An Investigation into the Teaching and Learning of Applications of Mathematics in Senior-Cycle Schools in Ireland

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A thesis submitted for the award of Doctor of Philosophy

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Declaration

The substance of this thesis is the original work of the author, and due reference and acknowledgement has been made, where necessary, to the work of others. No part of this thesis has been submitted in canditure for any degree.

____________________________________________________________________

Brian Carroll   (Candidate)
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Dedication

For those that believed in me:

"To strive, to seek, to find, and not to yield" - (Ulysses by Lord Alfred Tennyson).
Abstract

Current practises in Irish mathematics classrooms generally fail to make the necessary connections between mathematics and its place in real-life, as documents from the NCCA and the Chief Examiners Report have shown (NCCA, 2005; State Examinations Commission, 2005). This study focuses on the problem of improving the teaching and learning of mathematics, particularly upper secondary level in Ireland, by making a provision for the effective teaching of applications. The author anticipated the national focus on the use of applications in mathematics education with the inception of ‘Project Maths’.

APOS Theory was adapted for use in this study and the resultant approach has been field tested in a small scale intervention in Irish Senior-Cycle schools. APOS Theory is a purpose built theory for mathematics teaching which was developed by Dubinsky (1996) and his colleagues in the Research for Undergraduate Mathematics Education Community (RUMEC) for the purpose of mathematics education at third level.

The author harnessed the three-stage approach employed by APOS Theory (Exploratory, Implementation and Reflective Phase) to develop, pilot, implement and evaluate the subsequent teaching intervention. Bajpai’s Integrated Approach (1975) and the Harvard Calculus Approach (1991) influence the research design in that they offer a perspective on introducing mathematical concepts through multiple approaches (numerical, analytical, graphical, verbal) as opposed to the over-emphasis on analytical techniques widely practised in schools. Their emphasis on applications, modelling and case studies allows the author to achieve the overall aims of the research project. Results of the intervention showed that students find mathematics more interesting when taught through applications. In addition, the intervention highlighted the issue of assessment in the teaching of applications and modelling, where the participating students and teachers called for a review of assessment procedures in Senior-Cycle mathematics here in Ireland.
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Chapter 1

Introduction

1.1 Introduction

In recent years, there has been considerable concern about the low level of mathematical skills of students emerging from second-level education in Ireland and, in particular, of those proceeding to third-level education (NCCA, 2005a). The relatively poor take-up of Higher level mathematics (16% of students sitting their Leaving Certificate in 2010 completed the Higher level mathematics paper) rightly gives cause for concern, since it has implications for the follow-on study of mathematics to degree level (State Examinations Commission, 2008). Current practices in the teaching and learning of mathematics in post primary schools in Ireland generally fail to make the necessary connections between mathematics and its place in real-life, as documents from the National Council for Curriculum and Assessment (NCCA) and the Chief Examiners Report have shown (NCCA, 2005b; State Examinations Commission, 2005). Consequently, in the revised Leaving Certificate mathematics syllabus, written for a new curriculum initiative called ‘Project
Maths’, the Department of Education and Skills (D.E.S) outline the need for mathematics set in context where it states:

“By teaching mathematics in contexts that allow learners to see connections within mathematics, between mathematics and other subjects, and between mathematics and its applications to real life, it is envisaged that learners will develop a flexible, disciplined way of thinking and the enthusiasm to search for creative solutions.”

(An Roinn Oideachais agus Scileanna, 2010, p. 6)

This statement implies that the D.E.S are aware that the mathematics experience of upper secondary school students should cater for applications of mathematics. This is consistent with Government insistence on the need for the provision of opportunities for students to see the relevance of mathematics to their everyday lives and the world around them. The author of this research project anticipated these needs when he started his research study and consequently they have influenced the nature of his study. This thesis reports on research that is focused on improving the teaching of mathematics, particularly at upper secondary level in Ireland, by focusing on the teaching of applications to Senior-Cycle students.

### 1.2 Background to the Research

Traditionally, mathematics is taught by the practise and refinement of procedural skills and following perceived mastery of these skills students are then introduced to applications. The efficacy of this approach has been questioned and thus,
many mathematics educators have developed alternative approaches to the teaching and learning of mathematics so as to encompass the role of applications at an earlier stage of the learning process (Bajpai et al., 1975; Freudenthal, 1968; Hughes-Hallett, 1991; Meyer & Ludwig, 1999; Ormell, 1972). Applications have been at the core of these approaches and generally a shift away from the traditional approach to the teaching and learning of mathematics is presented by these mathematics educators. These approaches promote a strong movement away from “chalk and talk” style instruction, which is to be replaced by a greater emphasis on class activity e.g., calculator/computer exploration, discussion of problems, and group work.

Thus, the question of how to teach mathematics so as to provide for applications has been discussed for decades, particularly at an international level. At present, the need for students to learn mathematics through the medium of applications and modelling is widely accepted and its merits for inclusion are well recognised (Houston, 2007; Burkhardt, 2006; Ferrucci & Carter, 2003; James, 1985; Mustoe, 1992). However, its integration within mathematics curricula worldwide is still regarded as being in a period of transition. We have still to reach a stage of general classification, or a settled consensus on the role of applications within mathematics education (Burkhardt, 2006; Niss, 1987).

As it stands here in Ireland, the role of applications in the teaching and learning of mathematics is entering an exciting and innovative stage with the introduction of a new curriculum initiative, called ‘Project Maths’, to all second-level schools in September 2010. Applications have, dating back to the times of Pythagoras and Archimedes, been at the core of major mathematical thinking. However, mathematics curricula have not always demonstrated this successfully; particularly in
the curriculum we are now familiar with here in Ireland. ‘Project Maths’ has been primarily developed to place an increased emphasis on the use of applications in the teaching and learning of mathematics, as can be seen in one of its primary aims outlined in Section 1.1.

In the new revised Senior-Cycle, the mathematics curricula will enable students to further develop the knowledge and skills essential for their future lives, both in work related contexts and in their further studies that rely on a considerable background of mathematical understanding (NCCA, 2005). An uptake of 30% at higher level is targeted, so as to improve on the current figure of 16% (Project Maths, 2010).

The author himself has taught mathematics at both second and third level for the past six years. During this time he has been constantly faced with students who fail to see the relevance of mathematics to the world around them. These students generally display negative attitudes towards the subject and often question the need for mathematics in their overall educational experience. Given this personal experience, the author began to question the effectiveness of traditional pedagogical approaches to mathematics at second-level in Ireland as he found a significant number of students struggling to learn and appreciate mathematics. It was felt that current pedagogical approaches in Ireland needed to be improved by changing classroom practises. Thus, the potential effect of showing the relevance of mathematics to everyday life for secondary school students was a significant motivating factor for engaging in this study.
1.3 Research Aims

The study focuses on the problem of improving the teaching and learning of mathematics, particularly upper secondary level in Ireland, by making a provision for the effective teaching of applications. While approaches have been developed to cater for applications in mathematics to date, none have been developed for Senior-Cycle students in Ireland. The approaches developed by Bajpai (Bajpai et al., 1975), the Harvard Calculus Consortium (Hughes-Hallet, 1991) and others (Ormell, 1972; Meyer & Ludwig, 1991) are practise-led and lack a wider pedagogical perspective suited to secondary school mathematics teaching. APOS Theory offers such a perspective and theoretical pedagogical framework for mathematics teaching and learning based on constructivist principles. It is a purpose built theory for mathematics teaching and was developed by Dubinsky (Asiala et al., 1996) and his colleagues in the Research for Undergraduate Mathematics Education Community (RUME) for the purpose of mathematics education at third level. APOS Theory, in an effort to provide a perspective on what it means to learn and know something in mathematics, describes how actions become interiorised into processes and then encapsulated as mental objects, which take their place in more sophisticated cognitive schema’s (Dubinsky and McDonald, 2001; Asiala et al., 1996). The author has adapted this theory for use in this study and the resultant approach has been field tested in a small scale intervention in Irish Senior-Cycle schools. Hence, the main aim of the author’s study is:

- To develop a systematic approach to the teaching and learning of applications of mathematics for Senior-Cycle students in Irish schools

This study is guided by the following research questions:
• What are the key factors that contribute to the teaching and learning of applications in upper secondary schools?

• What theoretical pedagogical perspective is capable of integrating such factors appropriately? How can this be done effectively?

• How can the effectiveness of such an approach be evaluated?

As has been stressed by many researchers (Burkhardt, 2006; Niss 1987; Bajpai et al., 1976) success in teaching applications depends on the student’s grasp of the applications used in the classroom. Thus, the applications are culturally dependent and specific to the environment in which they are taught and the personal backgrounds/interests of the students involved. Consequently, it is necessary for each country to undertake relevant research and develop applications and pedagogical approaches that would be appropriate to the educational context in operation. This study is concerned with the development of approaches to improve mathematics teaching in Ireland by including a provision for applications and therefore, takes into consideration the Irish school system and educational objectives.

1.4 Theoretical Frameworks

To ensure the key research questions could be addressed, the author employed a number of theoretical frameworks.

APOS Theory (Asiala et al., 1996) forms the primary underlying theoretical framework used in this study and also influences the research design and methodology employed by the author. Furthermore, the author felt that the three-stage approach employed by APOS Theory (Exploratory, Implementation & Reflective Phases)
allowed the author to achieve the overall aims of the research project and therefore was deemed appropriate for use in this study.

The Harvard Approach (Hughes-Hallet, 1991) and Integrated Approach (Bajpai et al., 1975) both have significant influences on the research design. These approaches influenced the research design as the author felt that an adoption of these approaches would ensure that the students would be provided with opportunities to appreciate the fact that often a combination of techniques are used when solving problems arising in real-life contexts, both in industry and everyday experiences. Furthermore, these approaches provide an insight into current successful practises used in higher education mathematics courses throughout the world, thus providing a platform from which to approach the teaching and learning of applications in Senior-Cycle mathematics in Ireland.

A teaching intervention was designed, and implemented as part of this study so as to allow the author investigate the teaching and learning of applications in Senior-Cycle mathematics, while remaining consistent with the theoretical framework employed in this research study. In order to evaluate the success of the intervention, Shapiro’s model for evaluating an intervention was used (Shapiro, 1987). This model analysed four components including: treatment effectiveness; treatment integrity; social validity and treatment acceptability (See Section 7.4).

1.4.1 Research Methodology and Design

The research for this thesis adopts a pragmatic mixed-method methodology to investigate approaches to the development of pedagogical approaches that will support a provision for applications in mathematics education.
The author chose to analyse a certain concept that is taught on the Senior-Cycle mathematics syllabus in Ireland so as to narrow the focus of the research study. Consequently, the mathematical topic ‘Functions’ was chosen as the pedagogical focus of the intervention. The primary aim in doing so was to narrow the focus of the research project, while also ensuring an opportunity to highlight how the pedagogical approaches, applications and modelling problems may be used in this context within the Irish syllabus.

The research project integrates three different phases and the specific characteristics of each is discussed in detail in Section 4.8. To establish a true insight into the appropriateness of the teaching intervention and to help build a suitable framework the author has used a multi-method approach by combining both quantitative and qualitative methods of research. The use of both methods (quantitative and qualitative) at each distinct stage is discussed in more detail in Section 4.3. Figure 1.1 highlights their contribution to the overall design of the project. This research project was carried out over a three year period (October 2007- November 2010).

**Phase 1:**

- The Exploratory Phase Development (December 2008 – January 2009)
- Design of Teaching Intervention (February – April 2009)
- Analysis (May – June 2009)
Phase 2:

- The Implementation Phase Development (July – October 2009)
- Teaching Intervention (November – December 2009)

Phase 3: Reflective Phase

- Analysis (January – June 2010)
- Systematic Approach to Teaching Applications (January - June 2010)

1.5 Significance of the Research

Prior to conducting this research the author was aware of overarching concerns/hypotheses that impacted on the direction of the study and the research purpose aims and questions outlined in the Section 1.3. Given the author’s background as a mathematics teacher, he began to question the effectiveness of traditional pedagogical approaches to mathematics at secondary level as he found a significant number of students struggling to learn and appreciate mathematics and its applications. Evidence from reports in Ireland (NCCA, 2005; Chief Examiner’s Report, 2005) highlighted that the most common approach to learning mathematics for students in Ireland is a surface level approach involving rote learning and procedural knowledge. The author hypothesises that alternative approaches must be developed to improve understanding. He envisaged that providing a better experience for students through engagement with applications and mathematics set in
Phase 1
Exploratory Phase (Pilot Study)
- Literature Review
- Design & Implementation of Teaching Intervention
- Reflective Journal
- Student Questionnaire

Phase 2
Implementation Phase (Main Study)
- Redesign & Implementation of Teaching Intervention
- Teacher Interview
- Student Interview
- Student Questionnaire
- Reflective Journal

Phase 3
Reflective Phase
- Data Analysis
- Contributions & Recommendations

Figure 1.1: An overview of the three-phased research design implemented in the research project
context will address this issue and thus help in addressing the underpreparedness of students entering third level service mathematics courses.

In recent years there has been extensive awareness of the decline in the number of students undertaking undergraduate programs in Science and Engineering at third level institutes. Not alone are there fewer students than ever entering these programs, there is an increase in the number of students who drop out after their first year. Much of the problem is attributed to the students’ difficulties with mathematics in these courses (Liston, 2008). The ‘Mathematics Problem’ and issues involving the transition from school mathematics to service mathematics in Irish higher education have been issues highlighted over the last decade (O’Donoghue, 1999; Gill, 2006). As a result, there is considerable concern about the low level of mathematical skills of students, particularly in the area of problem solving and applications of mathematics, emerging from second-level education and, in particular, of those proceeding to third-level education (NCCA, 2005).

The research undertaken for this thesis investigates developing approaches to teaching upper secondary mathematics in Ireland by including a provision for applications. What is of concern to the author is that the mathematics curriculum in Ireland does not provide for applications of mathematics. Current practices in Ireland generally fail to make the necessary connections between mathematics and its place in real-life, as documents from the NCCA and the Chief Examiners Report have shown (NCCA, 2005; State Examinations Commission, 2005). The author anticipated the national focus on the use of applications in mathematics education (with the inception of ‘Project Maths’). Subsequently, this is the first study of its type to be undertaken in Ireland. In addition, the significance of the ‘Mathematics Problem’, combined with the growing concerns that higher
education graduates of engineering, science, business and computing are lacking required levels of mathematical proficiency for economic development are issues of concern both in Ireland and worldwide. This highlights the growing need for more students to make a successful transition from second-level mathematics education to third-level mathematics education so as improvements in quality and in retention and completion rates can be achieved in higher education mathematics courses.

1.6 Limitations of the Study

The author recognises that there are some limitations to the study. The author has adapted APOS Theory for use in this study, so as to improve the teaching and learning of mathematics, particularly upper secondary level in Ireland, by making a provision for the effective teaching of applications. The resultant approach has been field tested in a small scale intervention in Irish Senior-Cycle schools. Hence, the sample group used for this study will be drawn from a relatively small pool (n=68 students). Therefore, subjects not suitable for the project would include students in Junior-Cycle or those sitting Foundation level mathematics, as well as mathematics teachers not teaching the Senior-Cycle mathematics courses. The students involved must be preparing for Leaving Certificate mathematics at Higher or Ordinary level. Teachers involved must also be teaching mathematics at Senior-Cycle level. Secondly, due to nature of the Senior-Cycle mathematics curriculum in Ireland, there exists some issues regarding time constraints. It is envisaged that due to these time constraints, the majority of the participating students will be those currently in transition year. The author acknowledges this limitation.
However, the author believes that, as these students are preparing for Senior-Cycle mathematics courses and intend to take mathematics to Leaving Certificate level, they are valid participants. The author is aware of issues surrounding the generalisability of findings from such research and deals with them in his methodology chapter (Chapter 4).

1.7 Structure of the Thesis

Chapter 2

Chapter 2 is divided into two distinct parts. Part I examines General Issues in Mathematics Education. Part II examines Modelling and Applications in the Mathematics Curriculum.

Part I

The primary aim of this section was to review the key approaches to the development of mathematics curricula, particularly at secondary level, over the past half century, paying specific attention to the situation at both international and national level. In particular this chapter focused on issues such as: problem solving, constructivism and the instrumental vs. relational understanding debate, the ‘mathematics problem’ and the role of Information and Communications Technologies (ICT) in mathematics education.
Part II

Throughout this section the author looked in detail at the growth of modelling and applications in mathematics education. It focuses on what we recognise as applications and modelling in present day mathematics curricula, not only in Ireland but mathematics curricula worldwide.

Chapter 3

In this chapter a number of significant approaches developed by mathematics educators for both second and third-level mathematics education are analysed. The use of applications and modelling play a distinctive role in these approaches. This chapter also gives a detailed description of the APOS Theory framework that is used in this research project.

Chapter 4

This chapter provides the reader with an outline of the methods and methodologies employed in this study. The design and development of the theoretical framework is discussed in this chapter. The theoretical framework employed for this study is focused on improving the teaching of mathematics, particularly at upper secondary level in Ireland, by including a provision for the teaching of applications. Thus, an adaptation of the APOS Theory is proposed for the learning needs of the students and consequently influences the theoretical framework underpinning the research. Development of the research instruments and data collection methods is explained for both quantitative and qualitative approaches at the different phases.
of the research. Data analysis is discussed including ethical issues, validity, reliability and triangulation. Finally, the limitations of the study are outlined.

Chapter 5

The Exploratory Phase of the research is described in this chapter, highlighting key areas where an adaptation of APOS Theory has taken place. The Exploratory Phase is divided into three aspects: The Theoretical Analysis; Instructional Treatment; and Observations and Assessments. So as to remain consistent with APOS Theory, an initial theoretical analysis of ‘functions’ was developed after an extensive literature review, using the experience of the author as a mathematics teacher at both second and third level and his own experience of learning mathematics. The instructional treatments, which arose as a result of the theoretical analysis, are described, in the design and implementation of a teaching intervention. The intervention was administered to two separate cohorts of Senior-Cycle students (n = 35) where they participated in a two-week intervention aimed at showing the relevance of ‘functions’ to their everyday lives and the world around them. This intervention was developed, at this stage of the research, to assess the appropriateness of the initial design of the theoretical framework, and is described in detail in this chapter. Finally, an analysis of the Exploratory Phase is described at this stage of the study, so improvements in the intervention and theoretical framework can be undertaken at the Implementation Phase.
Chapter 6

The Implementation Phase of the research is described in this chapter. The chapter begins by highlighting key areas where appropriate changes from the Exploratory Phase have taken place. Again, the Implementation Phase is divided into three aspects: The Theoretical Analysis; Instructional Treatment; and Observations and Assessments. An initial theoretical analysis of ‘linear functions’ was developed to aid the design and implementation of the teaching intervention. The teaching intervention was designed to assess the appropriateness of the theoretical framework employed for this research study. The intervention was administered to three separate cohorts of Senior-Cycle students (n = 68) where they participated in a two-week intervention aimed at showing the relevance of ‘linear functions’ to their everyday lives and the world around them.

Chapter 7

This chapter outlines the Reflective Phase of the overall research study. In this phase, the author created a synthesis on the various themes and understandings that emerged during the previous two phases. The quantitative and qualitative analysis of the data collected at the Implementation Phase is discussed at this stage, including the dissemination of relevant findings.

Chapter 8

This chapter concludes the thesis by summarising the conclusions of the study, discussing the contribution of the work to the field of mathematics education, making recommendations and putting forward suggestions for future research.
based on the findings of this investigation. Insights into the teaching and learning of applications in Senior-Cycle mathematics are provided at this phase, with particular emphasis placed on the development of a ‘systematic approach to the teaching and learning of applications’.

### 1.8 Summary

This introductory chapter was designed to set out the nature and the scope of the thesis. It began with a description of the background to the research from both an international and Irish context. It identified the gap in the research with regards the teaching and learning of applications in Senior-Cycle mathematics in Ireland and the need for such research in Ireland was made clear in Section 1.1. Chapter 2 proceeds to set out in greater detail, the background for the research by examining the literature on the role of applications and modelling in mathematics education as well as reviewing the major developments in mathematics education over the past half century.
Chapter 2

Literature Review - Issues in Mathematics Education

2.1 Introduction

This research study is concerned predominantly with the teaching and learning of applications in Senior-Cycle mathematics. Thus, in this chapter the author will review the literature by exploring current thinking and previous research in the field of mathematics education, focusing on the development of mathematics curricula and the role of applications and modelling in mathematics education over the past half century. The first section reviews the key approaches to the development of mathematics curricula, particularly at Senior-Cycle, over the past half century, paying specific attention to the situation at both international and national level. The historical support provided allows an insight into the distinctive factors which have contributed to the growth and development witnessed in mathematics education in the last two decades. In addition, this chapter will examine the
pure vs. applied debate that has dominated mathematics education throughout this
time, thereby focusing on the role of applications and mathematical modelling
from a school perspective. In order to determine an understanding of the role
of modelling in mathematics education, the definition of modelling is explored,
and consequently the classification of modelling approaches commonly practised
in mathematics education are examined. Finally, this chapter will identify ex-
ternal and internal factors affecting the teaching and learning of applications at
Senior-Cycle level, namely: time constraints; students’ concepts of applications;
assessment procedures; limited professional development; and the role of ICT in
mathematics education.

2.2 The Background: International Perspective

Mathematics has undergone some of its most dramatic changes in terms of the
development of entire new branches of mathematics and applications to already
existing mathematics in the last century and a half. For instance in topics such
as: linear programming, topology, set theory, vectors and not to mention work
with computers (Smith, 2004). Traditionally school mathematics was dominated
by mathematics created thousands of years ago by the Greeks, Arabs and more
recently (300 years ago) the discovery of Calculus in the 17th Century (Newton
and Leibniz are both credited with the discovery of Calculus independently of
each other, with Leibniz starting first with integration and Newton with differenti-
ation. While Newton and Leibniz are credited with the development of Calculus,
mathematicians such as Cauchy, Riemann, Euler and Gauss provided further sig-
nificant ideas of Calculus throughout the 18th and 19th Century (Boyer, 1991)).
Therefore, such new developments must have implications for the teaching of mathematics within our school systems. By and large, most countries have enjoyed a rather stable structure in their school and university mathematics curricula throughout the 20th Century. Arithmetic and basic concepts of informal geometry have formed the core of what students have encountered as initial groundings in mathematical thinking and understanding within the traditional primary school curriculum over the last century (Kilpatrick, 1996; Steen, 1990). According to this position, algebra appears in the middle grades so as to enable a more formal and complex approach to mathematical encounters, with a gradual move towards the study of trigonometric functions and further geometry, and finally, to calculus and higher mathematics. Also, in recent years we have seen an increased emphasis on statistics within the mathematics curriculum. This development is underlined in the Irish context by the introduction of statistics and probability as a major compulsory strand in the new national curriculum entitled ‘Project Maths’. There is an increase in the amount of statistics and probability to be studied at both Junior Certificate and Leaving Certificate level. With this formation Steen (1990) and Kilpatrick (1996) observe the arranging of topics horizontally to shape the curriculum, recognised as ‘horizontal mathematics’, as each topic is designed primarily to prepare for the next. This established composition of mathematics curricula emerged in Western Europe after the Industrial Revolution, and by the early 20th Century was widely adopted and accepted almost unanimously (Kilpatrick, 1996).

However, social and economic factors determined that in most countries in the first half of the 20th Century, upper secondary and tertiary education was largely reserved for the elite which were primed to fill positions of importance within
society (Niss, 1996). Beforehand, the majority of students did not have the opportunities to study post-elementary mathematics education. This had obvious implications for the role and position of mathematics in the upper secondary and tertiary mathematics curricula at this time, where logical thinking and development of personality were offered as sufficient rationale for its inclusion (Niss, 1996).

Nevertheless, notwithstanding the structural stability of the mathematics curriculum at both school and university level over the last century, two major reallocations in emphasis have occurred: the ‘modern’ mathematics movement of the 1950’s and 1960’s, and the recent movement towards applications and real-life contexts within mathematics education, including the problem solving debate. Educational philosophies have been to the fore of these movements, where problem solving, constructivism and the instrumental vs relational understanding debate have been major emphasises throughout this time. At third level there has been an emphasis placed on the problems associated with the transition from second-level to university mathematics known as the ‘mathematics problem’. The use of computers in mathematics education must not be ignored as it has had a profound impact on the subject, particularly at third level and in industry. Each of these key mathematics education events are discussed in brief below. Thus, the mathematics curriculum has gradually shifted away from an emphasis on abstract structures, based largely around traditional aspects of the syllabus, towards efforts to include more realistic applications, with an emphasis on mathematics used in daily and professional life (Kilpatrick, 1996; de Lange, 1996; Askey, 1999). Research has fuelled the debate on many major topics. Indeed, the future for mathematics education is quite bright with the recent 11th International Congress on Mathematics
Education (ICME 11) covering thirty eight well documented topics, covering all topical issues and concerns including any advances, new trends, and important work done in the last few years on the topic.

2.2.1 ‘Modern/New’ Mathematics

During the first half of the last century, arithmetic, traditional geometry and algebra formed the basis of the mathematics curriculum in many countries worldwide with particular emphasis on rote learning of procedures and the acquisition of appropriate algorithms (Niss, 1996). This over-emphasis on algorithms and drill procedures eventually led to international calls for reform and restructuring of the mathematics curriculum. This was largely due to its over-emphasis on procedures and the noticeable lack of understanding of rules and processes it promoted (Niss, 1996; Howson et al., 1981). With this recognition giving an impetus for a focus on new teaching and learning outcomes within mathematics education, radical changes occurred in school mathematics curricula in the 1950’s and 60’s. In particular the movement away from traditional mathematics and towards the field of pure mathematics was widely pursued. The momentum for such efforts was largely supplied by leading mathematicians and educationalists, and in particular the work of the Bourbaki Group. This group of French mathematicians published a series of books promoting the theory of structures, which formed the basis of their work, entitled: ‘Elements de Mathematique’. Furthermore, an external determinant of change at the time was the early Soviet technical superiority in space and the launching of Sputnik in 1958. This caused concern in the U.S. in particular, which similarly prompted abrupt response to educational reform, especially in the field of science and mathematics (Griffiths & Howson, 1974).
The fundamental principle of the ‘modern’ mathematics movement was to attract students to the study of mathematics by emphasising its abstract structures (Kilpatrick, 1996). It was intended that students would understand and appreciate mathematics more if they could become familiar with its laws that denoted the fundamental structures. Therefore pupils would be provided with a more thorough understanding of the mathematics that they needed to learn in later life and, in particular mathematics at university level. The effect of the establishment of ‘modern’ mathematics in our society caused the “speeding up” (Willoughby, 1967, p. 31) of learning through the removal of topics considered no longer relevant to the needs of a modern day mathematician. While this speeding up ensured a smoother transition for those pupils wishing to become mathematicians, the vast majority of pupils who encounter mathematics within the school system do not wish to follow this path. In particular, problems were encountered when pupils were introduced to quite complex and abstract mathematics at an early age, therefore only leading to less understanding by those pupils in the mainstream.

2.2.2 Realistic Mathematics Education

Freudenthal, particularly, was very vocal in his opposition to the ‘modern/new’ mathematics movement and was highly critical of its perceived merits. As expected, with such support for the modern mathematics movement and such minimal opposition Freudenthal’s ideas were considered somewhat marginal at the time (NCCA, 2004). But he was not the only opposition to the ‘modern’ mathematics with Kline (1978), in his book ‘Why Johnny Can’t Add: The failure of the new math’, expressing serious reservations about the changes that were occurring (Askey, 1999; deLange, 1996). In addition a group of 64 mathematicians
signed an article, which was published in *American Mathematical Monthly* and in *The Mathematics Teacher*, objecting to the direction the new reforms were taking (Askey, 1999).

Thus, Realistic Mathematics Education (RME) arose as a response to ‘modern’ mathematics and the world-wide reform of mathematics teaching and learning (Oldham, 2002). In RME mathematics can be regarded as a human activity, devoid of the abstractive nature found in mathematics consistent with ‘modern’ mathematics (Freudenthal, 1968). RME was to the fore in the promotion of a movement of mathematics towards applications. Mathematics educationalists witnessed a movement towards mathematics problems set in real life context, in the way of modern and realistic problems that could be encountered in all aspects of mathematics (Oldham, 2002). Specifically, Freudenthal (1973), and further developed by Steen (1990) and deLange (1996), desired that mathematics curricula consider multiple parallel strands (vertical mathematics), with each inclusive of appropriate childhood experiences, that would enable the student to develop mathematical insight into the many different branches of mathematics (deLange, 1996). Steen (1990) outlined five profound mathematical ideas which would underpin this philosophy, which were later adapted and reshaped by the OECD PISA mathematics group in to four categories (OECD, 2003, p. 35):

- quantity
- space and shape
- change and relationships
- uncertainty
Unprecedented growth has occurred over the past 50 years as international conferences, seminars and journal papers have been devoted entirely to the promotion of mathematics education. Gradually the emphasis has shifted towards specific themes and pedagogical issues, encompassing an extensive array of trends and research ideas. In recent times, the dominant theme at the Eight International Congress on Mathematical Education (ICME 8) in 1996 was that of RME and the acquisition of mathematical literacy, an entire three decades since its initial formation.

Two major initiatives in Germany and Austria that focus on the connection between mathematics and its place in everyday life include MUED and ISTRON. The German-Austrian ISTRON-network consists of about 60 people including teachers, professors, lecturers, developers of curricula, authors of textbooks, teacher trainers, and publishers of journals where the main objective is to contributing to the improvement of the teaching of mathematics by means of innovation of resources and applications (Blum, 2010). The German teacher group MUED (Storing Mathematics Teaching Units; English translation) has about 800 members where its main focus is to emphasise problems from daily life and environmental situations (Cooney & Krainer, 1996). However, these superb initiatives are German language based and as of yet have been slow to reach an English speaking audience. A new book entitled ‘Real-World Problems for Secondary School Mathematics Students – Case studies (in press)’ edited by Prof. John O’Donoghue and Prof. Juergen Maasz bridges this gap where it provides English contributions from existing members under the guidance of Prof. Juergen Maasz.

RME also underpins the mathematical component of the OECD Programme for International Student Assessment, PISA 2000 and PISA 2003 (Oldham, 2002: 25)
p. 42). The National Council of Teachers of Mathematics (NCTM) Standards outlined in 2000 have illustrated the current trend in educational research in mathematics and its concerns with pedagogy in the U.S. The endeavour to promote teaching for understanding is paramount in the Standards outlined by the NCTM, where they have adopted many of the principles which underline RME and a fundamental aim in improving mathematical literacy (Askey, 1999). Furthermore, the development of Mathematics in Context (MiC), a result of collaboration between the Freudenthal Institute and the University of Wisconsin, further enhances the ideal of teaching for understanding and the role of mathematics set in real life context. MiC fundamentally aims to afford students opportunities to view mathematics as an interesting, powerful tool that enables them to better understand their world (Stevens, 2001). RME and its influence on mathematics education worldwide is discussed in more detail in Chapter 3.

2.2.3 Problem Solving

Problem solving in mathematics curricula throughout the world has received extensive consideration, particularly in the last thirty years (English et al., 2008; Chapman 2008). Polya’s (1945) seminal work on how to solve problems led the initial debate on this complex mathematical education topic. Polya’s four stages (understanding, devising a plan, carrying out the plan, and looking back) emphasised that mathematics was more than just a set of rules or procedures to be followed. At the end of the 1970s, problem solving gained greater acceptance around the world. In 1976, at the 3rd International Congress on Mathematical Education (ICME 3), in Karlsruhe, Germany, problem solving was one of the themes addressed (Allevato & Onuchic, 2008). Problem solving was, as predicted
in the 1980 Yearbook of the National Council of Teachers of Mathematics (Kru-l
lik, 1980, p. xiv), the dominant theme of the 1980’s (Schoenfeld, 1992). The
NCTM’s widely heralded statement that "problem solving must be the focus of
school mathematics" (NCTM, 1980, p. 1) set the scene for the NCTM’s exten-
sive exposure of problem solving in reform documents as a key factor of change
in mathematics education (NCTM, 1989). Current longstanding perspectives on
problem solving have treated it as an isolated topic in mathematics education with
conferences and research papers devoted to the topic. In mathematics education,
research on problem solving has focused primarily on ‘word problems’. The cat-
egorisation of problems used in school mathematics has evoked numerous differ-
et points of view with Polya (1945) describing problems as having four different
characteristics: Rule under your nose; Application with some choice; Choice of
a Combination; Approaching a research level. Numerous other researchers have
advanced these characterises to include Real, Realistic, Fantasy and Purely Math-
ematical problems (Diaz & Poblete, 2000). Orton (2004) echoes these classifi-
cations when he categorises problems into four types: Routine Problems, Novel
Problems, Word Problems and Real Life Applications. While the classification
of problems has yet to reach general consensus, so too has the pedagogical ap-
proaches used in problem solving. Problem solving abilities are assumed to de-
velop linearly through a number of stages as outlined below. The problem solving
approach can be categorised generally as:

- initial learning of concepts and procedures
- practice on “story problems”
- exposure to a range of strategies, e.g. draw a diagram, guess and check etc.
• applying these competencies to solving “novel” or “non-routine problems”.

(English et al., 2008)

English et al., (2008) argues that research on mathematical problem solving has stagnated for much of the 1990s and early part of this century and as a result new perspectives are required to advance approaches to problem solving in mathematics education.

