2),
\end{equation}
for any $x \in B_r(\bar{x})$, where the geodesic distance $d$ induced by $g_0$ satisfies $d(x, \bar{x}) = |x - \bar{x}|$ on $B_r$. Therefore
\begin{equation}
    \|du\|_{g_0}^2 = g_0^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j}
    = \left( \delta_{ij} + O(d(x, \bar{x})^2)^{ij} \right) \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j}
    \leq |Du|^2 + O(d(x, \bar{x})^2)|Du|^2
    \leq C |Du|^2
    \leq C d(x, \bar{x})^{2-2(n+m)},
\end{equation}
where $C > 0$ is a constant depending on $n$, $\text{diam}(N)$ and $Du$ is the gradient of $u$ in $\mathbb{R}^n$.
By combining (43) with [3, Lemma 3.5],
\begin{equation}
    \|du\|_{g_0}^2 > \frac{1}{4} d(x, \bar{x})^{2-2(n-m)}, \quad \text{on} \quad B_r(\bar{x})
\end{equation}
and this concludes the proof. \hfill □
4. Proofs of main results

We will only give the proofs of Theorems 2.3, 2.4 as proofs of Theorem 2.5, and Corollary 2.6 follow the same line of these proofs and arguments in [2] and [3].

Since the boundary $\partial M$ is Lipschitz, the normal unit vector field might not be defined on $\partial M$. Therefore, we consider the vector field $\tilde{\nu}$ introduced in Definition 2.3, instead. $\tilde{\nu}$ is locally defined near $\partial M$, it is $C^\infty$ smooth and it is not tangential to $\partial M$. With the same arguments used in [3, Section 3] we can state the following

LEMMA 4.1 For any $x^0 \in \partial M$, let $C_{r,h}$ be the cylinder introduced in Definition 2.3, such that $x^0 \in C_{r,h}$, then the point

$$z_\sigma = \rho_{x^0, \tilde{\nu}}(\sigma)$$

satisfies

$$C \tau \leq d(z_\tau, \partial M) \leq \tau,$$

for any $0 \leq \tau \leq \tau^0$, (47)

where $\tau^0$ and $C$ depend only on $L, r, h$.

Proof. The proof follows by rephrasing arguments of [3, Lemma 3.3] and by substituting the Euclidean distance with the geodesic one. ■

LEMMA 4.2 If $g \in \mathcal{H}$ and $a$ is a function satisfying conditions (20), (21), we have

$$|g(\cdot, a(\cdot))|^{\frac{1}{2}}g^{-1}(\cdot, a(\cdot)) \in W^{1,p}(M, T_0^2(M)).$$

(48)

Proof. The proof is a straightforward consequence of [3, Lemma 3.6] and conditions (15)-(17) of Remark 2.2. ■

Proof of Theorem 2.3. We start by recalling the identity (see [2, (b), p. 253], [6, (6.35), p.99])

$$\langle (\Lambda_{g(x,a(x))} - \Lambda_{g(x,b(x))}) u, v \rangle = (-1)^{n-1} \int_M (\ast_{g(x,a)} - \ast_{g(x,b)}) du \wedge dv,$$

which holds for any $u, v$ solutions to the Laplace-Beltrami equations

$$\Delta_{g(x,a)} u = \Delta_{g(x,b)} v = 0,$$

in $M$. (50)

With no loss of generality we suppose that $(-1)^{n-1} = 1$, the case when $(-1)^{n-1} = -1$ can be treated in a very similar way. Let $x^0 \in \partial M$ be a point such that

$$(a - b)(x^0) = \| a - b \|_{L^\infty(\partial M)}.$$

Let $0 < \tau \leq \{\tau_0, \frac{r_0}{4}\}$, where $\tau_0$ is the number fixed in (47) and $r_0$ is the number appearing in (41). We consider $z_\tau = \rho_{x^0, \tilde{\nu}}(\tau)$, where $\tilde{\nu}$ is the outer unit vector field at the boundary $\partial M$ introduced in Definition 2.3 and $\rho_{x^0, \tilde{\nu}}(\tau)$ is the geodesic introduced in Definition 2.1. Any point in the geodesic ball $B_\eta(z_\tau)$, with $\eta = r_0$ and $r_0$ small enough
so that there are no cut points in \( B_\eta(z_\tau) \), is uniquely connected with the center \( z_\tau \) by the unique shortest geodesic. By fixing \( m \), let \( u_m, v_m \) be the two singular solutions of
\[
\Delta g(x, a) u_m = \Delta g(x, b) v_m = 0, \quad \text{in} \quad B_\eta(z_\tau) \setminus \{z_\tau\}.
\]

obtained in Theorem 3.1. The manifold \( M \) can be enlarged by introducing
\[
M_{\tau/2} := \{ x \in N \mid d(x, \partial M) < \tau/2 \}.
\]

