Microlocal analysis of SAR imaging of a dynamic reflectivity function

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July 2, 2013

Abstract

In this article we consider four particular cases of Synthetic Aperture Radar imaging with moving objects. In each case, we analyze the forward operator F and the normal operator F^*F, which appear in the mathematical expression for the recovered reflectivity function (i.e. the image). In general, by applying the backprojection operator F^* to the scattered waveform (i.e. the data), artifacts appear in the reconstructed image. In the first case, the full data case, we show that F^*F is a pseudodifferential operator which implies that there is no artifact. In the other three cases, which have less data, we show that F^*F belongs to a class of distributions associated to two cleanly intersecting Lagrangians $I^{p,l}(\Delta, \Lambda)$, where Λ is associated to a strong artifact. At the and of the article, we show how to microlocally reduce the strength of the artifact.

1 Introduction

1.1 Background to the problem

In the Synthetic Aperture Radar (SAR) problem, an airplane carries an antenna that emits electromagnetic waves and records echoes as it moves along a flight path. More generally, the antenna emits electromagnetic waves which scatter from the ground and the reflected waves are detected with either the same antenna, or else by an independent (trans)ceiver located elsewhere. The received signals are used to produce an image of the ground (including objects on it).We do not attempt to image points that lie directly beneath the antenna because the imaging process involves application of a backprojection operator that is not well defined at such points. The ground and the objects on it manifest as inhomogeneities in the speed of propagation of electromagnetic waves and are therefore modeled by a singular perturbation in the light speed, $\frac{1}{c^2(x)} - \frac{1}{c_0^2} = q(x)$, where q is the “scene” on the ground we wish to image and c_0 is the speed of light in air, assumed to be constant. We assume that $supp(q)$ is approximately confined to a locally flat surface and express this as $q(x_1, x_2, x_3) = f(x_1, x_2)\delta(x_3)$.

In this article, we are interested in SAR imaging in the case that we have objects moving on the ground as time elapses. This means that q is now considered a function of space and time, i.e., $q = q(x, t)$. This is somewhat related to the problem considered in [2]. The latter paper considers a special case of what we consider here in that the authors consider a volume density of scatterers that move linearly in time. Our model is different, as we consider a more general reflectivity function and we are also able to microlocally diminish the associated artifacts from the image.

In addition, if an antenna is located with coordinates $y = (y_1, y_2, y_3)$, we emit a signal from it at a time $-T_y$. The activation time parameter T_y can be different for each source location y and adds flexibility. For example, a facet of a moving object may not be visible in the data unless the activation time is set appropriately. This is reflected in the description of the canonical relation C of the scattering operator F, as will be seen later. As is usual in the literature, we will only consider a scalar model for the electromagnetic wave. Thus, following the approach in [2], the mathematical model for the radio waves due to a point source, located at y and activated at time $-T_y$ is

$$\left(\Delta - \frac{1}{c^2(x,t)}\partial_t^2\right) u(y; x, t) = \delta(t + T_y)\delta(x - y), \tag{1}$$

where $u(y; x, t)$ is the wave field at (x, t) due to a point source located at y.

1.2 Derivation of the model

We write $u = u^{in} + u^{sc}$, where u^{in} is the incident field that satisfies

$$\left(\Delta - \frac{1}{c_0^2}\partial_t^2\right) u^{in}(y; x, t) = \delta(t + T_y)\, \delta(x - y). \tag{2}$$

We note that u^{in} is the fundamental solution for the wave equation. We linearize about $q = c_0^{-2}, u = u^{in}$ and a standard calculation shows that u^{sc} satisfies

$$\left(\Delta - \frac{1}{c_0^2}\partial_t^2\right) u^{sc}(y; x, t) = q(x, t)\, \frac{\partial^2 u}{\partial^2 t}\,(y; x, t). \tag{3}$$

Employing the Born approximation, i.e., replacing the total field u with the incident field u^{in} in the latter equation, we obtain

$$\left(\Delta - \frac{1}{c_0^2}\partial_t^2\right) u^{sc}(y; x, t) = q(x, t)\, \frac{\partial^2 u^{in}}{\partial^2 t}\,(y; x, t) \tag{4}$$

and convolve the right-hand side of the equation (4) with the fundamental solution to obtain the following integral expression for the scattered field

$$u^{sc}(y; z, t) = \int \frac{\delta(t - t' - \frac{|x-z|}{c_0})}{4\pi|x - z|} \frac{\partial_{t'}^2 \delta(t' + T_y - \frac{|x-y|}{c_0})}{4\pi|x - y|}\, q(x, t')\, dx\, dt'. \tag{5}$$

Note that we are implicitly assuming that the product on the right-hand side of (4) is well defined; this would be true for example if q has space-like singularities; $q(x, t) = \delta(t - v \cdot x)$ with $|v| \neq c_0^{-1}$.

In our set up, we think of y_1 as being a "slow time" parameter that describes the location of the antenna point source, as it moves along the flight track while t is considered as a "fast time" variable that is used for recording the radio wave echo time variation. There is nothing special about picking the y_1 variable for slow time, but in order to make concrete calculations, we will proceed with this choice.

Assumption 1.1. For the rest of the paper, we shall consider $T_y = \alpha y_1$, with $c_0\alpha > 1$, which reflects a set up where sources are activated monotonically along the y_1 axis at different times and the speed at which the sources are activated is the parameter α. This set up eases calculations and the reader will notice that the assumption $c_0\alpha > 1$ only excludes radar systems that move incredibly slowly.

Hence, rewriting the delta functions appearing in (5) as oscillatory integrals and rescaling time and the temporal frequencies reciprocally by a factor of c_0, we rewrite (5) and use it to define the forward operator F (which maps the model (q) to the data (u^{sc})) as follows:

$$Fq(y,z,t) = \int e^{i\,\phi(y,\,z,\,x,\,t,\,t',\,\omega,\,\omega')} a(x,y,z,t,\omega') q(x,t')\, d\omega d\omega' dx dt', \qquad (6)$$

where y is the transmitter location, z is the receiver location, t is the fast time at which the signal is recorded at the receiver, the phase $\phi(y,z,x,t,t',\omega,\omega')$ is explicitly given by

$$\phi(y,z,x,t,t',\omega,\omega') = \omega(t - t' - |x - z|) + \omega'(t' + c_0 \alpha y_1 - |x - y|)$$

and the amplitude $a(x,y,z,t,\omega')$ is given by

$$a(x,y,z,t,\omega') = \frac{c_0^2(\omega')^2}{|x-y||x-z|}\, m(y,z,t)\chi(\omega'). \qquad (7)$$

Here $(x,t') \in X \subset \mathbb{R}^4$, where X is the domain of the (generalized) function q; $(y,z,t) \in Y \subset \mathbb{R}^n$, where Y is the domain of the data (generalized) function Fq and n is a positive integer that depends on how much data we collect. The term $m(y,z,t)$ appearing in (7) is a smooth taper function which prevents artifacts that could arise in the image if the data recording was suddenly stopped either in time or location and χ is a cut-off function determined by using a band limited source whose temporal Fourier transform is supported away from zero. From now on it is understood that $\omega' \neq 0$ but ω can be 0 (see case 1).

1.3 Summary of results

We use microlocal methods to study the properties of the operator F which is a Fourier integral operator (FIO) associated to a canonical relation $C \subset T^*Y \times T^*X$ [10]. We consider the left and right projections of C; $\pi_L : C \longrightarrow T^*Y$, $\pi_R : C \longrightarrow T^*X$. Our ability to obtain a reasonable image of $q(x,t)$ depends strongly on the type of singularities that π_L, π_R can have. This in turn depends on the choice of the data set Y and in particular on its dimension. We consider the following 4 cases.

In the first case, also known as the full data case, we consider $y, z, x \in \mathbb{R}^3$. Thus we have a 7-dimensional data set from which we reconstruct a 4-dimensional image, i.e., an estimate of $q(x,t)$. In this case, we show that F is a FIO, π_L an injective immersion and π_R a submersion.

In the second case, we assume that the receiver and transmitter are the same, i.e., $y = z \in \mathbb{R}^3$, so we have a 4-dimensional data set and a 4-dimensional image. In this case, we show that both projections π_L and π_R have singularities called blowdowns (see section 2). Such canonical relations were studied before in [12] and in [7], [4] (where only one projection has blowdown singularities).

In the third case, the receiver and transmitter are both located at a distance h from the ground, i.e, $y = z$ and $y_3 = z_3 = h$ and the image location is on the surface $x_3 = 0$. Hence we have a 3-dimensional data set and we reconstruct a 3-dimensional image $q(x_1, x_2, t)$. In this case, both π_L and π_R have blowdown singularities.

In the last case, we assume that $y_3 = z_3 = h,\ y_2 = z_2,\ y_1 \neq z_1$. Thus we have a 4-dimensional data set and we reconstruct a 4-dimensional image and again both π_L and π_R have blowdown singularities.

To reconstruct the singularities of q (or, more precisely, its wavefront set), standard techniques suggest applying F^* to the data Fq to obtain F^*Fq, which often has the same wavefront set as q and is therefore a reasonable candidate for a reconstructed image. The normal operator F^*F that appears here is qualitatively different in each of the above 4 cases. In the first case, it is a pseudodifferential operator which indicates that by applying the backprojection operator no new singularities appear and the singularities in q that are visible in the data are faithfully reconstructed.

In the cases 2, 3, 4, the distribution kernel of the normal operator belongs to a class of operators whose wavefront relation consists of two Lagrangians: Δ, the diagonal in $T^*X \times T^*X$ which is responsible for the bona-fide part of the image and Λ, a flow-out from $\pi_R(\Sigma)$, where Σ represents the common set of singularities of projections π_L and π_R. This class of distributions is called $I^{p,l}(\Delta, \Lambda)$ and it will be described in section 2. The second Lagrangian is responsible for artifacts in the image. We can find the strength of the artifacts by finding the order of F^*F on $\Lambda \setminus \Delta$. It is shown in section 3 that these artifacts are stronger (case 2) or have equal strength with Δ (cases 3, 4).

A similar geometry with our case 1 appeared in the linearized inverse scattering problems studied by Nolan and Symes [15], where acoustic waves generated at the surface of the earth scatter off the heterogeneities in the subsurface and return to the surface. There, the goal is to use the pressure field at the surface to reconstruct an image of the subsurface.

In the case of a single source and receiver ranging over an open subset of the surface, Rakesh [17] showed that F is a FIO. Beylkin [1] showed that if caustics do not occur for the background soundspeed, F^*F is a pseudodifferential operator. For more general data acquisition geometries, the canonical relation of F depends on the sets of sources and receivers. It was proved in [15] and [16] that, if both sources and receivers vary over open and bounded subsets of the surface, then under the traveltime injectivity condition (TIC), generalizing the no-caustic assumption, F^*F is still a pseudodifferential operator.

In the last section, we will illustrate a microlocal method to decrease the strength of the artifacts. Similar work was done in [5] for canonical relations with a different structure. The idea is to apply a pseudodifferential operator Q to the data Fq before we apply the backprojection operator F^*, with the property that its principal symbol vanishes to some order $s > 1$ on $\pi_L(\Sigma)$. In this case, the artifact is still present but is weaker.