2.2.4 Constructivism

Constructivism is a theory of knowledge and learning that has been fundamental to many movements in education throughout the past half century, particularly in mathematics education (Warrick, 2001). Constructivist perspectives on learning and understanding have given rise to numerous theoretical and empirical works in mathematics education of late and as a result have given shape to many reform movements (Simon, 1995, Steffe & Gale, 1995). Central to the theory of constructivism is constructivists’ belief that learners actively construct their own understanding rather than becoming passive recipients of knowledge (Snowman & Biehler, 2006). Piaget is seen as the original constructivist. His theory of knowledge, published in 1954, portrayed the child as a ‘lone scientist’ creating his or her own sense of the world (Piaget, 1952 & 1971; Oxford, 1997; Warrick, 2001). Piaget maintained his focus on the individual learner (Piaget, 1960, 1972). Vygotsky’s views diverge from Piaget’s in this respect. While both would agree that learning occurs in the activities and experiences of the learner, Vygotsky places emphasis on the interaction with social groups (Vygotsky, 1978). Other
researchers have contributed much to the debate surrounding the role of constructivism in education, where Dewey called for education to be grounded in real experience. Bruner initiated curriculum change based on the notion that learning is an active, social process in which students construct new ideas or concepts based on their current knowledge (Warrick, 2001). While constructivism is at the heart of all mathematics understanding, a widespread acknowledgment of the role it can play in the teaching and learning of mathematics is yet to be expressed. This is partially due to the fact that it does not provide a model of instruction from which mathematics educators can begin to teach mathematics (Simon, 1995).

2.2.5 Instrumental vs. Relational Understanding

In the mid-1970’s Richard Skemp fuelled the debate on the need for students to be presented with mathematics that developed relational understanding - “knowing what to do and why” (Skemp, 1976, p. 2). He highlighted the reliance of mathematics teaching on instrumental understanding (“rules without reasons” (Skemp, 1976, p. 2)) and, thus, the existence of two quite distinct mathematics curricula. The educational concept of instrumental understanding has its roots in formalism (Ernest, 1985) in that formalism is the view that mathematics is a meaningless game played with marks on paper, following rules. From a practical and pedagogical perspective, Skemp (1976, 1978) argued that while instrumental understanding is attained more rapidly and efficiently than relational understanding this is not a sufficient argument to warrant a dominance of this approach. Skemp contended that the key rationale for relational understanding is in that it leads to longer retention and greater transferability, thus in essence a mathematics student will have a more comprehensive understanding of a mathematical concept.
Fundamentally, Skemp highlights that understanding can, depending on the instructional approach and the beliefs of the teacher and/or curriculum, be largely instrumental, largely relational, or some combination (Watson & Mason, 2007). Skemp’s (1976) innovative research on instrumental understanding was further developed by Mellin-Olsen (1981) and has laid the foundations for a shift in emphasis in the opposite direction (towards relational understanding) or at least a more balanced understanding of the mathematics taught in schools.

2.2.6 The Mathematics Problem

In mathematics education, the 1990’s are noteworthy for the emphasis on problems regarding the transition from school to university mathematics in many higher education institutions in Ireland (O’Donoghue, 1996; Hourigan & O’Donoghue, 2006), the UK (Croft, 2001), the US (Evensky et al., 1997) and in Australia (Barry & Davis, 1999). The ‘Mathematics Problem’ was originally characterised in a UK report entitled ‘Tackling the Mathematics Problem’ (LMS, 1995) which highlighted problems concerning mathematics education in both England and Wales. By 2000 the report, ‘Measuring the Mathematics Problem’ presented evidence of serious decline in students’ mathematical competencies and preparedness for mathematics based degree-courses (Engineering Council, 2002). This under-preparedness focuses attention on the inadequacies of the second level mathematics experience in preparing students for the prerequisites of third level service mathematics. Students demonstrating unsatisfactory understanding of fundamental mathematical concepts are categorised as ‘at risk’ (O’Donoghue, 1999) upon entry to third level service mathematics courses, and this is a growing concern among third level institutes.
To address the Mathematics Problem many third level institutes in Ireland and the UK have responded through the introduction of diagnostic testing to assess the students’ mathematical under-preparedness upon entry to third level service mathematics courses (Gill, 2006; Lawson, 1997). In addition the setting-up of Mathematics Learning Centres is now widespread to alleviate students’ problems while attempting to provide necessary support and one-to-one tuition (Ni Fhloinn & Nolan, 2007; Lawson, 2004). Subsequently peer-supported learning programmes are being introduced in the University of Limerick and other institutes to encourage students to support each other in learning, while increasing the level of student involvement in, and ownership of their learning (Madhi, 2007; Bradley, 2007).

Bridging courses are a worldwide phenomenon and have been provided to mature students and other students who are inadequately prepared to learn mathematics e.g. Harvard University, US; University of Auckland, New Zealand etc. (Wood, 2001).

Here in Ireland, the University of Limerick have undertaken similar interventions in the form of bridging courses. Support tutorials are also a popular source of extra provision for mature students and students who perform inadequately in the previously administered diagnostic test (Gill, 2006). Tralee Institute of Technology has introduced ‘Essential Maths’ to ensure students must have an adequate standard of fundamental understating of mathematical concepts to progress onto mathematics aimed at third level students (Cleary, 2007). In addition to these correctives measure there is a continual need to provide innovative methods of support and guidance for first year service mathematics students in higher education so as to stimulate students’ interest in mathematics and ensure success (Lawson, 2007).
2.2.7 ICT (Information and Communications Technologies) Policy

Even though computers have been around since the late 1940’s, it was only in the late 1960’s that educators began to consider and believe that technology could have significant effects on education. Technology has expanded beyond anyone’s perceptions of its usage, and education has had to adapt to this significant development. The influx of technology within our educational settings has been rather slow in occurring on a full-scale basis, largely due to economic, social and cultural reasons. However, educators can no longer ignore the possibilities technology brings to the paradigm of teaching and learning.

The growing digital divide in our society has been cited as a principle rationale for the inclusion of ICT within our education system. The gap in achievement between those who have access to technology in the home, and those who do not is a growing concern (NCCA, 2004). Society is becoming increasingly dependant on its ability to communicate through the medium of advanced technological prototypes, which we are gradually accepting as standard in our daily lives. The use of the web and the internet in society has created important higher-order skills such as “electronic, visual and media literacies” (Leask, 2001, p.15). Furthermore, the influence of ICT within our social lives has significantly increased our capacity to generate, manipulate, store and transmit large quantities of information cheaply and to communicate with others almost instantaneously (NCCA, 2004).

Loveless (2003) argues that the domestic market for technology has outstripped any prospect of schools keeping up. While students are in contact with high-tech televisions, game machines, PCs, video recorders, phones and music players
outside the realm of school, schools and educators alike must strive to create situations which are somewhat familiar to these students, thereby creating additional enthusiasm and interest in their chosen subject matter. Learning materials are greatly expanding, where textbooks are being supplemented by an abundance of multimedia materials in print, audio, video, and digital form. The growing perception of the inconsistency between the uses of ICT in the school and in the ‘real world’ outside of school, has ensured there has been increasing interest in the role of ICT in education at policy level both from the European Union and from National governments (NCCA, 2004).

ICT development over the past 50 years has been unprecedented. ICT development in mathematics education has been no different. Since the initial introduction of technology to mathematics education in the 1960’s, numerous developments have been utilised and upgraded. While the use of ICT in mathematics education hasn’t quite reflected the developments of technology in the real world, it has undergone many significant changes throughout this time (drill and practise; computer based tutorials (CBT); intelligent tutoring system (ITS); Logo programming; computer algebra systems (CAS); graphic calculators, GeoGebra etc.).

Snir (1995) argues that computers can make a unique contribution to the clarification and correction of commonly held misconceptions of phenomena by visualising those ideas. Instead of spending considerable time learning by hand routines, West (1995) argues that the computer can alleviate such time constraints, allowing the students to move more swiftly to higher-level conceptual matters and a variety of practice problems. In addition, West (1995) acknowledges that students who learn mathematics through the use of visual techniques have shown that, in comparison with traditional courses, they understand the basic concepts
better, can remember the information longer and can apply the concepts to practical uses more effectively. However, Daugherty (2007) identified that we have not seen widespread adoption of visualisation techniques in mathematics teaching and learning. Eisenberg and Dreyfus (1991) further this debate, where they analysed why there is such widespread reluctance on the part of both teachers and students to choose visual methods in problem solving and in establishing a basic understanding of fundamental notions. They concluded that visualisation techniques are cognitively more demanding of the learner than analytical techniques, which are more algorithmic in nature, thus providing negative attitudes (fear, anxiety etc.) for both teachers and students to using visualisation methods.

Another plausible explanation offered by Eisenberg (1994) is that visualisation techniques are simply not part of the disposition of what is accepted as proof in mathematics. In order to overcome such fears expressed by Eisenberg and Dreyfus, mathematics educators should try to build a bridge between visual and analytic or algebraic representations of the same mathematical concept. By doing so, the student will be provided with a more comprehensible and in-depth insight into mathematical concepts. Teachers can afford, much more effortlessly, to demonstrate both the origin of such mathematical concepts and their application, applications not just to the real-world but as insight into the inter-connectedness of mathematical ideas and domains.

### 2.3 The Background: At Home in Ireland

Ireland reflected the changes in world-wide reform in the 1950’s where they underwent changes in both the syllabus and assessment procedures experienced in
mathematics education over the last half-century. A “top-down” (Oldham, 1992) approach of change ensued where the Department of Education, in conjunction with assistance from third level institutes, devised a number of significant alterations to the existing syllabi. The first changes in the mathematics syllabi were in 1964 with the introduction of a new Leaving Certificate for examination in 1966, which was followed closely by a new junior cycle syllabus at two levels, namely Higher and Lower, for first examination in 1969 (An Roinn Oideachas, 2003). The influence of the ‘modern’ mathematics movement was highly visible in the modifications to the syllabi at this time, and change was undertaken in Ireland largely to ensure that the country kept in touch with modernisation in both society and education that occurred throughout the world at this period (Oldham, 1992).

Ireland adopted this philosophy to a greater extent than many of its partners, and it is still easily recognisable within the examination process with the presence and focus on precise terminology and abstraction that was characteristic of the movement (Oldham, 2002). These syllabi were largely untouched until the late 1980’s when it was quite apparent that large cohorts of pupils were not having their mathematical needs catered for by these syllabi. However, the changes of the 1980’s focused on content rather than pedagogy and “on the removal of particular problems, always associated with course content” (Oldham, 1996: p38). Consequently, any changes since have been as a result of courses being considered too long, or too abstract, where the process as a whole has been one of “evolution rather than revolution” (Oldham, 1991, p. 126). Traditionally, there has been an over-emphasis on the content of mathematics which establishes the syllabus within the school systems in Ireland, rather than a move towards a better understanding of the role of pedagogy in ensuring a more balanced experience for
pupils and teachers alike. Nonetheless, syllabus development in mathematics in Ireland has moved towards ‘involvement and negotiation’ (Oldham, 1992, p. 138) which could be described as ‘bottom-up’ development, in stark contrast to that development initiated by the Department of Education in the 1960’s.

The ‘involvement’ stage of curriculum design in Ireland has been provided through the contribution of teachers in the system and, especially, the Irish Mathematics Teachers’ Association (IMTA). The IMTA has been to the fore in curriculum design and implementation over the past twenty years, working closely with the National Council for Curriculum and Assessment (NCCA), thereby ensuring a move towards appropriate ‘negotiation’ of the mathematics curriculum in Ireland (Oldham, 1992). The syllabi have been tidied up considerably since the initial alterations, yet there must be further recognition of the current gap between follow-on material at all three levels (Primary, Junior & Senior). Efforts in the manner of providing four distinct levels at Senior-Cycle (Higher, Ordinary, Foundation, and Applied Leaving Certificate Mathematics) have ensured there has been a more realistic approach to catering for the needs of all pupils, as opposed to the Bourbaki approach.

A revised curriculum for primary schools in all subjects was introduced in 1999 and improvements in mathematics were to the fore. The philosophy underlying this was quite similar to that of RME (interestingly without formal contact with the literature on RME), where the focus was on real-life problems and on the use of contexts (Oldham, 2002). Significant developments included the introduction of the calculator to pupils from fourth class up and a focus on real-life problems and examples (Close et al., 2003). It was intended that teachers of the Junior cycle could build on material covered in the revised primary curriculum therefore,
ensuring a smoother transition between primary and post-primary mathematics (NCCA, 2004).

The new Junior Certificate replaced the Inter Certificate in 1989. There were three distinct levels known as syllabuses A, B and C, which were renamed as Honours, Ordinary and Foundation respectively and first examined under these names in 1992 (NCCA, 2005). To follow this, the Junior Certificate revised curriculum, was introduced in 2001. This was less insightful than the revision at primary level, and it was considered as an ‘evolution’ of the existing curriculum aimed at tidying up problem areas within the course, rather than a “root-and-branch” review (Oldham, 2002, p. 43). Revisions focused on content, which was in essence a minor update, by removing topics. Topics removed included the use of tables (coinciding with the introduction of the calculator), logarithms, composition and inverse of functions (NCCA, 2004).

Evaluation measures were by and large concerned solely with the structure of the examination process, where the revised syllabus was closely aligned to that used in the Leaving Certificate and the three-part structure: (a), (b), (c), that leads from easier work to more challenging tasks (NCCA, 2004). The professional development programme for existing mathematics teachers, consisting of in-service programmes, which accompanied the revised syllabus attempted to move towards teaching for understanding. This could be considered comparable to the NCTM Standards and, more significantly, away from the mechanist approach which dominated the ‘modern’ approach to mathematics education in second level education in Ireland (Oldham, 2002).
2.3.1 Project Maths

In an effort to promote international standards and principles outlined by the OECD, NCTM Standards and the PISA studies, a new initiative in mathematics education called ‘Project Maths’, was undertaken in 24 pilot schools in Ireland in September 2008. Widespread integration of all schools in Ireland took place in September 2010 (Project Maths, 2010). Fundamental to this change, is the greater emphasis being placed on student understanding of mathematical concepts, with increased use of contexts and applications that enables students to relate mathematics to everyday experience. Problem-solving skills are viewed as a key component in this new development where students come into contact with mathematics in an innovative approach, using applications that are meaningful for them (Project Maths, 2010). As an alternative to the current system, where mathematics syllabuses incorporate topics in a horizontal approach based on arithmetic, algebra, trigonometry, geometry, statistics and calculus (Kilpatrick, 1996), ‘Project Maths’ will be developed under five distinct strands:

- statistics and probability
- geometry and trigonometry
- number
- algebra
- functions.

Considerable changes will occur at both Junior and Senior level, where the Junior-Cycle, will become open to a more investigative approach which will be used to
build on and extend students’ experience of mathematics from primary school. In order to ensure improved continuity with primary school mathematics, a bridging framework is being developed that will connect the various strands of mathematics in the primary school to topics in the Junior Certificate mathematics syllabuses (Project Maths, 2010). An uptake of 60% of the student cohort for higher-level mathematics will be targeted to assist the increased uptake of higher level mathematics at senior level.

In the Senior-Cycle, the mathematics curricula will enable students to further develop the knowledge and skills essential for their future lives, both in work related contexts and in their further studies that rely on a considerable background in mathematical understanding (Project Maths, 2010). An uptake of 30% at higher level is targeted, so as to improve on the current figure of 16% (Project Maths, 2010). Certain aspects of the material offered under the present system will be incorporated under the relevant strands in the revised syllabuses, and, radically students will be required to study all five strands (Project Maths, 2010). The presence of options in current syllabi is viewed as an integral component of the ‘Mathematics Problem’ witnessed at third-level institutions worldwide (Gill, 2006). Thus, it is intended that the revised syllabuses will not have optional topics, thus ensuring that all students experience all aspects of the syllabus, which is in stark contrast to the current system in Ireland.

2.3.2 Programme for International Student Assessment (PISA)

The author examined the results of PISA so as to determine our (Ireland’s) standing from an international perspective in mathematical terms. The OECD Programme for International Student Assessment (PISA) is an international survey of
15-year-old students that takes place every three years. Students’ literacy in science, mathematics and reading is assessed in PISA, where students are assessed using a paper-and-pencil test containing a mixture of multiple-choice items and items where students need to write their own answers (Eivers et al., 2007). The term literacy is used to emphasise the ability to apply knowledge, rather than simply to reproduce facts that have been studied in a curriculum (Eivers et al., 2007). Fifteen-year-olds are the target group because this age marks the end of compulsory schooling in many countries. The overarching ideas governing the mathematical element of PISA 2003, in an effort to meet the historical developments and reflection of the major threads of current school curricula, were outlined as:

- quantity
- space and shape
- change and relationships
- uncertainty.

(OECD, 2003, p. 35)

Quantity

- deals with quantitative reasoning, recognition of numerical patterns and the processing and understanding of number as that are presented to us in different ways.

Space and Shape

- deals with the recognition of geometric patterns, the understanding of the relationship between shapes and images or visual representations, and the understand-
ing of two and three dimensional objects and their properties.

**Change and Relationships**

– deals with the recognition of change processes found in everyday natural phenomenon and their correlation with mathematical functions and the ability of these relationships to form a variety of different representations.

**Uncertainty**

– deals specifically with the two related topics of data and chance

(de Lange, 2003; OECD, 2003).

The PISA conceptualisation of mathematical literacy (see Figure 2.1) is based on the Realistic Mathematics Education movement, which emphasises the notion of “mathematising” (Eivers et al., 2007). This process involves taking a problem in a real-world context, organising it according to mathematical concepts, and gradually trimming away the reality. Once the key features of the problem are recognised, it can be solved mathematically. The final step in mathematising a problem is to make sense of the mathematical solution in terms of the real situation (OECD, 2003).
PISA established a range of levels associated with the scores which explain what a student can typically be expected to achieve at each level. For mathematics there are 6 levels, beginning at Level 1 with questions that require only the most basic skills to complete and increasing in difficulty with each level. For 2006, Ireland’s mean score in the mathematics domain is the 22nd highest of the 57 participating countries, and the 16th highest of the 30 OECD countries. While fewer students in Ireland scored at or below Level 1 (16%) on the PISA 2006 mathematics proficiency scale compared with the OECD average (21%), fewer also performed at or above Level 5 (Ireland’s score was 10% versus the average of 13%). These figures indicate that Ireland’s average mathematics performance is attributable to lower-achieving students doing reasonably well, and higher-achieving students doing relatively less well.
2.4 Applications from a School Mathematics Perspective

Niss (1987) argues that the elementary curriculum, while dealing with less abstract and more concrete conditions in mathematics than its successor, has provided a playing field conducive to applications. Its developments have mirrored those at post-elementary level, however, from rather different perspectives and at altered timeframes. The phases implemented at both levels are complicated to date, and are rather structural in nature as opposed to strictly chronological. Besides, they occur at various times in separate regions of the world, and, critically, at different strengths of implementation (Niss, 1987). The modern maths reform, in focusing on rather abstract and academic issues, assigned low priority to the realms of social and real life, which, consequently amounted to demands for adequate problem-solving skills. It followed that the primary focus was the fostering of pupil’s capacity to engage in and solve extra-mathematical problems by means of mathematics techniques, which was widely acknowledged as ‘problem-solving’ techniques. However, pedagogical difficulties and lack of appeal to students, led to an attempt to achieve these aims by providing situations within which students could detect the mathematics involved in the real world surrounding them (Niss, 1987). The current system in many countries underpins the philosophy of this ‘real world surrounding them’, which is highlighted in current practices at international level and recognised by the PISA study. The post-elementary mathematics curricula have encountered a number of phases in applicational aspects also. It became quite clear after the modern mathematics reform that the applicational capacity of mathematics had to be demonstrated in the teaching, not just assumed (Niss,
2.5 Why include Modelling in the Teaching and Learning of Mathematics?

The benefits of including modelling within mathematics programs are widely recognised (Ferrucci & Carter, 2003; James, 1988; Mustoe, 1992), with many countries calling for increased instructional emphasis on mathematical modelling techniques (Ferrucci & Carter, 2003). In the United States, the ‘Principles and Standards of School Mathematics’ outlined by the NCTM (2000) and the OECD Program for International Student Assessment in Mathematics (PISA 1999, 2003) have produced significant modelling demands. Mathematics classrooms are by and large dominated by students who can be described as ‘non-specialists’, in that they are not fully committed to mathematical study, or entirely convinced of the value of mathematics to their overall education and its place in their future lives (Ormell, 1984). With current practices in mathematics education catering almost exclusively for the small minority of ‘specialists’, many modern children are unable to recognise the need for mathematical thinking when they enter third level education or the workplace. Consequently, Ormell (1984) deduces that we need courses which are Capable of Appealing to the Non-Specialist (CANS). CANS courses aim to contrast present courses which are no longer appealing to the ‘non-specialist’, and in return provide opportunities for the ‘non-specialist’ to perceive value in mathematics and mathematical thinking, thus, alleviating their alienation from mathematics. Burghes (1985) identifies modelling as an activity that encourages the use of discussion, investigation and application in the teaching of
mathematics and for this reason merits inclusion within a mathematics program. Nonetheless, concern is expressed regarding the difficulties of implementation within a classroom setting, and the problems that may arise as a direct result (Burghes, 1985; James, 1985). Caution is advised regarding the introduction of new mathematical material, ideas or techniques, where mathematical modelling is not recommended as an appropriate vehicle for such activities as it can in fact hinder the understanding process in such circumstances (James, 1985; Burghes & Huntley, 1982).

Kaiser & Sriraman (2006, p. 302) offers a range of goals associated with an overall view of the separate approaches as outlined:

- **Pedagogical goals:** imparting abilities that enable students to understand central aspects of our world in a better way;

- **Psychological goals:** fostering and enhancement of the motivation and attitude of learners towards mathematics and mathematics teaching;

- **Subject-related goals:** structuring of learning processes, introduction of new mathematical concepts and methods including their illustration;

- **Science-related goals:** imparting a realistic image of mathematics as science, giving insight into the overlapping of mathematical and extra-mathematical considerations of the historical development of mathematics.

With the range of goals and aims concerning applications and modelling differing substantially, so to has the discussion on applications and modelling. In the next subsections, a number of issues will be discussed which arose as a result of examining why to include modelling in the teaching and learning of mathematics.
2.6 Pure vs. Applied Mathematics

The desirability of including applications in mathematics education, while always present in the 20th Century, had gained significant momentum by the early 1970’s. Schools had struggled with ‘modern’ mathematics and they had come to recognise the obvious pitfalls of ‘modern’ mathematics (Walker, 1970; Niss, 1987; Burghes et al., 1982).

Applied mathematics had warranted the image of ‘just mechanics’ and as a result applied mathematics progress had stagnated both at school and university levels (Ford & Hall, 1970; Burghes & Huntley, 1982). Many contributing factors were present in ensuring the gradual move towards the unification of mathematics. At the outset, many students began to question the integrity of the mathematical content which they were expected to study, particularly the isolation of the subject from other subjects, the sciences and from the world in general (Niss, 1987). With this revolution came the evolution of appropriate applications in more disciplines than ever before where mathematics had come to be recognised as a useful tool – economics, biology, geography, medicine, archaeology etc. (Burghes & Huntley, 1982; deLange, 1996). The emergence of effective applications of mathematics in problems associated with familiar everyday problems described by Davis & Hersh (1982, p. 83) as “common utility” was considered essential in ensuring successful implementation of applications in mathematics education. In the first half of the 20th Century the utility of mathematics was at the origin of curricula debate and deliberations at this level. The modern mathematics reform altered the course of the role and focus on applicational aspects of the elementary curriculum, with the curricula entering unprecedented avenues of change. The reaction against the ‘modern maths’ movement witnessed in many countries, as outlined in Kline’s
‘Why Johnny Cant Add’, can be considered as a contributor to progress in the acquisition of appropriate applications in new mathematics disciplines (de-Lange, 1996; Burghes & Huntley, 1982). Additionally, it must be noted that the increase in the availability of information technology was a major contributing factor in providing access to more and more realistic applications in other topics treatable by suitable mathematics (Davis & Hersh; 1982; deLange, 1996; Blum 2002). As a result, the rate of innovation in genuine applications in mathematics prospered, thus allowing for greater emphasis on new and original ideas to be explored at a rate unheralded previously.

The 1970’s witnessed a shift in emphasis towards applied mathematics as opposed to the traditional discipline of pure mathematics. This was evident in the requirements for entering university posts and research departments in mathematics (Niss, 1987; Davis & Hersh, 1982). The career opportunities for a considerable segment of graduates and PhD’s in mathematics were becoming further associated with applied mathematics, which in turn resulted in a visible attempt by many mathematicians to find a link between their own speciality and some area of application.

In essence, social requirements, relevance and the role of technology have driven the process of acquiring essential mathematical knowledge, thus “indicating that applied mathematics is the pre-eminent for mathematical growth in society” (de-Lange, 1996, p. 53). However, many educators consider the distinction between pure and applied mathematics to be lessening. The modern maths movement and the Mathematical Maoism doctrine witnessed in China could be best described as end-points of a pure-applied spectrum. The present movement away from these ideological phases towards a more balanced curriculum incorporating the
key aspects of both approaches is welcomed by many. (Davis & Hersh, 1982; de-
Lange, 1996; Niss, 1987). Ormell (1975, 1977) maintains that pure mathematics,
as is recognised in schools currently, offers a formalistic approach to the teaching
and learning of mathematics in post-primary schools. Applicable Mathematics
contrasts these conditions presenting, as an alternative, a naturalistic approach to
mathematics, whereby students will become exposed to a form of mathematics
in many different ways and on several levels, which has numerous contacts with
the real world. The contrast between a naturalistic approach and a formalistic ap-
proach to school mathematics can only be described as “end points on a spectrum
of styles of mathematics” (Omell, 1975, p. 349), involving the use of intuition vs.
abstract rules, practical imagination vs. formalisation, and informal vs. symbolic
methods of argument.

2.7 Models & Modelling

It is important to distinguish between mathematical modelling and mathematical
models. At ICME 6, in 1988 a model was defined as a set of objects and relation-
ships, chosen to represent and reflect different aspects of an extra-mathematics
165) describes encountering a mathematical model as the process when the stu-
dent is concerned with ‘applying a mathematical technique to a set-piece model
with emphasis on obtaining a solution to the model’. Primarily, the use of ap-
lications and models in mathematics teaching was to provide opportunities for
the student to actively engage in mathematics themselves in dealing with prob-
lems from other disciplines and the world surrounding them (Niss, 1987). How-
ever, as observed by Huntley & Burghes (1982) and James (1985), experiencing a mathematical model was a passive experience of seeing someone else’s completed model, as opposed to the active and demanding experience of designing and developing the model oneself.

The acknowledgment of the importance of students not being presented with models as objects of learning, but the need to arrive at them as an outcome of their own efforts was considered paramount in what became known as the ‘modelling’ process (Niss, 1987). The method leading from the initial problem condition to a mathematical model is generally accepted as mathematical modelling. However, it is well established to use that concept also for the entire process, consisting of structuring, mathematising, working mathematically and interpreting/validating (Blum, 2002, p. 153). The term mathematical modelling has numerous interpretations that have materialised from various research perspectives. Ogborn, as cited in Carter & Ferrucci, (2003), considers mathematical modelling as a type of simulated world with the characteristic that all of the components of the event are identified and taken into account prior to solving the problem. Mustoe (1992, p. 576) refers simply to the process of mathematical modelling as the four standard approaches of ‘formulation, solution, interpretation and validation’. Haines and Crouch (2007) view mathematical modelling as the process which involves moving from a real-world situation to a model, working with that model and using it to understand and to develop or solve real-world problems. Matos (1998, p. 26) reinforces these views with a description of mathematical modelling as “an activity where students give meaning to ideas, concepts, problems [and] mathematical and non-mathematical concepts”. While, most interpretations refer to being faced with a model, to which mathematical techniques can be applied. James (1985)
noted that mathematical modelling does not comprise finding a unique answer to a well-defined mathematical problem, as previously encountered. Instead a student is likely presented with a problem and has to strive to find relevant questions and answers from a condition that may initially look chaotic. Several educators provided a flow-chart overview of the entire modelling process in an effort to categorise the continuing process of adaptation and sophistication until a compromise is reached between reliability and ease of analysis (Bajpai et al., 1975). Penrose (1978) offers seven phases of the modelling cycle:

<table>
<thead>
<tr>
<th>Table 2.1: Seven phases of the modelling cycle (Penrose, 1978)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Specify the real problem</td>
</tr>
<tr>
<td>3. Specify the math problem</td>
</tr>
<tr>
<td>5. Interpret the math solution</td>
</tr>
<tr>
<td>7a. Revise</td>
</tr>
</tbody>
</table>

Further examples of these approaches are presented in Figures 2.2 and Figure 2.3. The 1982 Cockcroft Report, while attempting to describe essential elements of effective teaching, gave strong emphasis to the role of applications. Its most quoted paragraph (243) outlined a blueprint that implies a modelling approach (Burkhardt, 1989):

- Exposition by the teacher
- Discussion between the teacher and pupils and between pupils themselves
- Appropriate practical work
- Consolidation and practise of fundamental skills and routines
- Problem solving, including the application of mathematics in everyday situations
Figure 2.2: The iterative nature of model construction (Giordano & Weir, 1985)
Figure 2.3: Flow chart for the modelling process (Bajpai et al, 1975)
- Investigational work.

Despite the relative diversity in approaches to applications and modelling worldwide, the focus of modelling courses has consistently remained within the structure of the existing mathematics curricula. The conventional approaches to instruction and course work are almost entirely altered in this modelling approach, yet the mathematical substance of models remains largely unchanged (Niss, 1987). The integrated approach of instruction which was proposed by Bajpai and his colleagues at Loughborough University, was one such approach which in effect suggests “that those analytical, statistical, numerical and computer techniques which are relevant to a particular topic are all discussed or mentioned when the topic is being taught” (Bajpai et al., 1976, p. 350). Approaches in teaching mathematics traditionally contrast this approach where these approaches are experienced at staggered phases, with the mastery of one approach allowing access to another. This integrated approach ensures that all viable methods of solution to a particular problem are encountered as early as possible rather than initially encountering the perceived straightforward and/or significant technique. Projects, case-studies, individual work, small group work, written reports, oral reports (video and interview respectively) are all widely recognised as important tools and worthy of inclusion in successful modelling approaches (Houston, 2007, 1997; Clatworthy, 1989; Niss 1987; Ormell, 1984; Bajpai et al., 1975). Additionally, when the conventional aspect of the modelling course is added, Clatworthy (1989) argues that all components of paragraph 243 of Cockcroft’s report are catered for.
2.8 Classification of Modelling Approaches

The various backgrounds and philosophy behind modelling vary significantly due to their aims concerning the distinct approaches outlined by Kaiser and Sriraman (2006) in Table 2.2. Currently, there does not exist a uniform approach or consideration with respect to the teaching and learning of the applicability of mathematics (Burghes & Huntley, 1982), however, educators are exposed to a uniform usage of terminology (Kaiser & Wilander, 2006). In an effort to suitably identify the significant research employed at an international stage, a classification system for presenting modelling approaches has been suggested by Kaiser and Sriraman (2006). This classification system considers the current developments of the modelling discussion, while distinguishing various perspectives of modelling based on their initial perceptions.

A central characteristic of the realistic or applied perspective is that modelling is understood as “an activity to solve authentic problems and not as a development of mathematical theory” (Kaiser & Sriraman, 2006, p. 305). Authentic examples from industry and science tend to play an important role within this perspective.

The contextual perspective has its roots firmly in the perspective of solving word problems, a tradition particularly evident in American education. However, this process is deemed as much more than a problem solving activity, where the model eliciting perspective is based on the premise that “modelling research should take into account findings from the realm of psychological concept development to develop activities which motivate and naturally allow students to develop the mathematics needed to make sense of such situations” (Kaiser & Sriraman, 2006, p. 305).
Table 2.2: Classification of current modelling approaches (Kaiser & Sriraman, 2006)

<table>
<thead>
<tr>
<th>Name of the Perspective</th>
<th>Central Aims</th>
<th>Relation to earlier perspectives</th>
<th>Background</th>
</tr>
</thead>
<tbody>
<tr>
<td>Realistic or applied modelling</td>
<td>Pragmatic-utilitarian goals i.e. solving real-world problems, understanding of the real world, promotion of modelling competencies</td>
<td>Pragmatic perspective of Pollak</td>
<td>Anglo-Saxon pragmatism and applied mathematics</td>
</tr>
<tr>
<td>Contextual modelling</td>
<td>Subject-related and psychological goals i.e. solving word problems</td>
<td>Information processing approaches leading to system approaches</td>
<td>American problem solving debate, as well as everyday school practice and psychological lab experiments</td>
</tr>
<tr>
<td>Educational modelling: differentiated in a) didactical modelling b) conceptual modelling</td>
<td>Pedagogical and subject-related goals</td>
<td>a) Structuring of learning processes and its promotion b) Concept introduction and development Integrative perspectives (Blum, Niss) and further developments of the scientific-humanistic approach</td>
<td>Didactical theories and learning theories</td>
</tr>
<tr>
<td>Socio-critical modelling</td>
<td>Pedagogical goals i.e. critical understanding of the surrounding world</td>
<td>Emancipatory perspective</td>
<td>Socio-critical approaches in political sociology</td>
</tr>
<tr>
<td>Epistemological or theoretical modelling</td>
<td>Theory-oriented goals i.e. promotion of theory development</td>
<td>Scientific-humanistic perspective of “early” Freudenthal</td>
<td>Roman epistemology</td>
</tr>
</tbody>
</table>
The educational perspective fundamentally provides a continuation of integrative approaches which promotes the structuring of learning processes and fostering the understanding of concepts above all other goals. The majority of approaches developed with respect to modelling and mathematics education lie within this perspective (Kaiser & Sriraman, 2006).