Therefore
\[
\Delta g(x, a) u = 0, \quad \text{in} \quad M
\]
\[
u = u_m + \tilde{w}, \quad \text{in} \quad B_{\eta/2}(z_\tau) \cap M
\]
\[
u = \tilde{w}, \quad \text{in} \quad M \setminus B_\eta(z_\tau),
\]

where
\[
\| \tilde{w} \|_{W^{1,2}(M)} \leq C
\]

and \( C > 0 \) is a constant which depends on \( n, m, L, r \) and \( h \) only. The same argument can be applied to the singular solution \( v_m \) and, by setting \( m = 0 \), (49) leads to
\[
\langle (\Lambda g(x, a(x)) - \Lambda g(x, b(x))) u, v \rangle = \int_{M \setminus B_\eta(z_\tau)} (\ast g(x, a) - \ast g(x, b)) du \wedge dv
\]
\[
+ \int_{M \setminus B_\eta(z_\tau)} (\ast g(x, a) - \ast g(x, b)) du \wedge dv, \quad (53)
\]

where \( u \) and \( v \) are the solutions (51) of (50) for \( m = 0 \). By possibly reducing \( \eta \)
\[
u = u_0 + \tilde{w}, \quad v = v_0 + \tilde{w}, \quad \text{in} \quad B_\eta(z_\tau),
\]

where \( \tilde{w} \) satisfies (52). (53) leads in any coordinate system to
\[
\left\| \Lambda_{g(x, a(x))} - \Lambda_{g(x, b(x))} \right\|_* \leq C_1, \quad J_a \left( |g(x, b)|^{\frac{1}{2}} g^{-1}(x, a) - |g(x, b)|^{\frac{1}{2}} g^{-1}(x, b) \right) J^2_a (x - z_r) \cdot (x - z_r)
\]

where \(C_1\) is a positive constant depending on \(n, m, L, r, h\) and \(\text{diam}(M)\) only. By choosing normal coordinates centered in \(z_r\) on \(B_\eta(z_r)\) and by combining (55) with (37), we obtain

\[
\left\| \Lambda_{g(x, a(x))} - \Lambda_{g(x, b(x))} \right\|_* \leq C_1, \quad J_a \left( |g(x, b)|^{\frac{1}{2}} g^{-1}(x, a) - |g(x, b)|^{\frac{1}{2}} g^{-1}(x, b) \right) J^2_a (x - z_r) \cdot (x - z_r)
\]

By recalling that \(|g(x, a)|^{\frac{1}{2}} g^{-1}(x, a)\) is Hölder continuous (see Lemma 4.2 and [3]),

\[
\left\| \Lambda_{g(x, a(x))} - \Lambda_{g(x, b(x))} \right\|_* \leq C_1, \quad J_a \left( |g(x, b)|^{\frac{1}{2}} g^{-1}(x, a) - |g(x, b)|^{\frac{1}{2}} g^{-1}(x, b) \right) J^2_a (x - z_r) \cdot (x - z_r)
\]

By recalling that \(J_a^2 = g(z_r, a)|g(z_r, a)|^{-\frac{1}{2}}\) and similarly \(J_b^2 = g(z_r, b)|g(z_r, b)|^{-\frac{1}{2}}\), we get (see [3])

\[
J_b^2 \left( |g(x, b)|^{\frac{1}{2}} g^{-1}(x, b) - |g(x, b)|^{\frac{1}{2}} g^{-1}(x, b) \right) J^2_b (x - z_r) \cdot (x - z_r)
\]