Note that we are implicitly assuming that q belongs to a Sobolev space, which can be justified on physical grounds. The strength of a singularity of such a distribution refers to the index of the Sobolev space to which the distribution belongs; the smaller the index, the stronger the singularity.

The paper is structured as follows. The second section is dedicated to the description of the composition of FIOs, the definition of the blowdowns and a review of the $I^{p,l}$ classes. In section 3 we prove our main results for the four cases and in the last section we show how we can reduce the strength of the artifacts in cases 2, 3 and 4 (there are no artifacts to be removed from case 1). Reducing the strength of the artifact will help an interpreter of the image to decide what is real and what is an artifact.

2 Singularities and $I^{p,l}$ classes

As seen in the introduction, we need to study the operator F^*F in order to recover the image. This is a composition of two FIOs and in general the composition of two FIOs is not a FIO, therefore we start this section by recalling some arguments regarding FIOs. We begin by recalling the notions of transversal and clean intersection between manifolds followed by the definition of some classes of FIOs.

Definition 2.1. Two submanifolds $M,\ N \subset X$ intersect *transversally* if $TM + TN = TX$.

Definition 2.2. Two submanifolds $M,\ N \subset X$ intersect *cleanly* if $M \cap N$ is a smooth submanifold and if $T(M \cap N) = TM \cap TN$.

Definition 2.3. We denote by $I^m(C)$ the class of FIOs, $F : \mathcal{E}'(\mathrm{X}) \to \mathcal{D}'(\mathrm{Y})$, of order m associated to a canonical relation $C \subset T^*(Y \times X) \setminus 0$.

We recall in the next proposition a result due to Hörmander [10].

Proposition 2.4. Under a transversal intersection condition, if $F_1 \in I^{m_1}(C_1)$, with $C_1 \subset T^*Y \times T^*Z$ and $F_2 \in I^{m_2}(C_2)$, with $C_2 \subset T^*Z \times T^*X$, then $F_1 \circ F_2 \in I^{m_1+m_2}(C_1 \circ C_2)$, where $C_1 \circ C_2 = \{(y, \eta; x, \xi) \in T^*Y \times T^*X |\ \exists (z, \tau) \in T^*Z; (y, \eta; z, \tau) \in C_1 \text{ and } (z, \tau; x, \xi) \in C_2\}$.

Duistermaat and Guillemin [3] and then Weinstein [18] extended the above result to a clean intersection condition as follows.

Proposition 2.5. Let F_i and C_i, $i = 1, 2$ be as above. Then, under a clean intersection condition we have that $F_1 \circ F_2 \in I^{m_1+m_2+\frac{e}{2}}(C_1 \circ C_2)$, where the number e is called the *excess* and measures how many dimensions we are away from having transversal intersection.

When the conditions of propositions 2.4, 2.5 are not satisfied, then the analysis of the so-called left and right projections of a canonical relation C plays an important role in the study of the normal operator F^*F. We recall their definition for sake of completeness.

Definition 2.6. If C is the canonical relation associated to an FIO $F : \mathcal{E}'(\mathrm{X}) \to \mathcal{D}'(\mathrm{Y})$, then we denote by π_L and π_R the so-called *left* and *right projections* of C, $\pi_L : C \to T^*Y \setminus 0$, $\pi_R : C \to T^*X \setminus 0$ respectively.

The following well-known result (see [11]) holds.

Proposition 2.7. Suppose dim X = dim Y, then we have

1. if either π_L or π_R is a local diffeomorphism, then C is a local canonical graph;

2. if one of the projections π_R or π_L is singular (i.e. its differential drops rank) then so is the other one. They may have different types of singularities even though they drop rank on the same set:

$$\Sigma = \{(y, \eta, x, \xi) \in C |\ det\ d\pi_L = 0\} = \{(y, \eta, x, \xi) \in C |\ det\ d\pi_R = 0\}. \tag{8}$$

Remark 2.8. Case 1 in proposition 2.7 is particularly useful since if we have two canonical relations C_i, $i = 1, 2$, where at least one of them is a local canonical graph, then the composition calculus is covered by the transverse intersection condition.

Next, we define similar concepts for dim $X \neq$ dim Y.

Definition 2.9. Suppose dim $X <$ dim Y and $f : X \to Y$, a smooth function. Then f is an immersion if df is injective.

Definition 2.10. Suppose dim $X >$ dim Y and $f : X \to Y$, a smooth function. Then f is a submersion if df is surjective.

Remark 2.11. Let $C \subset (T^*Y \setminus 0) \times (T^*X \setminus 0)$, with dim $X <$ dim Y, be a canonical relation. Similar to case 2 from proposition 2.7, π_L and π_R drop rank over the same set, thus if π_L is an immersion then π_R is a submersion and vice-versa. Moreover, if π_L is injective and $F \in I^m(C)$, then $C^t \circ C$ is covered by the clean intersection calculus and $F^*F \in I^{2m}(\Delta)$, i.e., F^*F is a pseudodifferential operator [8]. Here C^t denotes the transpose of C.

In the next section we will show that in case 2, the projections drop rank by 2 and in cases 3 and 4, they drop rank by 1. In all of these three cases the projections exhibit *blowdown* type singularities. Under these geometries, the composition operator F^*F is not an FIO anymore but its kernel belongs to a class of distributions associated to two *cleanly intersecting* Lagrangians $I^{p,l}(\cdot, \cdot)$, which will also be described at the end of this section.

To describe the other cases, we next define the blowdown singularities.

Definition 2.12. [12] Let M and N be manifolds of dimension n and let $f : N \to M$ be a C^∞ function. f is said to have a *blowdown* singularity along a smooth hypersurface $\Sigma \subset M$ if f is a local diffeomorphism away from Σ, df drops rank by k at Σ for some $k \in \mathbb{N}$, $\ker df \subset T(\Sigma)$ and the determinant of the jacobian matrix of f vanishes to order k at Σ.

Remark 2.13. The local canonical form of a blowdown for f is [12]
$f(x_1, x_2, \ldots, x_{n-k}, x_{n-k+1}, \ldots, x_n) = (x_1, x_2, \ldots, x_{n-k}, x_{n-k+1}x_1, \ldots, x_n x_1)$

We also need the following definitions.

Definition 2.14. A submanifold $M \subset T^*X$ is *non-radial* if $\rho \notin (TM)^\perp$, where $\rho = \sum \xi_i \partial_{\xi_i}$.

Definition 2.15. A submanifold $M \subset T^*X$, $M = \{(x,\xi) \mid p_i(x,\xi) = 0, 1 \leq i \leq k\}$ is *involutive* if the differentials $dp_i, i = 1, \ldots, k$ are linearly independent and the Poisson brackets satisfy $\{p_i, p_j\} = 0,\ i \neq j$.

Remark 2.16. For example, $M = \{(x,\xi) | \xi_1 = \xi_2 = 0\}$ is involutive.

Definition 2.17. [6] Let $\Gamma = \{(x,\xi) \mid p_i(x,\xi) = 0,\ 1 \leq i \leq k\}$ be a submanifold of T^*X. Then the *flowout of* Γ is given by $\{(x,\xi;y,\eta) \in T^*X \times T^*X \mid (x,\xi) \in \Gamma,\ (y,\eta) = \exp\left(\sum_{i=1}^{k} t_i H_{p_i}\right)(x,\xi),\ t \in \mathbb{R}^k\}$, where H_{p_i} is the Hamiltonian vector field of p_i.

Remark 2.18. For example, $\tilde{\Lambda}_0 = \{(x', x_n, \xi', 0; x', y_n, \xi', 0) \mid x' \in \mathbb{R}^{n-1},\ \xi' \in \mathbb{R}^{n-1} \setminus 0\}$ is the flowout of $\Gamma = \{(x,\xi)|\xi_n = 0\}$.

Theorem 2.19. [12] Let $C \subset (T^*Y \setminus 0) \times (T^*X \setminus 0)$ be a canonical relation satisfying: away from a hypersurface $\Sigma \subset C$, both projections π_L and π_R are diffeomorphisms and at Σ they are both blowdowns dropping rank by k and both $\pi_L(\Sigma)$ and $\pi_R(\Sigma)$ are non-radial and involutive. If $A \in I^m(C)$ and $B \in I^{m'}(C^t)$, then $BA \in I^{m+m'+\frac{k-1}{2}, -\frac{k-1}{2}}(\Delta, \Lambda_{\pi_R(\Sigma)})$, where Δ is the diagonal in $T^*X \times T^*X$ and $\Lambda_{\pi_R(\Sigma)}$ is the flow-out from $\pi_R(\Sigma)$.

We conclude this section by defining $I^{p,l}$ classes. They were first introduced by Melrose and Uhlmann [13], Guillemin and Uhlmann [9] and Greenleaf and Uhlmann [6], [7].

We will consider Lagrangian submanifolds in the product space $T^*X \times T^*Y$ with respect to the difference symplectic form $\omega_{T^*X} - \omega_{T^*Y}$. It is proved in [9] that any two pairs of cleanly intersecting Lagrangians $(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ and (Λ_0, Λ_1) are equivalent in the sense that we can find microlocally (in a conic neighborhood) a canonical transformation χ which takes (Λ_0, Λ_1) into $(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ and $(\Lambda_0 \cap \Lambda_1)$ into $(\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1)$. Thus, we may consider the following model case:

$$\tilde{\Lambda}_0 = \Delta_{T^*\mathbb{R}^n} = \{(x,\xi;x,\xi)|\ x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n \setminus 0\}, \tag{9}$$

which is the diagonal in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ and

$$\tilde{\Lambda}_1 = \{(x', x_n, \xi', 0; x', y_n, \xi', 0)|\ x' \in \mathbb{R}^{n-1},\ \xi' \in \mathbb{R}^{n-1} \setminus 0\}. \tag{10}$$

Notice that $\tilde{\Lambda}_0$ intersects $\tilde{\Lambda}_1$ cleanly in codimension 1. Next we will define the class of *product-type* symbols $S^{p,l}(m,n,k)$.

Definition 2.20. [9] $S^{p,l}(m,n,k)$ is the set of all functions $a(z,\xi,\sigma) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^k)$ such that for every $K \subset \mathbb{R}^m$ and every $\alpha \in \mathbb{Z}^m_+, \beta \in \mathbb{Z}^n_+, \gamma \in \mathbb{Z}^k_+$, there is $c_{K,\alpha,\beta,\gamma}$ such that

$$|\partial_z^\alpha \partial_\xi^\beta \partial_\sigma^\gamma a(z,\xi,\sigma)| \leq c_{K,\alpha,\beta,\gamma}(1+|\xi|)^{p-|\beta|}(1+|\sigma|)^{l-|\gamma|}, \forall (z,\xi,\sigma) \in K \times \mathbb{R}^n \times \mathbb{R}^k.$$

Definition 2.21. [7] For the model case $(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$, we define $I^{p,l}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ to be the set of all distributions u such that $u = u_1 + u_2$, with $u_1 \in C_0^\infty$ and

$$u_2(x,y) = \int e^{i((x'-y')\cdot\xi' + (x_n - y_n - s)\xi_n + s\sigma)} a(x,y,s;\xi,\sigma)\, d\xi\, d\sigma\, ds,$$

with $a \in S^{p',l'}(2n+1,n,1)$, where $p' = p - \frac{n}{2} + \frac{1}{2}$ and $l' = l - \frac{1}{2}$.