The socio-critical perspective endorses the socio-cultural dimension of mathematics, which closely related to ethno-mathematics as promoted by D’Ambrosio (1999). This perspective emphasises the role of mathematics in society, including the identification and differentiation between modelling activities within industry and those activities within school mathematics, and the necessity to support critical thinking about the role of mathematics within society (Kaiser & Sriraman, 2006).

The epistemological perspective has further developed the ideas of Freudenthal’s (1973) early work regarding mathematising and modelling as part of theory development, towards Treffers (1987), and other Freudenthal’s successors, distinction of horizontal mathematising (Kaiser & Sriraman, 2006). The analysis provided by Kaiser and Sriraman (2006) in the form of a classification system of the distinct approaches to modelling, as provided in Table 2.2, demonstrates that significant further developments are taking place within the debate on applications and modelling throughout the world. Often these approaches enhance traditional approaches. However, while there exist a frequent commonality of usage of terms and concepts within the discipline of applications and modelling, the underlying assumptions often differ widely.
2.9 Current Research Trends

The first International Congress on Mathematical Education (ICME) in 1969 was the initial catalyst for recognition of the role of mathematics educators at all levels in making mathematics more understandable and interesting for all (Bajpai, 1970). Applications and modelling have been an important theme in mathematics education over the last 40 years, as can be recognised through the extent of literature on the topic, and in particular from the whole host of influential conferences held at both national and international level (Blum, 2002, Burkhardt, 2006). In particular the ICME conferences and the series of ICTMAs (the International Conferences on the Teaching of Mathematical Modelling and Applications), which have been held biannually since 1983, have provided a gateway for considerable deliberations regarding the role of applications in mathematics education. Some of the significant proceedings of each ICME will now be discussed. For example, at ICME 2, in 1973, computer science was proposed as a field of applications for the first time. ICME 3, in 1976, witnessed Ormell’s ‘Mathematics Applicable’ proposals as outlined in his 1972 paper ‘Mathematics, The Science of Possibilities’, among other significant contributions in the discipline of applications. In addition, the co-operation between science teachers and mathematics teachers was largely documented at that time at two separate conferences in 1978: the Unesco Conference and the conference on ‘Co-operation between Science teachers and Mathematics Teachers’ in Bielefeld. The ICME 4-7 conferences were motivated by the conviction that there were significant benefits to be gained for those students who may not consider mathematics a desirable discipline, through the inclusion of many applications and the stressing of the usefulness of mathematics in real life situations. Modelling and Applications was to the fore-front at ICME 6 in 1988.
as it was becoming apparent that there were many obstacles that prevented the implementation of applications at a practical level (deLange, 1996). The subsequent ICME conferences 8-10 have created extensive awareness of the complexity of the problems of teaching and learning modelling and applications, with the particular trend of recognition of modelling and its role in developing mathematical literacy becoming apparent (Kaiser & Wilander, 2004). ICME 11 which took place in Mexico, in 2008, has again provided a forum devoted entirely to mathematical applications and modelling in the teaching and learning of mathematics.

The Institute of Mathematics and its Applications (IMA) journal of *Teaching Mathematics and Its Applications* and the *International Journal of Mathematical Education in Science and Technology* (iJMEST) have also provided forums for the exchange of ideas and experiences which contribute to the improvement of mathematics teaching and learning for students at all levels. A distinctive feature of the IMA journal of *Teaching Mathematics and Its Applications* has been its emphasis on the applications of mathematics and mathematical modelling within the perspective of mathematics education world-wide (Lawson, 2008).

Altogether, it is clear that substantial research in mathematics education has been centred on modelling. The fundamental focus of many findings has been on practise, for example, the “constructing and trying out mathematical modelling examples for teaching and examinations, writing application-oriented textbooks, implementing applications and modelling into existing curricula or developing innovative, modelling-oriented curricula” (Blum, 2002, p. 150). These activities have contained many research components, such as:

“classification of relevant concepts; investigation of competencies and identification of difficulties and strategies activated by students"
when dealing with application problems; observation and analysis of teaching, and study of learning and communication processes in modelling-oriented lessons; evaluation of alternative approaches used to assess performance in applications and modelling”

(Blum, 2002, p. 150).

The diligence in recognising the diversity of approaches to applications and modelling is an effort to highlight the contemporary state of the educational debate concerning the teaching and learning of applications and modelling. So as to determine and identify the wide variety of categories proposed, the following section offers an insight into the factors determining the expansion of applications and modelling as established at the present time.

2.10 Factors Affecting Current Trends

The integration of applications and modelling within the mathematics curricula worldwide is still regarded as being in a period of transition. We have still to reach a stage of general classification, or a settled consensus on its role within mathematics education (Niss, 1987; Burkhardt, 2006). However, it cannot be overlooked that applications and modelling have secured a grip in post-elementary curricula in most countries (Niss, 1987; Burkhardt, 2006). The need for students to learn mathematics through the medium of applications and modelling is widely accepted and as already discussed it’s merits for inclusion are widely recognised (Burkhardt, 2006; Ferrucci & Carter, 2003; James, 1988; Mustoe, 1992). In countries, where planning and implementation of the mathematics curriculum are
controlled by central authorities and/or government sources, the inclusion of ade-quate applications, models and model building has been relatively slow, with wide-scale procedures for change considered problematic and slow-moving. In contrast, countries where curriculum development is decentralised, applications and modelling activities are plentiful, with a far more delicate response to concerns within the curricula (Niss, 1987). The process of change is considered most problematic when occurring on a large scale, and is a relatively slow process, incorporating various factors. The large-scale implementation of modelling is no different, with the existence of several distinct obstacles which will now be discussed by the author. These obstacles include:

1. Time constraints
2. Students’ conceptions of applications
3. Assessment procedures
4. Limited professional development.

2.10.1 Time Constraints

Applications and modelling activities are substantially different from traditional mathematical activities and are certainly more time consuming. The more realistic the application and modelling process the more extensive time is required to successfully complement this approach. Current educational programmes are precisely outlined so as to incorporate many disciplines. The time allotted to mathematical studies can be altered little so as to facilitate all existing and prospective aspects, thereby providing modest flexibility for change. The time allotted to
the pure elements of the mathematic curriculum would have to be diluted or re-
duced to allow room for applications and modelling to be facilitated (Niss, 1987,
Burkhardt, 2006). Grattan-Guinness (2001), while acknowledging the formalistic
approach of pure mathematics and the necessity to employ an approach driven by
applications, is cautious as regards the over emphasis on an applicational style
of approach. Central to his argument is that many skills would be required to
master, not only the mathematics involved in this approach, but also the extent to
which new material will be experienced when encountering the other disciplines
in which it is used or modelled. Many mathematics teachers and policy makers
would be unwilling to allow such dilution as it would represent to them a decline
in the quality of the mathematics education provided to students, a decline not
compensated by the significant mathematical understanding gained trough appli-
cations and modelling activities (Niss, 1987; Burkhardt, 2006).

However, time constraints alone cannot be used as a valid opponent of change.
The time taken to employ new and alternative approaches (especially with the
teaching of applications and modelling) in the mathematics classroom will be
significant, however, teachers should not see this as a deterrent but rather teachers
must embrace it and consider the added learning and pedagogical benefits of such
approaches. Any additional time devoted to such activities in classroom practise
will be redeemed with interest by the value added to the student experience in
terms of interest and enjoyment.

2.10.2 Students Conceptions of Applications

While applications are extensively recognisable for elementary mathematics activ-
ities, there does not exist to the same degree such applications for post-elementary
level. The perceived irrelevance of current mathematical activities has fuelled the debate for the inclusion of appropriate applications and modelling (Burkhardt, 2006). Numerous examples of applications are available, yet few really connect with students’ perceptions of realistic applications. In fact it will add to the complexity and misunderstanding of the influence of mathematics in tackling situations and problems from the world around us (Niss, 1987). It is imperative to ensure that appropriate applications are provided, as access to excessively specialised problems would result in over-complication of material and marginalised understanding by the student. Non-trivial examples, which do not demand too high a level of mathematical expertise, are required to guarantee suitable applications are supplied to the students (Bajpai et al., 1976). Rather than putting forth examples and case studies that are dependant on issues that in solving them mathematics plays a marginal role, considerable research is needed in providing cases for which mathematics provides crucial assistance in their solving, and therefore mathematics will provide answers that cannot be obtained by other means, or only with rigorous efforts deemed inappropriate (Niss, 1987). The aesthetic nature of mathematics is frequently proposed as a paradigm from which many mathematicians summarise their interest in mathematics. Ormell (1972) disputes this concept referring to the fact that one cannot derive a sufficient justification for mathematics in the curriculum from aesthetic factors alone. To do so would enable mathematics to find its status in a weaker position in the curriculum than other subjects such as music and art. Blaire (1973) argues for the focus on the applicative nature of the activities within mathematics as a focal point from which to justify the place of mathematics within education per se. Fundamentally, Blaire (1973, p. 419) emphasises that “if one accepts that mathematics is a simulating
activity for problem-solving in life-mathematics contexts, then this is the most practically relevant way a child can see what mathematics is about”.

### 2.10.3 Assessment Procedures

The responsibility of assessment is conflict-ridden in relation to many issues in mathematics education, and certainly the field of applications and modelling is no different. Crucial skills and techniques deemed appropriate for successful modelling are difficult to assess in environments where end-of-term written examinations are provided (Burton, 1997; Clatworthy, 1989). There are many alternative examples proposed by mathematics educators.

The current assessment process widely recognised in many countries, which is largely dominated by end-of-term examination, merely requires regurgitation of standard techniques applied to standard examples (Burton, 1997; Clatworthy, 1989; Bajpai et al., 1975). While most students obtain good marks in examination questions, it became apparent that understanding and interpreting solutions was by and large non-evident in students. Quite clearly, an effective evaluative strategy for the inclusion of modelling techniques and skills is unlikely to be an end-of-term written examination (Houston, 1997; Burton, 1997; Clatworthy, 1989). Bajpai et al. (1975) argued for the provision of situations within which students could appreciate the concept of a mathematical model and the methods of obtaining solutions to such a model, known as ‘case studies’. This introduction to a perspective of modelling should allow for the opportunity of presenting case studies which illustrate the role of mathematics in solving real-life problems. Bajpai et al. (1976, p. 353/354) observed that:

“if possible the case study should involve several mathematical techniques so that
the lecturer can make the very important point that in real-life the solution of a problem requires all the engineer’s (mathematical student’s) mathematical expertise and he cannot expect the problem to fit nicely into a standard drill example pattern.”

Clatworthy (1989) further develops this approach with a firm concentration on developing student’s skills in written and oral reports, in presenting solutions to peers or defending a case study based on mathematical argument. Fundamental to this practice is acknowledging the higher process level within which assessment for modelling techniques should aim at. To ensure this appropriate feedback is crucial, through the use of ‘feedback sheets’ to not only assess their work but also provide information on the variety of modelling skills and different levels at which they are operating (Clatworthy, 1989). An alternative to written and oral reports is outlined by Houston (1997) in which the use of posters is promoted. Houston (1997, p. 136) suggested that poster sessions for students:

- are an excellent alternative for developing communication skills
- encourage the thorough investigation and concise reporting of a topic
- provide opportunities for self and peer assessment and for peer learning
- promote a positive attitude in students.

As an adequate form of evaluation, Houston (1997) outlined certain criteria and conditions that are recommended for inclusion in suitable posters. At present in the UK, project work in the shape of investigations, or coursework tasks, account for 20% of the total assessment for the GSCE mathematics examination (Glaister & Glaister, 2000). Undoubtedly, there will be numerous deliberations as to
the desirable forms of evaluation considered most useful in assessing modelling
and applications within mathematics education. However, there exists almost
uniform agreement that current practises are outdated and need to be revitalised
(Burkhardt, 2006).

2.10.4 Limited Professional Development

Mathematics teacher education programmes, including teacher pre-service and
in-service, are relatively alienated from the field of modelling, and the use of
modelling processes within mathematics courses (Blum, 2002; Burkhardt, 2006).
The execution of curricula witnessed in mathematics classrooms arise as a result
of the skills that mathematics teacher acquire almost entirely in their pre-service
education, and enhanced during their early years of classroom practice. Continu-
ing professional development is somewhat minimal in Europe and the U.S. There
are very few opportunities for professional development provided for or under-
taken by teachers, where often those who need them most are slow to avail of any
of the existing opportunities to further their training (Burkhardt, 2006). Teaching
modelling or applications to the real world in mathematics classrooms may in-
voke a negative outlook in mathematics teachers towards this style of instruction,
as they may find it difficult to teach for a number of reasons (Ferrucci & Carter
, 2003). Firstly, this approach forces teachers to teach with new concepts of in-
stutional style, quite removed from the traditional approaches most recognised
Furthermore, this new approach will impact significantly on how they determine
the usefulness of integrating technology into existing practises. Teachers of math-
ematics will be presented with new approaches to teaching styles and instruction,
and with this they need to think about all possible avenues of success, with the integration of technology to the forefront (Ferrucci & Carter, 2003). To ensure that modelling and an applicational style of approach to teaching mathematics is catered for at all levels, future mathematics teachers must be taught in a similar way to the way they will teach later (Martinez-Luaces, 2005, Crouch & Haines, 2004; O’Donoghue, 1978). The desired success of this approach is heavily dependant on and influenced by both the teaching approach and motivation of such teachers, and significantly, if we want our teachers to be able to provide suitable conditions susceptible to the modelling process, we must first provide them with adequate training and instruction (Martinez-Luaces, 2005, Crouch & Haines, 2004; O’Donoghue, 1978).

2.11 The Role of ICT in Modelling

Over the last forty years, the use of technology has been central to solving mathematical problems set in real-life context. At first the role of the computer was limited, corresponding to ongoing developments of technology, with limited accessibility and a lack of availability of easy to use software (James, 1985). The increasing availability of inexpensive hand-held graphic calculators and computer algebra systems has widened the range and potential of problems and models which can be treated in educational contexts (Niss, 1987; James, 1985). However, it seems that even now when widely available, mathematics teachers rarely use computers in their educational practise. This can be attributed to not only the lack of teacher education programmes, but also the lack of awareness and understanding of the power of computer-based components of their teaching (Kadilevich et
al., 2005; Manoucherhri, 1999).

But what must be recognised at this stage is that in general, modelling fosters the students ability to think and learn, “computer based modelling amplifies this empowerment through utilising computers as versatile mindtools” (Kadilevich et al., 2005, p. 114). The positive impact technology has had on student’s education is extensively documented, with these studies also providing evidence which supports current mathematics education reform movements and in particular mathematical modelling (Galbraith, 2006; Kadilevich et al., 2005; Ferrucci & Carter, 2003; Manoucherhri, 1999; Kaput, 1992; Hoyles et al., 2001). Indeed, while there remains unclear advances, (such as uncertainty regarding attitudes and beliefs), perceived improvement in conceptual understanding (Galbraith, 2006), regarding the technologically enriched approaches to mathematics learning, there is now a heightened realisation among mathematics educators, that the search for definitive answers will require far superior investigation of “machine-mathematics-learner relationships” (Galbraith, 2006, p. 277). In relation to the use of technology in modelling practises, essentially, regardless of the learning environment utilised, there will always exist difficulties in moving between the real and mathematical world (Crouch & Haines, 2006; Kadilevich et al., 2005). Technology however, can help reduce such difficulties, and allow the teacher to focus on those aspects which cause most difficulties when moving between the two worlds (Kadilevich et al., 2005).
2.12 Conclusion

This research study is concerned predominantly with the teaching and learning of applications in Senior-Cycle mathematics. Thus, in this chapter the author reviewed the literature by exploring current thinking and previous research in the field of mathematics education, focusing on the development of mathematics curricula and the role of applications and modelling in mathematics education over the past half century.

The first section reviewed the key approaches to the development of mathematics curricula, particularly at Senior-Cycle, over the past half century, paying specific attention to the situation at both international and national level. The pure vs. applied debate that has dominated mathematics education throughout this time was also examined, thereby providing an insight into the role of applications and mathematical modelling from a school perspective. The definition of modelling was explored, and consequently the classification of modelling approaches commonly practised in mathematics education were examined. Finally, this chapter identifies the external and internal factors affecting the teaching and learning of applications at Senior-Cycle level, namely: time constraints; students’ concepts of applications; assessment procedures; limited professional development; and the role of ICT in mathematics education.

At present, the need for students to learn mathematics through the medium of applications and modelling is widely accepted and it’s merits for inclusion are well recognised (Houston, 2007; Burkhardt, 2006; Ferrucci & Carter, 2003; James, 1985; Mustoe, 1992). However, its integration within mathematics curricula worldwide is still regarded as being in a period of transition. We have still to reach a
stage of general classification, or a settled consensus on the role of applications within mathematics education (Burkhardt, 2006; Niss, 1987).

What is of concern to the author is that the mathematics curriculum in Ireland does not provide for mathematics set in context. Current practices in Ireland generally fail to make the necessary connections between mathematics and its place in real-life, as documents from the NCCA and the Chief Examiners Report have shown (NCCA, 2005; State Examinations Commission, 2005). However, as it stands here in Ireland, the role of applications in the teaching and learning of mathematics is entering an exciting and innovative stage with the introduction of a new curriculum initiative, called ‘Project Maths’, to all second-level schools in September 2010.

In the next Chapter the author examines a number of significant approaches to the teaching and learning of applications developed by mathematics educators for both second and third-level mathematics education. These were examined as applications and modelling plays a distinctive role in each of these approaches, thus providing the reader with an insight into appropriate approaches to the teaching and learning of applications in mathematics education.
Chapter 3

Teaching Approaches to Mathematics, Applications and Modelling

3.1 Introduction

The study focuses on the problem of improving the teaching and learning of mathematics, particularly upper secondary level in Ireland, by making a provision for the effective teaching of applications.

While approaches have been developed to cater for applications in mathematics to date, none have been developed for Senior-Cycle students in Ireland. Thus, this chapter examines a number of distinct approaches in which applications play a leading role. It is intended that these approaches will provide insights into current successful practises used in both primary and higher education mathematics courses throughout the world in this way providing a platform from which the
author can begin to think about the teaching and learning of applications for the purpose of this research study. These approaches include: *Realistic Mathematics Education (RME)*; *Mathematics in Context (MiC)*, Steen’s *Quantitative Literacy (QL)*; Applicable Mathematics – Ormell; the Integrated Approach – Bajpai and the Harvard Calculus Approach. However, while these approaches provide an insight into practices within the mathematics classroom, these approaches are practise-led and lack a wider pedagogical perspective suited to secondary school mathematics teaching.

As a result, in the concluding section of this chapter, a theoretical framework known as APOS Theory is examined (Asiala et al., 1996). The theory has been developed, by Dubinsky and his colleagues in the Research in Undergraduate Mathematics Education Community (RUMEC), in a bid to expand Piaget’s concept of reflective abstraction so as to encompass undergraduate mathematics education. APOS Theory is a purpose built theory for mathematics teaching and offers just such a perspective and theoretical pedagogical framework for mathematics teaching and learning based on constructivist principles. APOS Theory is the overarching framework that combines the approaches to teaching applications chosen by the author and provides the vehicle for the subsequent research design.

This chapter will provide the foundation for the subsequent chapters in relation to the methodology employed in this investigation and for the discussion of significant insights/findings generated from the data.
3.2 Realistic Mathematics Education (RME)

One of the major developments to mathematics curricula over the past half century is Realistic Mathematics Education (RME). RME involves the use of applications, models and real life problems and also underpins the PISA mathematical literacy framework, and thus, the author decided to analyse the approach in view of the overall objectives of this research study.

Since 1971, the IOWO (Instituut Ontwikkeling Wiskundeonderwijs, English translation: Institute for Development of Mathematics Education) and then the Freudenthal Institute have been developing a theoretical approach towards the learning and teaching of mathematics known as Realistic Mathematics Education (RME). RME is based on Hans Freudenthal’s view of mathematics as a human activity and his desire to improve mathematics education for all students of mathematics. The fundamental focus in RME is on providing all students with experiences of the full mathematisation cycle in a context that sees mathematics as a human activity (NCCA, 2004). Freudenthal felt that students should not be considered as passive recipients of ready made mathematics, in the form of algorithms, but rather that education should guide the students towards using opportunities to reinvent mathematics by doing it themselves. With a view to understanding and accepting Freudenthal’s view of mathematics education, an appreciation of his resistance to the formal and abstract ideas at the core of the ‘new/ modern’ mathematics movement must first be explored.

Freudenthal’s ideas were without doubt marginal when ‘new/ modern’ mathematics was dominant, but the tide has turned, in part due to the emergence of new understandings of teaching and learning procedures, and the growth in stature of
research in mathematics education across so many domains over the past thirty years (NCCA, 2004). While the ‘new/modern’ mathematics movement deemed abstraction as its greatest value, Freudenthal saw this as its primary weakness (NCCA, 2004), stating that:

“In an objective sense the most abstract mathematics is without doubt also the most flexible. But not subjectively, since it is wasted on individuals who are not able to avail themselves of this flexibility”

(Freudenthal, 1968, p. 5)

Freudenthal, in stark contrast to this movement, viewed mathematics as a human activity deeply embedded in real situations (Freudenthal, 1991; 1983). Furthermore, he promoted the development of curricula embedded in mathematics set in context and its relation to social and cultural situations (NCCA, 2004). While Freudenthal was indeed opposed to constructivism, viewing it as no more than “empty sloganising” (NCCA, 2004, p. 126), his own conception of reality has a noticeably constructivist feel (Cobb & Yackel, 1996; de Lange, 1996; Streefland, 1991). Freudenthal’s idea of reality and its emphasis on the learner’s perspective is constructivist in that he recognised that certain contexts have the potential to be more real, or to mean something different, from one learner to another (de Lange, 1996). According to Treffers and Beishuizen (1999) RME involves a complete reversal of the teaching / learning process. Freudenthal (1979, 1968) proposes that rather than starting with certain abstractions or definitions to be applied later, one must start with contexts that can be mathematised. Thus, context problems are used as both a starting point and the medium through which pupils develop understanding (Hough & Gough, 2007). Hough and Gough (2007), stress that
this relates strongly to Freudenthal’s (1977) view that ‘mathematics must be connected to reality, stay close to children, and be relevant to society in order to be of human value’.

Treffers (1987) formulated the idea of two types of mathematisation explicitly in an educational context and distinguished "horizontal" and "vertical" mathematisation. Van den Heuvel-Panhuizen (2010, p. 4) describes horizontal mathematisation as involving “going from the world of real-life into the world of mathematics” while, “vertical mathematisation means moving within the world of mathematics”. Freudenthal (1991, p. 71). described the distinction between horizontal and vertical mathematisation as:

“Horizontal mathematization leads from the world of life to the world of symbols. In the world of life one lives, acts (and suffers); in the other one symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly: this is vertical mathematization.”

Treffers’ (1987, p. 251) scheme included in Table 3.1 shows how the four different approaches to mathematics education diverge.

<table>
<thead>
<tr>
<th>Approach to Mathematics Education</th>
<th>Horizontal</th>
<th>Vertical</th>
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<tbody>
<tr>
<td>Realistic</td>
<td>+</td>
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</tr>
<tr>
<td>Empiricist</td>
<td>+</td>
<td>-</td>
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<tr>
<td>Structuralist</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Mechanistic</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

+ includes the specific type of mathematisation
- excludes the specific type of mathematisation

The notion of applications or mathematical modelling has been particularly dom-
inant in the theory of RME (NCCA, 2004; deLange, 1996; Cobb & Yackel, 1996; Treffers, 1987). Cobb and Yackel (1996) suggest that the theory behind RME relies strongly on real world applications and modelling, and constitutes a domain specific instructional theory. Van den Heuvel-Panhuizen (2010, p. 10) sees the role of models in RME to “serve as an important device for bridging the gap between informal, context-related mathematics and the more formal mathematics”. Therefore, in order to bridge this gap, models have to shift from a ‘model of’ a particular situation to a ‘model for’ all kinds of other situations (Streefland, 1993, 1985; Van den Heuvel-Panhuizen, 2003).

In terms of assessment, RME seeks to design assessment and learning opportunities that consist of genuine problems i.e. ‘rich, non-mathematical contexts that are open to mathematization’ (Van den Heuvel-Panhuizen, 1996, p. 19). Van den Heuvel-Panhuizen (1996, p. 20) distinguishes RME assessment items from traditional word story problems which are often ‘rather unappealing, dressed up problems in which context is merely window dressing for the mathematics put there’. Thus, RME textbooks include practical application problems rather than artificial word story problems. Verschaffel et al. (2000) note the defining characteristics of RME assessments as follows:

- extensive use of visual elements
- provision of various types of materials
- all the information may not be provided
- there is a general rather than single answer
- a focus on relevant and essential contexts
• asking questions to which students might want to know the answer

• using questions that involve computations before formal techniques for those computations have been taught

(NCCA, 2005, p. 117)

3.3 Mathematics in Context (MiC)

*Mathematics in Context* (MiC) has been examined by the author as it provides an insight into the teaching and learning of high school (post-primary school) mathematics in the U.S.. Given that MiC promotes an extensive use of applications across the mathematics curriculum, it was deemed appropriate for inclusion in this chapter.

*Mathematics in Context* (MiC) represents a broad mathematics curriculum for the middle grades (6-8) (similar to late-primary and early post-primary) consistent with the content and pedagogy suggested by the NCTM *Curriculum and Evaluation Standards for School Mathematics, and Professional Standards for Teaching Mathematics* (Conway et al., 2001). The development of the curriculum units is as a result of collaboration between research teams at the Freudenthal Institute at the University of Utrecht, and research teams at the University of Wisconsin, while it also includes contributions from a group of middle school teachers in the U.S. (Holt et al., 2005).

A total of forty units have been developed, which are considered innovative and unique in that they make extensive use of realistic contexts (Conway et al., 2001; Holt et al., 2005; NCCA, 2004). Units attempt to provide particular emphasis
on the inter-relationships between the specific mathematical domains outlined in MiC: number, algebra, geometry and statistics. The underlying rationale behind each of the units is to “connect mathematical content both across mathematical domains and to the real world” (Conway et al., p. 143). The domains in outline are:

**Number** (whole numbers, common fractions, ratio, decimal fractions, percents, and integers)

**Algebra** (creation of expressions, tables, graphs, and formulae from patterns and functions)

**Geometry** (measurement, spatial visualisation, synthetic geometry, and coordinate and transformational geometry)

**Statistics** (data visualisation, chance, distribution and variability, and quantification of expectations).

The philosophy behind the units is that of teaching mathematics for understanding, a significant ideal of the NCTM Standards. Such an approach provides the curriculum with substantial and recognisable benefits for both students and teachers (Stevens, 2001). MiC aims to afford students with opportunities to view mathematics as an interesting, powerful tool that enables them to better understand their world. The real-world settings outlined in MiC promote mathematical connections across the domains, where these domains are otherwise viewed as a set of disjointed rules and concepts by and large independent of one another (Stevens, 2001). Although many units may emphasise the principles within a particular mathematical domain, most involve ideas from several domains, emphasising the
interconnectedness of mathematical ideas. However, the MiC curriculum requires students to do much more than just give answers (as is prevalent in current assessment and evaluation methods worldwide), instead it looks for students to provide reasoning and justification for their answers, thus encouraging student’s understanding of concepts as opposed to mastery of concepts devoid of understanding (Meyer & Ludwig, 1999).

It is desired that students of mathematics should be able to reason mathematically, and more importantly this should not be restricted to the more able cohort. Thus, units of the MiC curriculum were designed so as to present multiple levels so that the able student can go into more depth while a student having trouble can still make sense out of the activity (Meyer, 2001). This distinct approach, reminiscent of catering for mixed abilities, was devised so as to result in innovative approaches to instruction, increased enthusiasm for teaching, and a more positive image not only with students, but society at large (Conway et al., 2001). However, what must be noted, and not disregarded, is that the success of MiC is in large part dependant on the efforts of mathematics teachers within the classroom (Meyer & Ludwig, 1999). The extent of energy displayed in ensuring significant changes occur, combined with their openness to change and willingness to listen to students is paramount in determining this success.

The Ames Community School District (CSD) was a field test site located in a university community, which used MiC experimentally in a few classrooms from 1993 to 1995. In 1995, the district formally adopted MiC as its curriculum for grades 5–8 and began to use MiC district-wide during the 1995–96 school year (Webb & Meyer, 2002). Table 3.2 compares national percentile rankings for students in grades six, seven, and eight on three sub-sections of the Iowa Test of Basic
Skills (ITBS) before and since MiC was adopted. In all cases, the percentile rankings of students’ scores improved after the full implementation of MiC (Webb & Meyer, 2000). However, the Ames CSD results are limited by the inability of the ITBS to capture improvement in student reasoning and problem solving, which are primary instructional goals of MiC (Webb & Meyer, 2002).

Table 3.2: Student Achievement Results, Iowa Test of Basic Skills (ITBS)

<table>
<thead>
<tr>
<th></th>
<th>1993</th>
<th>1996</th>
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<tbody>
<tr>
<td></td>
<td>National Percentile</td>
<td>National Percentile</td>
</tr>
<tr>
<td><strong>Computation</strong></td>
<td></td>
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</tr>
<tr>
<td>Grade 6</td>
<td>62</td>
<td>81</td>
</tr>
<tr>
<td>Grade 7</td>
<td>60</td>
<td>72</td>
</tr>
<tr>
<td>Grade 8</td>
<td>59</td>
<td>79</td>
</tr>
<tr>
<td><strong>Concepts</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Grade 6</td>
<td>79</td>
<td>93</td>
</tr>
<tr>
<td>Grade 7</td>
<td>81</td>
<td>90</td>
</tr>
<tr>
<td>Grade 8</td>
<td>84</td>
<td>90</td>
</tr>
<tr>
<td><strong>Problem Solving</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Grade 6</td>
<td>87</td>
<td>96</td>
</tr>
<tr>
<td>Grade 7</td>
<td>89</td>
<td>94</td>
</tr>
<tr>
<td>Grade 8</td>
<td>93</td>
<td>94</td>
</tr>
</tbody>
</table>

3.4 Applications and Mathematical Literacy

Mathematical Literacy was examined by the author as it is based on applications and modelling. Furthermore, the idea of mathematical literacy has underpinned many approaches to numeracy and quantitative literacy over the past half century and thus will allow the author to examine its position and influence on mathematics education throughout this time.

The theoretical approach to mathematical literacy, which was derived out of the
concept of numeracy outlined in the 1959 “Crowther Report” in the UK (Ministry of Education, 1959), relies strongly on applications and modelling (Kaiser & Wilander, 2001). Current reform movements in mathematics education identify and acknowledge the relevance of the concept of mathematical literacy that is mainly based on applications and modelling (Kaiser & Wilander, 2001). Essentially, the concept of numeracy has undergone three phases of evolution: formative, mathematical and integrative (Maguire & O’Donoghue, 2002).

Initially the formative stage comprised of the origin of this concept of numeracy established in the “Crowther Report”, which was largely based on the ability of students to communicate mathematically at an extensive level, particularly about issues that arise in everyday life (Steen, 2008). Over two decades later, the mathematical stage, the “Cockroft Report” (1982) sought to revive the concept of numeracy and its place within mathematics education, and is widely regarded as the first major document to urge that numeracy be given a priority position in mathematics education. In recent times, the integrative stage, the Organisation for Economic Cooperation and Development (OECD) attempted to define and assess student knowledge and skills in mathematical literacy through its Program for International Student Assessment (PISA) (Steen, 2003). This has provided a means of highlighting the importance of mathematical literacy (derived from the concept of numeracy) worldwide and its position required by our global technological society (Steen, 2008). Mathematical literacy has been defined as:

“an individual’s capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements, and to engage in mathematics in ways that meet the needs of an individual’s current and future life as a constructive, con-
Mathematical literacy is considered to be the application of mathematical understanding to various problems set in real life context, where an array of fundamental mathematical knowledge and skills is required (Meaney, 2007). Mathematical literacy includes not only the ability to solve real-world problems but also the inclination to do so (Kaiser & Wilander, 2001). A decisive capacity of mathematical literacy is the ability of the individual to “pose, formulate, solve, and interpret problems using mathematics within a variety of situations or contexts” (OECD, 2003. p. 25), which can be considered an offshoot of mathematical modelling techniques (Doyle, 2007; Kaiser & Wilander, 2001).

One underlying aspect of mathematical literacy is the inclination of mathematics educators to interpret it differently, namely: quantitative literacy, numeracy, mathematical literacy, and mathematical competencies (Niss, 2003). However, irrespective of the labels used, what people are referring to is something other than expertise in pure, theoretical mathematics. Significantly, de Lange (2003) views mathematical literacy as the overarching literacy comprising all others, specifically: spatial literacy, numeracy, and quantitative literacy. This can be seen in Figure 3.1.
Mathematical literacy is fast becoming a recognised pre-requisite in the preparation of all students as it is seen as underpinning educational experience of all students of mathematics, and at all levels from elementary, through secondary and third-level, to the workplace. In the U.S. in 2003, the National Assessment of Adult Literacy (NAAL) survey, was administered to college students, with results showing over 20% of students completing four-year degrees operating only at the basic level of quantitative literacy (Steen, 2008). In response the National Numeracy Network (NNN), which was founded in 2000, has organised a consortium of numeracy programs and mathematical literacy centres at colleges and universities aimed at promoting quantitative literacy across all disciplines (Steen, 2008).
3.4.1 Steen’s Quantitative Literacy (QL)

Quantitative Literacy (QL) is also given different interpretations by different authors, where some use the word ‘quantitative’ in a much broader sense than numbers and data only (Niss, 2003). We could extend this viewpoint to assert that there is no clear understanding of what is meant by numeracy or quantitative literacy (Sutcliffe, 2003). Although QL does not have a clear definition with general consensus, its goal is widely shared: “to prepare students with the ability to think quantitatively in a variety of contexts” (Sutcliffe, 2003, p. 191).