The function \(t \rightarrow g(x^0, t)|g(x^0, t)|^{-\frac{1}{2}}\) is absolutely continuous (see [18, Lemma 3.1.1]) and by combining it with (11),
\[
\left( g_{ij}(x^0, b)|g(x^0, b)|^{-\frac{1}{2}} - g_{ij}(x^0, a)|g(x^0, a)|^{-\frac{1}{2}} \right) (x - z_\tau)^i(x - z_\tau)^j
\]
\[
= \int_{a(x^0)}^{b(x^0)} \left[ D_t \left( g(x^0, t)|g(x^0, t)|^{-\frac{1}{2}} \right) \right] (x - z_\tau)^i(x - z_\tau)^j dt
\]
\[
= \int_{a(x^0)}^{b(x^0)} -|g(x^0, t)|^{-1} g_{ij} D_t \left( g^{jk}(x^0, t)|g(x^0, t)|^{\frac{1}{2}} \right) \cdot
\]
\[
\cdot g_{kj}(x^0, t)(x - z_\tau)^i(x - z_\tau)^j dt
\]
\[
= \int_{b(x^0)}^{a(x^0)} |g(x^0, t)|^{-1} D_t \left( g^{jk}(x^0, t)|g(x^0, t)|^{\frac{1}{2}} \right) \cdot
\]
\[
\cdot (g_{ik}(x^0, t)(x - z_\tau)^k)(g_{ji}(x^0, t)(x - z_\tau)^j) dt
\]
\[
\geq E^{-1} E^{-1} \int_{b(x^0)}^{a(x^0)} \parallel \theta \parallel_{g_0}^2 dt,
\]
where \( \theta = \theta(x^0, z_\tau, x, t) \) and \( \theta = \theta_t(x^0, z_\tau, x, t) dx^i \in \Omega^1(M) \). If we recall that in normal coordinates we have
\[
\parallel \theta \parallel_{g_0}^2 = |\theta|^2 + \left[ d(x, z_\tau)^2 \right]_{ij} \theta_i \theta_j > \frac{1}{2} |\theta|^2
\]
and we combine together (57), (58) with (10), we obtain
\[
\left( g_{ij}(x^0, b)|g(x^0, b)|^{-\frac{1}{2}} - g_{ij}(x^0, a)|g(x^0, a)|^{-\frac{1}{2}} \right) (x - z_\tau)^i(x - z_\tau)^j
\]
\[
\geq \frac{1}{2} E^{-1} \lambda^{-2} E^{-3} (a - b)(x^0)|x - z_\tau|^2.
\]
Hence, we have
\[
J_b^2 \left( |g(x^0, a)|^{\frac{1}{2}} g^{-1}(x^0, a) - |g(x^0, b)|^{\frac{1}{2}} g^{-1}(x^0, b) \right) J_a^2
\]
\[
\geq \left( \frac{1}{2} E^{-1} \lambda^{-2} E^{-3} - C \tau^3 \right) (a - b)(x^0)|x - z_\tau|^2
\]
and, choosing
\[
\tau \leq \left( \frac{1}{4 C} E^{-1} \lambda^{-2} E^{-2} \right)^\frac{1}{3},
\]
we obtain
\[
J_b^2 \left( |g(x^0, a)|^{\frac{1}{2}} g^{-1}(x^0, a) - |g(x^0, b)|^{\frac{1}{2}} g^{-1}(x^0, b) \right) J_a^2 \geq C(a - b)(x^0)|x - z_\tau|^2.
\]
Therefore
\[ \| a - b \|_{L^\infty(\partial M)} \int_{M \cap B_\tau(z_\tau)} |x - z_\tau|^{2n} \, dx \]
\[ \leq C \left\{ \int_{M \cap B_\tau(z_\tau)} |x - z_\tau|^{2n+\alpha} \, dx + \int_{M \cap B_\tau(z_\tau)} |x - z_\tau|^{2n} |x - x_0|^{\beta} \, dx + C_1 + C_2 + \| \Lambda g(x, a(x)) - \Lambda g(x, b(x)) \|_* \| u \|_{H_\frac{1}{2}(\partial M)} \| v \|_{H_\frac{1}{2}(\partial M)} \right\} \]
and by estimating the above integrals and the \( H_{\frac{1}{2}}(\partial M) \) norms of \( u \) and \( v \) (see [2], [3]) we finally obtain
\[ \| a - b \|_{L^\infty(\partial M)} \tau^{2-n} \leq C \left\{ \tau^{2-n+\alpha} + \tau^{2-n+\beta} + C_1 + C_2 \right\}. \]
If we let \( \tau \to 0 \) we obtain the following inequality
\[ \| a - b \|_{L^\infty(\partial M)} \leq C \| \Lambda g(x, a) - \Lambda g(x, b) \|_* \cdot \] (60)
Recalling that, for almost every \( x \in \Omega \), the function
\[ t \to g(x, t) \]
is absolutely continuous on \([\lambda^{-1}, \lambda]\) we may write
\[ |g(x, a(x)) - g(x, b(x))| = \left| \int_{b(x)}^{a(x)} D_t g(x, t) \, dt \right| \]
\[ \leq \int_{b(x)}^{a(x)} \sup_{t, x} |D_t g(x, t)| \, dt \]
\[ \leq C |(a(x) - b(x))|, \]
for every \( x \in M \). Taking the \( L^\infty \)-norm on both sides, we obtain
\[ \| g(x, a) - g(x, b) \|_{L^\infty(\partial M)} \leq C \| a - b \|_{L^\infty(\partial M)} \cdot \] (61)
By combining (60) and (61) we conclude the proof. ■