Now we can generalize this and give an invariant definition of the $I^{p,l}(\Lambda_0, \Lambda_1)$ class for any two cleanly intersecting Lagrangians in codimension 1.

Definition 2.22. [9] Let $I^{p,l}(\Lambda_0, \Lambda_1)$ be the set of all distributions u such that $u = u_1 + u_2 + \sum v_i$, where $u_1 \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$, $u_2 \in I^p(\Lambda_1 \setminus \Lambda_0)$, the sum $\sum v_i$ is locally finite and $v_i = Fw_i$, where F is a zero order FIO associated to χ^{-1}, the canonical transformation from above, and $w_i \in I^{p,l}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$.

This class of distributions is invariant under FIOs associated to canonical transformations which map the pair (Λ_0, Λ_1) to itself and preserve the intersection of Λ_0 and Λ_1. If $u \in I^{p,l}(\Lambda_0, \Lambda_1)$, then microlocally, $u \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$ and $u \in I^p(\Lambda_1 \setminus \Lambda_0)$ [9].

3 Main results

3.1 CASE 1

In the first case, the full data set, $x, y, z \in \mathbb{R}^3$ and the forward operator is given by

$$Fq(y,z,t) = \int e^{i\,\phi(y,\,z,\,x,\,t,\,t',\,\omega,\,\omega')}\, a(x,y,z,t,\omega')\, q(x,t')\, d\omega\, d\omega' dx\, dt', \quad (11)$$

where $\phi(y,z,x,t,t',\omega,\omega') = \omega(t - t' - |x-z|) + \omega'(t' + c_0\alpha y_1 - |x-y|)$ and the symbol $a \in S^2$ includes factors like geometrical spreading, etc, that we had before. It is easy to check that the F in (11) defines a FIO of order $2 + 2/2 - 11/4 = 1/4$, according to the formula for the order of an FIO from [10]. We make the following assumption.

Assumption 3.1. $x_3 \neq z_3$. Notice that this is a reasonable assumption given that sources and receivers will be obviously well separated from the scene on the ground.

We have the following result.

Theorem 3.2. Let $C \subset (T^*\mathbb{R}^7\backslash 0)\times(T^*\mathbb{R}^4\backslash 0)$ be the canonical relation of F. For each $(y,z,x,t,\omega') \in supp(a)$, assume $1 + c_0\alpha\frac{x_1-y_1}{|x-y|} > 0$. Then C has π_L an injective immersion and π_R a submersion and F^*F is a pseudodifferential operator of order $\frac{1}{2}$.

Proof of theorem 3.2: The canonical relation of F given in (11) is

$$\begin{aligned} C &= \Big\{(y,z,t,c_0\alpha\omega' - \partial_{y_1}|x-y|\omega', -\partial_{y_2}|x-y|\omega', -\partial_{y_3}|x-y|\omega', \\ &\quad -\partial_z|x-z|\omega,\omega;\ \ x,t',\partial_x|x-y|\omega' + \partial_x|x-z|\omega, \omega-\omega'), \\ &\quad t' = |x-y| - c_0\alpha y_1;\ t = |x-z| + |x-y| - c_0\alpha y_1\Big\} \\ &= \Big\{(y_1,y_2,y_3,z_1,z_2,z_3,t,c_0\alpha\omega' + \frac{x_1-y_1}{|x-y|}\omega', \frac{x_2-y_2}{|x-y|}\omega', \frac{x_3-y_3}{|x-y|}\omega', \\ &\quad \frac{x_1-z_1}{|x-z|}\omega, \frac{x_2-z_2}{|x-z|}\omega, \frac{x_3-z_3}{|x-z|}\omega, \omega; \\ &\quad x_1,x_2,x_3,t', \frac{x_1-z_1}{|x-z|}\omega + \frac{x_1-y_1}{|x-y|}\omega', \frac{x_2-z_2}{|x-z|}\omega + \frac{x_2-y_2}{|x-y|}\omega', \\ &\quad \frac{x_3-z_3}{|x-z|}\omega + \frac{x_3-y_3}{|x-y|}\omega', \omega-\omega'), \\ &\quad t' = |x-y| - c_0\alpha y_1;\ t = |x-z| + |x-y| - c_0\alpha y_1\Big\}, \end{aligned} \tag{12}$$

Notice that (x,y,z,ω,ω') constitute local coordinates on C. We check the singularities of the projections π_L,π_R. In considering the Jacobian matrices of π_L and π_R, it is often helpful to reorder the independent variables to identify sub-blocks of the Jacobian which are the identity matrix and then concentrate on the remainder of the Jacobian. We will do that without further comment from now on. We have

$$\pi_L(y,z,\omega,\omega',x) = \Bigg(y,z,\omega,-c_0\alpha y_1+|x-y|+|x-z|, c_0\alpha\omega'+\omega'\frac{x_1-y_1}{|x-y|}, \omega'\frac{x_2-y_2}{|x-y|},\omega'\frac{x_3-y_3}{|x-y|},\omega\frac{x_1-z_1}{|x-z|},\omega\frac{x_2-z_2}{|x-z|},\omega\frac{x_3-z_3}{|x-z|}\Bigg). \tag{13}$$

The Jacobian of $d\pi_L$ is a 14×11 matrix with the 7×7 identity block in the y,z,ω variables. Thus, to find the rank of π_L, it suffices to find the rank of the 7×4 matrix in ω', x:

$$\begin{pmatrix}
0 & \frac{x_1-z_1}{|x-z|}+\frac{x_1-y_1}{|x-y|} & \frac{x_2-z_2}{|x-z|}+\frac{x_2-y_2}{|x-y|} & \frac{x_3-z_3}{|x-z|}+\frac{x_3-y_3}{|x-y|} \\
c_0\alpha+\frac{x_1-y_1}{|x-y|} & \omega'\frac{(x_2-y_2)^2+(x_3-y_3)^2}{|x-y|^3} & -\omega'\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} & -\omega'\frac{(x_1-y_1)(x_3-y_3)}{|x-y|^3} \\
\frac{x_2-y_2}{|x-y|} & -\omega'\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} & \omega'\frac{(x_1-y_1)^2+(x_3-y_3)^2}{|x-y|^3} & -\omega'\frac{(x_3-y_3)(x_2-y_2)}{|x-y|^3} \\
\frac{x_3-y_3}{|x-y|} & -\omega'\frac{(x_1-y_1)(x_3-y_3)}{|x-y|^3} & -\omega'\frac{(x_3-y_3)(x_2-y_2)}{|x-y|^3} & \omega'\frac{(x_2-y_2)^2+(x_1-y_1)^2}{|x-y|^3} \\
0 & \omega\frac{(x_2-z_2)^2+(x_3-z_3)^2}{|x-z|^3} & -\omega\frac{(x_1-z_1)(x_2-z_2)}{|x-z|^3} & -\omega\frac{(x_1-z_1)(x_3-z_3)}{|x-z|^3} \\
0 & -\omega\frac{(x_1-z_1)(x_2-z_2)}{|x-z|^3} & \omega\frac{(x_1-z_1)^2+(x_3-z_3)^2}{|x-z|^3} & -\omega\frac{(x_3-z_3)(x_2-z_2)}{|x-z|^3} \\
0 & -\omega\frac{(x_1-z_1)(x_3-z_3)}{|x-z|^3} & -\omega\frac{(x_3-z_3)(x_2-z_2)}{|x-z|^3} & \omega\frac{(x_2-z_2)^2+(x_1-z_1)^2}{|x-z|^3}
\end{pmatrix}.$$

We consider first the case when $\omega\neq 0$. We have that $c_0\alpha+\frac{x_1-y_1}{|x-y|}\neq 0$ since $c_0\alpha>1$ and $\frac{x_1-y_1}{|x-y|}<1$. We consider the following 4×4 submatrix

$$\begin{pmatrix}
0 & \frac{x_1-z_1}{|x-z|}+\frac{x_1-y_1}{|x-y|} & \frac{x_2-z_2}{|x-z|}+\frac{x_2-y_2}{|x-y|} & \frac{x_3-z_3}{|x-z|}+\frac{x_3-y_3}{|x-y|} \\
c_0\alpha+\frac{x_1-y_1}{|x-y|} & \omega'\frac{(x_2-y_2)^2+(x_3-y_3)^2}{|x-y|^3} & -\omega'\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} & -\omega'\frac{(x_1-y_1)(x_3-y_3)}{|x-y|^3} \\
0 & \omega\frac{(x_2-z_2)^2+(x_3-z_3)^2}{|x-z|^3} & -\omega\frac{(x_1-z_1)(x_2-z_2)}{|x-z|^3} & -\omega\frac{(x_1-z_1)(x_3-z_3)}{|x-z|^3} \\
0 & -\omega\frac{(x_1-z_1)(x_2-z_2)}{|x-z|^3} & \omega\frac{(x_1-z_1)^2+(x_3-z_3)^2}{|x-z|^3} & -\omega\frac{(x_3-z_3)(x_2-z_2)}{|x-z|^3}
\end{pmatrix},$$

whose determinant is

$$\frac{\omega^2}{|x-z|^3}\,(x_3-z_3)\left(c_0\alpha+\frac{x_1-y_1}{|x-y|}\right)\left(1+\frac{(x-y)\cdot(x-z)}{|x-y||x-z|}\right).$$

We have that $x_3\neq z_3$, $c_0\alpha+\frac{x_1-y_1}{|x-y|}\neq 0$ as before and $1+\frac{(x-y)\cdot(x-z)}{|x-y||x-z|}\neq 0$ since otherwise the vectors $x-y$ and $x-z$ must be collinear and point in opposite directions which is not possible. Thus, this determinant is nonzero, which means that the matrix $d\pi_L$ has maximal rank 11. It follows from definition 2.9 that π_L is an immersion and from Remark 2.11 that π_R is a submersion.