One such activist in the promotion of Quantitative Literacy is Steen (2008; 2003; 2002; 2001; 1999). He has employed the development of a core curriculum that embraces the access to and understanding of quantitative information. This extensive contact with quantitative information is grounded in authentic, existing tasks in which meaningful and pragmatic problems from life and work are embedded (Steen, 2003).

The realisation is that numeracy once incorporated into mathematics curricula gives access to students and teachers to “make sophisticated use of elementary mathematics rather than, as in the classroom, elementary use of sophisticated mathematics” (Steen, 2001b). This awareness has fostered the current conception of numeracy known as the integrative stage outlined by Maguire & O’Donoghue, (2002) and Steen (2008).

Steen (2002) argues that numeracy must enter the curriculum, not as a stand alone subject, but saturating all aspects of the curriculum in an attempt to help students learn to cope with the quantitative demands of modern society. Significantly, numeracy is to be viewed not the same as mathematics in the curriculum, nor as an...
alternative either, but as an equal and supporting partner (Steen, 2002).

Fundamental to Steen’s argument for ‘Quantitative Literacy’ is the need for new approaches to the teaching and learning of mathematics (Steen, 1999). Ultimately, Steen sees the narrow arithmetic-algebra track as out of date. In replacement, Steen (2001b) proposes that mathematics education at post-primary level should focus on horizontal breadth and connectedness that offers multiple points of entry and numerous opportunities for catching up.

Virtually every public issue depends on data, assumptions, forecasts and systematic thinking, thereby contributing to a data-drenched society in which we live and work (Steen, 1999). This data-drenched society that we are surrounded by offers opportunities for students to experience ‘mathematics in the workplace’ in a classroom setting. Mathematics in the workplace is generally rich in data, dependent on technology, and tied to useful applications, allowing for multi-step procedures in solving open-ended problems and a need for a higher degree of accuracy. The primary source of this data is the computer and its potential can be compared to the influence of the printing press to the public’s need for literacy (Steen, 2001a). Computers have the power to represent and visualise all types of information (words, graphics, music etc.) as numbers, the new literacy of our age, and the ability to comprehend and transform quantitative data (Steen, 2001a; 2001b).

3.5 Applicable Mathematics – Ormell

The Ormell Approach, namely “Applicable Mathematics”, has emerged from two initial articles written by C.P. Ormell in 1972, entitled “Mathematics, Science of
"Possibilities” and “Mathematics – Applicable versus Pure and Applied”, and has been a core theme developed by the author throughout his career. With regard to an analysis of this unique outlook, from the outset Ormell’s work can be considered a somewhat philosophical and ideological approach to mathematics *per se*. In relation to mathematical activity as viewed from the outside world i.e. the non-specialist mathematician or the layman, Ormell suggests that mathematics is predominantly concerned with the investigation of the possibilities of the real world, in particular that: “*mathematics is the science of possibilities*” (Ormell, 1972, p. 329). It is quite evident that Ormell is primarily concerned with the relevance of mathematics and, specifically, its relevance to hypothetical rather than actual situations.

‘Applicable Mathematics’ is, in essence, a modelling approach to mathematics aimed at improving the mathematical experience of the students at upper secondary school; however, it imposes a focus on the appreciation of the purpose and relevance of mathematics, and more significantly the nature of the mathematics-reality relationship. It allows the students to appreciate, at a time when their mathematical and conceptual understanding is at a significant juncture i.e. the preparation for the transition from second-level mathematics to university mathematics, what mathematics is essentially about, and quite literally why we study mathematics at all.

Central to this approach (and to many other pedagogical approaches i.e. RME, MiC etc.) is the use of applications, but what makes this approach distinctive from all other approaches is the realisation that mathematics is fundamentally concerned with hypothetical situations. Furthermore, ‘Applicable Mathematics’ attempts to break down the distinctive pure and applied mathematics approaches,
as is traditional, thus allowing the student to experience mathematics as an approach concerned with a wider field of representation encompassing both pure and applied approaches as an entire entity.

Ormell’s ideas were current at a period in which the role of applications and modelling was widely discussed, where numerous innovative and telling ideas were offered, and mathematics education became saturated with such approaches. However, Ormell’s approach was generally considered marginal, largely because of his philosophical approach, where more traditional methods of instruction and pedagogical approaches were favoured. However, there still exists genuine value in his approach and it is not reasonable to say that his ideas should not be considered. In fact, when we currently locate ourselves at a stage when the integration of applications and modelling within mathematics curricula is still regarded as being in a period of transition, Ormell’s approach offers a framework of thinking within which the student can appreciate the usefulness and relevance of mathematics.

Ormell’s ideas stemmed from an attempt to offer an explanation of the relationship of mathematical activity to the outside world (non-specialists), which he hoped, was in contrast to the inward view offered by mathematicians at that time. Certainly, mathematical activity has throughout the ages provided either logistic, formalist or intuitionist accounts of what mathematics is about and what it offers as a discipline of study. Although well documented and discussed, these accounts offered no concrete evidence on how mathematics relates to other activities, at least to the outside world (Ormell, 1972a). Significantly, these accounts provide “little illumination” to the layman approaching mathematics from the outside (Ormell, 1972a). In fact, these approaches can serve to cloud one’s understanding of the role of mathematics in education, thus affecting and demotivating students
of mathematics (Ormell, 1972a).

Given that Ormell suggests that mathematics is concerned with the possibilities, rather than the actualities, of the world, the author appreciates and certainly lays merit in such an approach. However, while Ormell was of the view that this offered an explanation to the non-specialist, and while indeed an attempt at explanation was presented in his articles, it could not be considered an insightful explanation offering a pathway for the layman to understand. In fact, because of its philosophical terminology and ideology, Ormell added further mystification to the nature of mathematics. A mystification he stressed the importance of avoiding (Ormell, 1972b).

Ormell, as previously mentioned, suggests that mathematics is concerned with hypothetical situations, rather than the actualities of the world. The author expands this view where we must realise when we experience mathematics we: predict; forecast; analyse; examine; draw conclusions; prove; show that etc. All of these actions are only made possible through the use of mathematical techniques, and central to such attempts is that these actions are concerned with problems in the physical world. While they might in turn have a bearing on the physical world, they do not initially. Mathematics in the physical world exists only on paper, as a record of techniques, manipulations, rules and/or systems. It does not, at the outset, do anything in the physical world, although it may result in actions taking place depending on results. Blaire (1973), in a critical review of Ormell’s initial paper, expands this view, where he argues that the objects of the mathematical world are passive in function, as opposed to the world of science where these objects take on an active role. Quite simply he surmises: “Pi has a purpose, but electrons do things” (Blaire, 1973, p. 414).
When we discuss problems set in real life context from a mathematical perspective (otherwise known as applications of mathematics to real life), we tend to discuss them as possibilities without actually realising it. If we expand this view, we can assess that the mathematician in his vocation tends to deal with the hypothetical, not the actual (that is the job of the scientist, the engineer etc.). This view of what mathematics is about allows the student (and teacher) to see mathematics less as a body of results but rather more as an activity, an activity which has the capacity to have an impact on the physical world. It then has the capacity to provide the student with a clear conception of what we do mathematics for, and recognise mathematics as a discipline which we use to understand, handle and explore hypothetical situations of the real world.

Nonetheless, it is an offhand statement by Ormell in an auxiliary article in 1977, which offers a more realistic and genuine chance of convincing the non-specialist of the true nature of the mathematics-reality relationship. It simply states that: “mathematics is the science of ifs” (Ormell, 1977, p. 257). Upon interpretation it can be easily identified as using simple and concise language, thereby offering an uncomplicated, unpretentious and transparent explanation of what mathematics is about. However, upon reflection, to the mathematician this might offer an insightful explanation, but to the layman, this might add another element of obscurity to the debate. Furthermore, not only does this statement relate to the layman, thus providing an authentic foundation upon which one can draw their own conclusions as to the relationship of mathematics to the real world, but it provides a statement void of ambiguous philosophical, theoretical or technical terminology. In doing so, it prevents a demystification of the oft complex and intricate definition of mathematics, thus providing an open and honest affirmation of mathematical activity.
which can be effortlessly interpreted by both the specialist mathematician and the non-specialist layman.

Essentially this approach offers relevance. And it is this relevance which will hold the key to this approach’s success. If mathematics is not presented in a fashion-able way which highlights its relevance, to those non-specialists a major lack of motivation will result. By providing relevance, with which the student and teacher can now use to propel and truly deepen their understanding of vital mathematical concepts and methods, it also provides a strong sense of motivation for the student to strengthen their mathematical beliefs and understanding regarding mathematics’ relevance to the real world. As Ormell (1972b) suggests the most natural way to create motivation is to let the subject speak for itself i.e. to show what it does, or can do. In essence, Ormell’s approach attempts to provide a medium within which one can appreciate what mathematics is really about, its ability to relate to the real world, the nature of this relationship and most of all why we expand such effort in this intellectually demanding way.

Applicable Mathematics is primarily based on Peirce’s ideas, where in the first half of the twentieth century, he offered the initial pronouncement of mathematics as the science of possibilities. Ironically, Peirce’s work somewhat mirrored Ormell’s in that both offered an approach at a time when society was unable to comprehend and grasp effectively such philosophical thought (Grattan-Guiness, 1997).

The pure-applied relationship of mathematics has had a strangle-hold over almost all mathematical activity over the past half century. The traditional values of pure mathematics and its justification within our educational system is a significant argument with which applied mathematicians have dismissed the importance of
such an intellectually demanding and abstract domain of mathematics. However, it is not sufficient to say that pure mathematics ought not to be taught and abandoned in favour of a utilitarian approach. In fact, the author’s attitude is that we need now more than ever to be able to understand why it should be part of the mathematics we teach.

As was Ormell’s view, if aesthetic appeal was deemed sufficient to warrant inclusion, surely such activities as art and music would hold a higher degree of importance within our educational system (Ormell, 1977). As the current situation stands the attitude of the lay public is captured in the belief that mathematics is an extremely important subject, but they lack the necessary understanding (as do many teachers and students of mathematics) to explain why. ‘Applicable Mathematics’ is designed as an attempt at enlightenment. One of the key differences between ‘Applicable Mathematics’ and traditional approaches is the realisation that there is little or no need for the pure/applied distinction of mathematics. ‘Applicable Mathematics’ is concerned with a wider field of representation than traditional approaches. The pure-applied dichotomy of mathematics which has dominated the classification of distinct approaches all these years is embraced as a whole entity in this approach, as opposed to the tendency to display one’s beliefs directly left or right of a dividing line (which is not easily recognisable, if at all), between pure and applied mathematics.

### 3.6 Integrated Approach - Bajpai

Bajpai and his colleagues in Loughborough University, in an attempt to improve students understanding and learning of aspects of engineering mathematics at un-
dergraduate level, suggested an approach encompassing analytical, statistical, numerical and computer solutions to real problems (Bajpai et al., 1970; 1975; 1976). While initially designed to alter the methods of teaching and understanding at service level engineering mathematics (Bajpai et al., 1970), Bajpai recognised the opportunities for incorporating these techniques and the exploitation of the computer across mathematics education at all levels (Bajpai et al., 1984). Essentially the integrated approach which was proposed by Bajpai and his colleagues (1976, p. 350) means “that those analytical, statistical, numerical and computer techniques which are relevant to a particular topic are all discussed or mentioned when the topic is being taught”. Approaches in teaching mathematics traditionally contrast this approach where these approaches are experienced at staggered phases, with the mastery of one approach allowing access to another. Traditional methods of teaching and learning also often leave the students with much doubt as “to where the topics fitted in the overall solution of problems” (Bajpai, 1985, p. 419). This approach ensures the students are provided with opportunities to appreciate the fact that often a combination of techniques is used when solving problems arising in real-life contexts, both in industry and everyday experiences. The integrated approach ensures that all viable methods of solution to a particular problem are encountered as early as possible rather than firstly encountering the perceived uncomplicated and/or significant technique (Bajpai, 1985). Using such a teaching approach, allows students to compare more effectively which method or combination of methods is required to solve particular problems and hence “improve the students understanding of the mathematical problem” (Bajpai et al., 1976, p. 350).
3.6.1 Case Studies & Modelling

Bajpai et al. (1975) argued for the provision of situations within which students could appreciate the concept of a mathematical model and the methods of obtaining solutions to such a model. This introduction to a perspective of modelling should allow for the opportunity of presenting case studies which illustrate the application of mathematics to solve real-life problems. Bajpai et al. (1975) offered an approach to modelling which included 10 stages, as outlined in Figure 3.2.

The mathematics teacher is faced with the predicament of achieving a balance between imparting the basic knowledge to the student and showing them the applications of the mathematics which are relevant to the student’s needs (Bajpai, 1985). Bajpai (1976, p. 353/354) observed that

"if possible the case study should involve several mathematical techniques so that the lecturer can make the very important point that in real-life the solution of a problem requires all the engineer’s (mathematical student’s) mathematical expertise and he cannot expect the problem to fit nicely into a standard drill example pattern".

It is imperative in that successful and appropriate case studies are provided, as access to excessively specialised problems would result in over-complication of material and marginalised understanding by the student. Non-trivial examples which do not demand too high a level of mathematical expertise is required to guarantee suitable case studies are supplied to the students, and it is essential that the time-consuming calculations should be avoided (or exploited through the power of the computer where possible) (Bajpai et al., 1976). Explicitly, the introduction of case studies provide benefits in maintaining a high motivation among students
Figure 3.2: Flow chart for the modelling process (Bajpai et al, 1975)
and provide an appreciation of the role of mathematics and its utility in real-life contexts (Bajpai et al., 1976).

3.7 Harvard Calculus Approach

In the late 1980’s, the problems of undergraduate calculus education in the United States were of sufficient scale that the finding of a suitable solution was funded by the National Science Foundation (Hughes-Hallett & Gleason, 1992). Faculty at Harvard University and seven other institutions came together under the auspices of the Calculus Consortium based at Harvard University in an effort to design a new syllabus (Hughes-Hallett, 1991). The authors focused on a small number of key concepts, emphasising depth of understanding rather than breadth of coverage. Essentially, the curriculum was prepared by starting with a clean slate where faculty in engineering, physics, chemistry, biology and economics had considerable input into these texts, both in choice of topics and choice of applications (Hughes-Hallett & Gleason, 2005).

Guiding Principles:

The main guiding principle behind this approach was teaching mathematics using the ‘Rule of Three’ (Tall, 2003). The ‘Rule of Three’ ultimately starts with the way topics are represented. The Harvard approach to mathematics states:

“that wherever possible topics should be taught graphically and numerically, as well as analytically. The aim is to produce a course
where the three points of view are balanced, and where students see each major idea from several angles”

(Hughes-Hallett, 1991, p. 121).

The ‘Rule of Three’ later became the ‘Rule of Four’, extending the representations to include the verbal, giving four basic modes. Thus, The guiding principles underlying the Calculus Consortium at Harvard University can be summarised by the following:

- **Rule of Four:** Where appropriate, topics should be presented graphically, numerically, symbolically and verbally (Tall, 2003).

- **Problem Driven:** Formal definitions and procedures evolve from the investigation of practical problems. Whenever possible, the texts start with a practical problem and derive the general results from it. These practical problems are usually, but not always, real world applications.

- **Open-Ended Real World Problems:** The real world problems are open-ended, meaning that there may be more than one solution depending on a student’s analysis. Many times, solving a problem relies on common sense ideas that are not stated in the problem but which students will know from everyday life.

- **Plain English:** These books present the main ideas of calculus in plain English to encourage the students to read it in detail, rather than just reading the worked out examples.

(Hughes-Hallett, 2005)
If students are to think graphically and numerically, they must be as familiar with concepts represented by graphs and tables as by formulas and equations (Hughes-Hallett, 1991). The Harvard Calculus approach implies that all concepts are introduced numerically, graphically, symbolically and verbally. The order in which these approaches are used varies, and all are regarded as important (O’Keeffe, 1994). In such an approach, immense emphasis is placed on reading, writing, and modelling of real world situations, not just in exercises but also in the development of the theory. For example, the derivative is introduced through the concept of average speed, and the integral is introduced to solve the problem of measuring distance travelled (O’Keeffe, 1994). Problems are commonly taken from the real world, where students have to interpret what mathematics is required, and act upon these appropriately with regards to the specific problem. Consequently, the problems are often open-ended, thus the students are expected to write logical explanations of their solutions, as opposed to the traditional methods of providing numerical answers void of meaning. Essential to this approach, is that students routinely make use of a graphics calculator and/or a computer package as they develop the concepts.

In keeping with this emphasis, the developers of the Harvard program consulted extensively with client disciplines, e.g. engineers, biologists, and economists, and have listened to them (O’Keeffe, 1994; Hughes-Hallett & Gleason, 1992). These discussions have indicated that two of the major areas that warranted improvement, according to the client disciplines, were (i) dealing with functions given in the form of tabular data, and (ii) given $f'(x)$, to investigate the properties of $f(x)$ (O’Keeffe, 1994). Upon prudent analysis, it can be recognised that (i) relates to the introduction of concepts numerically, while (ii) relates to the graphical ap-
The Harvard approach ensures that there has been a strong movement away from “chalk and talk” style instruction, which has been duly replaced by a greater emphasis on class activity e.g., calculator/computer exploration, discussion of problems, group work, etc. This is a natural outcome of the reform approach, because of both the open-ended nature of so many of the problems, and the emphasis placed on communication. For the successful implementation of the reform program, such an approach cannot be implanted into a traditional course just by introducing some graphical work, or using a computer. Using diagrams, sketching graphs, making calculations, etc, does not constitute a reform program unless the student instinctively uses these methods without prompting whenever it is appropriate.

Fundamental to the reform movement, was the Harvard Consortium Calculus text, written by Hughes-Hallett and Gleason (1994) entitled: “Calculus”. MacLane (1996) described the text as user friendly, while providing a large collection of real world exercises. Brosnan and Rally (1995) observed that at every stage the book emphasises the meaning (in practical, graphical or numerical terms) of the symbols being used. Also, Brosnan and Rally (1995) noted that not only are many of the problems in the text open-ended, but the authors assume that you have access to a graphics calculator or computer with appropriate CAS or graphics software. However, reservations have been expressed regarding the overall approach by Rosen and Klein (1996) who argued that the Harvard Calculus approach is deficient in exercises involving algebraic manipulation. The Harvard approach provides students with less practice in standard algebraic manipulations than traditional approaches. Furthermore, Rosen and Klein argue that the de-emphasis of high school level algebra is a disservice to students and is consistent with the
"dumbing down" of mathematics curricula.

3.8 APOS Theory

The approaches examined in the previous sections provide an insight into practices within the mathematics classroom. However, these approaches are practice-led and lack a wider pedagogical perspective suited to secondary school mathematics teaching. As a result, in the concluding section of this chapter, a theoretical framework known as APOS Theory is examined (Asiala et al., 1996).

APOS Theory is, primarily, a constructivist theory that deals with the approaches an individual uses in learning mathematics. The theory has been developed as a result of the work of Dubinsky and his colleagues in the Research in Undergraduate Mathematics Education Community (RUMEC), in a bid to expand Piaget’s concept of reflective abstraction so as to encompass undergraduate mathematics education. Essentially APOS Theory is based on the following hypothesis offered by RUMEC which is presented to offer a perspective on what it means to learn and know something in mathematics:

“An individual’s mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organising these into schemas to use in dealing with the situations”

(Asiala et al., 1996, p. 7)

To ensure an appropriate and noteworthy appraisal of APOS Theory is conducted,
its relationship with Piaget’s notion of reflective abstraction must be first taken into account. The next section will offer an insight into this relationship and acknowledge the successful attempts of RUMEC to further develop this concept so as to include mathematics at third level.

### 3.8.1 APOS Theory and Reflective Abstraction

APOS Theory has been developed in an attempt to understand the ideas of Piaget concerning reflective abstraction, with particular emphasis being placed on its role in the context of college level mathematics (Asiala et al., 1996). Piaget proposed the concept of reflective abstraction in an effort to describe the construction of logico-mathematical structures by an individual during the course of cognitive development (Dubinsky, 1991). Piaget observed that reflective abstraction has no absolute beginning but is present at the very earliest stages in the coordination of sensorimotor structures (Beth & Piaget, 1966, p. 203-208). Furthermore, Piaget deduced that reflective abstraction continues through to higher mathematics to the degree that the entire history of the development of mathematics from ancient times to the present day may be considered as an example of the process of reflective abstraction (Piaget, 1985, p. 149-150). Central to the concept of reflective abstraction is Piaget’s notion that it consists of drawing properties from mental or physical actions at a particular level of thought. This may involve consciousness of the actions or the act of separating a form from its content. Thus, whatever is “abstracted” is projected onto a higher plane of thought where other actions are present and may be accompanied by more powerful modes of thought (Beth & Piaget, 1966; Piaget, 1971, 1972, 1985). APOS Theory clearly has its foundations in Piaget’s preference for the construction aspect of reflective abstraction, and it is
from this perspective that comparisons can be drawn between the APOS Theory proposed through the work of Dubinsky and RUMEC and the notion of reflective abstraction by Piaget. However, while Piaget believed that reflective abstraction was as important for higher mathematics as it was for children’s logical thinking, his research was mainly concerned with the latter (Dubinsky, 1991). APOS Theory, in turn, has been designed to offer a framework for research and curriculum development in undergraduate mathematics.

3.8.2 Overview of the APOS Theory Framework

The framework used in this research consists of three components (Theoretical Analysis, Instructional Design, Observations and Assessments). Figure 3.3 illustrates each of these components and the relationships among them. A study of the cognitive growth of an individual trying to learn a particular mathematical concept takes place by means of several refinements, as the investigator/researcher repeatedly cycles through the component activities (Asiala et al., 1996).

![Figure 3.3: Overview of the APOS Theory framework (Asiala et al., 1996, p. 5)](image-url)
It is intended that research begins with a theoretical analysis of the concept in question so as to gain an insight into what it means to understand the concept and how that understanding can be constructed by a learner. Significantly, this initial analysis is based primarily on the researchers’ understanding of the concept in question and on their experiences as learners and teachers of the concept. Thus, the analysis informs the design of instruction. The design and implementation of the instruction provides a means for gathering data or for reconsidering the theoretical analysis with respect to the data. Repetitions of the cycle may be continued for as long as appears necessary to achieve stability in the researchers’ understanding of the epistemology of the concept (Dubinsky & McDonald, 2001; Dubinsky, 2001; Asiala et al., 1996, Dubinsky, 1991). It is important to develop more fully the description of the three components of the framework and their interconnections, thus, the following section will provide such an insight.

**Theoretical Analysis**

The purpose of the theoretical analysis of a concept is to provide a model of cognition that a learner may undertake. It essentially offers a description of “specific mental constructions that a learner might make in order to develop her or his understanding of the concept” (Asiala et al., 1996, p.7). The student generally undergoes a cognitive conflict when they are presented with new material. In order for the student to move to higher levels of cognitive understanding the student must encapsulate the required mental constructions. Dubinsky (2001) refers to this analysis as a genetic decomposition of the concept. The analysis is primarily prepared by applying a general theory of learning that is greatly influenced by the researchers’ own understanding and interpretation of the concept, coupled
with their previous experiences in learning and teaching it. As already referred to, APOS Theory is based on the hypothesis offered by RUMEC which is presented to offer a perspective on what it means to learn and know something in mathematics. Figure 3.4 exhibits Dubinsky’s proposed APOS Theory and how it describes how actions become interiorised into processes and then encapsulated as mental objects, which take their place in more sophisticated cognitive schemas (Dubinsky and McDonald, 2001; Asiala et al., 1996). A more detailed description of each of these mental constructions will be given below.

**Figure 3.4: Constructions for mathematical knowledge (Asiala et al., 1996, p. 9)**

**Action**

An action or action conception can be described as a transformation that a student undertakes as a direct result of some external stimuli (Dubinsky and McDonald, 2001). The individual’s understanding of a transformation is limited to an action
conception when he/she limits his/her reaction to external cues that give precise details on what steps to take (Asiala et al., 1996, p. 10). Generally the students will follow a set of instructions, apply a formula or adapt a step-by-step approach. For example, a student who cannot interpret a condition as a quadratic unless he or she has been provided with the quadratic formula is restricted to an action concept of a quadratic. The student can see the result of the formula calculation but does not have an understanding of the concept of quadratic roots. In essence, an action conception is very limited, however, they form the crucial beginning of understanding a concept.

Process

Asiala et al., (1996, p. 10/11) describe a ‘process’ as “when an action is repeated by an individual, and he or she reflects upon it, it may be interiorized into a process”. In essence, the learner performs the same action but now, not necessarily directed by external stimuli. Thus, “a process is perceived by the individual as being internal, and under one’s control, as opposed to being influenced in response to external cues” Asiala et al., (1996, p. 11). Achieving this status is when a learner can internalise the action steps. The individual can now visualise what the solution may be or run through them mentally without actually performing the operations.

Object

When an individual is able to reverse the process, can reflect on the process, becomes aware of the process as a totality, and realises that transformations (be they actions or processes) can act on it, then he or she is thinking of the process as an
object (Dubinsky and McDonald, 2001; Asiala et al., 1996). Dubinsky (2001) describes this as when the process has been encapsulated to an object. Significantly, it is important to note that it is often necessary to “de-encapsulate the object back to the process from which it came in order to use its properties in manipulating it” (Asiala et al., 1996, p. 11). In the example of quadratics, the student fully understands the concept of a quadratic, and furthermore, can visualise the quadratic regardless of the format in which the quadratic is presented.

**Schema**

Fundamentally, a collection of processes and objects can be organised in a structured way to form a schema. Asiala et al., (1996, p. 13) describe an individual’s schema as “the totality of knowledge and understanding (consciously or subconsciously) which accompanies a particular mathematical topic”. Dubinsky and his colleagues suggest that:

> “an individual’s schema for a concept includes her or his version of the concept that is described by the genetic decomposition, as well as other concepts that are perceived to be linked to the concept in the context of problems situations”

(Asiala et al., 1996, p. 12).

When a student is faced with problems involving quadratics the student has a number of inherent structured approaches that he or she can call upon to solve the problem.
Instructional Treatments

The second component of the research framework offered by Dubinsky is centred around designing and implementing instruction based on the theoretical analysis. Dubinsky and McDonald (2001) propose that an individual’s mathematical knowledge consists in a propensity to use particular constructions, however, what constructions are utilised is dependant on the problem in hand. Furthermore, Dubinsky suggests that we cannot expect students to learn mathematics in the logical order in which it can be laid out, that an acceptance must be realised of the non-linear growth of understanding within an individual:

“the student develops partial understandings, repeatedly returns to the same piece of knowledge, and periodically summarises and ties related ideas together”

(Asiala et al., 1996, p. 13).

The work of RUMEC in recognising this growth informs the general instructional approach in which they refer to an holistic spray. In this approach, the students are presented with an environment which contains as much as possible about the material being studied, as opposed to being sequentially organised as is traditionally practised within mathematics classrooms. RUMEC propose a particular pedagogical approach to help students make the desired mental constructions of a particular concept which they refer to as the ACE Teaching Cycle. The three components of the ACE Cycle are: Activities, Class Discussion, and Exercises.

Activities are centred around students working in small groups in the computer lab on computer tasks designed to foster specific mental constructions suggested by the theoretical analysis.
Class Discussion provides an opportunity for these same groups to work on paper and pencil tasks based on the computer activities. The teacher can also avail of the opportunity to provide definitions, explanations and overviews of the concepts being discussed and worked on.

Exercises are presented in relatively traditional fashion for students to work on in teams. They are generally expected to be completed as homework.

Observations and Assessments

The third component of the framework is concerned with collection and analysis of data. RUMEC promote the use of two distinct approaches to the gathering of this data. Firstly they feel the written assessments are necessary but also audio-recorded interviews. A combination of both different assessments in both written and interview format are deemed most suitable as a student may perform well in a written assessment, yet the transcript may reveal little understanding, while for another student the reverse may be true. The methods employed throughout this phase of the framework do not provide precise information that lead to indisputable conclusions. According to RUMEC the best that can be hoped for is data that is illustrative and suggestive (Asiala et al., 2004).

3.8.3 Summary

The author felt that the three-stage approach employed by APOS Theory (Exploratory, Implementation and Reflective Phases) allowed the author to pilot, implement and evaluate the subsequent teaching intervention. Chapters 5, 6 and 7 deal with each phase respectively. Furthermore, the three-stage cyclical process
(Theoretical Analysis, Instructional Design, Observations and Assessments) allowed the author to field test his approach in adapting APOS Theory so as to include a provision for the teaching of applications.

However, while APOS Theory offers a perspective on how learning a mathematical concept might take place, it offers no specific foundation for the role of applications in the learning of a mathematical concept, and as a result an attempt to do so ensures the uniqueness of the research project. The author demonstrates how a genetic decomposition can be used not only to predict how the mathematical constructs of the learner develops, but will take the process further and use the genetic decomposition as a tool for highlighting the role of applications in the mathematical knowledge of the learners (see Section 5.4 and Section 6.3).

Much of the research carried out by RUMEC was primarily focused on students who were studying mathematics as mathematical specialists. Cooperative learning and computer programming were activities that were promoted so as to foster the mental constructions called for by the theoretical analysis. While computer programming would be a familiar activity for the students that APOS Theory was originally designed to help, the students participating in the teaching intervention undertaken in this research project would not be familiar with such an activity. The author believes that the model can be adapted in a way that the beneficial activities like cooperative learning and computer activities remain as core activities. However, in the place of computer programming the author believes that GeoGebra can be used so as to promote learning in a student-friendly manner.

The ACE Teaching Cycle was subsequently adapted to remain consistent with the overall aims of the research study (see Section 5.6). The Activities were centred around students working in small groups in the computer lab on computer
tasks in GeoGebra designed to foster specific mental constructions suggested by the theoretical analysis. Furthermore, the activities were designed so as to ensure the students became familiar with mathematical modelling. In the Classroom Discussion the teacher could avail of the opportunity to provide definitions, explanations and overviews of the concepts being discussed and worked on through the medium of applications. GeoGebra was used, where appropriate, in the classroom discussion to visualise the concepts. Exercises were presented in relatively traditional fashion for students to work on. However, often the students are asked to explain their answers in words or graphs, an approach not traditionally employed in Senior-Cycle mathematics in Ireland.

A comprehensive rationale for adapting APOS Theory for the teaching of applications of mathematics at Senior-Cycle Level in Ireland is presented in Section 5.2, which deals with these issues in more detail.

3.9 Conclusion

A detailed analysis of a number of significant approaches developed by mathematics educators for both second and third-level mathematics education in which the use of applications and modelling play a distinctive role was provided in Chapter 3. This work provided a basis for the design of the theoretical framework employed in this research project.

The study focuses on the problem of improving the teaching and learning of mathematics, particularly upper secondary level in Ireland, by making a provision for the effective teaching of applications. While approaches analysed have been developed to cater for applications in mathematics to date, none have been developed
Chapter 3 provides a description of the theoretical frameworks employed in this research study and the influence that these frameworks have had on the research design and subsequent teaching intervention. In particular, the author has focused on Bajpai’s Integrated Approach (1975) and the Harvard Calculus Approach (1991). These approaches influenced the research design as the author felt that an adoption of these approaches would ensure that the students would be provided with opportunities to appreciate the fact that often a combination of techniques are used when solving problems arising in real-life contexts, both in industry and everyday experiences. Furthermore, these approaches provide an insight into current successful practises used in higher education mathematics courses throughout the world, thus providing a platform from which to approach the teaching and learning of applications in Senior-Cycle mathematics in Ireland. However, both Bajpai’s Integrated Approach (1975) and the Harvard Calculus Approach (1991) are practise-led and lack a wider pedagogical perspective suited to secondary school mathematics teaching. They do not offer a perspective on how learning a mathematical concept might take place, and thus, an adaption of APOS Theory is proposed as it offers just such a perspective and theoretical pedagogical framework for mathematics teaching.

The author felt that the three-stage approach employed by APOS Theory (Exploratory, Implementation and Reflective Phases) allowed the author to pilot, implement and evaluate the subsequent teaching intervention. The three-stage cyclical process (Theoretical Analysis, Instructional Design, Observations and Assessments) allowed the author to field test his approach in adapting APOS Theory so as to include a provision for the teaching of applications.

In the following chapter (Chapter 4), the rationale and intent for devising and
implementing a multi-method research design for this study in three phases is outlined. The chapter will describe the rationale for the overall research design as well as the methodology used (qualitative and quantitative). Consideration is also given to concerns relating to ethics, validity, and reliability.
Chapter 4

Research Methodology

4.1 Introduction

This chapter outlines the rationale and intent for devising and implementing a multi-method research design for this study in three phases. The chapter describes the rationale for the overall research design as well as the methodology used (qualitative and quantitative). The specific details of each of the methods employed such as data collection and analysis are dealt with as each of the three phases of the research is presented. Individual methods employed in all three phases are integral components of the larger design. Consideration is also given to concerns relating to ethics, validity, and reliability within this chapter.