**Proof of Theorem 2.4.** Let \( \tilde{\nu} \) be the vector field introduced in Definition 2.3. By following the same line of [3, proof of Theorem 2.2] and arguments of the proof of Theorem 2.3, we obtain
\[ \| \frac{\partial^j}{\partial \nu^j} (a - b) \|_{L^\infty(\partial M \cap W)} \leq C \| \Lambda g(x, a) - \Lambda g(x, b) \|_{\delta_j} \]
for any \( j \leq k \). (62)
in boundary normal coordinates on $\partial M \cap \bar{W}$, where $\delta_j = \prod_{i=0}^{j-1} \frac{\alpha}{\alpha + j}$. By recalling the interpolation inequality

$$\|Df\|_{L^\infty(\partial M \cap \bar{W})} \leq C \left\{ \| \frac{\partial}{\partial D} f\|_{L^\infty(\partial M)} + \| f\|_{C^{\alpha/(1+\alpha)}(\partial M \cap \bar{W})} \right\},$$

(63)

for any $f \in C^{1,\alpha}(M)$ (see [2, Lemma 3.2], [3, estimate (3.38)]) and combining it with (62), we obtain

$$\|D^k(a - b)\|_{L^\infty(\partial M \cap \bar{W})} \leq C \| \Lambda_{g(x, a)} - \Lambda_{g(x, b)}\|_{\delta_k}^{\delta_k}.$$

(64)

Here $D$ denotes the Euclidean gradient and if we observe that

$$D\beta g(x, a(x)) = \sum_{\gamma + \delta \leq \beta} P_\gamma \delta(a(x), \ldots, D|\delta| a(x)) \cdot D^\beta x D|\delta| g(x, a(x)),$$

where $\beta$ is any multiindex and $P_\gamma \delta$ is a polynomial in the variables $p = (p_\eta), |\eta| \leq |\delta|$ (see [3, equality (3.40)]), we obtain

$$\|D^k(g(x, a) - g(x, b))\|_{L^\infty(\partial M \cap \bar{W})} \leq C \| a - b\|_{C^\alpha(\partial M \cap \bar{W})}.$$

(65)

(65) and (64) leads to

$$\|D^k(g(x, a) - g(x, b))\|_{L^\infty(\partial M \cap \bar{W})} \leq C \| \Lambda_{g(x, a)} - \Lambda_{g(x, b)}\|_{\delta_k}^{\delta_k \alpha}.$$

(66)

in boundary normal coordinates. Let $Z$ be a smooth vector field on $M$, and if $Z = Z_k \frac{\partial}{\partial x^k}$ in boundary normal coordinates on $\bar{W}$ and the Einstein convention on indices summation has been applied, then

$$\nabla_Z \left( g_{ij}(x, a) - g_{ij}(x, b) \right) = Z^k \left\{ \frac{\partial}{\partial x^k} \left( g_{ij}(x, a) - g_{ij}(x, b) \right) 
- \Gamma^l_{ik} \left( g_{lj}(x, a) - g_{lj}(x, b) \right) 
- \Gamma^l_{jk} \left( g_{il}(x, a) - g_{il}(x, b) \right) \right\},$$