Next we show the injectivity of π_L. For this it is enough to show that $\pi_L(y, z, \omega, x, \omega') = \pi_L(y', z', \hat{\omega}, x', \hat{\omega}')$ implies that

$$(y, z, \omega, x, \omega') = (y', z', \hat{\omega}, x', \hat{\omega}')$$

or, more precisely, that $x = x'$ and $\omega' = \hat{\omega}'$. We use the following relations

$$|x - y| + |x - z| = |x' - y| + |x' - z| \tag{14}$$

$$\frac{x_1 - z_1}{|x - z|} = \frac{x_1' - z_1}{|x' - z|} \tag{15}$$

$$\frac{x_2 - z_2}{|x - z|} = \frac{x_2' - z_2}{|x' - z|} \tag{16}$$

$$\frac{x_3 - z_3}{|x - z|} = \frac{x_3' - z_3}{|x' - z|} \tag{17}$$

$$\omega' \frac{x_3 - y_3}{|x - y|} = \hat{\omega}' \frac{x_3' - y_3}{|x' - y|} \tag{18}$$

and rewrite them in prolate spherical coordinates. Via a translation, we can assume that $y = (a, 0, h)$ and $z = (-a, 0, h)$. Thus we get

$$\begin{aligned} x_1 &= a \cosh \rho \cos \phi \\ x_2 &= a \sinh \rho \sin \phi \cos \theta \\ x_3 &= h + a \sinh \rho \sin \phi \sin \theta, \end{aligned}$$

where $\rho > 0,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi$ and similarly for x' we get

$$\begin{aligned} x_1' &= a \cosh \rho' \cos \phi' \\ x_2' &= a \sinh \rho' \sin \phi' \cos \theta' \\ x_3' &= h + a \sinh \rho' \sin \phi' \sin \theta'. \end{aligned}$$

We will show that $\theta = \theta',\ \rho = \rho',\ \phi = \phi', \omega' = \hat{\omega}'$. Relation (14) becomes

$$a(\cosh \rho - \cos \phi) + a(\cosh \rho + \cos \phi) = a(\cosh \rho' - \cos \phi') + a(\cosh \rho' + \cos \phi'),$$

hence $\cosh\rho = \cosh\rho' \Rightarrow \rho = \rho'$. Relation (15) becomes

$$\frac{\cosh\rho\cos\phi + 1}{\cosh\rho + \cos\phi} = \frac{\cosh\rho\cos\phi' + 1}{\cosh\rho + \cos\phi'},$$

from which we get $\cos\phi = \cos\phi'$ and $\phi = \phi'$. Relation (16) becomes

$$\frac{\sinh\rho\sin\phi\cos\theta}{\cosh\rho + \cos\phi} = \frac{\sinh\rho\sin\phi'\cos\theta'}{\cosh\rho + \cos\phi'},$$

from which $\cos\theta = \cos\theta'$. Relation (17) becomes

$$\frac{\sinh\rho\sin\phi\sin\theta}{\cosh\rho + \cos\phi} = \frac{\sinh\rho\sin\phi'\sin\theta'}{\cosh\rho + \cos\phi'}$$

and we get $\sin\theta = \sin\theta'$. Thus $\theta = \theta'$ and $x = x'$. From relation (18) we get $\omega' = \hat{\omega}'$. Thus π_L is injective.

Now we consider the case when $\omega = 0$ and from the matrix of $d\pi_L$ we consider the 4×4 submatrix

$$\begin{pmatrix} 0 & \frac{x_1-z_1}{|x-z|} + \frac{x_1-y_1}{|x-y|} & \frac{x_2-z_2}{|x-z|} + \frac{x_2-y_2}{|x-y|} & \frac{x_3-z_3}{|x-z|} + \frac{x_3-y_3}{|x-y|} \\ c_0\alpha + \frac{x_1-y_1}{|x-y|} & \omega'\frac{(x_2-y_2)^2+(x_3-y_3)^2}{|x-y|^3} & -\omega'\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} & -\omega'\frac{(x_1-y_1)(x_3-y_3)}{|x-y|^3} \\ \frac{x_2-y_2}{|x-y|} & -\omega'\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} & \omega'\frac{(x_1-y_1)^2+(x_3-y_3)^2}{|x-y|^3} & -\omega'\frac{(x_3-y_3)(x_2-y_2)}{|x-y|^3} \\ \frac{x_3-y_3}{|x-y|} & -\omega'\frac{(x_1-y_1)(x_3-y_3)}{|x-y|^3} & -\omega'\frac{(x_3-y_3)(x_2-y_2)}{|x-y|^3} & \omega'\frac{(x_2-y_2)^2+(x_1-y_1)^2}{|x-y|^3} \end{pmatrix},$$

whose determinant is

$$-\frac{\omega'^2}{|x-y|^2}\left(1 + \frac{(x-y)\cdot(x'-y)}{|x-y||x'-y|}\right)\left(1 + c_0\alpha\frac{x_1-y_1}{|x-y|}\right),$$

which is nonzero since $1 + c_0\alpha\frac{x_1-y_1}{|x-y|} > 0$ by the assumption in the theorem, $\omega' \neq 0$, and $1 + \frac{(x-y)\cdot(x'-y)}{|x-y||x'-y|} \neq 0$ by the same argument described above. Thus, π_L is an immersion.

To prove the injectivity of π_L we consider the relations:

$|x-y| + |x-z| = |x'-y| + |x'-z| \quad (1')$

$\omega' c_0\alpha + \omega'\frac{x_1-y_1}{|x-y|} = \hat{\omega}' c_0\alpha + \hat{\omega}'\frac{x_1'-y_1}{|x'-y|} \quad (2')$

$\omega'\frac{x_2-y_2}{|x-y|} = \hat{\omega}'\frac{x_2'-y_2}{|x'-y|} \quad (3')$

$\omega'\frac{x_3-y_3}{|x-y|} = \hat{\omega}'\frac{x_3'-y_3}{|x'-y|}. \quad (4')$

We square the last three relations and add them term by term and after simplification we obtain:

$$(c_0\alpha\omega')^2+(\omega')^2+2c_0\alpha(\omega')^2\frac{x_1-y_1}{|x-y|}=(c_0\alpha\hat{\omega}')^2+(\hat{\omega}')^2+2c_0\alpha(\hat{\omega}')^2\frac{x_1'-y_1}{|x'-y|}. \quad (5')$$

From the second relation we get:

$$\hat{\omega}'\frac{x_1'-y_1}{|x'-y|}=\omega' c_0\alpha+\omega'\frac{x_1-y_1}{|x-y|}-\hat{\omega}' c_0\alpha \quad (6')$$

and we substitute this in $(5')$ and get:

$$(c_0\alpha\omega')^2+(\omega')^2+2c_0\alpha(\omega')^2\frac{x_1-y_1}{|x-y|}=-(c_0\alpha\hat{\omega}')^2+(\hat{\omega}')^2+2(c_0\alpha)^2\omega'\hat{\omega}'+2c_0\alpha\omega'\hat{\omega}'\frac{x_1-y_1}{|x-y|} \quad (7')$$

After putting all the terms on the left hand side we have:

$$(c_0\alpha)^2(\omega'-\hat{\omega}')^2+(\omega'-\hat{\omega}')(\omega'+\hat{\omega}')+(\omega'-\hat{\omega}')\omega' 2c_0\alpha\frac{x_1-y_1}{|x-y|}=0 \quad (8')$$

or

$$(\omega'-\hat{\omega}')[(c_0\alpha)^2(\omega'-\hat{\omega}')+\omega'+\hat{\omega}'+2c_0\omega'\alpha\frac{x_1-y_1}{|x-y|}]=0. \quad (9')$$

If $\omega'=\hat{\omega}'$, then we can use prolate coordinates to get $x=x'$ as we did above for $\omega\neq 0$.

If $\omega'\neq\hat{\omega}'$, we show that the second paranthesis cannot be 0. We suppose that it is 0 and we obtain:

$$\omega'((c_0\alpha)^2+2c_0\alpha\frac{x_1-y_1}{|x-y|}+1)=\hat{\omega}'((c_0\alpha)^2-1). \quad (10')$$

By completing the square in the first paranthesis we have that it is always positive. So is the other paranthesis since we assumed $c_0\alpha>1$. Thus ω' and $\hat{\omega}'$ have the same sign. Next we complete the square in the first paranthesis and use relations $(2')$, $(3')$, $(4')$ again. We get:

$$\omega'((c_0\alpha+\frac{x_1-y_1}{|x-y|})^2+(\frac{x_2-y_2}{|x-y|})^2+(\frac{x_3-y_3}{|x-y|})^2)=\hat{\omega}'((c_0\alpha)^2-1)$$

and

$$\hat{\omega}'((c_0\alpha+\frac{x_1'-y_1}{|x'-y|})^2+(\frac{x_2'-y_2}{|x'-y|})^2+(\frac{x_3'-y_3}{|x'-y|})^2)=\omega'((c_0\alpha)^2-1)$$

and

$$\hat{\omega}'((c_0\alpha)^2 + 2c_0\alpha\frac{x_1' - y_1}{|x' - y|} + 1) = \omega'((c_0\alpha)^2 - 1). \quad (11')$$

Now we subtract relations ($10'$) and ($11'$) and we get:

$$\omega'(1 + c_0\alpha\frac{x_1 - y_1}{|x - y|}) = -\hat{\omega}'(1 + c_0\alpha\frac{x_1' - y_1}{|x' - y|}).$$

Now, ω' and $\hat{\omega}'$ have the same signs and both $1+c_0\alpha\frac{x_1-y_1}{|x-y|}$ and $1+c_0\alpha\frac{x_1'-y_1}{|x'-y|}$ are positive so we get a contradiction. Thus the second paranthesis in ($9'$) cannot be 0 and hence $\omega' = \hat{\omega}'$ and π_L is injective.

Finally from [8], we can conclude that $C^t \circ C$ and F^*F are covered by the clean intersection calculus and hence F^*F is a pseudodifferential operator of order 1/2.

3.2 CASE 2

In this case we assume that $y = z$, resulting in less data than in case 1. In this case, the transmitter and receiver are coincident and so we are collecting back-scattered data. The forward operator is here given by

$$Fq(y,t) = \int e^{i\,\phi(y,\,x,\,t,\,t',\,\omega,\,\omega')}\, a(x,y,t,\omega')\, q(x,t')\, d\omega\, d\omega'\, dx\, dt', \qquad (19)$$

where $\phi(y,\,x,\,t,\,t',\,\omega,\,\omega') = \omega(t-t'-|x-y|)+\omega'(t'+c_0\alpha y_1-|x-y|)$, $a \in S^2$ and F is an FIO of order $2 + 2/2 - 8/4 = 1$. Analogous to assumption 3.1, we assume that

Assumption 3.3. $x_3 \neq y_3$.

We have the following result.

Theorem 3.4. Let $C \subset (T^*\mathbb{R}^4 \setminus 0) \times (T^*\mathbb{R}^4 \setminus 0)$ be the canonical relation of F. For each $(y,x,t,\omega') \in \mathit{supp}(a)$, assume $1 + c_0\alpha\frac{x_1-y_1}{|x-y|} > 0$. Then both projections of C have blowdown singularities along a codimension 2 submanifold, Σ, $\pi_L(\Sigma)$ and $\pi_R(\Sigma)$ are involutive and non-radial and $F^*F \in I^{\frac{5}{2},-\frac{1}{2}}(\Delta,\Lambda_1)$, where Δ is the diagonal in $T^*\mathbb{R}^4 \times T^*\mathbb{R}^4$ and Λ_1 is the flowout from $\pi_R(\Sigma)$.