4.2 Research Paradigms

Due to the diverse nature of the topic being investigated, this study employs a mixed methods research approach. Johnson and Onwuegbuzie (2004, p. 17) ad-
vocate mixed methods research as “an expansive and creative form of research, not a limiting form of research”. The combination of the qualitative and quantitative research paradigms evident in mixed methods is one of its primary characteristics and often contributes to better-quality research in comparison to single method research (Johnson & Onwuegbuzie, 2004). The quantitative and qualitative paradigms are distinguished primarily by their ontology (assumptions concerning reality), epistemology (knowledge of that reality) and methodology (the particular ways of knowing that reality). Table 4.1 highlights the strengths and weaknesses of each research paradigm. An insight into each paradigm is provided in order to gain a deeper understanding of their characteristics before discussing the merit of combining these two paradigms for this research project.

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<th>Strengths of quantitative methods research</th>
<th>Strengths of qualitative methods research</th>
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<tr>
<td>• Provide wide coverage of the range of situations.</td>
<td>• Ability to look at how change processes over time.</td>
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<td>• Can be fast and economical.</td>
<td>• Ability to understand meanings.</td>
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<td>• May be of considerable relevance to policy decisions.</td>
<td>• To adjust to new issues and ideas as they emerge.</td>
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<td>• Contributes to the evolution of theories.</td>
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<td>• Provides a way of gathering data that is seen as natural rather than artificial.</td>
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<th>Weaknesses of quantitative methods research</th>
<th>Weaknesses of qualitative methods research</th>
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<td>• Tend to be flexible and artificial.</td>
<td>• Data collection takes a great deal of time and resources.</td>
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<td>• Not very effective at understanding processes or the significance that people attach to actions.</td>
<td>• Can be difficult to analysis and interpret data.</td>
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<td>• Not very helpful for generating theories.</td>
<td>• Considered to be untidy as it is hard to control their pace, progress and end points.</td>
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<td>• They focus on what is, which makes it hard for the policy maker to infer what changes and actions should take place in the future.</td>
<td>• Policy makers are given low creditability.</td>
</tr>
</tbody>
</table>
4.2.1 Quantitative Research

According to Bryman and Bell (2003), quantitative research can be described as a method of investigation that stresses quantification in the compilation and study of information that involves a reasoned attitude to the bond concerning theory and research, wherein the prominence lies on the examination of theories. Studies suggest that quantitative is appropriate to analyse data which incorporates “... the practices and norms of the natural scientific model” and in particular positivism (Bryman & Bell, 2003). This policy is likewise suitable for the breakdown of facts that embodies a notion “of social reality as an external, objective reality” (Bryman & Bell, 2003). The quantitative paradigm is based on an ontological position of there being only one truth, an impartial reality that is independent of human awareness (Lincoln & Guba, 2000). Advocates of the quantitative research paradigm contend that social science investigation should be objective i.e. conclusions can be determined reliably and validly, and that much sought-after-time and context-free generalisations are feasible (Johnson & Onwuegbuzie, 2004). Therefore, educational researchers are expected to remove any biases while also remaining detached and uninvolved with the participants (Tashakkori & Teddlie, 1998).

4.2.2 Qualitative Research

Qualitative research is described as an understanding of human conduct and permits the researcher to observe actions and humankind, from the viewpoint of a person involved in the study (Bryman & Bell, 2003). Marshall (1984) explains “qualitative research is concerned with capturing other people’s meanings” (cited
in Bryman and Bell 2003, p. 294). Additional enquiries propose qualitative research permits the investigator to acquire a thorough appreciation of “social phenomena” which could not be achieved through quantitative data alone (Silverman, 2000, p. 8). Engaging in qualitative research allows one to focus on the viewpoint of the person being surveyed, and enables an emphasis to be placed on what that particular individual considers essential, thus providing a sense of direction to the researcher (Bryman & Bell, 2003). Bryman and Bell also suggest that this particular type of research method may be employed for the telling of various events over time. The ontological position of the qualitative paradigm proposes that there are many realities/truths based on ones construction of reality.

4.2.3 Triangulation

A significant proportion of social research is based on a single research approach and as a result may suffer from the limitations associated with that approach. Triangulation is described as more than one approach to study a research question in order to provide a greater confidence in the research finding. Cohen et al., (2000) propose that relying on one method of observation is disadvantageous due to the fact that it may distort the researcher’s picture of the investigation. They believe the more the methods contrast with each other, the greater the researcher’s confidence about the findings. The idea of triangulation was extended further than its conventional relationship with research methods by Denzin (1970), who highlighted four forms of triangulation:

1. **Data triangulation** involves gathering data through a series of sampling strategies, so that data is gathered at different times and social locations, on a variety of people.
2. **Investigator triangulation** uses multiple researchers instead of a single source to gather and interpret data.

3. **Theoretical triangulation** uses more than one theoretical approach to interpret data.

4. **Methodological triangulation** uses more than one approach to gather data. This may consist of using within or between method strategies.

### 4.3 Description of Mixed Methods Employed

Mixed-methods research is defined as “the class of research where the researcher mixes or combines qualitative and quantitative research techniques, methods, approaches, concepts or language into a single study” (Johnson & Onwuegbuzie, 2004, p. 17). Some similarities exist between the two paradigms (quantitative and qualitative), which facilitates the combination of both research methods. According to Sechrest and Sidana (1995, p. 78) these similarities include:

- The ultimate goal of understanding the world in which we live.
- The use of empirical observations to tackle research questions.
- Data is described.
- Explanatory arguments are constructed.
- Outcomes are conjectured.
- Precaution is taken to minimise bias and other sources of validity.
For this research project a mixed methods approach is evident in, and justified by, the use of methodological triangulation (Denzin, 1970). Data was obtained from a selection of students in three participating schools and the three co-operating teachers. Throughout this research more than one research instrument was used to collect data. A questionnaire was distributed to 68 students that participated in the Implementation Phase of the intervention. Follow-up interviews took place with 10 students chosen by method of stratified random sampling, while the three co-operating teachers were also interviewed subsequent to the completion of the intervention. Contrasting methodological approaches and tools ensure triangulation occurred. Table 4.2 summarises the between-methods application and implementation in this research project. It also demonstrates how mixing is occurring at both the data collection and data analysis level.

4.4 Validity and Reliability

According to Bryman and Bell (2003) reliability and validity are among the most important criteria for the evaluation of research. Saunders et al. (2003), maintains that for a researcher to diminish the likelihood of reaching the wrong answer in a study requires paying particular attention to research design that involves reliability and validity. Thus, the degree of validity and reliability of a given body of research is the extent to which meaningful conclusions can be drawn from the data.
<table>
<thead>
<tr>
<th>Phase</th>
<th>Sub-Part</th>
<th>Qualitative Data Collection</th>
<th>Quantitative Data Collection</th>
<th>Qualitative Analysis</th>
<th>Quantitative Analysis</th>
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<tbody>
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<td>1</td>
<td>Review of Literature</td>
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<td></td>
<td>Theoretical Analysis at the Exploratory Phase</td>
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<td></td>
<td>Design/Piloting of Teaching Intervention</td>
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<td>Design/Piloting of Test Instruments</td>
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<td>2</td>
<td>Theoretical Analysis at the Implementation Phase</td>
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<td>Design/Implementation of Teaching Intervention</td>
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<td>Interviews</td>
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<td>Questionnaires</td>
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<td>Reflective Journals</td>
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</table>
4.4.1 Validity

Validity is an essential component of effective research. If a piece of research is invalid then it has little significance, thus, in essence it is essentially worthless (Cohen et al., 2000). However, one must realise that is impossible for research to be entirely valid and, by this means, validity should be seen as a matter of degree rather than a complete state (Cohen et al., 2000). In quantitative research attempts at validity are ensured through careful sampling, appropriate instrumentation and suitable statistical analysis of the data (Cohen et al., 2000). In qualitative research there is no certainty, only confidence in the results. Confidence is significantly increased if the researcher is not reliant on one source or method for collecting data where a multi-method approach or triangulation is required. Furthermore, validity in qualitative research is addressed through the honesty, depth and scope of the data achieved and importantly through the objectivity of the researcher (Cohen et al., 2000).

**Internal Validity**

Internal validity emphasises that the findings must accurately describe the phenomenon being researched. The fundamental purpose of validating research is to convince those interested that the data has not been misinterpreted or misrepresented. This is addressed in the context of this research study by drawing from multiple sources of data:

- teacher debriefing and validation of researcher’s interpretations,
- analysis of student questionnaires,
• teacher’s and student’s taped and transcribed interviews,

• teacher’s reflective journal.

External Validity

External validity relates to whether the findings of a study can be generalised outside the definite exploration environment (Bryman & Bell, 2003; Cohen et al., 2000). The author administered questionnaires to three different cohorts at Senior-Cycle level and completed interviews with a sample from each group so as to ensure that the views and opinions of each group are represented. The sample size and composition of the sample, as regards age, background and ability, gives a reasonable basis for limited generalisation.

4.4.2 Reliability (Quantitative Research)

Reliability in quantitative research is primarily concerned with precision and accuracy: Cohen et al., (2000, p. 117) defines reliability in quantitative research as “essentially a synonym for consistency and replicability over time, over instruments and over groups of respondents”. In essence, they argue that for quantitative research to be reliable it must demonstrate that if it were carried out on a similar group under similar constraints then similar results would be found. The anonymous feature of the student questionnaire used in this research encourages greater honesty and thus ensures more reliable results. There was no threat to the reliability of the research findings as one observer carried out the research for this study. The structure of the questionnaire was revised several times and was also the subject of a pilot test, in the guise of the Exploratory Phase, to make the answering
as easy as possible, thereby contributing to the trustworthiness of the observed information.

4.4.3 Reliability (Qualitative Research)

Reliability in qualitative research is described as being a fit between what researchers’ record as data and what actually occurs in the natural setting that is being researched (Cohen et al., 2000; Bogdan & Biklen, 1992). Reliability is insured throughout this research study by recording honestly and comprehensively events as they happen in the field. Reliability was promoted in this research as the interviewee was presented with a set of pre-defined questions drafted for the purpose of obtaining information, impacting precisely upon the research objectives of this report and the variables outlined in the theoretical structure. The questions used for the research could allow another researcher to obtain the same information by using the same set of questions, thus supporting the external reliability of this research.

4.5 Educational Interventions

A teaching intervention was designed, and implemented as part of this study so as to allow the author investigate the teaching and learning of applications in Senior-Cycle mathematics, while remaining consistent with the theoretical framework employed in this research study. This section examines the definition of the term ‘intervention’ from a variety of perspectives and backgrounds, in particular educational interventions. This enables the author to identify the general approaches to evaluating an educational intervention and consequently facilitate the evaluation
strategy employed in the Implementation Stage of the research. Such an approach enables the author to establish a description of the term and construct for use in the context of this research study.

4.5.1 Defining Interventions

In an effort to gain an insight into and an understanding of the term ‘intervention’, the author will define the term for use in the context of this research study, by examining existing definitions of the expression found in a variety of sources. According to the Oxford Concise Dictionary (1996), an intervention is “the act or an instance of intervening” so as “to prevent or modify the result or course of events”. Notably, there are numerous definitions provided from a variety of sources and in order to pin down an effective meaning of the term, a number of these definitions were explored. An intervention can be defined as:

*Any action that is taken to change a situation, generally following an analysis of data and other evidence. This term is useful as it emphasises that to change students’ achievement, you will have to change something about the situation that lies behind achievement or non-achievement.*

(New Zealand Ministry of Education, 2010)

*An activity or set of activities aimed at modifying a process, course of action or sequence of events, in order to change one or several of their characteristics such as performance or expected outcome.*

(World Health Organisation, 2004)
From an educational perspective the following definition was proposed:

*By intervention we mean any change or programme of change which is implemented in the classroom, it could be formal or informal and might be initiated by the individual teacher, the school or another organisation from outside.*

(Teaching Expertise, 2010)

General understanding and appreciation for the term ‘intervention’ has arisen out of many of these definitions. However, the underlying principles of the intervention are key to defining the nature and purpose of the intervention utilised. Interventions are widely used in educational and healthcare issues, where an intervention is aimed at a positive outcome rather than the perceived negative one that would ordinarily occur (Wilkes & Bligh, 1999). Essentially, an intervention means imposing a change, usually something novel (an activity, strategy, or approach), in an already ongoing relationship with the primary goal of improving it. The traditional approaches to the teaching and learning of mathematics warrant appropriate action to try and influence the pedagogical practices of mathematics teachers, not only in Irish classrooms but extensively throughout the world. This action is necessary to prevent stagnation in beliefs and attitudes towards mathematics at Senior-Cycle, often found among mathematics teachers and students alike in Irish schools. The decisive action in this research study involves the utilisation of applications of mathematics at Senior-Cycle level to address the problem of declining standards in mathematical competence in this area and problem solving, while ensuring one of the key objectives of the Leaving Certificate, that of
relational understanding, is catered for in the mathematical experience of Senior-Cycle students. The intervention is employed in order to change beliefs, attitudes and understanding with regards to the relevance of mathematics. It is designed primarily to produce a positive outcome, rather than the negative situation that occurs at present. Therefore, from the perspective of this thesis the author defines the term intervention for use in the context of this study as:

*An innovative manipulation of existing pedagogical techniques designed to enhance current classroom practise and subsequent views, attitudes and understanding, relating to applications of mathematics.*

The following section gives a brief description of the classifications of interventions, and in particular mathematics interventions, within the discipline of school psychology. In addition, four general approaches to the evaluation of educational interventions that have emerged over the last decade will be examined. These classifications will provide an insight into preparing the model employed in evaluating the intervention employed in this research study.

### 4.5.2 Evaluation of Interventions

A review of the Mathematics Education journals failed to uncover any distinct theories on interventions within the school context (Codding et al., 2005; Leiss, 2005). However, research on school psychology certainly shows evidence of the use of interventions, most notably through evaluations of particular interventions in the treatment of academic (achievement), behavioural and mental problems. Activities such as child assessment, consultation with teachers, education and system analysis have formed the basis for creating interventions within the school
setting so far. Thus, central to the success of any intervention is the quality of data on which to base the particular intervention. This provides background information to the intervention and opportunities to evaluate the effectiveness of the program (Regan, 2005; Sandoval, 1993). Research on interventions has largely been directed towards evaluation of particular interventions and the acceptability of interventions. It has emerged that instructional interventions are determined by two distinct approaches:

1. Process-based Interventions

2. Behavioural Interventions (Skill/Curriculum theorists)

(Sandoval, 1993)

Process-based interventions are generally directed at an individual child’s pattern of cognitive processes or abilities. Evaluating a child’s reasoning provides an initial setting/starting point to begin appropriate educational intervention (Sandoval, 1993; Regan, 2005). According to Sandoval (1993), process-based interventions have not been shown to be particularly effective due to the complex nature of deducing an individual’s reasoning ability. Behavioural interventions enable curriculum/skill theorists to educate children according to their identified needs for specific skills, rather than focusing on individual differences. Behaviour-theory based interventions have become well validated over the past thirty years, where Gagne’s (1970) learning hierarchies have formed the theoretical approach to such interventions (Regan, 2005; Sandoval, 1993). The author consulted research on the evaluation of educational interventions in the discipline of ‘medical education’. In the absence of literature from mathematics education on which to base
the design and evaluation of a school-based intervention this was deemed an appropriate substitute. Wilkes and Bligh (1999, p. 1269) acknowledge educational evaluation as the: “the systematic appraisal of the quality of teaching and learning”. In essence they view evaluation as an attempt to improve education from two perspectives: formative evaluation and summative evaluation. Formative evaluation identifies areas where teaching can be improved, while summative evaluation judges the effectiveness of the teaching (Wilkes & Bligh, 1999). Shapiro (1987) outlines four key parameters by which intervention research can be evaluated. These are:

- Treatment effectiveness
- Treatment integrity
- Social validity
- Treatment acceptability.

**Treatment effectiveness**

The effectiveness of the intervention is an important measure in the evaluation of any strategy employed. The effectiveness of an intervention is typically a quantitative measure of the change observed as a result of the strategy.

**Treatment integrity**

To ensure the intervention can be implemented with replicable results the integrity of the intervention strategy is of utmost importance. That is the extent to which the
specified scheme is actually executed in the manner prescribed in the intervention documentation.

**Social validity**

Social validity was defined by Shapiro (1987, p. 293) as the ‘evaluation of the intervention by the clients or consumers’. Consequently, the question ‘to what extent did the project meet its overall goals?’ needs to be answered by analysing the participant’s views.

**Treatment acceptability**

Kazdin (1981, p. 494) defined treatment acceptability as: “judgements by laypersons, clients and others on whether treatment procedures are appropriate, fair and reasonable for the problem or client”. Thus, it is important to determine if significant unintended impacts resulted from the intervention strategy.

Edwards (1991) proposes a four stage cyclical process when evaluating an educational intervention in Figure 4.1:
Furthermore, Wilkes & Bligh (1999, p. 1270) propose four general approaches to educational evaluations that have emerged over the last decade:

- **Student Oriented** – predominantly uses measures of student performance as the principal indicator

- **Programme Oriented** – compares the performance of the course as a whole to its overall objectives and often involves descriptions of curriculum or teaching activities
• Institution Oriented – usually carried out by external organisations and aimed at grading the quality of teaching for comparative purposes

• Stakeholder Oriented – takes into account the concerns and claims of those involved and affected by the course or programme of education.

With respect to this research study, it is clear that the intervention would be evaluated primarily from a ‘programme oriented’ perspective but will include measures of student performance. As described above, such an approach is concerned with the overall objectives of the intervention and teaching activities employed in the intervention. As outlined in the theoretical framework for this research study, the intervention focuses explicitly on the key aspects outlined in a ‘programme oriented’ approach (overall objectives, guiding principles, teaching activities etc.) to evaluating an educational intervention. Chapters 5 and 6 describe in more detail the objectives, guiding principles and subsequent instructional techniques employed in this research study. Furthermore, the theoretical framework employed throughout this study enables a clear evaluation process that takes into account the key stages outlined by Edwards (1999). While the author has determined that the evaluation process is ‘programme oriented’, it must also be noted that the intervention is concerned with students’ performance and views on mathematics at Senior-Cycle. While these are important aspects of the intervention, it does not enable the author to declare that the programme is also ‘student oriented’ as this is not the key guiding principle underpinning the intervention. Thus, the evaluation process is ‘programme oriented’ but supported by ‘student orientation’.
4.6 Theoretical Frameworks Employed in this Research Project

Chapter 3 outlined the primary theoretical framework employed in this study – APOS Theory (Asiala et al., 1996). The author felt that the three-stage approach employed by APOS Theory (Exploratory, Implementation and Reflective Phases) allowed the author to pilot, implement and evaluate the subsequent teaching intervention. The three-stage cyclical process (Theoretical Analysis, Instructional Design, Observations and Assessments) allowed the author to field test his approach in adapting APOS Theory so as to include a provision for the teaching of applications. Bajpai’s Integrated Approach (1975) and the Harvard Calculus Approach (1991) influence the research design in that they offer a perspective on introducing mathematical concepts through multiple approaches (numerical, analytical, graphical, verbal) as opposed to the over-emphasis on analytical techniques widely practised in schools. Their emphasis on applications, modelling and case studies allows the author to achieve the overall aims of the research project. The relationship between these theoretical frameworks and their influence on the research design is demonstrated in Table 4.3.
Table 4.3: Relationship between the theoretical frameworks and their influence on the research design.

<table>
<thead>
<tr>
<th>Theoretical Framework</th>
<th>Influence on Research Design</th>
<th>Influence on Teaching Intervention</th>
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</thead>
<tbody>
<tr>
<td>APOS Theory</td>
<td>3 – Phase approach:</td>
<td>ACE Teaching Cycle:</td>
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<td></td>
<td>• Exploratory Phase</td>
<td>• Activities</td>
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<td></td>
<td>• Implementation Phase</td>
<td>• Classroom Discussion</td>
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<td></td>
<td>• Reflective Phase</td>
<td>• Exercises</td>
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<td></td>
<td><strong>Cyclical process:</strong></td>
<td></td>
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<tr>
<td></td>
<td>• Theoretical Analysis</td>
<td></td>
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<tr>
<td></td>
<td>• Instructional design</td>
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</tr>
<tr>
<td></td>
<td>• Observations and Assessments</td>
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<tr>
<td>Bajpai Integrated Approach (1975)</td>
<td>Distinct Approaches to solving a real-life problem:</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Analytical</td>
<td>- Case Studies</td>
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<tr>
<td></td>
<td>• Computer</td>
<td>- Real world problems</td>
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<tr>
<td></td>
<td>• Numerical</td>
<td>- Mathematical modelling</td>
</tr>
<tr>
<td></td>
<td>• Statistical</td>
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<tr>
<td>Harvard Calculus Approach (1991)</td>
<td><strong>The Rule of Four:</strong></td>
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<td></td>
<td>Topics should be presented:</td>
<td>- Problem Driven</td>
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<td></td>
<td>• Graphically</td>
<td>- Open-Ended Real World Problems</td>
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<td></td>
<td>• Verbally</td>
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4.7 Research Questions

The study focuses on the problem of improving the teaching and learning of mathematics, particularly upper secondary level in Ireland, by making a provision for the effective teaching of applications. While approaches have been developed to cater for applications in mathematics to date, none have been developed for Senior-Cycle students in Ireland. The approaches developed by Bajpai (1975), the Harvard Calculus Consortium (1991) and others (Ormell, 1972; Meyer & Ludwig, 1999) are practice-led and lack a wider pedagogical perspective suited to secondary school mathematics teaching. APOS Theory offers just such a perspective and theoretical pedagogical framework for mathematics teaching and learning based on constructivist principles. It is a purpose built theory for mathematics teaching and was developed by Dubinsky (1996) and his colleagues in the Research for Undergraduate Mathematics Education Community (RUMEC). The author has adapted this theory for use in this study and the resultant approach has been field tested in a small scale intervention in Irish Senior-Cycle schools. Hence, the main aim of the author’s study is:

- to develop a systematic approach to the teaching and learning of applications of mathematics for Senior-Cycle students in Irish schools

This study is guided by the following research questions:

- what are the key factors that contribute to the teaching and learning of applications in upper secondary schools?

- what theoretical pedagogical perspective is capable of integrating such factors appropriately? How can this be done effectively?
• how can the effectiveness of such an approach be evaluated?

4.8 Project Design – A Three-Phased Approach

The primary aim for this research is to focus on the problem of improving the teaching of mathematics, particularly at upper secondary level in Ireland, by including a provision for the teaching of applications. Given the diversity of investigation within this project and given due consideration to the available methodologies, the author chose a mixed-methods approach that incorporated both elements of the qualitative and quantitative research paradigms. Effective methodological triangulation is achieved through the use of a ‘between methods’ approach in that different types of qualitative and quantitative research is employed. This study originated when it became apparent that no research in this specific area had been carried out in Ireland prior to this project. Thus, there was no available data to work with so after an extensive review of literature had been carried out a pilot study incorporating a teaching intervention, known as the ‘Exploratory Phase’, was undertaken. This was the first phase of the research project. The primary aim of this first phase was to investigate:

1. the role of applications and modelling in mathematics education,

2. to examine the factors affecting classroom practise,

3. to establish and clarify the key issues facing the teaching of mathematics by including a provision for applications.

This was to ensure that subsequent research carried out by the author would address the relevant issues and contribute to development in this research domain.
Phase I also informed the design of the methodology to be employed in the remainder of the project, as well as the development of the teaching intervention, questionnaires and reflective journals used. The second phase of the research, known as the ‘Implementation Phase’, was concerned with the execution of the teaching intervention and subsequent data collection. Analysis of the data and the derivation of conclusions and recommendations form the third phase of this research project.

When combined, Phase 1, 2, and 3 of the research project provide compelling insights into the issue of applications in the teaching of upper secondary mathematics in Ireland. Although the phases are separated for the purpose of the description of the individual methodologies utilised, their natural relationship is evident in the overall research design. Figure 4.2 gives an overview of the three-phase approach to this research project.

### 4.8.1 Phase 1

This phase incorporates a comprehensive review of the literature on general issues in mathematics education, and the role of applications in mathematics education (Chapter 2). A detailed analysis of a number of significant approaches developed by mathematics educators for both second and third-level mathematics education are analysed in which the use of applications and modelling play a distinctive role was provided in Chapter 3. This work provided a basis for the design of the theoretical framework employed in this research project and subsequent testing through the design and implementation of a small scale teaching intervention which is reported and discussed in Chapter 5 and 6 of this thesis. By using multiple sources of data in Phase 1 validity is ensured, which is of particular importance
Figure 4.2: An overview of the three-phased research design implemented in the research project
since this is the first study of its kind to be carried out in Ireland. Figure 4.3 outlines the stages of Phase 1 of this research study.

**Data Collection**

In February 2009, participating schools were located in the mid-west region (1 mixed-gender and 1 all girls school). There were two participating teachers and a total of 35 students (n=35) who took part in the intervention. 82.9% of the sample were females (29 out of 35 students), while the breakdown of ages was as follows: 15 years = 8.6%, 16 years = 51.4% and 17 years = 40%. In the Exploratory Phase data was collected using the following instruments:

- Participating teachers’ reflective journals (Qualitative)
- Questionnaire administered to the participating students which measured their perceptions on applicable mathematics (Quantitative)

**Reflective Journals**

Piaget (1960, 1952), among many other educators (Shoenfeld, 1998; Skemp, 1986), stressed the significance of reflection in their theories of cognitive development. Reflection plays a considerable role in problem solving strategies, while Tomlin (1995) and Pescoff (2000) maintain that reflection can be also used as a coping mechanism for math anxiety. Reflection has been put forward as a vital part of the overall teaching process, where Kyriacou (1998) contends that reflection becomes inherent in the job. The use of reflective journals is widely emphasised in pre-service teaching courses and is often referred to as ‘reflective teaching’
Figure 4.3: Outline of Stages of Phase 1 of the Research Study
(Calderhead & Gates, 1993; Russel & Munby, 1992). ‘Reflective teaching’ has been widely advocated and thus has been encouraged as part of teachers’ normal practice and professional development (Kyriacou, 1998).

Exploratory Phase

In the Exploratory Phase, the participating teacher was encouraged to keep a journal of his/her personal thoughts and feelings regarding key components of the intervention for each individual class. This allows the teacher to revisit the activities of the class and reflect on the teaching experience in his/her own time. The teacher was expected to complete the journal (see Appendix H) after each class, give a brief outline and overall reflection of the class, while they were also expected to extend their comments based on specific headings related to the intervention, such as:

- Students’ work
- Accompanying material
- Real-life problems
- GeoGebra Applets.

Questionnaire

The questionnaire was selected as one of the primary research instruments of the research study as it was a required method of observation outlined by RUMEC with respect to the theoretical framework of APOS Theory. In particular, the use
of a questionnaire is an efficient instrument for gathering information and facts on the sample.

**Piloting the questionnaire**

The questionnaire for the Exploratory Phase was piloted with six leaving certificate students in February 2009. The questionnaire for the Implementation Phase was piloted with five leaving certificate students and five research students in mathematics education in September 2009. The author spoke to each participant following their completion of the questionnaire, discussing the content and the wording of questions. Piloting ensured an increase in validity and reliability as the statements were revisited and revised accordingly to make them suitable for the sample.

**Questionnaire (Exploratory Phase)**

It provided the author with demographic information including the student’s age, Junior Certificate grade and level of mathematics study (see Appendix J). Following the initial demographic information, the questionnaire was aimed at discovering which mathematical topics the students considered applicable to real-life contexts. The questionnaire consisted of two-open ended questions to begin with to determine the students initial reactions to what they consider to be applicable mathematics. This was an opportunity for the students to express unequivocally their thoughts on applicable mathematics without any external influence such as a list of topics or subjects. It was intended that the students would express the first ideas that came into their minds immediately after reading the question. In question 3 the students were to tick the mathematical topics (from a list of 18 mathe-
matical topics) according to what they considered applicable to real-life contexts. The students could tick more than one answer. Question 4 took the same format as Question 3, where the students were to tick the mathematical topics (from a list of 18 mathematical topics) according to what they considered least applicable to real-life contexts. This section was based on a study undertaken in Vienna by Humenberger (2000) in which students, student teachers and in-service teachers were questioned on their opinions in relation to applications in mathematics education. Humenberger (2000) conducted a significant survey of students, student teachers and in-service teachers where they were questioned on their opinions in relation to applications in mathematics education. The study asked students (491 students mostly of age 17), student teachers (‘pre-service teachers’, 202 candidates), and in-service mathematics teachers (173 candidates) to give their opinion about “Teaching applications in mathematics education” by answering a written questionnaire.

**Data Analysis**

The data collected in the questionnaire was quantitative in nature and thus quantitative analysis was used. This was facilitated through the use of Excel and SPSS version 16.0, computer software designed for the analysis of quantitative data. The data consisted of the coded response of 35 questionnaires returned from the students participating in the Exploratory Phase of the teaching intervention. The data collected in the Reflective Journals is qualitative and thus qualitative analysis was used. For the Reflective Journals, computer assisted qualitative data analysis (CAQDA) was incorporated through the use of the software package NVivo, a tool used to aid the researcher in the analysis of the qualitative data. Each participating
4.8.2 Phase 2

This phase of the research included the re-design of the teaching intervention, subsequent theoretical analysis and implementation of instruction based on the theoretical analysis. Figure 4.4 outlines the key stages of Phase 2 of the Research.

Data Collection

In November 2009, participating schools were again located in the mid-west region (2 mixed-gender schools). There were three participating teachers and a total of 68 students (n=68) took part in the intervention. 42.6 % of the sample were aged 15 (29 out of 68), while the remainder were aged 16. Females make up 51.5 % of the sample (35 out of 68 students). In the Implementation Phase data was collected using the following instruments:

- Participating teachers’ reflective journals (Qualitative),
- Semi-structured interviews of the participating teachers (Qualitative),
- Semi-structured interviews of a sample of the students (n=10), (Qualitative),
- Questionnaire administered to the participating students which measured their views on the intervention and their perceptions on applicable mathematics (Quantitative).
Figure 4.4: Outline of Stages of Phase 2 of the Research Study
Each of these will be discussed in the following subsections. The findings will be used as a means of triangulation to support the theory employed in this research project.

**Reflective Journal**

The teacher journal at the Exploratory Phase had a section on lesson aims and outcomes. It was decided to remove this section as the lesson aims and outcomes were pre-decided for each lesson. The length of the journal was shortened from three pages to one so as to ensure that the teacher was focussed in their reflection (see Appendix I). They were expected to extend their comments based on specific headings such as:

- Accompanying material
- Real-life problems
- GeoGebra Applets.

**Interviews**

Interviews provide a number of advantages; firstly they present information that is appropriate and pertinent to the question. Subsequently, they provide the interviewer with the opportunity to gather information that is specific and not readily available from any other source to be analysed. Finally, interviews can give information that can be concealed by a written response. For example, if a respondent’s tone of voice or expression changes, it might mean that they are uncomfortable with the question. Interviews provide the opportunity for investigating attitudes
and emotions of the interviewee. Motivations and opposition in the direction of
certain aspects of the topic can be determined from this type of research. The
author formulated the interview questions based on the guidelines from Bryman
and Bell (2003) for formulating research questions, which is shown in Figure 4.5.

The author’s objective was to investigate both the students’ and co-operating teach-
ers’ feelings and views on the intervention and the impact that it would have on the
teaching and learning of applications at Senior-Cycle level. The semi-structured
interviews were organised into three sections:
This data gathering instrument or interview plan guided the 10 student and 3 teacher interviews that took place. The interview plan was adjusted slightly to examine the roles of the interviewee where there were separate student interviews (see Appendix L) and teacher interviews (see Appendix M). Each student interview was conducted on an individual basis and lasted approximately 5 minutes, while each teacher interview lasted approximately 7 to 8 minutes. The interviews were designed to be as flexible as possible. All interviews took place post completion of the intervention, within a time-frame of 5 days. Such a short time frame was chosen so as to alleviate any distractions that may occur through new material covered in the mathematics classroom subsequent to the intervention and/or any other academic or non-academic distractions that may have occurred.

**Student Questionnaire**

The demographic information including the student’s age, Junior Certificate grade and level of mathematics study was retained as a basis for background information. However, the student questionnaire was changed from two perspectives. Seven questions were put to the students to assess their views on the intervention in what was referred to as Section A. Section A contained statements based on a five point Likert scale (see Appendix K). According to Cohen et al. (2000) Likert scales offer the researcher freedom to use measurements with opinion, quantity and quality. As a result the author felt it both appropriate and useful to use this
approach in the student questionnaire. Strong feelings could be indicated on ei-
ther side of the scale and there was an option for respondents who were unsure of
statements: Section A:

\[
\begin{align*}
\text{SD} &= \text{Strongly Disagree} \\
\text{D} &= \text{Disagree} \\
\text{U} &= \text{Undecided} \\
\text{A} &= \text{Agree} \\
\text{SA} &= \text{Strongly Agree}.
\end{align*}
\]

In Section B, (similar to the questionnaire in the Exploratory Phase) the mathe-
matical topics that the students had to tick to assess their views on applicable and
least applicable mathematical topics were altered. The topics chosen were Junior
Certificate centred, where previously the topics were Leaving Certificate centred.
This was because it was felt that not all of the topics might have been studied at the
time of implementation and all the students had completed their Junior Certificate
the previous summer.

4.8.3 Phase 3 – Reflective Phase

This phase was the third and final phase of the research. In this phase, the author
had an opportunity to evaluate the success of the small scale intervention and con-
sequently draw conclusions on the appropriateness of the design of the theoretical
pedagogical framework employed in this research. The same methods of analysis
were used during this phase as during Phase 1 of the study except that the author
also included the use of computer assisted qualitative data analysis (CAQDA) to
analyse the Interviews. Interviewees were coded using pseudonyms. Chapter 7
provides an in-depth report of the analysis of the data collected at this phase.