(67)

where $\Gamma^l_{ik}$ are the Christoffel symbols of $\nabla$ with respect to the boundary normal coordinates. Therefore we obtain

$$\|\nabla_Z(g(x, a) - g(x, b))\|_{L^\infty(\partial M \cap \bar{W})} \leq \|Z\|_{L^\infty(\partial M \cap \bar{W})} \cdot \left\{ \|D(g(x, a) - g(x, b))\|_{L^\infty(\partial M \cap \bar{W})} 
+ 2C_1\|g(x, a) - g(x, b)\|_{L^\infty(\partial M \cap \bar{W})} \right\},$$

(68)

where $C_1$ is a positive constant depending on the Christoffel symbols appearing in (67). By combining (66) for $k = 1$ and (22) with (68) we obtain
\[ \| \nabla Z (g(x, a) - g(x, b)) \|_{L^\infty(\partial M\cap\bar{W})} \leq C_2 \| \Lambda_{g(x,a)} - \Lambda_{g(x,b)} \|_{*}^{\delta_1} \cdot \left( 1 + 2C_3 \| \Lambda_{g(x,a)} - \Lambda_{g(x,b)} \|_{*}^{1-\delta_1} \right) \]

where \( C_2 \) is a positive constant depending only on \( \| Z \|_{L^\infty(\partial M\cap W)} \), \( n, p, L, r, h, \text{diam}(M), \text{dist}(W \cap \partial M, M \setminus U) \), \( \lambda, E, \mathcal{E}, \alpha, k \) and \( E_k \) and \( C_3 \) is a positive constant depending on \( n, p, L, r, h, \text{diam}(M), \lambda, E, \mathcal{E} \) and the Christoffel symbols of the connection \( \nabla \) with respect to the boundary normal coordinates. Without loss of generality we can assume that

\[ \| \Lambda_{g(x,a)} - \Lambda_{g(x,b)} \|_{*} \leq 1, \]  

in fact, if the opposite inequality holds, then (68) would lead to

\[ \| \nabla Z (g(x, a) - g(x, b)) \|_{L^\infty(\partial M\cap W)} \leq \| Z \|_{L^\infty(\partial M\cap W)} (C(E_k)) \]
\[ \leq \| Z \|_{L^\infty(\partial M\cap W)} (C(E_k)) \| \Lambda_{g(x,a)} - \Lambda_{g(x,b)} \|_{*}, (71) \]

Therefore, by and (69) and (70) we finally obtain

\[ \| \nabla Z (g(x, a) - g(x, b)) \|_{L^\infty(\partial M\cap W)} \leq C \| \Lambda_{g(x,a)} - \Lambda_{g(x,b)} \|_{*}^{\delta_1}, \]  

where \( C > 0 \) is a constant depending on \( n, p, L, r, h, \text{diam}(M), \text{dist}(W \cap \partial M, M \setminus U) \), \( \lambda, E, \mathcal{E}, \alpha, k, E_k, \| Z \|_{L^\infty(\partial M\cap W)} \) and the Christoffel symbols of the connection \( \nabla \) with respect to the boundary normal coordinates. The desired estimate (27) for a general \( k \) can be obtained by applying the above argument to

\[ \nabla^k_{Z} (g(x, a) - g(x, b)) = \nabla (\ldots \nabla (g(x, a) - g(x, b))) \cdot \underbrace{\ldots \nabla}_{k\text{-times}} \]

5. Conclusions

In this study we improve the results obtained in [3] in the following aspects.

i) We give a geometric formulation of the inverse conductivity problem considered in [3], in dimension \( n > 2 \), where it is well known that the conductivity \( \sigma \) of a manifold uniquely determines a metric \( g \) such that \( \sigma = *_g \), where \( *_g \) is the Hodge star operator mapping 1-forms into \((n - 1)\)-forms (see [6], [13], [17]).

We prove results of uniqueness and stability at the boundary similar to [3, Theorems 2.1-2.3] and in the interior as in [3, Theorem 2.4], in the case where the body in question is a compact manifold with Lipschitz boundary embedded in an open \( C^\infty \) smooth Riemannian manifold \( N \) (Theorems 2.3-2.5 and Corollary 2.6 respectively);

ii) the so-called monotonicity assumption of [3, p.255] is here stated in terms of the Riemannian metric \( g_0 \) on \( N \). The case of a manifold with a flat metric \( g_0 \) will be still more general than the one treated in [3]. The case when \( (g_0)_{ij} = \delta_{ij} \) is the Euclidean metric on \( \mathbb{R}^n \) will lead to the monotonicity assumption given in [3].
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