Proof of theorem 3.4: We define $\sigma = \omega + \omega'$ and compute the canonical relation of F in (19) as follows.

$$\begin{aligned} C \;=\; & \Bigg\{ \Big(y_1,\, y_2,\, y_3,\, t,\, c_0\alpha\omega' + \sigma\frac{x_1 - y_1}{|x-y|},\, \sigma\frac{x_2 - y_2}{|x-y|},\, \sigma\frac{x_3 - y_3}{|x-y|},\, \omega; \\ & x,\, t',\, \sigma\frac{x_1 - y_1}{|x-y|},\, \sigma\frac{x_2 - y_2}{|x-y|},\, \sigma\frac{x_3 - y_3}{|x-y|},\, \omega - \omega' \Big) \\ & t' = -c_0\alpha y_1 + |x-y|;\; t = -c_0\alpha y_1 + 2|x-y| \Bigg\}. \end{aligned} \tag{20}$$

Thus $C \subset (T^*\mathbb{R}^4\backslash 0)\times(T^*\mathbb{R}^4\backslash 0)$ and the local coordinates are (x, y, ω, ω'). Let us look at the left projection π_L, which can be considered as a map $\mathbb{R}^8 \to \mathbb{R}^8$,

$$\begin{aligned} \pi_L(x, y, \omega, \omega') \;=\; & \Big(y_1,\, y_2,\, y_3,\, t,\, c_0\alpha\omega' + \sigma\frac{x_1 - y_1}{|x-y|}, \\ & \sigma\frac{x_2 - y_2}{|x-y|},\, \sigma\frac{x_3 - y_3}{|x-y|},\, \omega \Big). \end{aligned}$$

This can be rewritten (giving the identity in the y, ω variables) as

$$\begin{aligned} \pi_L(y,\, \omega,\, \omega',\, x) \;=\; & \Big(y_1,\, y_2,\, y_3,\, \omega,\, -\, c_0\alpha y_1 + 2|x-y|,\, c_0\alpha\omega' + \sigma\, \frac{x_1 - y_1}{|x-y|}; \\ & \sigma\frac{x_2 - y_2}{|x-y|},\, \sigma\frac{x_3 - y_3}{|x-y|} \Big), \end{aligned} \tag{21}$$

making the evaluation of its Jacobian determinant easier to compute. In this case, the Jacobian of $d\pi_L$ is a 8×8 matrix with the 4×4 identity block in the y, ω variables. Thus, to find the rank of π_L it suffices to find the rank of the 4×4 matrix in ω', x:

$$\begin{pmatrix} 0 & 2\frac{x_1-y_1}{|x-y|} & 2\frac{x_2-y_2}{|x-y|} & 2\frac{x_3-y_3}{|x-y|} \\ c_0\alpha + \frac{x_1-y_1}{|x-y|} & \sigma\frac{(x_2-y_2)^2+(x_3-y_3)^2}{|x-y|^3} & -\sigma\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} & -\sigma\frac{(x_1-y_1)(x_3-y_3)}{|x-y|^3} \\ \frac{x_2-y_2}{|x-y|} & -\sigma\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} & \sigma\frac{(x_1-y_1)^2+(x_3-y_3)^2}{|x-y|^3} & -\sigma\frac{(x_3-y_3)(x_2-y_2)}{|x-y|^3} \\ \frac{x_3-y_3}{|x-y|} & -\sigma\frac{(x_1-y_1)(x_3-y_3)}{|x-y|^3} & -\sigma\frac{(x_3-y_3)(x_2-y_2)}{|x-y|^3} & \sigma\frac{(x_2-y_2)^2+(x_1-y_1)^2}{|x-y|^3} \end{pmatrix},$$

whose determinant is

$$-\frac{2\sigma^2}{|x-y|^2}\left(1+c_0\alpha\frac{x_1-y_1}{|x-y|}\right).$$

Since by our assumption in the theorem, $1+c_0\alpha\frac{x_1-y_1}{|x-y|}>0$, $d\pi_L$ drops rank by 2 on

$$\Sigma=\{\sigma=0\}=\{\omega+\omega'=0\}. \tag{22}$$

Next, we compute the 2 dimensional kernel of $d\pi_L$ on Σ. We have that $v\in$ Ker $d\pi_L$, if $v=(0,0,0,0,\delta\,\omega',\ \delta x_1,\ \delta x_2,\ \delta x_3)=\delta\,\omega'\,\partial_{\omega'}+\delta x_1\partial_{x_1}+\delta x_2\partial_{x_2}+\delta x_3\partial_{x_3}$, with

$$\left(c_0\alpha+\frac{x_1-y_1}{|x-y|}\right)\delta\omega'=0 \tag{23}$$

and

$$\frac{x_1-y_1}{|x-y|}\,\delta x_1+\frac{x_2-y_2}{|x-y|}\,\delta x_2+\frac{x_3-y_3}{|x-y|}\,\delta x_3=0. \tag{24}$$

From equation (23) we get

$$\delta\omega'=0,$$

whereas from equation (24), it follows that

$$\delta x_3=-\frac{x_1-y_1}{x_3-y_3}\,\delta x_1-\frac{x_2-y_2}{x_3-y_3}\,\delta x_2,$$

which implies that

$$Ker(d\pi_L)=span\{\partial_{x_1},\partial_{x_2}\},$$

which is tangent to Σ. Hence, π_L has a blowdown singularity. Moreover

$$\pi_L(\Sigma)=\{\eta_2=\eta_3=\eta_1+\eta_4=0\ \} \tag{25}$$

is involutive and non-radial. Here, $\eta_1,\eta_2,\eta_3,\eta_4$ denote the dual variables to y_1,y_2,y_3,t respectively. We do the same analysis for the right projection

$$\begin{aligned}\pi_R(x,\ \omega,\ \omega',\ y_1,\ y_2,\ y_3) &= \Bigg(x,\ \omega-\omega',\ -c_0\alpha y_1+|x-y|,\ \sigma\frac{x_1-y_1}{|x-y|},\\ &\qquad \sigma\frac{x_2-y_2}{|x-y|},\ \sigma\frac{x_3-y_3}{|x-y|}\Bigg).\end{aligned} \tag{26}$$

We know that π_R also drops rank by 2 on the same set $\Sigma = \{\sigma = 0\} = \{\omega + \omega' = 0\}$. Since $d\pi_R$ has the identity block 3×3 matrix in x variables, we need the 5×5 submatrix in (ω, ω', y) variables to compute the kernel of $d\pi_R$ on Σ

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -c_0\alpha - \frac{x_1-y_1}{|x-y|} & -\frac{x_2-y_2}{|x-y|} & -\frac{x_3-y_3}{|x-y|} \\ \frac{x_1-y_1}{|x-y|} & \frac{x_1-y_1}{|x-y|} & 0 & 0 & 0 \\ \frac{x_2-y_2}{|x-y|} & \frac{x_2-y_2}{|x-y|} & 0 & 0 & 0 \\ \frac{x_3-y_3}{|x-y|} & \frac{x_3-y_3}{|x-y|} & 0 & 0 & 0 \end{pmatrix}.$$

We obtain that $v \in Ker(d\pi_R)$, if

$$\begin{aligned} v &= (0,\ 0,\ 0,\ \delta\omega,\ \delta\omega',\ \delta y_1,\ \delta y_2,\ \delta y_3) \\ &= \delta\omega\,\partial_\omega + \delta\omega'\,\partial_{\omega'} + \delta y_1\,\partial_{y_1} + \delta y_2\,\partial_{y_2} + \delta y_3\,\partial_{y_3}, \end{aligned}$$

with

$$\delta\omega - \delta\omega' = 0 \tag{27}$$

$$\frac{x-y}{|x-y|}\,(\delta\omega + \delta\omega') = 0 \tag{28}$$

and

$$\Big(c_0\alpha + \frac{x_1-y_1}{|x-y|}\Big)\delta y_1 + \frac{x_2-y_2}{|x-y|}\,\delta y_2 + \frac{x_3-y_3}{|x-y|}\,\delta y_3 = 0. \tag{29}$$

By combining equation (27) with $(\delta\omega + \delta\omega)\frac{x_3-y_3}{|x-y|} = 0$, we obtain

$$\delta\omega' = \delta\omega = 0$$

and from equation (29) it follows that

$$\delta y_1 = -\frac{1}{c_0\alpha + \frac{x_1-y_1}{|x-y|}}\left(\frac{x_1-y_1}{|x-y|}\,\delta y_2 + \frac{x_3-y_3}{|x-y|}\,\delta y_3\right),$$

which implies that

$$Ker(d\pi_R) = span\{\partial_{y_2}, \partial_{y_3}\},$$

which is tangent to Σ. Hence π_R has a blowdown singularity. Moreover

$$\pi_R(\Sigma) = \{\xi_1 = \xi_2 = \xi_3 = 0\} \tag{30}$$

is involutive and non-radial. In conclusion, by theorem 2.19 (see [12]), we have that $F^*F \in I^{\frac{5}{2},-\frac{1}{2}}(\Delta, \Lambda_1)$, where

$$\Lambda_1 = \Lambda_{\pi_R(\Sigma)} = \left\{(x_1,\, x_2,\, x_3,\, x_4,\, 0,\, 0,\, 0,\, \xi_4;\, \bar{x}_1,\, \bar{x}_2,\, \bar{x}_3,\, x_4,\, 0,\, 0,\, 0,\, \xi_4)\right\} \quad (31)$$

is the flowout from $\pi_R(\Sigma)$ in $T^*\mathbb{R}^4 \times T^*\mathbb{R}^4$, which intersects Δ cleanly in codimension 3. This means that $F^*F \in I^2(\Delta \setminus \Lambda_1)$ and $F^*F \in I^{\frac{5}{2}}(\Lambda_1 \setminus \Delta)$, so the artifact is stronger. In the last section we will show how to microlocally reduce the strength of the artifact.

3.3 CASE 3

Next we consider the case when $y = z$ and $x_3 = 0,\ y_3 = z_3 = h$. This is essentially the same as case 2, except for the fact that now we are assuming that the scatterer is confined to the surface $x_3 = 0$ and so we need less data, and we confine the radar system to a surface $x_3 = h > 0$. In this case the forward operator is given by

$$Fq(y,t) = \int e^{i\,\phi(y,\,x,\,t,\,t',\,\omega,\,\omega')}\, a(x,y,t,\omega')\, q(x_1,x_2,t')\, d\omega d\omega' dx dt', \qquad (32)$$

where $\phi(y,\, x,\, t,\, t',\, \omega,\, \omega') = \omega(t - t' - |x-y|) + \omega'(t' + c_0\alpha y_1 - |x-y|)$, $a(x,y,t,\omega') = a(x_1,x_2,y_1,y_2,t,\omega') \in S^2$, $q(x,t') = q(x_1,x_2,t')$ and F is an FIO of order $2 + 2/2 - 6/4 = 3/2$. We have the following result.

Theorem 3.5. Assume that the scatterer is located to one side of the projected flight path (straight line) on the ground, i.e., $x_1 - y_1 > 0$ and let $C \subset (T^*\mathbb{R}^3 \setminus 0) \times (T^*\mathbb{R}^3 \setminus 0)$ be the canonical relation of F. Then both projections of C have blowdown singularities along a codimension 1 submanifold, Σ, $\pi_L(\Sigma)$ and $\pi_R(\Sigma)$ are involutive and non-radial and $F^*F \in I^{3,0}(\Delta, \Lambda_2)$, where Δ is the diagonal in $T^*\mathbb{R}^3 \times T^*\mathbb{R}^3$ and Λ_2 is the flowout from $\pi_R(\Sigma)$.

Proof of theorem 3.5: We expected to have to make one-sided location assumption because in the case of the static radar this was a necessary assumption too [14].