4.9 Ethics

Cohen et al. (2000) stress the importance of taking into account the effects of
the research on the participant, and to act in such a way as to preserve their hu-
man dignity. Ethical issue were recognised and the guidelines set down by the
University of Limerick Research Ethics Committee (ULREC) were adhered to.
Various ethical considerations were taken into account and before the research
began ULREC approved the author’s application form. Ethical considerations for
this research project included:

• recognition of harm for the participating students and teachers,

• recognition of the effect of the researcher’s presence on the research setting,

• interpretation of the data being as non-biased as possible,

• preserving confidentiality and anonymity,

• making the research subjects aware of how the data will be used,

• giving participants an opportunity to withdraw at any point during the re-
search study.

Prior to the implementation of the intervention at both the Exploratory and Im-
plementation Phases the author issued a teacher information sheet and consent
form to the participating teachers (see Appendices F and G). Following that, the author issued a principal information sheet and consent form to the participating school (see Appendices A and B). Following consent from the principal, the author obtained written consent from the participating teacher and subsequently distributed subject information and consent forms for the students in the participating classes (see Appendices C, D and E). The primary objective of the information sheets was to provide the participant with the necessary information regarding the intervention, while also outlining the requirements of each participant at their designated level i.e. principal, teacher and student. Participation was voluntary with participants retaining the right to withdraw from the study at any time. At all times the researcher followed university guidelines in obtaining student data. Data analysis of the student questionnaire was carried out using the statistical package SPSS (version 16 for Windows). All entries into SPSS were coded. Analysis of the students and teacher interviews was carried out using NVivo. Students were picked at random using simple random sampling. All participating teachers were interviewed. Confidentiality was ensured throughout the interview and analysis process where each participant was given a pseudonym. It was explained that no information about the students would be identified in the final report and all data would be stored safely with access only to the investigators as specified by university guidelines for such data.

4.10 Conclusion

This chapter has provided an account of the theoretical framework employed in this research project, as well as the necessary theoretical and practical consid-
erations needed to be taken into account in the implementation of this research project. A mixed-method approach is adopted and a three phased approach is implemented in order to address the research questions. A detailed description of the research design is given so to facilitate the reader’s understanding of the complex nature of the investigation undertaken. Consideration is also given to concerns relating to ethics, validity, and reliability within this chapter.
Chapter 5

Teaching Applications of Mathematics: The Exploratory Phase

5.1 Introduction

In this chapter the author presents the Exploratory Phase of the research study in an attempt to remain consistent with the adaptation of APOS Theory framework. An adaption of APOS Theory has been undertaken for the purpose of this research project and is proposed for the learning needs of students learning mathematics through applications. The chapter begins with an explanation of why the approach taken by the original authors (Dubinsky et al., 1996) of APOS Theory was not deemed appropriate for the mathematical learning needs of the students. The three-stage cyclical process promoted by APOS Theory: theoretical analysis, instructional treatment and analysis is described in detail, paying particular
attention to how it was adapted for the purpose of this research study. The author chose to analyse a certain concept that is taught on the Senior-Cycle mathematics syllabus in Ireland so as to narrow the focus of the research study. The concept chosen was that of a ‘function’. The findings of the Exploratory Phase provide a basis from which the author assessed the appropriateness of the theoretical analysis and instructional design of the Exploratory Phase. These can then ensure that the necessary modifications are made where necessary for the Implementation Phase (main phase) of the research study.

5.2 Rationale for Adapting APOS Theory for the Teaching of Applications of Mathematics at Senior-Cycle Level in Ireland

The study focuses on the problem of improving the teaching and learning of mathematics, particularly upper secondary level in Ireland, by making a provision for the effective teaching of applications. While approaches have been developed to cater for applications in mathematics to date, none have been developed for Senior-Cycle students in Ireland. The approaches developed by Bajpai et al. (1975), the Harvard Calculus Consortium (Hughes-Hallett, 1991) and others (Ormell, 1972; Meyer & Ludwig, 1991) are practise-led and lack a wider pedagogical perspective suited to secondary school mathematics teaching. APOS Theory offers just such a perspective and theoretical pedagogical framework for mathematics teaching and learning based on constructivist principles. It is a purpose built theory for mathematics teaching at third-level and was developed by
Dubinsky (1996) and his colleagues in the Research for Undergraduate Mathematics Education Community (RUME).

Bajpai’s Integrated Approach (1975) and the Harvard Calculus Approach (1991) have influenced the design of the research project and subsequent design of the teaching intervention. These approaches influenced the research design as the author felt that an adoption of these approaches would ensure that the students would be provided with opportunities to appreciate the fact that often a combination of techniques are used when solving problems arising in real-life contexts, both in industry and everyday experiences. Furthermore, these approaches provide an insight into current successful practises used in higher education mathematics courses throughout the world, thus providing a platform from which to approach the teaching and learning of applications in Senior-Cycle mathematics in Ireland. However, both Bajpai’s Integrated Approach (1975) and the Harvard Calculus Approach (1991) are practise-led and lack a wider pedagogical perspective suited to secondary school mathematics teaching. They do not offer a perspective on how learning a mathematical concept might take place, and thus, an adaption of APOS Theory is proposed as it offers just such a perspective and theoretical pedagogical framework for mathematics teaching.

Thus, the author felt that the three-stage approach employed by APOS Theory (Exploratory, Implementation and Reflective Phases) allowed the author to pilot, implement and evaluate the subsequent teaching intervention. The three-stage cyclical process (Theoretical Analysis, Instructional Design, Observations and Assessments) allowed the author to field test his approach in adapting APOS Theory so as to include a provision for the teaching of applications. Bajpai’s Integrated Approach (1975) and the Harvard Calculus Approach (1991) influence the
research design in that they offer a perspective on introducing mathematical concepts through multiple approaches (numerical, analytical, graphical, verbal) as opposed to the over-emphasis on analytical techniques widely practised in schools. Their emphasis on applications, modelling and case studies allows the author to achieve the overall aims of the research project. The relationship between these the theoretical frameworks and their influence on the research design is demonstrated in Table 4.3.

APOS Theory is, primarily, a constructivist theory that deals with the approaches an individual uses in learning mathematics. The theory has been developed in a bid to expand Piaget’s concept of reflective abstraction so as to encompass undergraduate mathematics education (Weyer, 2010). Piaget proposed the concept of reflective abstraction in an effort to describe the construction of logico-mathematical structures by an individual during the course of cognitive development (Dubinsky, 1991). Central to the concept of reflective abstraction is Piaget’s notion that it consists of drawing properties from mental or physical actions at a particular level of thought (Piaget, 1985, 1972, 1971; Beth & Piaget, 1966). Essentially, APOS Theory is based on the following hypothesis offered by RUMEC which is presented to offer a perspective on what it means to learn and know something in mathematics:

“An individual’s mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organising these into schema’s to use in dealing with the situations”

(Asiala et al, 1996, p. 7)
While APOS Theory offers a perspective on how learning a mathematical concept might take place, it offers no specific foundation for the role of applications in the learning of a mathematical concept, and as a result an attempt to do so ensures the uniqueness of the research project.

A study of the cognitive growth of an individual trying to learn a particular mathematical concept takes place by means of several refinements, as the investigator/researcher repeatedly cycles through the component activities (Asiala et al., 1996). It is intended that research begins with a theoretical analysis of the concept in question so as to gain an insight into what it means to understand the concept and how that understanding can be constructed by a learner. Significantly, this initial analysis is based primarily on the researchers’ understanding of the concept in question and on their experiences as learners and teachers of the concept. Thus, the analysis informs the design of instruction. The design and implementation of the instruction provides a means for gathering data or for reconsidering the theoretical analysis with respect to the data. In essence, the purpose of the theoretical analysis of a concept is to provide a model of cognition that a learner may undertake and a suggested learning trajectory for the concept in question. It essentially offers a description of “specific mental constructions that a learner might make in order to develop her or his understanding of the concept” (Asiala et al., 1996, p. 7). The student generally undergoes a cognitive conflict when he/she is presented with new material. In order for the student to move to higher levels of cognitive understanding the student must ‘encapsulate’ the required mental constructions i.e. the student reverses the process, reflects on the process, and becomes aware of the process as a totality.

Dubinsky (2001) refers to this analysis as a genetic decomposition of the concept.
The analysis is primarily prepared by applying a general theory of learning that is greatly influenced by the researchers’ own understanding and interpretation of the concept, coupled with their previous experiences in learning and teaching it. The author’s examination of the literature on applications and modelling ensured a perspective on the teaching and learning of applications in mathematics education worldwide. Thus, the author deemed Bajpai’s Integrated Approach (1975) and the Harvard Calculus Approach (1991) as appropriate to achieve the overall aims of the research study. From the author’s experience as a mathematics teacher at both second and third level he found that many students are unable to apply their mathematics or to appreciate the role of mathematics within everyday life, because they fail to understand or realise the influence that mathematics can exhibit within one’s future education and/or work-life. Current practices in the teaching and learning of mathematics in Senior-Cycle schools in Ireland generally fail to make the necessary connections between mathematics and its place in real-life (NCCA, 2005). As a result students have displayed an unwillingness to accept any part of a mathematical concept without understanding the reasons why certain approaches are taken, and more notably, the students must become conscious of the relevance these concepts can play in real-life situations. The author aims to demonstrate how a genetic decomposition can be used not only to predict how the mathematical constructs of the learner develops, but will take the process further and use the genetic decomposition as a tool for highlighting the role of applications in the mathematical knowledge of the learners.

Thus, the author has adapted APOS Theory so that each concept begins with an introduction to applications to allow the learner interiorise the action stage into a process for each mathematical concept at that level. It is intended that the in-
troduction to applications at such an early stage in the learning process will pro-
vide opportunities for the acquisition of one of the key objectives of the Leaving
Certificate Mathematics syllabus – that of ‘relational understanding’. The author
feels that that there is a need to present the students with an environment which
contains as much as possible about the concept at the beginning, as opposed to be-
ing sequentially organised, as is traditionally practised within mathematics class-
rooms. Such an approach ensures the teaching and learning of mathematics is
approached from an alternative direction than is traditionally practised in mathe-
matics classrooms throughout the world. Rather than introducing a new mathe-
atical concept, learning the techniques and skills, and then applying these skills
to real life-problem, this approach begins with applications and uses these as a
basis for the learner to make connections between real-life situations and mathe-
matics. It allows the learners to see and appreciate the relevance of mathematics
to their everyday lives, while ensuring one of the key objectives of the Leaving
Certificate Mathematics Syllabus is catered for. As a result, it is intended that the
learner will then be better equipped to advance their mathematical knowledge.

The illustration of both the how and why of mathematics is a deep-rooted and well
appreciated phenomena. For the typical student the ‘how to do it?’ comes first and
this is an action stage. The ‘why to do it?’ part comes about as a result of moving
through the stages of process, to objects and finally to the schema stage where
a full understanding is displayed. The exposure of students to mathematics that
leads to genuine applications is an ideal vehicle for revealing to students the power
of mathematical thought, while ensuring the students can acknowledge both the
how and why of mathematical concepts.

As has been stressed by many researchers (Burkhardt, 2006; Niss 1987; Bajpai
et al., 1976) success in teaching applications depends on the student’s grasp of the applications used in the classroom. Thus, the applications are culturally dependent and specific to the environment in which they are taught and the personal backgrounds/interests of the students involved, although it is reasonable to expect ‘portability’ of applications in many instances. Consequently, it is necessary for each country to undertake relevant research and develop applications and pedagogical approaches that would be appropriate to the educational context in operation. This study is concerned with the development of approaches to improve mathematics teaching in Ireland by including a provision for applications and therefore, takes into consideration the Irish school system and educational objectives.

Much of the research carried out by RUMEC was primarily focused on students who were studying mathematics as mathematical specialists. Cooperative learning and computer programming were activities that were promoted so as to foster the mental constructions called for by the theoretical analysis. While computer programming would be a familiar activity for the students that APOS Theory was originally designed to help, the students participating in the teaching intervention undertaken in this research project would not be familiar with such an activity. The author believes that the model can be adapted in a way that the beneficial activities like cooperative learning and computer activities remain as core activities. However, in the place of computer programming the author believes that GeoGebra can be used so as to promote learning in a student-friendly manner. GeoGebra is freely available to download under an open-source license and also runs from a web browser, so that there are few barriers (technical or financial) to its use (Sangwin, 2007). GeoGebra provides a versatile tool for visualising math-
ematical ideas (Brophy & Gill, 2007), thus, the author felt that GeoGebra could be used in both the classroom discussion and modelling activities as outlined by the adapted ACE Teaching Cycle. Subsequently, several avenues were explored during the Exploratory Phase to find a suitable replacement strategy that would foster the mental constructions called for by the theoretical analysis. This approach was consistent with the APOS Theory framework.

The ACE Teaching Cycle was subsequently adapted to remain consistent with the overall aims of the research study. The Activities were centred around students working in small groups in the computer lab on computer tasks in GeoGebra designed to foster specific mental constructions suggested by the theoretical analysis. Furthermore, the activities were designed so as to ensure the students became familiar with mathematical modelling. In the Classroom Discussion the teacher could avail of the opportunity to provide definitions, explanations and overviews of the concepts being discussed and worked on through the medium of applications. GeoGebra was used, where appropriate, in the classroom discussion to visualise the concepts. Exercises were presented in relatively traditional fashion for students to work on. However, often the students are asked to explain their answers in words or graphs, an approach not traditionally employed in Senior-Cycle mathematics in Ireland.

The dominance of the adoption of a teach-to-the-examination approach, as opposed to teaching for understanding, in order to maximise points (Gill, 2006) ensures that the study of genuine applications is minimal or non-existent. Consequently, focusing on the teaching of mathematical concepts through applications is a major change in direction from the normal pedagogical approaches taken by traditional text books. This has significant implications for teacher knowledge. A
number of different types of knowledge have been cited by researchers as those critical for teaching, including: subject matter knowledge; pedagogical knowledge; knowledge of students, curricular knowledge and knowledge of other subjects (O’Meara, 2010). Various researchers including Shulman (1986), Ernest (1989), Fennema & Franke (1992) and Rowland et al. (2005) developed models which combined a number of these types of knowledge and highlighted that teachers should become competent in each domain in order to carry out their duties effectively. In order to alleviate this problem for the participating teachers in this research study, the theoretical analysis and subsequent instructional materials (including all applications, applets, exercises etc.) are provided. Teachers’ need both knowledge of the mathematical concepts concerned and the ‘know how’ to teach these concepts. In addition, the teachers will need knowledge of the applications area. Thus, the author provides an inclusive package that in a sense is ‘self-contained’ as regards teacher knowledge for applications. Such a self-contained approach ensured the participating teachers could focus on the quality of their instruction, rather than the acquisition of knowledge of other subject areas needed to teach the applications.

The three-stage cyclical process (Theoretical Analysis, Instructional Design, Observations and Assessments) undertaken for the Exploratory Phase is now discussed in the following sections.

5.3 The Mathematical Focus of the Intervention

The Exploratory Phase was designed to test the success of modifying APOS Theory to include applications. Subsequently, a small scale teaching intervention was
designed to offer support for the adaption of APOS Theory. Appropriate peda-
gogical approaches, applications, exercises and modelling problems were made
available for the subsequent teaching intervention. The author chose to analyse a
certain concept that is taught on the Senior-Cycle mathematics syllabus in Ireland
so as to narrow the focus of the research study. Consequently, the mathematical
topic ‘Functions’ was chosen as the pedagogical focus of the intervention. The
primary aim in doing so was to narrow the focus of the research project, while
also ensuring an opportunity to highlight how the pedagogical approaches, ap-
plications and modelling problems may be used in this context within the Irish
syllabus. Furthermore, analysis of Leaving Certificate results and selected results
of the diagnostic test administered by the University of Limerick to students tak-
ing service mathematics modules, highlighted the under-preparedness of students
entering third-level service mathematics courses, particularly when it came to the
topic of ‘Functions’.

The Leaving Certificate mathematics examination, at both Ordinary and Higher
Level, consists of two papers, each of two and a half hours duration. Three hun-
dred marks are allocated to each paper, giving a total of 600 marks. On paper 1,
candidates attempt any six from a range of eight questions. All questions relate to
material on the Core part of the syllabus. Each question carries fifty marks. On
Paper 2, candidates attempt any five questions from the seven available in Section
A, and one question from the four available in Section B. All questions in Section
A relate to material on the Core part of the syllabus. Section B consists of one
question on each of the four Options on the syllabus. Each question carries fifty
marks. At Ordinary level, there are three questions in Paper 1 which concentrate
on ‘Functions & Calculus’. At Higher Level, there are two questions in Paper
1 which concentrate on ‘Functions & Calculus’. Note: This structure will be replaced in the context of ‘Project Maths’ from 2012 onwards (with transition in arrangement in the intervening years). The most important changes to exam structure is that there are no examinable options in ‘Project Maths’. Table 5.1 gives a breakdown of the number of students answering each question and the average mark awarded for each question, according to the most recent Chief Examiner’s Report of mathematics in 2005.

Table 5.1: Breakdown of Results in 2005 and 2010 for ‘Functions & Calculus’ Questions

<table>
<thead>
<tr>
<th>Level</th>
<th>Question Number</th>
<th>Topic</th>
<th>Popularity</th>
<th>Average Marks (out of 50)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ordinary</td>
<td>6</td>
<td>Functions &amp; Calculus</td>
<td>85%</td>
<td>30</td>
</tr>
<tr>
<td>Ordinary</td>
<td>7</td>
<td>Functions &amp; Calculus</td>
<td>95%</td>
<td>31</td>
</tr>
<tr>
<td>Ordinary</td>
<td>8</td>
<td>Functions &amp; Calculus</td>
<td>80%</td>
<td>28</td>
</tr>
<tr>
<td>Higher</td>
<td>6</td>
<td>Functions &amp; Calculus</td>
<td>92%</td>
<td>35</td>
</tr>
<tr>
<td>Higher</td>
<td>7</td>
<td>Functions &amp; Calculus</td>
<td>89%</td>
<td>35</td>
</tr>
</tbody>
</table>

With the inclusion of 3 questions at Ordinary level and 2 questions at Higher level, the syllabus clearly outlines ‘Functions and Calculus’ as key topics. However, considering, that only 18.9% of students sat the Higher Level paper and 70.5% of students sat the Ordinary Level paper in 2005, this highlights that a significant percentage of students entered third-level mathematics under-prepared. The University of Limerick in order to assess the mathematical under-preparedness of students’ upon entry to third level service mathematics courses carry out a diagnostic test in the first lecture of two major service mathematics modules. This test includes 40 questions based on fundamental mathematics concepts. Included in this test are two questions related to graphing functions (Q31 Sketch $y = 3x + 2$; Q32 Sketch $y = x^2 + 2$). The result, shown in Table 5.2, highlight the lack of ba-
sic skills in the topic of ‘Functions’ exhibited by those students entering service mathematics modules in 2005 (n = 497) and the up-to-date figure exhibited by those students entering service mathematics modules in 2010 (n = 738).

Table 5.2: Selected Results of Diagnostic Test in the University of Limerick in 2005

<table>
<thead>
<tr>
<th>Year</th>
<th>Question Number</th>
<th>Correctly</th>
<th>Incorrectly</th>
</tr>
</thead>
<tbody>
<tr>
<td>2005</td>
<td>31</td>
<td>38%</td>
<td>62%</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>9.3%</td>
<td>90.7%</td>
</tr>
<tr>
<td>2010</td>
<td>31</td>
<td>31.6%</td>
<td>68.4%</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>6.4%</td>
<td>93.6%</td>
</tr>
</tbody>
</table>

The traditional method of introducing functions on the Higher Level Leaving Certificate syllabus in Ireland begins with exploring the limits of a function, progressing to the limits of a trigonometric function, onto differentiating from first principles. Generally, it is assumed that the student is aware of the definition and classification of a typical function. By and large the significance of the topic in relation to applications is left to the individual teacher. The dominance of the adoption of a teach-to-the-examination approach, as opposed to teaching for understanding, in order to maximise points (Gill, 2006) ensures that the study of genuine applications is minimal or non-existent.

5.4 The Exploratory Phase

The study focuses on the problem of improving the teaching and learning of mathematics, particularly upper secondary level in Ireland, by making a provision for the effective teaching of applications. The author has adapted APOS Theory for use in this study and the resultant approach has been field tested in a small scale
intervention in Irish Senior-Cycle schools. The framework used in this research consists of three components. Figure 5.1 illustrates each of these components and the relationships among them. A study of the cognitive growth of an individual trying to learn a particular mathematical concept takes place by means of several refinements, as the investigator/researcher repeatedly cycles through the component activities (Asiala et al., 1996).

Figure 5.1: Overview of the APOS Theory Framework

For the Exploratory Phase the author was faced with three initial challenges:

1. To design a theoretical analysis of functions that would best predict how a student might engage in practical applications to real-life situation, while forming the necessary constructs during the learning period.

2. To find suitable student-friendly practical applications that would foster the mental constructions called for by the theoretical analysis.
3. To find suitable methods of observation and assessment to test if the adaptation of the theory was indeed beneficial to students learning mathematics at Senior-Cycle grade.

The method adopted in the Exploratory Phase is based on an extensive review of the literature and the experience of the author as a teacher of mathematics at both second and third level institutions. The remaining sections of this chapter describe in detail each stage of the cyclical process as undertaken for the purpose of this research project.

5.5 Theoretical Analysis of Functions for the Exploratory Phase

The theoretical analysis is a genetic decomposition of functions into actions, processes and objects. Figure 5.2 is an adaptation of the original illustrative representation of a theoretical analysis and shows how function actions are interiorised into function processes and these in turn are encapsulated into function objects (Asiala et al., 1996, p. 9). At later phases the function objects become function actions and the process continues in an upward spiral until a complete function object, called a function schema, is attained.

To construct the initial theoretical analysis for the teaching of functions to Senior-Cycle mathematics students, the author used his experience as a mathematics teacher at both second and third level and his own experience of learning mathematics, coupled with an extensive literature review of other approaches. In this theoretical analysis three levels were identified and each is described below.
As previously stated this theoretical analysis is designed for Higher Level students and there are issues with prerequisite material that needs to be highlighted. In the Higher Level mathematics course it is accepted that a student has prior knowledge and a clear understanding of functions. This is not the general case in practise and therefore an approach had to be designed to include prerequisite material. This was achieved by treating prerequisite material as objects that are treated as actions at the lowest level. It is at this level that these objects need to be interiorised if the learner is to progress to the process stage. The author feels that many of the difficulties that the student is likely to experience learning calculus can be traced back to a lack of understanding of the basic concepts e.g. functions. Figure 5.3 illustrates the actions and processes that need to be encapsulated to form a basic function representation object. A colour scheme has been adopted so that the reader can distinguish between actions, process and objects. The colour scheme
Grey = Actions
Purple = Processes
Green = Objects

Grey with Green Border = An object, which appears as an action at a more advanced level

Figure 5.3: Basic Function Representation Object
Definition of a Function Object

This object is the keystone of the theoretical analysis in that it provides the student with an opportunity to become aware of the distinct approaches to mathematics - graphically, numerically and algebraically, which provides a knowledge base that is necessary for a successful move to higher level objects and provides a platform for future learning. The students start the learning process with a demonstration of the idea of a function encompassing all three approaches mentioned above. In addition, applications are the medium through which the definition and classification of a function are presented. The purpose of this is to ensure that any misconception of what place a function has in the real world in the context of mathematics is rectified. The first cognitive conflict that the student is likely to encounter is interiorising the application’s action into an application’s process. A student who has not already encapsulated the required algebraic, numerical, and graphical objects will not pass beyond the action state. This problem will then escalate as the student attempts to move to higher levels. A successful interiorisation of the applications action into an application process will mean that the student can start to assimilate and accommodate each of the other actions at this level into processes, and eventually encapsulate all three into a basic function representation object. This object will then appear as an action/object at the next level.

5.5.2 Genetic Decomposition Level II

Level II contains a “linear function properties object”. Figure 5.4 describes the genetic decomposition of the linear function object.
This object allows the student to accommodate all the basic properties of a linear function before moving onto operations involving non-linear functions. Each process at this level is preceded by an action of the same name. Again, a student who has not already encapsulated the required algebraic, numerical, and graph-
ical objects will not pass beyond the action state. Each action is governed by a particular assumption and the student needs to progress from applying this assumption to any given linear function, to a more general application. As indicated in Figure 5.4 the student begins with the introduction of the concepts of slope and y-intercept through an application to real-life. The process is two-dimensional in that the student, in order to fully understand the concepts, should be able to make connections between both the mathematical and applicable representation of the concept. Interaction between all processes is then necessary to achieve encapsulation of the complete object. Once more this object will then appear as an action/object at the next level.

5.5.3 Genetic Decomposition Level III

Level III contains a “quadratic function properties object”. Figure 5.5 describes the genetic decomposition of the quadratic function object.

**Quadratic Function Properties**

This object allows the student to accommodate all the basic properties of a quadratic function before moving onto operations involving additional non-linear functions, including trigonometric, exponential and logarithmic functions. Each process at this level is preceded by an action of the same name. Again, a student who has not already encapsulated the required algebraic, numerical, and graphical objects will not pass beyond the action state. Each action is governed by a particular assumption and the student needs to progress from applying this assumption to any given quadratic function, to more general applications.
Figure 5.5: Quadratic Function Properties
5.5.4 Summary

The theoretical analysis described above was the first component of the adaptation of the APOS framework in the Exploratory Phase. Due to the restricted time commitments of the participating schools the entire function schema was not observed. The theoretical analysis was developed by the author prior to the implementation of the intervention in the participating schools. Following this, the instructional treatment was designed so as to correspond with the theoretical analysis providing sufficient material and exercises for the participating teachers and students. This instructional treatment was designed so as to explore areas where potential mathematical problems might occur and following the analysis and observation of the intervention, this knowledge could be combined with the theoretical analysis in order to develop a refined instructional treatment for the implementation phase. The instructional treatment employed in the Exploratory Phase will be discussed in the following section.

5.6 Instructional Treatment for the Exploratory Phase

The initial instructional treatment was developed prior to the intervention and presented to the participating teachers so as to enable the teachers to become familiar with the material. As already stated time was a key factor in the design of the theoretical analysis, thus ensuring the instructional treatment was developed accordingly. Regardless of the time constraints the author made the decision that it was more beneficial to the study that whatever was possible to cover in that time period should be dealt with in a well rounded manner. The author decided that the entire schema would be taught and thereby ensuring the instructional treatment
was developed accordingly. To reduce the risk of error and also to save time a set of printed notes was given as a form of mini-textbook to each student (Appendix N). A teacher copy was also provided which had additional notes on mathematical modelling, while also including the required solutions for the real-life problems (Appendix R). It was envisaged that the pace and structuring of each lesson would be determined by the participating teachers. The ACE Teaching Cycle proposed by Dubinsky (1996) has been employed in this intervention and is expected to be undertaken where possible by the teachers/students. The three components of the ACE Cycle are: Activities, Class Discussion, and Exercises.

5.6.1 Activities

The activities were centred around students working in small groups in the computer lab on computer tasks in GeoGebra designed to foster specific mental constructions suggested by the theoretical analysis. Furthermore, the activities were designed so as to ensure the students became familiar with mathematical modelling. The students were also provided with basic training in the use of GeoGebra to ensure that they could utilise the graphing utility provided by the dynamic software. An influencing factor in the exclusive use of GeoGebra was not only the degree of user-friendliness but the fact that the software is completely free to download. The problems explored in the mathematical modelling sessions were designed to supplement the instructional treatment, while providing the students with genuine applications that they can relate to. Where appropriate, the problems were chosen to reinforce the students appreciation and acknowledgment of the position of mathematics within their own lives as demonstrated in the classroom discussions. The problems presented are open-ended in that there is more
than one approach and more than one solution, depending on the students analysis of the problem. The use of an approach favouring common sense is routinely required. This is not stated explicitly in the problem but should be deduced from everyday life experiences and utilised to solve the mathematics problem. The students were presented with a sample report based on a real-life report entitled: ‘A Report On: The Possibility of a Temperature Being the Same Number Using Both Fahrenheit & Centigrade Scales of Temperature’ (See Appendix O) in order to familiarise the students with the modelling process, including writing the report. The modelling problem that was explicitly explored in this session was:

**O2 Phone Customers**

Currently, 2009, O2 Ireland Limited Phone Company is the second largest mobile phone operator in Ireland with approximately 40% market share or 1.646 million customers. Assume the equipment they now have can service up to 1.7 million customers, while the number of customers has been increasing by about 1% each year. Assuming the growth continues at the same pace, what will the situation be in 2020? 2025? How long will it be before additional equipment will be needed?

5.6.2 Classroom Discussion

The Classroom Discussion provided an opportunity for these same groups to work on paper and pencil tasks based on the computer activities. The teacher could also avail of the opportunity to provide definitions, explanations and overviews of the concepts being discussed and worked on.

GeoGebra applets were employed to introduce the concept of the linear and quadratic functions so as to ensure the students could understand and appreciate the graphing qualities they possess. The applets provided a visual stimulus through which
the students could observe the effects that altering \( m \), the coefficients of \( x \), and the constant \( c \) would have on the graph. Furthermore, the applets ensured the teacher could provide this stimulus quite effortlessly through the movement of a mouse as opposed to a time-consuming demonstration on the board. Figure 5.6 shows an example of the applet used for a linear function:

![Figure 5.6: Straight Line Applet](image)

All new concepts, rules and definitions were presented (where appropriate) through the medium of an application to real-life as opposed to the traditional method of introducing new concepts, rules and definitions, then learning the algorithms and procedures associated with them while finally applying these procedures to an oft out-dated real-life problem. This procedure was employed so as to ensure the students could appreciate and acknowledge the position of mathematics within their own lives. An example of this included the definition of a function (Also see Appendix N):
**Definition of a Function**

In mathematics, a function is used to represent the dependence of one quantity upon another. We can define a function as follows:

Let X and Y be sets of real numbers. A function f is a rule that assigns to each number x in X a single number y in Y. X is called the domain of f. The number y is the image of x under f and is denoted by f(x). The set of images of elements of X is called the range of f.

Or more simply put:

A function is a rule that takes certain numbers as inputs and assigns to each a definite output. The set of all input numbers is called the domain of the function and the set of resulting output numbers is called the range of the function.

The input variable (x) is called the independent variable, and the output variable (y or f(x)) is called the dependent variable.

Functions are often depicted pictorially as below. The diagram illustrates, for example, that the function f takes a value, a, in the domain and assigns it to a unique value f(a) in its range.

A function could, for example, indicate the take-home pay of an employee, (y) given the employee earns the minimum wage of 8.65 Euro for each hour worked (t). This would allow us to express the function as: f(t) = 8.65 t
The primary focus of this intervention was on conceptual understanding and it implies that where appropriate, topics should be presented graphically, numerically, and algebraically. Ultimately, the order in which these approaches are used varies, and all are regarded as important. This approach ensures the students are provided with an opportunity to appreciate the fact that often a combination of techniques are used when solving problems arising in real-life contexts, both in industry and everyday experiences.

5.6.3 Exercises

Exercises were presented in relatively traditional fashion for students to work on. They are generally expected to be completed as homework. The problems attempt to probe the students understanding, where possible, from all three perspectives - graphically, numerically, and algebraically. Often the students are asked to explain their answers in words or graphs, an approach not traditionally employed in Senior-Cycle mathematics in Ireland. An example of the student exercises include:

<table>
<thead>
<tr>
<th>Q2</th>
<th>The value of a car, V = f(a), in thousands of euro, is a function of the age of the car, a, in years.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>Interpret the statement f(7) = 8</td>
</tr>
<tr>
<td>b)</td>
<td>Sketch a possible graph of V against a. Is f an increasing or decreasing function. Explain.</td>
</tr>
<tr>
<td>c)</td>
<td>Explain the significance of the horizontal and vertical intercepts in terms of the value of the car.</td>
</tr>
</tbody>
</table>

Furthermore, there are few examples in the text that are exactly like the homework
problems. This ensures the students cannot just look at a homework problem and search for a similar worked-out example. Success with the homework will come by grappling with the ideas presented.

5.6.4 Summary

As previously discussed this instructional treatment was designed so as to explore potential problem areas that could then be addressed in the Implementation Phase. The third and final phase of the Exploratory Phase was the assessment and observation of the intervention. The methods of assessment, accompanied with the findings, experience and knowledge gained, that have been used in the Exploratory Phase are described in the following sections.

5.7 Observations and Assessments for the Exploratory Phase

The participating teachers for the Exploratory Phase were observed and assessed as follows:

- Reflective Journals

The participating students for the Exploratory Phase were observed and assessed as follows:

- Questionnaires
Data Analysis

The data collected in the Reflective Journals is qualitative and thus qualitative analysis was used. For the Reflective Journals, computer assisted qualitative data analysis (CAQDA) was incorporated through the use of the software package NVivo, a tool used to aid the researcher in the analysis of the qualitative data. Each participating teacher was coded using pseudonyms.

The data collected in the questionnaire was quantitative in nature and thus quantitative analysis was used. This was facilitated through the use of Excel and SPSS version 16.0, computer software designed for the analysis of quantitative data. The data consisted of the coded response of 35 questionnaires returned from the students participating in the Exploratory Phase of the teaching intervention.

5.7.1 Teacher Reflective Journals

The Reflective Journal was designed to allow opportunities for the participating teachers to reflect on their overall experience of each lesson and provide an insight to the intervention from their perspective. Each journal in the Exploratory Phase was three pages per lesson. They were expected to extend their comments based on specific titles such as:

- Class aims
- Learning Outcomes
- Summary of Class
- Student’s work
• Accompanying material
• Real-life problems
• GeoGebra Applets
• Overall Reflection.