Like in the proof of theorem 3.4 it is convenient to define $\sigma = \omega + \omega'$, then the canonical relation is here given by

$$C = \left\{ \left(y_1,\, y_2,\, t,\, c_0\alpha\omega' + \sigma\, \tfrac{x_1-y_1}{|x-y|},\, \sigma \tfrac{x_2-y_2}{|x-y|},\, \omega; x,\, t',\, \sigma\, \tfrac{x_1-y_1}{|x-y|},\, \sigma\, \tfrac{x_2-y_2}{|x-y|},\, \omega - \omega'\right),\right.$$

$$t' = -\,c_0\alpha y_1 + |x-y|;\ t = -\,c_0\alpha y_1 + 2|x-y| \Big\}.$$

where $|x-y| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + h^2}$. Notice that $C \subset (T^*\mathbb{R}^3 \setminus 0) \times (T^*\mathbb{R}^3 \setminus 0)$ and the local coordinates are $(x_1,\ x_2,\ y_1,\ y_2,\ \omega,\ \omega')$, thus

$$\pi_L(y,\ \omega,\ \omega',\ x) = \left(y_1, y_2, \omega, -\,c_0\alpha y_1 + 2|x-y|, c_0\alpha\omega' + \sigma\frac{x_1-y_1}{|x-y|}; \sigma\frac{x_2-y_2}{|x-y|} \right). \tag{33}$$

To check the singularities of π_L we consider the Jacobian of $d\pi_L$. This is a 6×6 matrix with the 3×3 identity block in the y, ω variables. Thus, to find the rank of π_L it suffices to consider the 3×3 matrix in ω', x

$$\begin{pmatrix} 0 & 2\frac{x_1-y_1}{|x-y|} & 2\frac{x_2-y_2}{|x-y|} \\ c_0\alpha + \frac{x_1-y_1}{|x-y|} & \sigma\frac{(x_2-y_2)^2+h^2}{|x-y|^3} & -\sigma\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} \\ \frac{x_2-y_2}{|x-y|} & -\sigma\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} & \sigma\frac{(x_1-y_1)^2+h^2}{|x-y|^3} \end{pmatrix},$$

whose determinant is

$$-2\,\frac{\sigma}{|x-y|^2}\left(c_0\alpha(x_1-y_1) + \frac{(x_1-y_1)^2}{|x-y|} + \frac{(x_2-y_2)^2}{|x-y|} \right).$$

Since $c_0\alpha > 1$ and $x_1 - y_1 > 0$, the parenthesis above is different than 0, therefore π_L drops rank by 1 at

$$\Sigma = \{\sigma = 0\} = \{\omega + \omega' = 0\} \tag{34}$$

and from the matrix of $d\pi_L$ we can see that $v \in \mathrm{Ker}\ (d\pi_L|_\Sigma)$ if

$$v = (0,\ 0,\ 0,\ \delta\,\omega',\ \delta x_1,\ \delta x_2) = \delta\omega'\partial_{\omega'} + \delta x_1 \partial_{x_1} + \delta x_2 \partial_{x_2},$$

with

$$\left(c_0\alpha + \frac{x_1-y_1}{|x-y|} \right)\delta\omega' = 0 \tag{35}$$

and

$$\frac{x_1-y_1}{|x-y|}\,\delta x_1 + \frac{x_2-y_2}{|x-y|}\,\delta x_2 = 0. \tag{36}$$

Equation (35) leads to $\delta\omega' = 0$, whereas equation (36) leads to $\delta x_1 = -\frac{x_2-y_2}{x_1-y_1}\,\delta x_2$. Thus $\mathrm{Ker}\ (d\pi_L) = \mathrm{span}\{\partial_{x_2}\}$ and is tangent to Σ. Hence π_L has a blowdown singularity. Moreover

$$\pi_L(\Sigma) = \{\eta_2 = \eta_1 + \eta_3 = 0\}, \tag{37}$$

which is involutive and non-radial. Similarly, the right projection is given by

$$\pi_R(x,\ \omega',\ \omega,\ y) = \left(x,\ \omega-\omega',\ -c_0\alpha y_1 + |x-y|,\ \sigma\frac{x_1-y_1}{|x-y|},\ \sigma\frac{x_2-y_2}{|x-y|}\right), \tag{38}$$

with the identity in x variables and the Jacobian matrix in $\omega, \omega', y_1, y_2$ given by

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -c_0\alpha - \frac{x_1-y_1}{|x-y|} & -\frac{x_2-y_2}{|x-y|} \\ \frac{x_1-y_1}{|x-y|} & \frac{x_1-y_1}{|x-y|} & -\sigma\frac{(x_2-y_2)^2+h^2}{|x-y|^3} & \sigma\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} \\ \frac{x_2-y_2}{|x-y|} & \frac{x_2-y_2}{|x-y|} & \sigma\frac{(x_2-y_2)(x_1-y_1)}{|x-y|^3} & -\sigma\frac{(x_1-y_1)^2+h^2}{|x-y|^3} \end{pmatrix}.$$

Thus, $d\pi_R$ drops rank by 1 on Σ and Ker $(d\pi_R) = \partial_{y_2}$, which is tangent to Σ. Thus π_R has a blowdown singularity. We have that

$$\pi_R(\Sigma) = \{\xi_1 = \xi_2 = 0\}, \tag{39}$$

which is involutive and non-radial. In conclusion, by theorem 2.19, $F^*F \in I^{3,0}(\Delta, \Lambda_2)$, with

$$\Lambda_2 = \Lambda_{\pi_R(\Sigma)} = \{(x_1, x_2, x_3, 0, 0, \xi_3; \bar{x}_1, \bar{x}_2, x_3, 0, 0, \xi_3)\}$$

which intersects Δ cleanly in codimension 2. This means that $F^*F \in I^3(\Delta \setminus \Lambda_2)$ and $F^*F \in I^3(\Lambda_2 \setminus \Delta)$ so the artifact is as strong as the image in the true singularity location. In the last section, we will show how to microlocally reduce the strength of this artifact.

3.4 CASE 4

The last case is the one when $y_3 = z_3 = h,\ y_2 = z_2,\ y_1 \neq z_1$. This corresponds to a transmitter/receiver pair located at the same height and at two separate points of a common line. In this case, the forward operator is

$$Fq(y, z, t) = \int e^{i\,\phi(y,\,z,\,x,\,t,\,t',\,\omega,\,\omega')}\, a(x, y, z, t, \omega')\ \ q(x, t')\, dx\, d\omega\, d\omega'\, dt', \tag{40}$$

where $\phi(y,\ z,\ x,\ t,\ t',\ \omega,\ \omega') = \omega(t - t' - |x-z|) + \omega'(t' + c_0\alpha y_1 - |x-y|)$ and

$$
\begin{aligned}
|x-z| &= \sqrt{(x_1-z_1)^2+(x_2-y_2)^2+(x_3-h)^2}\\
|x-y| &= \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-h)^2}.
\end{aligned}
$$

Here $a(x,y,z,t,\omega') = a(x,y_1,y_2,z_1,t,\omega') \in S^2$ and F is an FIO of order $2+2/2-8/4=1$. Similarly to cases 1,2, where we assumed $x_3 \neq z_3$ and $x_3 \neq y_3$, we make the following assumption.

Assumption 3.6. $x_3 \neq h$.

Remark 3.7. In this case we convolve the data with a temporal function, whose Fourier transform is supported away from $\omega = 0$ hence we do not image the singularities in q that would propagate into this part of the wavefront set in the data.

The main result is

Theorem 3.8. *Let $C \subset (T^*\mathbb{R}^4 \setminus 0) \times (T^*\mathbb{R}^4 \setminus 0)$ be the canonical relation of F. Then both projections of C have blowdown singularities along a codimension 1 set, Σ, $\pi_L(\Sigma)$ and $\pi_R(\Sigma)$ are involutive and non-radial, and $F^*F \in I^{2,0}(\Delta,\Lambda_3)$, where Δ is the diagonal in $T^*\mathbb{R}^4 \times T^*\mathbb{R}^4$ and Λ_3 is the flowout from $\pi_R(\Sigma)$.*

Proof of theorem 3.8: We calculate

$$
\begin{aligned}
C = \Bigg\{ &\Big(y_1,\ y_2,\ z_1,\ t,\ c_0\alpha\omega' + \omega'\frac{x_1-y_1}{|x-y|},\ \omega'\frac{x_2-y_2}{|x-y|} + \omega\frac{x_2-y_2}{|x-z|},\ \omega\frac{x_1-z_1}{|x-z|},\ \omega;\\
&x_1,\ x_2,\ x_3,\ t',\ \omega'\frac{x_1-y_1}{|x-y|} + \omega\frac{x_1-z_1}{|x-z|},\ \omega'\frac{x_2-y_2}{|x-y|} + \omega\frac{x_2-y_2}{|x-z|},\\
&\omega'\frac{x_3-h}{|x-y|} + \omega\frac{x_3-h}{|x-z|},\ \omega-\omega'\Big),\\
&t' = -c_0\alpha y_1 + |x-y|;\ t = -c_0\alpha y_1 + |x-y| + |x-z| \Bigg\}.
\end{aligned}
$$

In this case, $C \subset (T^*\mathbb{R}^4 \setminus 0) \times (T^*\mathbb{R}^4 \setminus 0)$ and the local coordinates on C are

$$
(y_1,\ y_2,\ z_1,\ x,\ \omega,\ \omega').
$$

As before, we consider the left projection

$$\begin{aligned}\pi_L(y_1,\, y_2,\, z_1,\, \omega,\, \omega',\, x) \;&=\; \Big(y_1,\, y_2,\, z_1,\, \omega,\, t,\, c_0\alpha\omega' + \omega'\, \frac{x_1-y_1}{|x-y|}, \\ &\qquad \omega'\,\frac{x_2-y_2}{|x-y|} + \omega\frac{x_2-y_2}{|x-z|},\, \omega\frac{x_1-z_1}{|x-z|}\Big).\end{aligned} \tag{41}$$

The Jacobian of $d\pi_L$ is a 8×8 matrix, with the 4×4 identity block in the y_1, y_2, z_1, ω variables. Thus, to find the rank of π_L, it suffices to find the rank of the 4×4 matrix in ω', x

$$\begin{pmatrix} 0 & \frac{x_1-y_1}{|x-y|} + \frac{x_1-z_1}{|x-z|} & \frac{x_2-y_2}{|x-y|} + \frac{x_2-y_2}{|x-z|} & \frac{x_3-h}{|x-y|} + \frac{x_3-h}{|x-z|} \\ c_0\alpha + \frac{x_1-y_1}{|x-y|} & \omega'\frac{(x_2-y_2)^2+(x_3-h)^2}{|x-y|^3} & -\omega'\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} & -\omega'\frac{(x_1-y_1)(x_3-h)}{|x-y|^3} \\ \frac{x_2-y_2}{|x-y|} & A & B & C \\ 0 & \omega\frac{(x_2-y_2)^2+(x_3-h)^2}{|x-z|^3} & -\omega\frac{(x_1-z_1)(x_2-y_2)}{|x-z|^3} & -\omega\frac{(x_1-z_1)(x_3-h)}{|x-z|^3} \end{pmatrix},$$

where

$$\begin{aligned} A &= -\omega'\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} - \omega\frac{(x_1-z_1)(x_2-y_2)}{|x-z|^3}; \\ B &= \frac{\omega'}{|x-y|} + \frac{\omega}{|x-z|} - (x_2-y_2)^2\Big(\frac{\omega'}{|x-y|3} + \frac{\omega}{|x-z|^3}\Big) \\ &= \omega'\frac{(x_1-y_1)^2+(x_3-h)^2}{|x-y|^3} + \omega\frac{(x_1-z_1)^2+(x_3-h)^2}{|x-z|^3}; \\ C &= -\omega'\frac{(x_3-h)(x_2-y_2)}{|x-y|^3} - \omega\frac{(x_3-h)(x_2-y_2)}{|x-z|^3}. \end{aligned}$$

The determinant is

$$\frac{\omega}{|x-z|^2}\left(c_0\alpha + \frac{x_1-y_1}{|x-y|}\right)(x_3-h)\left(1+\frac{(x-y)\cdot(x-z)}{|x-y||x-z|}\right)\left(\frac{\omega'}{|x-y|} + \frac{\omega}{|x-z|}\right).$$

Thus by Remark 3.7, $\omega \neq 0$ and π_L and π_R drop rank by 1 on

$$\Sigma = \left\{\frac{\omega'}{|x-y|} + \frac{\omega}{|x-z|} = 0\right\}. \tag{42}$$

Next, we find the singularities of π_L. Notice that on Σ, the third row of the matrix has a common factor (x_2-y_2). We will consider the case when

$x_2 - y_2 = 0$ and when $x_2 - y_2 \neq 0$. When $x_2 - y_2 = 0$, the 4×4 matrix becomes

$$\begin{pmatrix} 0 & \frac{x_1-y_1}{|x-y|} + \frac{x_1-z_1}{|x-z|} & 0 & \frac{x_3-h}{|x-y|} + \frac{x_3-h}{|x-z|} \\ c_0\alpha + \frac{x_1-y_1}{|x-y|} & \omega' \frac{(x_3-h)^2}{|x-y|^3} & 0 & -\omega' \frac{(x_1-y_1)(x_3-h)}{|x-y|^3} \\ 0 & 0 & 0 & 0 \\ 0 & \omega \frac{(x_3-h)^2}{|x-z|^3} & 0 & -\omega \frac{(x_1-z_1)(x_3-h)}{|x-z|^3} \end{pmatrix}.$$

Hence $v \in \text{Ker } (d\pi_L)$, if

$$v = (0,\, 0,\, 0,\, 0,\, \delta\omega',\, \delta x_1,\, \delta x_2,\, \delta x_3) = \delta\omega' \, \partial_{\omega'} + \delta x_1 \, \partial_{x_1} + \delta x_2 \, \partial_{x_2} + \delta x_3 \, \partial_{x_3},$$

with

$$\left(\frac{x_1 - y_1}{|x-y|} + \frac{x_1 - z_1}{|x-z|}\right) \delta x_1 + \left(\frac{x_3 - h}{|x-y|} + \frac{x_3 - h}{|x-z|}\right) \delta x_3 = 0, \tag{43}$$

$$\left(c_0\alpha + \frac{x_1 - y_1}{|x-y|}\right) \delta\omega' + \omega' \frac{(x_3 - h)^2}{|x-y|^3} \, \delta x_1 - \omega' \frac{(x_1 - y_1)(x_3 - h)}{|x-y|^3} \, \delta x_3 = 0 \tag{44}$$

and

$$\omega \frac{(x_3 - h)^2}{|x-z|^3} \, \delta x_1 - \omega \frac{(x_1 - z_1)(x_3 - h)}{|x-z|^3} \, \delta x_3 = 0. \tag{45}$$

From equation (45) we get $\delta x_1 = \frac{x_1 - z_1}{x_3 - h} \, \delta x_3$ and from equation (43),

$$\left(\left(\frac{x_1 - y_1}{|x-y|} + \frac{x_1 - z_1}{|x-z|}\right) \frac{x_1 - z_1}{x_3 - h} + \frac{x_3 - h}{|x-y|} + \frac{x_3 - h}{|x-z|}\right) \delta x_3 = 0$$

i.e.

$$\left(\frac{(x_1 - y_1)(x_1 - z_1) + (x_3 - h)^2}{|x-y|} + \frac{(x_1 - z_1)^2 + (x_3 - h)^2}{|x-z|}\right) \delta x_3 = 0,$$

which leads to

$$|x-z| \left(1 + \frac{(x-y) \cdot (x-z)}{|x-y||x-z|}\right) \delta x_3 = 0 \tag{46}$$

and from equation (46) we get $\delta x_3 = 0$ and then $\delta x_1 = 0$ and $\delta\omega' = 0$. Thus Ker $(d\pi_L) = \text{span}\{\partial_{x_2}\}$, which is tangent to Σ. Hence π_L has a blowdown singularity. When $x_2 - y_2 \neq 0$, we can factor $x_2 - y_2$ from the third row and we get $v \in \mathrm{Ker}\ (d\pi_L)$, if

$$v = (0,\ 0,\ 0,\ 0,\ \delta\omega',\ \delta x_1,\ \delta x_2,\ \delta x_3) = \delta\omega'\,\partial_{\omega'} + \delta x_1\,\partial_{x_1} + \delta x_2\,\partial_{x_2} + \delta x_3\,\partial_{x_3},$$

with

$$\begin{aligned}\left(\frac{x_1 - y_1}{|x-y|} + \frac{x_1 - z_1}{|x-z|}\right)\delta x_1 + \left(\frac{x_2 - y_2}{|x-y|} + \frac{x_2 - y_2}{|x-z|}\right)\delta x_2 \\ + \left(\frac{x_3 - h}{|x-y|} + \frac{x_3 - h}{|x-z|}\right)\delta x_3 = 0,\end{aligned} \tag{47}$$

$$\begin{aligned}\left(c_0\alpha + \frac{x_1 - y_1}{|x-y|}\right)\delta\omega' + \omega'\frac{(x_2-y_2)^2 + (x_3-h)^2}{|x-y|^3}\,\delta x_1 \\ -\omega'\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3}\,\delta x_2 - \omega'\frac{(x_1-y_1)(x_3-h)}{|x-y|^3}\,\delta x_3 = 0,\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{1}{|x-y|}\delta\omega' - \left(\omega'\frac{x_1-y_1}{|x-y|^3} + \omega\frac{x_1-z_1}{|x-z|^3}\right)\delta x_1 \\ -(x_2-y_2)\left(\frac{\omega'}{|x-y|^3} + \frac{\omega}{|x-z|^3}\right)\delta x_2 \\ -\left(\omega'\frac{x_3-h}{|x-y|^3} + \omega\frac{x_3-h}{|x-z|^3}\right)\delta x_3 = 0,\end{aligned} \tag{49}$$

$$\begin{aligned}\omega\frac{(x_2-y_2)^2 + (x_3-h)^2}{|x-z|^3}\,\delta x_1 - \omega\frac{(x_1-z_1)(x_2-y_2)}{|x-z|^3}\,\delta x_2 \\ -\omega\frac{(x_1-z_1)(x_3-h)}{|x-z|^3}\,\delta x_3 = 0.\end{aligned} \tag{50}$$

From equation (47) we have that

$$\delta x_3 = -\frac{1}{\frac{x_3-h}{|x-y|} + \frac{x_3-h}{|x-z|}}\left(\frac{x_1-y_1}{|x-y|} + \frac{x_1-z_1}{|x-z|}\right)\delta x_1 - \frac{x_2-y_2}{x_3-h}\,\delta x_2$$

and, by combining the last with equation (50), we obtain

$$|x-z|\left(1 + \frac{(x-y)\cdot(x-z)}{|x-y||x-z|}\right)\delta x_1 = 0,$$

which leads to $\delta x_1 = 0$ and hence $\delta x_3 = -\frac{x_2-y_2}{x_3-h}\,\delta x_2$. From equation (49) we obtain $\delta\omega' = 0$. Thus Ker $(d\pi_L) = \text{span}\{-\frac{x_2-y_2}{x_3-h}\,\partial_{x_3} + \partial_{x_2}\}$, which is tangent to Σ. Equation (48), which we have not examined yet, is consistent with these results. Hence π_L has a blowdown singularity and

$$\pi_L(\Sigma) = \{\eta_2 = 0 = F\,(y_1,\ z_1,\ t,\ \eta_1,\ \eta_3,\ \eta_4)\}\,, \tag{51}$$

where F is determined below. To find F we use the following relations between the variables and their dual relations:

$$t = -c_0\alpha y_1 + |x-y| + |x-z| \tag{52}$$
$$\eta_1 = c_0\alpha\omega' + \omega'\frac{x_1-y_1}{|x-y|} \tag{53}$$
$$\eta_3 = \omega\frac{x_1-z_1}{|x-z|} \tag{54}$$
$$\eta_4 = \omega \tag{55}$$
$$\frac{\omega'}{|x-y|} + \frac{\omega}{|x-z|} = 0. \tag{56}$$
$$\tag{57}$$

If we solve relation (56) for $\omega' = -\omega\frac{|x-y|}{|x-z|}$ and we replace it in relation (53), we get

$$\eta_1 = -\omega\left(c_0\alpha + \frac{x_1-y_1}{|x-y|}\right)\frac{|x-y|}{|x-z|}.$$

Using (55), relations (53) and (54) become

$$\frac{\eta_1}{\eta_4} = -c_0\alpha\frac{|x-y|}{|x-z|} - \frac{x_1-y_1}{|x-z|} \tag{58}$$
$$\frac{\eta_3}{\eta_4} = \frac{x_1-z_1}{|x-z|} \tag{59}$$

respectively and (59) can be rewritten as

$$\frac{\eta_1+\eta_3}{\eta_4} = -c_0\alpha\frac{|x-y|}{|x-z|} + \frac{y_1-z_1}{|x-z|}. \tag{60}$$

(60), together with (52), can be used to solve for $|x-y|$ and $|x-z|$. From

(52) we get

$$|x-z| = t + c_0\alpha y_1 - |x-y|,$$

which replaced in (60) gives

$$(\eta_1+\eta_3)(t+c_0\alpha y_1 - |x-y|) = (y_1 - z_1 - c_0\alpha|x-y|)\eta_4.$$

In order to solve this for $|x-y|$, we assume

Assumption 3.9. *$\eta_1+\eta_3-\eta_4 c_0\alpha \neq 0$*

and obtain

$$|x-y| = \frac{(\eta_1+\eta_3)(t+c_0\alpha y_1)-(y_1-z_1)\eta_4}{\eta_1+\eta_3-c_0\alpha\eta_4},$$

therefore $|x-z|$ becomes

$$|x-z| = \frac{-(t+c_0\alpha y_1)c_0\alpha\eta_4+(y_1-z_1)\eta_4}{\eta_1+\eta_3-\eta_4 c_0\alpha}.$$