While both participating teachers completed the journals, to varying degrees of success, the responses were often descriptive in their nature as opposed to reflective (See Appendix H). Helpful comments provided in the reflective journals included:

(Teacher1) “Students were invigorated by the idea of working as a pair – discussion was stimulated.”

(Teacher2) “They (the students) worked out algebraically and graphically the x and y intercepts. Very few students previously realised the connection between both.”

The comments provided in the Reflective Journals are limited for the purpose of data analysis for this research study. However, Schoenfeld (2004) highlighted the issue of the gap between research objectives and actual practic in classroom settings and the fact that the researcher does not always get everything he wishes despite his best efforts. Schoenfeld (2004, p. 15) highlights the issue of “understanding what can happen to nice ideas when they enter the “real world,” and work in partnership with teachers and administrators to nurture the ideas in practice”. Thus, these issues can be addressed at the Implementation Phase and such alterations are discussed in Section 6.2.
5.7.2 Student Questionnaire

Demographic information including the student’s age, Junior Certificate grade and level of mathematics study is initially provided. Following that, the primary component of the questionnaire consisted of two-open ended questions to determine the students initial reactions to what they deem to be applicable mathematics. This was an opportunity for the students to express unequivocally their thoughts on applicable mathematics without any external influence such as a list of topics or subjects. It was intended that the students would express the first ideas that came into their minds immediately after reading the question. In question 3 the students were asked to tick the mathematical topics (from a list of 18) according to what they considered applicable to real-life contexts (see Appendix J). The students could tick more than one answer. Question 4 took the same format as Question 3, where the students were asked to tick the mathematical topics (from the same list of 18 topics as the previous question) according to what they considered least applicable to real-life contexts.

Student Profile

The questionnaire was administered to a purposive sample, and was designed to examine the attitudes of students in Senior-Cycle mathematics who are preparing for third level education. Therefore, the cohorts selected to take part in this study were two classes of Leaving Certificate mathematics students. In total 35 students (n=35) took part in the intervention. 82.9% of the sample were females (19 out of 35 students), while the breakdown of ages was as follows: 15 years = 8.6%, 16 years = 51.4% and 17 years = 40%.
Results

Students’ initial response to what mathematical topics they consider to be applicable

The students were first asked to write down their initial feelings regarding what mathematical topics they would consider applicable to real life contexts (see Appendix J). Their spontaneous reaction to the question was recorded. Arithmetic was referred to 17 times, Statistics 15 times and Trigonometry 10 times. Within Arithmetic, Basic Operations (addition, subtraction, division and multiplication) were referred to in some form by 9 different students. In total there were 53 references, with 7 different mathematical topics referred to. The data reveals that students have a limited opinion of the mathematics topics they consider to be applicable, where the students limit their responses to just seven topics. Perhaps this is due to the fact that these topics tend to be both taught and examined through the medium of applications traditionally. What is striking is the topics without reference, such as algebra, complex numbers and vectors, are of a more abstract and theoretical nature and as such are not considered applicable by the students in their own initial reaction.

Changes in students views when the mathematical topics are listed

Given that the students were then asked to select from a list of 18 mathematical topics (see Appendix J) according to what they considered applicable to real-life contexts, the data showed that when the mathematical topics are listed Statistics is the mathematical topic in the Leaving Certificate syllabus considered to be most applicable to real-life contexts. Table 5.3 shows the references to applicable and
least applicable mathematical topics according to the participating students. There were a total of 178 responses for ‘applicable mathematics’ and 170 for ‘least applicable mathematics’ because the students were allowed to tick more than one topic. The high number of references to Statistics is perhaps due to the fact (or at least partly) that this was a recent topic covered in their mathematics lessons. Topics such as Differentiation, Integration, Functions, and Linear and Quadratic equations, had a combined total of 13 references (half the number of references to Arithmetic alone). Considering the wealth of applications these topics encompass it seems remarkable that these topics would show such a low number of references.

When the students were also asked to tick from the same list of 18 mathematical topics according to what they considered least applicable to real-life contexts, Logarithms scored one of the highest with 15 references. Linear Equations had most references with 21 separate references, followed closely by Quadratic Equations with 20 references. It is notable that Algebra has 16 references from students when asked about what they consider least applicable to real-life contexts. However in the previous question (topics the students considered applicable mathematics), Algebra had 14 references. Furthermore, one student of the 21 did not answer this question. However, the same student ticked each mathematical topic when asked what mathematical topics they considered to be most applicable. This is perhaps due to the student considering all mathematics to be applicable.

It appears from the outset that Irish Senior-Cycle students consider the most obvious mathematical topics applicable to real-life contexts i.e. Arithmetic, Geometry, Statistics and Trigonometry when provided with a list of the topics covered at Leaving Certificate level. Although the students have been students of mathemat-
ics for up to five years they have not realised the broad applicability of many of the topics covered in Senior-Cycle mathematics. Mathematics is generally taught as a strictly logical discipline consisting of definitions, theorems and proofs.

Table 5.3: Applicable and least applicable mathematical topics - by students’ opinion (The Exploratory Phase)

<table>
<thead>
<tr>
<th>Topic</th>
<th>Applicable</th>
<th>Least Applicable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic</td>
<td>26</td>
<td>1</td>
</tr>
<tr>
<td>Algebra</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td>Linear equations</td>
<td>3</td>
<td>21</td>
</tr>
<tr>
<td>Quadratic equations</td>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>Inequalities</td>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>Logarithms</td>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>Complex numbers</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>Matrices *</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Geometry</td>
<td>24</td>
<td>8</td>
</tr>
<tr>
<td>Vectors *</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>Trigonometry</td>
<td>17</td>
<td>13</td>
</tr>
<tr>
<td>Sequences and Series*</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>Functions</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>Curve Sketching</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>Differentiation</td>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>Integration *</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Statistics</td>
<td>29</td>
<td>1</td>
</tr>
<tr>
<td>Probability</td>
<td>24</td>
<td>6</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>178</strong></td>
<td><strong>170</strong></td>
</tr>
</tbody>
</table>

* These denote topics which the students have yet to have studied extensively.

**Students views on what other subject areas they consider mathematics useful for**

In total there were 81 different responses, with Table 5.4 showing the breakdown of these responses. One student failed to provide at least one response to the question. In total 9 different subject areas were mentioned, with the Sciences being the most referred to with 29 references. Notably, a breakdown of Science
responses shows Chemistry had 9 distinct references, Science *per se* also with 9 references. Physics had 5 references and Biology had 3 references. Also, Business Studies had a breakdown of responses, where Accounting had 11 responses and Business Studies *per se* had 10 responses. Economics had a total of 4 responses.

It is encouraging to see quite a number of different references at this stage, particularly in more humanistic subjects such as art and music. However, the depth of understanding that the students have is not catered for in this question and as of such would need further investigation. While, the students clearly see that Science, and Business Studies have a link with mathematics, to what extent they see the applicability of mathematics in these subjects is unproven. It would be interesting to see whether students can distinguish beyond the basic operations of mathematics and its role in both these subjects.

Table 5.4: Most applicable subjects- by students’ opinion (The Exploratory Phase)

<table>
<thead>
<tr>
<th>Rank</th>
<th>Subject</th>
<th>Number of References</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>The Sciences</td>
<td>29</td>
</tr>
<tr>
<td>2</td>
<td>Business Studies</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>Design and Communications Graphics</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>Geography</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>Construction Studies</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>Computers</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>Engineering</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>Art</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>Music</td>
<td>1</td>
</tr>
</tbody>
</table>

**Findings Influence on the Implementation Phase**

In terms of the author’s larger research study these findings provided a basis for assessing the appropriateness of the theoretical analysis and instructional design of
the Exploratory Phase. The findings can then ensure that the necessary modifications are made where necessary for the Implementation Phase (main phase) of the research study. Most importantly, the findings highlighted the need for providing a greater variation in applications to ensure the students’ personal backgrounds and interests could be catered for. The perceived irrelevance of current mathematical activities has fuelled the debate for the inclusion of appropriate applications and modelling (Burkhardt, 2006). Numerous examples of applications are available, yet few really connect with students’ perceptions of realistic applications. In fact it will add to the complexity and misunderstanding of the influence of mathematics in tackling situations and problems from the world around us (Niss, 1987). It is imperative to ensure that appropriate applications are provided, as access to excessively specialised problems would result in over-complication of material and marginalised understanding by the student. There were a number of issues that needed addressing after completion and analysis of the Exploratory Phase. These issues included:

- Theoretical Analysis from the Exploratory Phase
- Re-design of applications used
- Teacher and Student Manual
- Layout and length of the Teacher Reflective Journal
- Appropriateness of the Student Questionnaire
- Provision of Teacher Guidelines

Each of these issues will be discussed at length in Chapter 6, where steps taken to address how these issues were addressed. Decisions made at the Implementation
Phase were made based on the formal and informal analysis of the Exploratory Phase.

5.8 Conclusion

This chapter began by proposing an adaption of APOS Theory for the learning needs of Senior-Cycle students learning mathematics through applications. The rationale for such an approach was provided, while at the same time the rationale for the author’s approach to teaching applications was provided. The mathematical content of the intervention was discussed, where the topic ‘functions’ was chosen as the main focus. The Exploratory Phase of the research project has been described, highlighting key areas where an adaption of the theory has taken place, and subsequent knowledge and experience gained during this phase of the research. An extensive theoretical analysis was proposed for the students’ learning of the concept of ‘functions’. Following that, the instructional treatment proposed for the teaching intervention was described, ensuring the ACE teaching cycle was catered for. Finally, suitable methods of observation and analysis were discussed; allowing the author to highlight the findings influence on the larger research study, thus ensuring appropriate action can be undertaken at the Implementation Phase. The Implementation Phase of the research study is described in the next chapter. Decisions made at the Implementation Phase were made based on the formal and informal analysis of the Exploratory Phase, particularly the theoretical analysis and instructional treatment, and will be subsequently discussed in Chapter 6.
Chapter 6

Teaching Applications of Mathematics: The Implementation Phase

6.1 Introduction

In Chapter 5 the author described the Exploratory Phase of the teaching intervention. An initial theoretical analysis of ‘functions’ was developed after an extensive review of the literature. This was supported by the experience of the author as a secondary school mathematics teacher and his own experience as a student learning advanced mathematics. An instructional treatment was consequently designed, consistent with APOS Theory while suitable methods of observation and analysis were undertaken. Firstly, the key issues highlighted from the Exploratory Phase will be discussed and the measures undertaken to address them will be identified. Throughout the remaining sections of this chapter, the author will describe
the revised theoretical analysis, instructional treatments and methods of observation and assessment. The author believes that this adaptation of the theoretical analysis combined with an instructional treatment developed around the use of applications in the teaching and learning of Senior-Cycle mathematics can lead to suitable pedagogical approaches that will support teachers in teaching applications. The author decided to concentrate exclusively on ‘linear functions’ when applying the instructional treatment to the teaching intervention, so as to ensure appropriate understanding is garnered at such an early stage of the student’s learning of the entire ‘function’ concept.

6.2 Issues Arising from the Exploratory Phase

After completion and analysis of the Exploratory Phase there were a number of issues that needed to be addressed. These issues included:

- Theoretical Analysis from the Exploratory Phase
- Re-design of Applications used
- Length and layout of Teacher and Student Manuals
- Appropriateness of the Teacher Reflective Journal
- Focus of the Student Questionnaire
- Provision of Teacher Guidelines
Theoretical Analysis from the Exploratory Phase

In order to remain consistent with APOS Theory, the Theoretical Analysis from the Exploratory Phase needed to be addressed before the Instructional Treatment or Analysis of the Implementation Phase could be implemented. After careful examination and analysis of the Exploratory Phase it was decided to alter the Theoretical Analysis slightly. The author decided to concentrate exclusively on ‘linear functions’ when applying the instructional treatment to the teaching intervention, so as to ensure appropriate understanding is garnered at an early stage of the student’s learning of the entire ‘function’ concept. This was based on the author’s judgement that solid learning trajectories proceed from linear to quadratic to higher functions. This in turn had an effect on the design of the theoretical analysis at the Implementation Stage. Each of the four levels are discussed at length in Section 6.3 of this chapter.

Re-design of Applications Used

The author felt the applications used at the Exploratory Phase, needed to be re-designed so as to provide the students with better opportunities to understand the concepts taught. As can be seen from the following example, it shows the use of an application involving ‘mobile phones pay-plan’ at the Exploratory Phase. It was initially designed to show how non-linear functions can be found in numerous aspects of our real lives.
A function is linear if its slope, or rate of change, is the same at every point. In particular, linear functions are used to model situations that show a constant rate of change between 2 variables.

In the example below a mobile phone bill tariff is examined where there is a fixed charge of 50 Euro for 300 minutes. By examining the graph we can deduce that for each 100 minutes used thereafter there is a 12.60 Euro charge.

The current assessment process widely recognised in many countries, which is an end-of-term examination in Ireland, merely requires regurgitation of standard techniques applied to standard examples (Burton, 1997; Clatworthy, 1989; Bajpai et al., 1975). While most students obtain good marks in examination questions, it became apparent that understanding and interpreting solutions was by and large non-evident in student’s work. Thus, the author redesigned this application to focus on the manner of questioning so the participating teachers could ensure the students interpreted, justified and validated answers, and did not just accept them without consideration of the application at hand. Often the students are asked to explain their answers in words or graphs, an approach not traditionally employed in Senior-Cycle mathematics in Ireland. This can be seen in the following example.
In the example below the graph represents a particular mobile phone bill tariff.

a) Identify the independent and dependent variables.
b) By examining the graph what is the charge for each 100 minutes used thereafter?
c) What is the charge for using 600 minutes in any given month?
d) Describe the function in words.

As has been stressed by many researchers (Burkhardt, 2006; Niss 1987; Bajpai et al., 1976) success in teaching applications depends on the student’s grasp of the applications used in the classroom. Thus, the applications are culturally de-
pendent and specific to the environment in which they are taught and the personal backgrounds/interests of the students involved. The author felt that as a result of narrowing the focus of the intervention to include only ‘linear functions’, there was a need to provide a greater variety of applications. This was to ensure the student and teacher could appreciate the range of applications available for each concept involved. While, the availability of a greater variety of applications also ensured the students’ personal backgrounds and interests could be catered for.

In addition, it was felt that it was useful to get the students to draw a diagram/sketch a graph and identify, if possible, the given and required quantities on the diagram. This allowed the learner to visualise the problem. Bajpai et al. (1975), Hughes-Hallett (1991), and West (1995) acknowledge that students who learn mathematics through the use of visual techniques have shown that, in comparison with traditional courses, they understand the basic concepts better, can remember the information longer and can apply the concepts to practical uses more effectively.

**Length and Layout of Teacher and Student Manuals**

It was decided that the student manual (so as to remain consistent with the pedagogical approaches) would be divided into five distinct lessons focusing on Linear Functions. Each of the lessons would have exclusive objectives and the layout is described in detail in Section 6.4. Furthermore, each lesson would have its own distinct exercises session, rather than a generic set of exercises at the back of the student manual. Because of the layout of the manual it was appropriate to have exercises for each distinct lesson, as opposed to having a revision section at the end of the manual. This was to allow the teacher to easily make connections between the lesson and the exercises provided. Solutions to all the exercises were
provided so as to limit the amount of preparation work the participating teachers needed to do. Also, this is consistent with the most popular mathematical textbooks (O’Keeffe, 2009). Furthermore, there are few examples in the text that are exactly like the homework problems. This ensures the students cannot just look at a homework problem and search for a similar worked-out example. Success with the homework will come by grappling with the ideas presented.

The author decided to provide a PowerPoint presentation for each lesson to supplement the teacher manual and also so as to ensure the GeoGebra applets were included seamlessly into each lesson. Each lesson in its entirety was provided on PowerPoint and a hyperlink was created when the applet was to be used. Each applet was also provided separately on a disc so as to ensure that unexpected problems with the internet could be catered for.

Snir (1995) argues that computers can make a unique contribution to the clarification and correction of commonly held misconceptions of phenomena by visualising those ideas. Instead of spending considerable time learning routines by hand, West (1995) argues that the computer can alleviate such time constraints, allowing the students to move more swiftly to higher-level conceptual matters and a variety of practice problems.

GeoGebra applets were employed to introduce the concept of the linear functions so as to ensure the students could understand and appreciate the graphing qualities they possess. The applets provided a visual stimulus through which the students could observe the dynamic capabilities of the applets. Furthermore, the applets ensured the teacher could provide this stimulus quite effortlessly through the movement of a mouse as opposed to a time-consuming demonstration on the board. An influencing factor in the exclusive use of GeoGebra was not only the
extent of degree of user-friendliness but the fact that the software is completely free to download.

**Appropriateness of the Teacher Reflective Journal**

The comments provided in the reflective journals for the Exploratory Phase were limited for the purpose of data analysis for this research study. The teacher reflective journal had a section on lesson aims and outcomes. This section was removed as the lesson aims and outcomes were pre-decided for each lesson. The remaining sections of the teacher journal were retained. Furthermore, the participating teachers completed the journals, to varying degrees of success, where the responses were often descriptive in their nature as opposed to reflective. Schoenfeld (2004) highlighted the issue of the gap between research objectives and actual practise in classroom settings and the fact that the researcher does not always get everything he wishes despite his best efforts. Thus, the length of the journal was shortened from three pages to one to ensure that the teacher was focussed in their reflection (Appendix I).

**Focus of the Student Questionnaire**

The Student questionnaire was changed from two perspectives. Seven questions were put to the student to assess their views on the intervention. A five point Likert scale was used so as strong opinions could be recorded. The mathematical topics that the students had to tick to assess their views on applicable and least applicable mathematical topics were also altered. The topics chosen were Junior Certificate (lower secondary school) centred, where previously the topics were Leaving Certificate (upper secondary school) centred. This was because it was felt
that not all of the topics might have been studied at the time of implementation and all the students had completed their Junior Certificate the previous summer. Thus, for the Implementation Phase, Section A of the student questionnaire was used to examine the views of students in Senior-Cycle mathematics who participated in the intervention regarding the appropriateness and success of the intervention, while Section B was aimed at discovering which mathematical topics the students considered applicable to real-life contexts. This will be discussed in more detail later in Chapter 7.

**Provision of Teacher Guidelines**

Teacher Guidelines were provided so as to ensure the participating teachers were aware of the key learning outcomes intended for the intervention for both the participating teacher and students. Furthermore, it provided an opportunity for the teacher to view the layout of each lesson, together with written instructions on how to use the GeoGebra applet and a brief introduction to Mathematical Modelling. The guidelines were designed in order to provide written support for the participating teachers (Appendix Q).

### 6.3 Revised Theoretical Analysis of Functions

The revised theoretical analysis of functions is consistent with the original version in the Exploratory Phase. The main difference is that each level is slightly altered based on the analysis of the instructional treatment of the Exploratory Phase. Furthermore, each concept begins with the introduction to applications to allow the learner interiorize the action stage into a process for each mathematical concept
at that level. The author will refer to this as “Applications”. This is the collection of applications that contains the genetic decomposition of the mathematics behind each action step. The applications were developed to match the age range and mathematical ability of the participating students. It is intended that the introduction to applications at such an early stage in the learning process will provide opportunities for the acquisition of one of the key objectives of the Leaving Certificate Mathematics syllabus – that of ‘relational understanding’. It approaches the teaching and learning of mathematics from an alternative direction than is traditionally practised in mathematics classrooms throughout the world. Rather than introducing a new mathematical concept, learning the techniques and skills, and then applying these skills to real life-problem, this approach begins with applications and uses these as a basis for the learner to make connections between real-life situations and mathematics. It allows the learners to see and appreciate the relevance of mathematics to their everyday lives, while ensuring one of the key objectives of the Leaving Certificate Mathematics Syllabus is catered for. As a result, it is intended that the learner will then be better equipped to advance their mathematical knowledge of functions.

Again a colour scheme has been adopted so that the reader can distinguish between actions, processes, and objects. Actions are described using the colour grey. Processes appear in navy, while objects appear in green. Applications are described using a blue colour. At certain levels, for the purpose of this intervention, the path undertaken by the learner follows the red arrows. This will be explained further at the necessary level. Each diagram is read from the bottom up. In this revised theoretical analysis four levels were identified and each is described in the following pages.
6.3.1 Genetic Decomposition Level I

This level contains a basic function representation. Figure 6.1 illustrates the applications, actions and processes that need to be encapsulated to form a basic function representation object. The basic function representation is once again the keystone of the theoretical analysis. There are, however, a number of changes to the original genetic decomposition. The idea of a function is retained. This contains no formal mathematics and is included only as introductory material. As is consistent with each stage of the theoretical analysis each concept, rule or definition is introduced through the medium of applications. The students are first introduced to writing functions. This focuses on the concept of function having dependent and independent variables and the general written form of a function: \( y = f(x) \). The students then go on to learn about the different representations of a function – namely algebraic, graphical and numerical. Research has identified the benefits of introducing students to the different representations of a mathematical concept (Dick & Edwards, 2008; Hughes-Hallett, 1991; Janvier, 1987; Bajpai et al., 1975). Traditionally, the algebraic approach has dominated the teaching and learning of mathematics at Senior-Cycle level in Ireland. Throughout the teaching intervention all three approaches are utilised and explored where possible.

To form a complete and thorough understanding of a basic function representation, the students are then provided with a general definition of a function. Once all three concepts at the action stage can be interiorized by the students into their relevant processes, it allows the student to encapsulate the basic function representation and he can now move up to the next level where the basic function representation will appear as an action/object. As can be seen from Figure 6.1, the sequence in which the learner is introduced to these concepts and rules is de-
Figure 6.1: Basic Function Representation Object
void of standardisation i.e. it is possible for the learner to form a basic function representation understanding through completion of only one (and indeed two) of these processes. However, the author feels that in order to form a complete and thorough understanding of a basic function representation the learner should be introduced to all three concepts.

6.3.2 Genetic Decomposition Level II

Level II contains a “function properties object”. Figure 6.2 describes the genetic decomposition of the function object.

Function Properties Object

This object allows the student to accommodate all the basic properties of a function before moving onto operations involving linear functions. The idea of elementary functions is explored by the student, where the teacher introduces the students to each type of elementary function. As mentioned earlier it will be specifically the route indicated by the red arrows (algebraic functions) which will be undertaken for the purpose of this intervention. However, the teacher will inform the students of the different types of elementary functions (algebraic, trigonometric, exponential and logarithmic) and provide a visual stimulus through the use of GeoGebra applets. Without an introduction to the different types of functions, and the necessary interiorization into an elementary functions process, the learner would have to deal with a cognitive conflict regarding the evaluation of functions. A student who has not already encapsulated the required elementary process stage will not pass beyond the evaluating function action stage. This problem will then
Figure 6.2: Function Properties Object
escalate as the student attempts to move to higher levels. As already stated the focus of this intervention centres entirely on algebraic functions. However, there is a further breakdown of algebraic functions which is depicted in Figure 6.3. So as to narrow the focus of the intervention and keep it manageable in terms of what can be expected from students and teachers in actual school settings during a trial, it was decided by the author to focus solely on linear functions. As a result the subsequent levels of genetic decomposition are concerned with linear functions.

6.3.3 Genetic Decomposition Level III

Level III contains a “linear function properties object”. Figure 6.4 describes the genetic decomposition of the linear function object.

Linear Function Properties

This object allows the student to accommodate all the basic properties of a linear function before moving onto operations involving non-linear functions. Each process at this level is preceded by an action of the same name. A student who has not already encapsulated the prior levels of genetic decomposition will find it difficult to pass beyond the action stage. Each action is governed by a particular assumption and the student needs to progress from applying this assumption to any given linear function, to a more general application. As indicated in Figure 6.4 the student begins with the introduction of the concepts of slope and y-intercept through an application to real-life. GeoGebra applets were employed to introduce the concept of the linear functions so as to ensure the students could understand and appreciate the graphing qualities they possess. The applets provided a visual
Figure 6.3: Elementary Functions Object
Figure 6.4: Linear Function Properties Object
stimulus through which the students could observe the effects that altering the slope and y-intercept would have on the graph.

At this stage of the learning process, it is vital that the appropriate connections between both the mathematical and applicable representation of the concept are presented. Thus, the learner is encouraged to make connections between what has just been learned through self and guided discovery and a general definition for a given mathematical concept. The process is two-dimensional in that the student, in order to fully understand the concepts, should be able make connections between both the mathematical and applicable representation of the concept. Interaction between all processes is then necessary to achieve encapsulation of the complete object. Once more this object will then appear as an action/object at the next level.

6.3.4 Genetic Decomposition Level IV

This is the most advanced level and when a learner has encapsulated this object, he or she is said to have a function schema. Figure 6.5 depicts this level.

When a learner reaches this level this means that they have encapsulated all the knowledge from previous levels and can freely call on this knowledge to solve a wide variety of problems. Essential to the overall function schema is that the process is two-dimensional in that the student, in order to fully understand the concepts, should be able to make connections between both the mathematical and applicable representation of the concept.
Figure 6.5: Function Schema

- Function Schema
- becomes the totality of knowledge that a learner has for the concept of a
- Basic Function Representation Object
- are the medium, through which the following concepts are introduced
- Function Object
- Elementary Function Object
- Applications
- The Idea of a Function
6.3.5 Summary

Using an approach consistent with the APOS Theory methodology the author has revised and extended the original theoretical analysis from the Exploratory Phase. Once again, due to the restricted time commitments of the participating schools the entire function schema was not observed, where the route followed was indicated by the red arrows. The primary focus of the intervention was on linear functions and the theoretical analysis above reflected this. Again, the theoretical analysis was developed by the author prior to the implementation of the intervention in the participating schools. The original idea of the genetic decomposition was an attempt to predict how the concept might be constructed in the mind of the learner (Dubinsky et al., 1996). The author has demonstrated how a genetic decomposition can be used not only to predict how the mathematical constructs of the learner develops, the author has taken the process further and used the genetic decomposition as a tool for highlighting the role of applications in the mathematical knowledge of the learners. Following this, the instructional treatment was designed to correspond with the theoretical analysis providing sufficient material and exercises for the participating teachers and students. The instructional treatment employed in the Implementation Phase will be discussed in the following section.
6.4 Instructional Treatment for the Implementation Phase

Potential gaps in the mathematical knowledge of the learners were highlighted in the genetic decomposition in the Exploratory Phase. In addition, there was significant knowledge gained through analysis and careful examination of the implementation of the initial instructional treatments at the Exploratory Phase. As a result the approach taken for the instructional treatments was a series of seven lessons that would highlight the use of applications in the teaching and learning of Linear Functions.

6.4.1 Key Aspects of the Teaching Intervention

The key aspects of this teaching intervention include:

- The intervention aims to show the students the different representations of a function (i.e. algebraic, graphical and numerical)
- All concepts, rules and definitions are introduced through the medium of applications.
- The applications are deemed to be appropriate for the age range of the students.
- The mathematical content is based on prior mathematical experiences of the students and the expected current level of Senior-Cycle students in Ireland.
- GeoGebra applets are provided to allow the teacher to demonstrate the visual aspect of a particular concept.
• A PowerPoint presentation is provided for the teacher so as to enable the teachers to incorporate the use of ICT into the lessons. In particular it allows the teacher to provide clear and concise diagrams easily.

• The modelling session is provided to allow the students to readily see the usefulness and importance of mathematics, while affording them an opportunity to work on a given real-life problem that utilises mathematics to solve the problem.

6.4.2 Teacher Outcomes

After participation in this intervention, it is intended that the teacher will achieve the following pedagogical outcomes:

• An enhanced understanding of the use of mathematics in other subject areas and in the real world.

• An extended range of teaching methodologies employed in the teaching of functions to Senior-Cycle level, which includes both the exposure to applications and the different representations of a function (i.e. algebraic, graphical and numerical)

• An extended range of ICT skills through the use of PowerPoint presentations and GeoGebra applets.

• An opportunity to prepare for the introduction of ‘Project Maths’, the new nationwide initiative in Irish secondary schools.
6.4.3 Student Outcomes

After participation in this intervention, it is intended that the student will achieve the following learning outcomes:

- An enhanced understanding of the use of mathematics in other subject areas and in the real world.
- An enhanced understanding of the different representations of functions (i.e. algebraic, graphical and numerical)
- An exposure to solving real-world applications and problems which use mathematics.
- An opportunity to appreciate the role of mathematical modelling and, in particular, writing reports.
- An opportunity to appreciate the role of mathematics in one's future education and/or work-life.

It was decided that the intervention manual (so as to remain consistent with the pedagogical approaches) would be divided into five distinct lessons focussing on Linear Functions. There were also to be two lessons subsequently focusing on mathematical modelling, including an opportunity for the students to engage in a mathematical modelling session in small groups. Each of the lessons have particular objectives and the layout is described in detail in the next subsection. Furthermore, it was decided that each lesson would have its own distinct exercises session, rather than a generic set of exercise at the back of the text-book. This approach is consistent with the criteria set out in the theoretical analysis for accommodation and assimilation of material. From a practical perspective, each
lesson as it is described below, takes up two 35/40 minutes periods in a classroom setting. Similar to the Exploratory Phase the instructional treatment was developed prior to the intervention and presented to the participating teachers so as to enable the teachers to become familiar with the material. To reduce the risk of error and also to save time a set of printed notes was given as a form of mini-textbook to each student (Appendix P). A teacher copy was also provided which had additional notes on mathematical modelling, while also including the required solutions for the real-life problems (Appendix R).

6.4.4 Outline of Lessons

The ACE Teaching Cycle is adhered to throughout this intervention and is visible in the design and layout of the lessons. Lessons 1-5 explore the Classroom Discussion of linear functions, while lessons 6 and 7 are based around modelling Activities. The Exercises supplement the Classroom Discussion as traditional homework exercises. An outline of each of the seven lessons is described below. A short description of the lesson is provided, while each of the applications and the stages when GeoGebra applets are used in the lesson is listed. So as to provide the reader with an insight into the applications used in the intervention an outline of each lesson is provided accompanied by the first page of each lesson. The complete lessons can be viewed in the teacher and student manuals (Appendix R and P respectively).
Lesson 1

This lesson is designed to teach the basic representations of a function, while providing the learner with a general definition of a function. The lesson begins by introducing the learner to writing functions, while identifying the general method of writing functions i.e. $y = f(x)$. The learner is then shown the different representations of a function (i.e. algebraic, graphical and numerical). This is one of the key aims of the intervention. Finally, the learner is provided with a general definition of a function and afforded opportunities to write a definition in their own words. Each concept covered in this lesson is introduced through application and these applications include:

1. Writing Functions Example: Hot Air Balloon/Cruise Boat

2. Representing Functions Example: Hot Air Balloon

3. Definition of a Function Example: Wages
Functions – Lesson 1

1. Writing Functions

The modern notation for a function is derived from the efforts of many 17th and 18th Century mathematicians, and in particular the work of Gottfried Wilhelm Leibniz and Leonhard Euler.

**Function Notation**

The word function was first used by Gottfried Wilhelm Leibniz in 1694 as a term to denote any quantity connected with a curve, such as the coordinates of a point on a curve or the slope of a curve. Forty years later, Leonhard Euler used the word function to describe any expression made up of a variable and some constants. He introduced the notation

\[ y = f(x) \]

The input variable (x) is called the *independent variable*, and the output variable (y or f(x)) is called the *dependent variable* as it depends on whatever value x holds.

**Example: Hot Air Balloon / Cruise Boat**

For each situation, express the given function in your own words. Write two variables as an ordered pair and write the function with the notation \( y = f(x) \).

1. You are riding in a hot air balloon. As the balloon rises, the surrounding atmospheric pressure decreases (causing your ears to pop!).
Lesson 2

This lesson is designed to introduce the learner to Elementary functions, while providing a visual stimulus in the form of GeoGebra applets. It is not intended that the learner is able to comprehend each type of function at this stage in the learning process, but they should be able to identify the different types of polynomial functions both algebraically and graphically. The learner is then introduced to evaluating functions. Both applications and standard mathematical examples are used at this stage of the lesson. The applications used in this lesson include:

1. Elementary Functions - GeoGebra applets

2. Evaluating Functions:
   - Example 1: Temperature Scales
   - Example 2: Preventing a Car Crash
   - Example 3: Pelican Eggs
1. Elementary Functions

By the end of the 18th Century, mathematicians and scientists had concluded that many real-world phenomena could be represented by mathematical models taken from a collection of functions called elementary functions.

Elementary functions fall into three categories:
1. Algebraic functions (polynomial)
2. Trigonometric functions (sine, cosine, tangent etc.)
3. Exponential and logarithmic functions (e.g. Log10, e^{-x})

For polynomial functions of low degree the following simpler forms are often used:

- **Zeroth Degree:** \( f(x) = a \)  
  Constant function
- **First Degree:** \( f(x) = ax + b \)  
  Linear Function
- **Second Degree:** \( f(x) = ax^2 + bx + c \)  
  Quadratic Function
- **Third Degree:** \( f(x) = ax^3 + bx^2 + cx + d \)  
  Cubic Function

By definition, trigonometric, exponential and logarithmic functions are not polynomial functions.