Observe that

$$x_1 = \frac{1}{2}\left(\frac{|x-y|^2-|x-z|^2}{z_1-y_1}+y_1+z_1\right),$$

therefore

$$|x-y|^2-|x-z|^2 = \frac{(t+c_0\alpha y_1)^2(\eta_1+\eta_3+c_0\alpha\eta_4)-2(y_1-z_1)\eta_4(t+c_0\alpha y_1)}{\eta_1+\eta_3-c_0\alpha\eta_4},$$

then we obtain

$$\begin{aligned}
\frac{\eta_3}{\eta_4} &= \frac{(t+c_0\alpha y_1)^2(\eta_1+\eta_3+c_0\alpha\eta_4)}{2(z_1-y_1)(-(t+c_0\alpha y_1)c_0\alpha\eta_4+(y_1-z_1)\eta_4)}\\
&- \frac{2(y_1-z_1)\eta_4(t+c_0\alpha y_1)}{2(z_1-y_1)(-(t+c_0\alpha y_1)c_0\alpha\eta_4+(y_1-z_1)\eta_4)}\\
&- \frac{(y_1-z_1)^2(\eta_1+\eta_3-c_0\alpha\eta_4)}{2(z_1-y_1)(-(t+c_0\alpha y_1)c_0\alpha\eta_4+(y_1-z_1)\eta_4)}
\end{aligned}$$

and

$$\begin{aligned}
F(y_1,z_1,t,\eta_1,\eta_3,\eta_4) &= (t+c_0\alpha y_1)^2(\eta_1+\eta_3+c_0\alpha\eta_4)\\
&- 2(y_1-z_1)\eta_4(t+c_0\alpha y_1)\\
&- (y_1-z_1)^2(\eta_1+\eta_3-c_0\alpha\eta_4)\\
&- 2\eta_3(z_1-y_1)\left(-(t+c_0\alpha y_1)c_0\alpha+(y_1-z_1)\right)=0.
\end{aligned}$$

One can verify that $\frac{\partial F}{\partial \eta_1}$ and $\frac{\partial F}{\partial \eta_3}$ cannot be simultaneously 0 and therefore $\pi_L(\Sigma)$ is a smooth submanifold. Because F does not depend on y_2, we have that $\pi_L(\Sigma)$ is involutive. Furthermore, $\rho \notin \text{Span}\{H_F, H_{\eta_2}\}$ and so $\pi_L(\Sigma)$ is non-radial.
We now examine π_R.

$$\begin{aligned}\pi_R(x,\,\omega,\,\omega',\,y_1,\,y_2,\,z_1) \;\; = \;\; & \Big(x_1,\,x_2,\,x_3,\,\omega-\omega',\,-c_0\alpha y_1+|x-y|,\\ & \omega'\frac{x_1-y_1}{|x-y|}+\omega\frac{x_1-z_1}{|x-z|},\,\omega'\frac{x_2-y_2}{|x-y|}+\omega\frac{x_2-y_2}{|x-z|},\\ & \omega'\frac{x_3-h}{|x-y|}+\omega\frac{x_3-h}{|x-z|}\Big). \end{aligned} \tag{61}$$

The Jacobian of $d\pi_R$ is a 8×8 matrix, with the 3×3 identity block in the x variables. Thus it is enough to consider the Jacobian D, in the variables $(\omega,\omega',y_1,y_2,z_1)$

$$D=\begin{pmatrix} 1 & -1 & 0 & 0 & 0\\ 0 & 0 & -c_0\alpha-\frac{x_1-y_1}{|x-y|} & -\frac{x_2-y_2}{|x-y|} & 0\\ \frac{x_1-z_1}{|x-y|} & \frac{x_1-y_1}{|x-y|} & -\omega'\frac{(x_2-y_2)^2+(x_3-h)^2}{|x-y|^3} & -A & -\omega\frac{(x_2-y_2)^2+(x_3-h)^2}{|x-z|^3}\\ \frac{x_2-y_2}{|x-z|} & \frac{x_2-y_2}{|x-y|} & \omega'\frac{(x_2-y_2)(x_1-y_1)}{|x-y|^3} & -B & \omega\frac{(x_2-y_2)(x_1-z_1)}{|x-z|^3}\\ \frac{x_3-h}{|x-z|} & \frac{x_3-h}{|x-y|} & \omega'\frac{(x_3-h)(x_1-y_1)}{|x-y|^3} & -C & \omega\frac{(x_3-h)(x_1-z_1)}{|x-z|^3} \end{pmatrix}.$$

Thus, $v\in \text{Ker }(d\pi_R)$ if

$$v=(0,\,0,\,0,\,\delta\omega,\,\delta\omega',\,\delta y_1,\,\delta y_2,\,\delta z_1)=\delta\omega\,\partial_\omega+\delta\omega'\,\partial_{\omega'}+\delta y_1\,\partial_{y_1}+\delta y_2\,\partial_{y_2}+\delta z_1\,\partial_{z_1},$$

with $Du=0$, where $u=(\delta\omega,\,\delta\omega',\,\delta y_1,\,\delta y_2,\,\delta z_1)$. Notice that the tangent space of Σ is spanned by $(\delta x_1,\,\delta x_2,\,\delta x_3,\,\delta\omega,\,\delta\omega',\,\delta y_1,\,\delta y_2,\,\delta z_1)$, which is annihilated by the gradient of $\frac{\omega'}{|x-y|}+\frac{\omega}{|x-z|}$:

$$(\cdot,\cdot,\cdot,\frac{1}{|x-z|},\,\frac{1}{|x-y|},\,\omega'\frac{x_1-y_1}{|x-y|^3},\,\omega'\frac{x_2-y_2}{|x-y|^3}+\omega\frac{x_2-y_2}{|x-z|^3},\,\omega\frac{x_1-z_1}{|x-z|^3}).$$

This is the last row of the matrix D multiplied by x_3-h, which means that Ker $(d\pi_R)\in T\Sigma$. Hence, π_R has a blowdown singularity. Moreover

$$\pi_R(\Sigma)=\{\xi_2=\xi_3=0\}, \tag{62}$$

which is involutive and non-radial.
In conclusion, by theorem 2.19, $F^*F \in I^{2,0}(\Delta, \Lambda_3)$, where

$$\Lambda_3 = \Lambda_{\pi_R(\Sigma)} = \{(x_1,\ x_2,\ x_3,\ x_4,\ \xi_1,\ 0,\ 0,\ \xi_4;\ x_1,\ \bar{x}_2,\ \bar{x}_3,\ x_4,\ \xi_1,\ 0,\ 0,\ \xi_4)\}$$

intersects Δ cleanly in codimension 2. As before, we have that $F^*F \in I^2(\Delta \setminus \Lambda_2)$ and $F^*F \in I^2(\Lambda_2 \setminus \Delta)$ so the artifact is as strong as the primary image. In the last section we will show how to microlocally reduce the strength of this artifact.

4 Reduction of the strength of artifacts

We consider the artifacts arising in cases 2, 3 and 4. We have that

$$F^*F \in I^{p,l}(\Delta, \Lambda),$$

where $\Lambda = \Lambda_1, \Lambda_2, \Lambda_3$ respectively. The idea is to apply a pseudodifferential operator

$$Q : \mathcal{E}'(Y) \to \mathcal{D}'(Y)$$

of order 0 to F before applying F^*, i.e.

$$F^*QF \in I^{p,l}(\Delta, \Lambda).$$

Note that the orders p and l do not change. To reduce the order of F^*QF on Λ we will choose Q such that its principal symbol σ_Q vanishes to some order s on $\pi_L(\Sigma)$. For example, denoting by Δ the Laplacian, we have in case 2

$$Q = (\partial_{y_2}^2 + \partial_{y_3}^2 + (\partial_{y_1} + \partial_{y_4})^2)(-\Delta)^{-1},$$

which vanishes to order 2 on

$$\pi_L(\Sigma) = \{\eta_2 = \eta_3 = \eta_1 + \eta_4 = 0\}.$$

For case 3 we have

$$Q = (\partial_{y_2}^2 + (\partial_{y_1} + \partial_{y_3})^2)(-\Delta)^{-1},$$

which also vanishes to order 2 on

$$\pi_L(\Sigma) = \{\eta_2 = \eta_1 + \eta_3 = 0\}.$$

We consider the general case in which both projections of the canonical relation of F are blowdowns which drop rank by k. Using the ideas of [7], p. 459 and [12], let a be the principal symbol of F and b the principal symbol of F^*. Then we can write $a = \alpha \times |\pi_L^*(\omega_Y^n)|^{\frac{1}{2}}$, where ω_Y^n is the symplectic form in T^*Y. Since π_L is a blowdown, α has a conormal singularity at Σ of order $-\frac{k}{2}$. Similarly, $b = \beta \times |\pi_L^*(\omega_Y^n)|^{\frac{1}{2}}$, where β has a conormal singularity at Σ of order $-\frac{k}{2}$. Now we introduce Q. We have $\sigma_{QF} = \pi_L^*(\sigma_Q)\sigma_F$. Since $\sigma_Q|\pi_L(\Sigma) = 0$ of order s, then $\pi_L^*(\sigma_Q|\pi_L(\Sigma)) = 0$ of order s because π_L is a blowdown. Then the new α has a singularity of order $s - \frac{k}{2}$. Thus $\beta \cdot \alpha$ has a conormal singularity of order $s - k$ above $\pi_L(\Sigma)$, which, pushed down by π_R (which is a blowdown), gives rise to a conormal singularity of the same order in the principal symbol of F^*QF. Thus, $l = s - \frac{k-1}{2}$ and $p = 2m - l = 2m - s + \frac{k-1}{2}$. By applying the above argument to cases 2, 3 and 4 we have:

i case 2: $k = 2$ and $s \geq 2$ hence $F^*QF \in I^{\frac{5}{2}-s, s-\frac{1}{2}}(\Delta, \Lambda_1)$, which means that $F^*QF \in I^2(\Delta \setminus \Lambda_1)$ and $F^*QF \in I^{\frac{5}{2}-s}(\Lambda_1 \setminus \Delta)$. Thus the artifact has lower order than the initial location of the singularities coming from Δ;

ii case 3: $k = 1$ and $s \geq 1$ hence $F^*QF \in I^{3-s,s}(\Delta, \Lambda_2)$, which means that $F^*QF \in I^3(\Delta \setminus \Lambda_2)$ and $F^*QF \in I^{3-s}(\Lambda_2 \setminus \Delta)$. Thus the artifact has lower order than the initial location of the singularities coming from Δ;

iii case 4: $k = 1$ and $s \geq 1$ hence $F^*QF \in I^{2-s,s}(\Delta, \Lambda_3)$, which means that $F^*QF \in I^2(\Delta \setminus \Lambda_3)$ and $F^*QF \in I^{2-s}(\Lambda_3 \setminus \Delta)$. Thus the artifact has lower order than the initial location of the singularities coming from Δ.

5 Acknowledgements

Raluca Felea thanks RIT for awarding her the sabbatical leave which enabled this research collaboration at University of Limerick, Ireland and Isaac Newton Institute, Cambridge, UK. She also thanks University of Limerick for the hospitality during her visit in October, 2011 and Isaac Newton Institute for the hospitality during her visit in November and December, 2011. She was supported by Simons Foundation grant 209850.

Romina Gaburro thanks the Isaac Newton Institute, Cambridge for the hospitality during a visit in December, 2011, which enabled part of this research collaboration as well as Science Foundation Ireland (Grant number 06/MI/005) for support.

Clifford Nolan thanks Science Foundation Ireland (Grant number 06/MI/005) for support as well as the Isaac Newton Institute, Cambridge for their hospitality during a visit in December, 2011.

All the authors thank the referees for the wonderful and helpful suggestions.

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