The general rule of a polynomial function states:

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \quad a_n \neq 0
\]

where the positive integer \( n \) is the degree of the polynomial. The numbers \( a_i \) are coefficients, with \( a_n \) the leading coefficient and \( a_0 \) the constant term of the polynomial function.
Lesson 3

In this lesson the learner is introduced to linear functions. Two applications looking at Olympic Events allow the learner to form their own understanding and mathematical knowledge of the idea of a linear function and the mathematical concepts involved with linear functions i.e. slope, increasing/decreasing functions, and intercepts. Building on the initial introduction to a linear function, the learner is provided with a definition of a linear function. The applications used in this lesson include:

1. Applications of Linear Functions - Example: Pole Vault / Mile Run

2. Definition of a Linear Function - GeoGebra applet

3. Uses of Linear Functions - Example: Mobile Phone Bill Tariff

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Functions – Lesson 3

1. Applications of Linear Functions

Example: Pole Vault / Mile Run

During the early years of the Olympics, the height of the men’s winning pole vault increased approximately 8 inches every four years. Table 2 shows that the height started at 130 inches in 1900, and increased by the equivalent of 2 inches per year. So the height was a linear function of time from 1900 to 1912. If \( y \) is the winning height and \( t \) is the number of years since 1900, we can write

\[
y = f(t) = 2t + 130
\]

Since \( y = f(t) \) increases with \( t \), we say that \( f \) is an increasing function. The coefficient 2 tells us the rate, in inches per year, at which the height increases.

<table>
<thead>
<tr>
<th>Year</th>
<th>1900</th>
<th>1904</th>
<th>1908</th>
<th>1912</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (inches)</td>
<td>130</td>
<td>138</td>
<td>146</td>
<td>154</td>
</tr>
</tbody>
</table>

Table 2: Men’s Olympic pole vault winning height (approximately)
Lesson 4

To further establish the concepts learned in the previous lesson, this lesson focuses specifically on the intercepts of a linear function. Again a visual stimulus in the form of GeoGebra applets is provided. So as to ensure the learner is aware of the mathematical capacities of x and y intercepts the learner is introduced to finding intercepts not only in linear functions, but also in quadratic and cubic functions.

The applications used in this lesson include:

1. Application of Intercepts of a Graph - Example: Demand Function

2. Intercepts of a Graph

   • GeoGebra applet

   • Example: Linear/Quadratic/Cubic Functions
Functions – Lesson 4

1. Application of Intercepts of a Linear Function

Example: Demand Function

The demand function is given by the equation \( P = 100 - 0.5Q \), where \( Q \) is the number of ipod shuffles demanded daily throughout Ireland; \( P \) is the price per ipod shuffle, in Euros.

Complete the following table which plots the quantity demanded over the range \( 0 \leq Q \leq 200 \):

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Price (in Euros)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>40</td>
<td>80</td>
</tr>
<tr>
<td>80</td>
<td>60</td>
</tr>
<tr>
<td>120</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td></td>
</tr>
</tbody>
</table>

Describe the table in your own words:

________________________________________________________________

________________________________________________________________

Sketch a graph of the function:
Lesson 5

This lesson focuses specifically on the slope concept of a linear function. The lesson begins by examining a real-life application of a rain gauge, thereby allowing the learner to form their own mental constructions of the slope of a linear function. Building on the initial introduction to the slope of a linear function, the learner is provided with a definition of the slope of a linear function using two data points. Again a visual stimulus in the form of GeoGebra applets is provided. Finally, the lesson focuses on functions not written in standard form and the mathematical procedures needed in such cases. The applications used in this lesson include:

1. Application of the Slope of a Linear Function - Example: Rain Gauge

2. Definition of the Slope of a Linear Function

- GeoGebra applet

- Example: Determining slope from a graph
Functions – Lesson 5

1. Application of the Slope of a Linear Function

Example: Rain Gauge

Imagine that we measure the depth of rain accumulating in a rain gauge as a steady rain falls. The rain stops after 6 hours, and we want to describe how the rain depth varied with time during the storm.

What is the dependent variable? ____________________________

What is the independent variable? __________________________

Based on the measurements with the rain gauge, we find the rain depth function as the following graph:
Lesson 6

This lesson focuses specifically on mathematical modelling and is a basic introductory lesson to mathematical modelling within the school setting. The learner is introduced to the benefits of mathematical modelling and report writing. Each of the 6 stages of the modelling process is outlined through examining a sample report provided (See Appendix O and R respectively). The lesson breakdown is as follows:

1. Outline of mathematical modelling
2. Example of mathematical modelling

Lesson 7

In this lesson the students work in small groups to solve a real-life mathematical problem. It is intended that each group begins the report of the procedures that they undertook in solving the problem. The report is to be provided as part of the assessment procedure for the next lesson.

1. Group modelling problem - O2 Phone Problem

6.4.5 Summary

This approach requires students to do much more than just give answers (as is prevalent in current assessment and evaluation methods worldwide), instead looking for students to provide reasoning and justification for their answers, thus encouraging student understanding of concepts as opposed to mastery of concepts
devoid of understanding. It approaches the teaching and learning of mathematics from an alternative direction than is traditionally practised in mathematics classrooms throughout the world. Rather than introducing a new mathematical concept, learning the techniques and skills, and then applying these skills to real life-problems, this approach begins with applications and uses these as a basis for the learner to make connections between real-life situations and mathematics. It allows the learners to see and appreciate the relevance of mathematics to their everyday lives, while ensuring one of the key objectives of the Leaving Certificate mathematics Syllabus is catered for – that of ‘relational understanding’.

6.5 Data Collection and Analysis for the Implementation Phase

The participating teachers for the Implementation Phase agreed to record their thoughts in a Reflective Journal for the entire duration of the intervention. The participating students for the Implementation Phase completed a post-intervention questionnaire. It was decided to conduct interviews with both the participating teachers and a sample of the students (n=10) to ensure triangulation within the larger study. The author’s objective was to investigate both the students’ and co-operating teachers’ feelings and views on the intervention and the impact that it would have on the teaching and learning of applications at Senior-Cycle level. While the quantitative study allowed no room for opinions and discussion, the interviews aimed to provide such opportunities. The Reflective Journals rubric and questionnaires were analysed and revised from the Exploratory Phase and, thus, were considered appropriate measures of observation for the Implementation
Phase. The analysis and the results of the Implementation Phase are discussed at length in Chapter 7 in what is known as the Reflective Phase of the research study.

6.6 Conclusion

In this chapter the author has presented a revised theoretical analysis of functions that can now be used to highlight the path a learner may undertake when learning the concept. The author demonstrated how a genetic decomposition can be used not only to predict how the mathematical constructs of the learner develops, but also as a tool for highlighting the role of applications in the mathematical knowledge of the learners. The revised instructional treatments based on the revised theoretical analysis consisted of seven distinct lessons. All of these lessons introduced the concepts to be learned through the medium of applications. The author believes that this adaptation of the theoretical analysis combined with an instructional treatment developed around the use of applications in the teaching and learning of Senior-Cycle mathematics can lead to suitable pedagogical approaches that will support teachers in teaching applications. The analysis and the results of the Implementation Phase are discussed at length in Chapter 7. A proposed systematic approach to the teaching and learning of applications is also described in the next chapter.
Chapter 7

Teaching Applications of Mathematics: The Reflective Phase

7.1 Introduction

The traditional approaches to the teaching and learning of mathematics warrant appropriate action to try and influence the pedagogical practices of mathematics teachers, not only in Irish classrooms but extensively throughout the world. This action is necessary to prevent stagnation in beliefs and attitudes towards mathematics at Senior-Cycle, often found among mathematics teachers and students alike in Irish schools. The decisive action in this research study involves the utilisation of applications of mathematics at Senior-Cycle level to address the problem of declining standards in mathematical competence in this area and problem solving, while ensuring one of the key objectives of the Leaving Certificate, that of relational understanding, is catered for in the mathematical experience of Senior-Cycle students.
The purpose of this chapter is to provide an insight into the teaching intervention employed in this research study and thus to promote a more in-depth knowledge of the teaching and learning of applications in Senior-Cycle mathematics. This chapter ensures the study remains consistent with the theoretical framework employed throughout (APOS Theory) and thus describes the ‘Reflective Phase’ of the research. This Reflective Phase consists of quantitative analysis of the student questionnaire to examine the views of students in Senior-Cycle mathematics who participated in the intervention regarding the appropriateness and success of the intervention, and qualitative analysis to promote a more in-depth knowledge of its impact on both the students and the co-operating teachers who participated in the intervention. The chapter includes a:

- Presentation of quantitative and qualitative findings
- Discussion of findings (quantitative and qualitative)
- Insights into the teaching and learning of applications

The chapter is developed with reference to the author’s objectives and research questions, and discusses the main or emerging themes/issues.

### 7.2 Analysis of Quantitative Data - Student Questionnaire

In this section of the chapter, the author presents and analyses the main findings from the student questionnaire administered at the Implementation Phase of the research study (see Appendix J). The questionnaire was selected as one of the
primary research instruments of the Implementation Phase as it was a required method of observation and assessment outlined by RUMEC with respect to the implementation of APOS Theory. Section A of the student questionnaire was used to examine the views of students in Senior-Cycle mathematics who participated in the intervention regarding the appropriateness and success of the intervention, while Section B was aimed at discovering which mathematical topics the students considered applicable to real-life contexts. The analysis of the student questionnaire proceeds in three stages as follows:

**Stage 1: Descriptive Statistics includes:**

- Analysis – methodology
- Student profiles
- Preliminary analysis of the questions.

**Stage 2: Inferential statistics includes:**

- Relationship between students’ gender and views of the intervention
- Relationship between students’ Junior Certificate Level of study and views of the intervention

**Stage 3: Students’ Perception of Applicable Mathematics.**

**Research questions (quantitative study)**

The following research questions guided this follow-up quantitative study.
Do students see the benefit of studying mathematics? Do they find it valuable in everyday life or future careers?

Do students see the role of mathematics in other subject areas?

What are the students’ perceptions of applicable mathematics?

Can the students identify the role of applications in mathematical topics they have studied at Junior Certificate Level?

More specific research questions were used to elicit information on the intervention itself. Specifically, did the intervention allow the students to:

- Find the mathematics to be more enjoyable?
- Find the mathematics to be more interesting?
- See the relevance of mathematics to their everyday lives?

Data Analysis - Methodology

The questionnaires were analysed using SPSS (Version 16.0 for Windows) software. The data consisted of the coded response of 68 questionnaires returned from the students participating in the Implementation Phase of the teaching intervention. Descriptive statistics revealed the median percentiles for all items.

Students’ profiles

The demographic profile of students is analysed to gain insight into the age distribution, gender dispersion, Junior Certificate grade and Junior Certificate level of the sample.
Students’ background

The questionnaire was administered to a purposive sample, and was designed to examine the views of students in Senior-Cycle mathematics who participated in the intervention regarding the appropriateness and success of the intervention. Therefore, the cohorts selected to take part in this study were two classes of Higher Level Leaving Certificate mathematics students and one class of Ordinary Level Leaving Certificate mathematics. All classes involved were transition year classes due to the nature of the school year and the time constraints placed on both teachers and students studying Senior-Cycle mathematics. In total 68 students took part in the intervention.

Age Distribution

42.6 % of the sample were aged 15 (29 out of 68), while the remainder were aged 16. It is important to realise that all these students are delaying the transition from Junior-Cycle to Senior-Cycle mathematics, by participating in a pre Senior-Cycle year known as Transition Year.

Gender Distribution

Females make up 51.5 % of the sample (35 out of 68 students).

Distribution of Junior Certificate Grades and Levels

72.1% of the respondents undertook their Junior Certificate mathematics examination at Higher Level. The remaining 27.9% undertook Ordinary Level mathematics at Junior Certificate. A cross tabulation was carried out to compare students’
Junior Certificate grade with the level at which they studied mathematics at Junior Certificate. The most frequent grade attained was a Grade B at both Higher Level (29.4%) and Ordinary Level (13.2%). 5.9% of Higher Level students recorded an A result in comparison to 4.4% at Ordinary Level. Figure 7.1 shows the graphical representation of Junior Certificate grades and levels for the entire sample.

| HA = Higher Level A Grade | OA = Ordinary Level A Grade |
| HB = Higher Level B Grade | OB = Ordinary Level B Grade |
| HC = Higher Level C Grade | OC = Ordinary Level C Grade |
| HD = Higher Level D Grade | OD = Ordinary Level D Grade |

Figure 7.1: Students’ Junior Certificate mathematics grade and level.
7.2.1 Preliminary Analysis of Section A of the Questionnaire

Section A of the questionnaire consists of 7 statements assessing students’ views and reactions to the intervention. The median and number of respondents for each item are presented in Table 7.1. As mentioned earlier, respondents were asked to indicate their level of agreement or disagreement with each item:

1 = Strongly Disagree,
2 = Disagree,
3 = Undecided,
4 = Agree,
5 = Strongly Agree.

Therefore, a low score was assigned to negative responses and a high score to positive responses. Negatively worded items included: Items 2, 4 and 6 and are scored in reverse.
Table 7.1: The median and number of respondents for each item of the questionnaire

<table>
<thead>
<tr>
<th>No.</th>
<th>Item</th>
<th>N</th>
<th>Median of scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>I believe studying mathematics helps me with problem solving in other areas/subjects</td>
<td>68</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>I do not like writing reports for mathematics activities e.g. a report on a problem.</td>
<td>68</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>Mathematics is more interesting when taught through applications to real-life situations.</td>
<td>68</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>Mathematics is not important in everyday life</td>
<td>68</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>Working in pairs or groups makes mathematics class more enjoyable.</td>
<td>68</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>I do not understand why we need to study mathematics in school.</td>
<td>68</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>I believe studying mathematics will help me when I go to college or when I am working.</td>
<td>68</td>
<td>4</td>
</tr>
</tbody>
</table>

Encouragingly, the results show that the students found that mathematics was more interesting when taught through applications to real-life (Item 3), with 39 (57.3%) of the students (out of 68) providing a positive response to this item. 58 (85.3%) of the students agreed or strongly agreed that working in groups or pairs makes mathematics class more enjoyable (Item 5), highlighting the role that this approach can play in creating positive attitudes to mathematics and mathematics class. Interestingly, 41 (60.3%) of the students agreed or strongly agreed that they did not like writing reports for a mathematics problem (Item 2), while 45 (66.2%) believe that studying mathematics helps with problem solving in other areas/subjects (Item 1). When questioned about their views on whether mathematics is important in everyday life (Item 4), 54 (79.4%) disagreed with a negatively worded statement, thus, the students were displaying a strong sense of the importance of mathematics in their everyday lives. Regarding Item 6, 65 (95.6%) of
the students disagreed or strongly disagreed with a negatively worded statement, assessing the students’ understanding of why they need to study mathematics in school. Item 7 explored the students’ opinions on their beliefs of the importance of mathematics when in work or college, where 58 (85.3%) of the students agreed or strongly agreed that studying mathematics will help them when in college or in work.

7.2.2 Relationship between Gender and Questionnaire items

The author wished to examine if there was a correlation between gender and the students’ views and reactions to the intervention. A cross tabulation was carried out to compare each item of the questionnaire and students’ gender. A statistical association was found in one of the items and is discussed below. A Chi-Squared test was used to measure the level of statistical association for each item.

Students’ views on not liking writing reports for a mathematical problem: trends by gender

Upon analysis of the relationship between gender and the negatively worded statement assessing students’ views on writing reports for a mathematical problem a statistical association was present \( (p < 0.05) \), with 25.7% of females strongly agreeing with the statement, as opposed to 3% of males. Table 7.2 shows the breakdown of each response based on gender. Furthermore, 42.4% of males are undecided as opposed to 14.3% of females.
Table 7.2: Breakdown of students’ views on writing reports for a mathematical problem by gender

<table>
<thead>
<tr>
<th></th>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Male</strong></td>
<td>3</td>
<td>1</td>
<td>14</td>
<td>14</td>
<td>1</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>9.1%</td>
<td>3.0%</td>
<td>42.4%</td>
<td>42.4%</td>
<td>3.0%</td>
<td>100%</td>
</tr>
<tr>
<td><strong>Female</strong></td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>17</td>
<td>9</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>2.9%</td>
<td>8.6%</td>
<td>14.3%</td>
<td>48.6%</td>
<td>25.7%</td>
<td>100%</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>4</td>
<td>4</td>
<td>19</td>
<td>31</td>
<td>10</td>
<td>68</td>
</tr>
<tr>
<td></td>
<td>5.9%</td>
<td>5.9%</td>
<td>27.9%</td>
<td>45.6%</td>
<td>14.7%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Mathematics is more interesting when taught through applications to real-life situations: trends by gender.

Upon analysis of the relationship between gender and students’ views on mathematics being more interesting when taught through applications to real-life situations, 37.1% of females disagree with the statement as opposed to 9.1% of males. Table 7.3 shows the breakdown of each response based on gender. Although the negative response rates differ considerably by gender there is no statistically significant association between the items explored. Significantly, the number of males (69.7%) who agree or strongly agree with this statement is somewhat larger than the responses of females (45.7%).
Table 7.3: Breakdown of students views on mathematics being more interesting when taught through applications to real-life situations by gender

<table>
<thead>
<tr>
<th></th>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>17</td>
<td>6</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>9.1%</td>
<td>9.1%</td>
<td>12.1%</td>
<td>51.5%</td>
<td>18.2%</td>
<td>100%</td>
</tr>
<tr>
<td>Female</td>
<td>3</td>
<td>13</td>
<td>3</td>
<td>12</td>
<td>4</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>8.6%</td>
<td>37.1%</td>
<td>8.6%</td>
<td>34.3%</td>
<td>11.4%</td>
<td>100%</td>
</tr>
<tr>
<td>Total</td>
<td>6</td>
<td>16</td>
<td>7</td>
<td>29</td>
<td>10</td>
<td>68</td>
</tr>
<tr>
<td></td>
<td>8.8%</td>
<td>23.5%</td>
<td>10.3%</td>
<td>42.6%</td>
<td>14.7%</td>
<td>100%</td>
</tr>
</tbody>
</table>

7.2.3 Relationship between Junior Certificate Level of Study and Questionnaire Items

The author was also interested if the students’ Junior Certificate Level had an effect on the students’ views and reactions to the intervention. A cross tabulation was carried out to compare each item of the questionnaire with students’ Junior Certificate Level. A statistical association was found in one of the items and is discussed below.

Mathematics provides help in problem solving in other areas/subjects by Junior Certificate Level: trends by Junior Certificate Level

Analysis of the relationship between Junior Certificate Level and students’ views on mathematics providing help in problem solving in other areas/subjects showed 94.7% of Ordinary Level students agree with the statement, while the remaining 5.3% disagree with the statement. 55.1% of Higher Level students agreed or strongly agreed that mathematics provides help in problem solving in other ar-
eas/subjects. Table 7.4 shows the breakdown of each response based on Junior Certificate Level. There is a statistically significant association between Junior Certificate Level and students’ views on mathematics providing help in problem solving in other areas/subjects (p = 0.004).

Table 7.4: Breakdown of students views on mathematics providing help in problem solving in other areas/subjects by Junior Certificate Level

<table>
<thead>
<tr>
<th></th>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Higher</td>
<td>4</td>
<td>13</td>
<td>5</td>
<td>21</td>
<td>6</td>
<td>49</td>
</tr>
<tr>
<td></td>
<td>8.2%</td>
<td>26.5%</td>
<td>10.2%</td>
<td>42.9%</td>
<td>12.2%</td>
<td>100%</td>
</tr>
<tr>
<td>Ordinary</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>18</td>
<td>0</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>5.3%</td>
<td>0%</td>
<td>94.7%</td>
<td>0%</td>
<td>100%</td>
</tr>
<tr>
<td>Total</td>
<td>4</td>
<td>14</td>
<td>5</td>
<td>39</td>
<td>6</td>
<td>68</td>
</tr>
<tr>
<td></td>
<td>5.9%</td>
<td>20.6%</td>
<td>7.4%</td>
<td>57.4%</td>
<td>8.8%</td>
<td>100%</td>
</tr>
</tbody>
</table>

7.2.4 Perceptions of Applicable Mathematics

The students were first asked to write down their initial feelings regarding what mathematical topics they would consider applicable to real life contexts. Their spontaneous reaction to the question was recorded and the students could provide more than one answer. Arithmetic was referred to 41 times, Measure (Area and Volume etc.) 39 times, Statistics 22 times and Graphs 12 times. Considering the questionnaire was administered post completion of the intervention it was surprising to see only 10 references to Functions. In total there were 166 references, with 13 different mathematical topics referred to. 3 of the 68 students did not answer this question while at the other end of the spectrum one student stated: “All topics to do with mathematics are applicable to real-life contexts, whether it be buying something in the shop, estimating how many will attend a party or when converting mile to kilometres when driving past old (road) signs”.

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Changes in students views when the mathematical topics are listed.

The students were then asked to select from a list of 10 mathematical topics according to what they considered applicable to real-life contexts. The data showed that when the mathematical topics are listed, Measure is the mathematical topic in the Junior Certificate syllabus considered to be most applicable to real-life contexts. There were a total of 333 responses for ‘applicable mathematics’ and 241 responses for ‘least applicable mathematics’ because the students were allowed to tick more than one topic. Table 7.5 shows the references of applicable and least applicable mathematical topics according to the participating students. Arithmetic was placed 2nd, with 53 references. Sets recorded 16 references, which could be considered quite a high number of references when compared to Trigonometry which received the same number of references. Geometry had a total of 18 references and Functions had 24 references. Considering the wealth of applications these topics encompass it seems remarkable that these topics recorded such a low number of references.

The students were also asked to tick from the same list of 10 mathematical topics according to what they considered least applicable to real-life contexts. It is surprising to see Trigonometry score the highest with 45 references. Furthermore, Geometry (36 references) and Functions (31 references) are two of the highest references to topics considered least applicable to real-life contexts. This is consistent with the low number of references recorded for each of these three topics for the previous question (topics the students considered applicable mathematics). It is also notable that Algebra has 42 references from students when asked about what they consider least applicable to real-life contexts. However in the previous question (topics the students considered applicable mathematics), Algebra had 20
Table 7.5: Applicable and least applicable mathematical topics - by students’ opinion (The Implementation Phase)

<table>
<thead>
<tr>
<th>Topic</th>
<th>Applicable Responses</th>
<th>Least Applicable Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic</td>
<td>53</td>
<td>8</td>
</tr>
<tr>
<td>Algebra</td>
<td>20</td>
<td>42</td>
</tr>
<tr>
<td>Sets</td>
<td>16</td>
<td>38</td>
</tr>
<tr>
<td>Number Systems</td>
<td>27</td>
<td>19</td>
</tr>
<tr>
<td>Measure</td>
<td>62</td>
<td>2</td>
</tr>
<tr>
<td>Statistics</td>
<td>50</td>
<td>11</td>
</tr>
<tr>
<td>Functions</td>
<td>24</td>
<td>31</td>
</tr>
<tr>
<td>Trigonometry</td>
<td>16</td>
<td>45</td>
</tr>
<tr>
<td>Geometry</td>
<td>18</td>
<td>36</td>
</tr>
<tr>
<td>Graphs</td>
<td>47</td>
<td>9</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>333</strong></td>
<td><strong>241</strong></td>
</tr>
</tbody>
</table>

Students views on what other subject areas they consider mathematics useful for

Another aim of the student questionnaire was to clarify what other subject areas (which the students study in school) they would consider provide the opportunity to use mathematics. In total there were 240 different responses, with Table 7.6 showing the breakdown of these responses. Two students failed to provide at least one response to the question. In total 12 different subject areas were mentioned, with the Sciences being the most referred to with 60 references. Business Studies was referred to 52 times, while Design and Communication Graphics (which is the new revised Technical Graphics syllabus) was referred to 40 times. A more in-depth breakdown of the Sciences and Business Studies is discussed below. Construction Studies, Engineering and Home Economics had a total of 30, 27, and 14
references respectively. Geography and Art had 9 and 3 references respectively. Some of the more unanticipated subjects referred to were History with 2 references and Music, C.S.P.E. (Civic, Social and Political Education), and religion with 1 reference each.

Table 7.6: Most applicable subjects- by students’ opinion (The Implementation Phase)

<table>
<thead>
<tr>
<th>Rank</th>
<th>Subject</th>
<th>Number of References</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>The Sciences</td>
<td>60</td>
</tr>
<tr>
<td>2</td>
<td>Business Studies</td>
<td>52</td>
</tr>
<tr>
<td>3</td>
<td>Design and Communications Graphics</td>
<td>40</td>
</tr>
<tr>
<td>4</td>
<td>Construction Studies</td>
<td>30</td>
</tr>
<tr>
<td>5</td>
<td>Engineering</td>
<td>27</td>
</tr>
<tr>
<td>6</td>
<td>Home Economics</td>
<td>14</td>
</tr>
<tr>
<td>7</td>
<td>Geography</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>Art</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>History</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>Music</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>C.S.P.E. (Civic, Social and Political Education)</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>Religion</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>240</td>
</tr>
</tbody>
</table>

Notably, a breakdown of the Science responses, which is presented in Table 7.7, shows Science with 29 references, and Physics with 26 references. Chemistry had 4 distinct references while Biology had 1 reference. Also, Business Studies had a breakdown of responses, which is presented in Table 7.8, where Business Studies had 36 responses, and Accounting had 16 responses. Economics had no responses.
Table 7.7: Breakdown of Science references- by students’ opinion (The Implementation Phase)

<table>
<thead>
<tr>
<th>Rank</th>
<th>Breakdown of Science References</th>
<th>Number of References</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Science*</td>
<td>29</td>
</tr>
<tr>
<td>2</td>
<td>Physics</td>
<td>26</td>
</tr>
<tr>
<td>3</td>
<td>Chemistry</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>Biology</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td><strong>Total</strong></td>
<td><strong>60</strong></td>
</tr>
</tbody>
</table>

* The Science syllabus at Junior Certificate level has three major components: biology, chemistry and physics; It is examined as a single subject encompassing all three components.

Table 7.8: Breakdown of Business Studies references- by students’ opinion (The Implementation Phase)

<table>
<thead>
<tr>
<th>Rank</th>
<th>Breakdown of Science References</th>
<th>Number of References</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Business Studies</td>
<td>36</td>
</tr>
<tr>
<td>2</td>
<td>Accounting</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td><strong>Total</strong></td>
<td><strong>52</strong></td>
</tr>
</tbody>
</table>

7.2.5 Discussion of Findings

The questionnaire was selected as one of the primary research instruments of the Implementation Phase as it was a required method of data collection outlined by RUMEC with respect to the theoretical framework of APOS Theory. In particular, the use of a questionnaire is an efficient instrument for gathering information and facts on the sample. It provided the author with demographic information including the students’ age, Junior Certificate grade and level of mathematics study. Section A of the student questionnaire was designed to examine the views of students in Senior-Cycle mathematics who participated in the intervention, while Section B was aimed at discovering which mathematical topics the students considered applicable to real-life contexts. Each of the sections is discussed individually.
Limitations of Section A of the questionnaire

The short time-frame (2-3 weeks) of the teaching intervention and the age range (15-16 years) of the students prevents the author from drawing generalisable conclusions from this section of the questionnaire. While the findings cannot be generalised, this does not prevent them from being useful in other contexts and pedagogical settings. These factors had an influence in the approach of administering the questionnaire post-completion of the intervention only.

Section A – Students Views of the Intervention

The median and percentiles were chosen as measures of centrality and variability as the data for each item was skewed. Each of these items is examined in more detail in the questions posed to the students in the follow-up interviews and is discussed later in this chapter in Section 7.3. This allows for more confidence that Section A of the questionnaire has provided an insight into the students’ views of the teaching intervention, thus highlighting some noteworthy findings. Although, the nature of the intervention does not permit the author to draw generalisable conclusions for all Irish Senior-Cycle students it has shown that the students who took part in the intervention:

- find mathematics more interesting when taught through applications
- prefer to work in pairs or small groups within the mathematics classroom
- do not like writing reports for a mathematics problem
- believe that mathematics is important in everyday life
• believe that studying mathematics helps in problem solving in other subjects/areas

• understand why they study mathematics in school

• believe that the studying of mathematics will help them in college or in work.

Following this initial analysis, a cross tabulation was carried out to compare each item of the questionnaire and students’ gender, and to compare each item of the questionnaire and students’ Junior Certificate Level. A statistical association was found between two items. When comparing gender with students’ views on writing reports for a mathematical problem, 25.7% of the females who answered the questionnaire strongly agreed with the statement, as opposed to 3% of males. When comparing Junior Certificate Level and students’ views on mathematics providing help in problem solving in other areas/subjects, 94.7% of Ordinary Level students that answered the questionnaire agreed with the statement, while the remaining 5.3% disagreed with the statement. 42.9% of Higher Level students also agreed that mathematics provides help in problem solving in other areas/subjects. Once again, the nature of the intervention does not permit the author to generalise for all Irish Senior-Cycle students.

Section B – Perceptions of Applicable Mathematics

It appears from the outset that the students surveyed consider the most obvious mathematical topics applicable to real-life contexts i.e. Arithmetic, Geometry, Statistics and Trigonometry. This is perhaps due to the traditional instructional
approaches practised by teachers in the Irish education system. Teachers are affected by the continued emphasis placed by pupils on acquiring as many points as possible, where they are required to help pupils achieve these goals. They are forced to adopt a teach-to-the-examination approach in order to maximise points, as opposed to teaching for understanding (Gill, 2006). Furthermore, time allocated to any one subject at Senior-Cycle is low in international terms (NCCA, 2005), thus, time allocated to teaching-for-understanding is minimal in Irish mathematics classrooms, where emphasis on rote-learning and cramming is paramount (O’Donoghue, 1999). Section B of the student questionnaire has shown that many students are unable to appreciate the role of mathematics in everyday life, where they fail to understand or realise the influence that mathematics can exert on one’s future education and/or work-life. Current practices in the teaching and learning of mathematics in Senior-Cycle schools in Ireland generally fail to make the necessary connections between mathematics and its place in real-life, as documents from the NCCA and the Chief Examiners Report have shown (NCCA, 2005; State Examinations Commission, 2005).

7.3 Analysis of Qualitative Data - Student and Teacher Interviews, Reflective Journals

7.3.1 Introduction

The purpose of this section is to develop a more in-depth knowledge of the intervention’s impact on both the students and the co-operating teachers who participated in the intervention. Like the quantitative study, this qualitative study focuses
on issues identified throughout this thesis, namely the teaching and learning of applications and the role of modelling in mathematics education, that contribute to the Senior-Cycle mathematics experience of students in Irish schools. This section also progresses the discussion in the previous section, Section 7.2, by focusing on the consequences of these issues on the students in comparison to the co-operating teachers.

**Qualitative Study (semi-structured interviews)**

The author’s objective was to investigate both the students’ and co-operating teachers’ feelings and views on the intervention and the impact that it would have on the teaching and learning of applications at Senior-Cycle level. While the quantitative study allowed no room for opinions and discussion, the interviews aimed to provide such opportunities. The sample which incorporated the participating teachers and the participating students was chosen by a method known as simple random sampling. A simple random sample gives each member of the population an equal chance of being chosen (Hunt & Tyrell, 2001).

**Research Instrument (qualitative study)**

10 semi-structured interviews (see Appendix K) were conducted with students selected randomly from each of the three participating schools. Also the three co-operating teachers were interviewed (see Appendix L). All interviews took place post completion of the intervention, within a time-frame of 5 days. Such a short time frame was chosen so as to alleviate any distractions that may occur due to new material covered in the mathematics classroom subsequent to the intervention.
and/or any other academic or non-academic distractions that may have occurred. Computer assisted qualitative data analysis (CAQDA) was incorporated through the use of the software package NVivo, a tool used to aid the researcher in the analysis of the qualitative data. Interviewees were coded using pseudonyms. A list of the pseudonyms is provided below for each participating student. A more detailed discussion of these aspects is discussed in the methodology in Chapter 4.

**Student1** – Ron

**Student2** – Jack

**Student3** – Ger

**Student4** – Helen

**Student5** – Hugh

**Student6** – Amy

**Student7** – Anne

**Student8** – Declan

**Student9** – Tommy

**Student10** – Lisa

**Research questions (qualitative study)**

The following research questions guided this follow-up qualitative study:

- Can the students relate to the real-life problems provided? Why/Why not?
• Can the students’ see the relevance of mathematics to their everyday lives?

• Can the students’ understand the topic (Linear Functions) and/or mathematics as a discipline more comprehensively?

• What barriers to the students’ understanding existed? i.e. new learning techniques, time constraints etc.

• What impact did the intervention have on the teachers? Would it influence their teaching of mathematics in the future?

Data analysis of qualitative study

The semi-structured interviews were organised into three sections outlined below:

Section1 Applications

Section2 Modelling

Section3 GeoGebra

This data gathering instrument or interview plan guided the 10 student and 3 teacher interviews that took place. The interview plan was slightly adjusted to examine the roles of the interviewee where there were separate student interviews (see Appendix K) and teacher interviews (see Appendix L). Each student interview was conducted on an individual basis and lasted approximately 5 minutes, while each teacher interview lasted approximately 7 to 8 minutes. The interviews were designed to be as flexible as possible. The consequent duration of both student and teacher interviews was determined by their engagement and did not result in interviews of longer duration. Pseudonyms were used for all interviewees and
any teachers who might have been named throughout the course of the interviews. All interviews were transcribed using Voice Editor 3 software package and NVivo software was used to facilitate the analysis of the interview transcripts. The data was coded based on a list of nodes drawn up for each section of the interview outlined above. The nodes which emerged from the data for the student and teacher interviews can be seen in Table 7.9.

Table 7.9: Coding of data for student and teacher semi-structured interviews

<table>
<thead>
<tr>
<th>Section</th>
<th>Nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Applications</td>
<td>Relate to Problems</td>
</tr>
<tr>
<td></td>
<td>Couldn’t Relate</td>
</tr>
<tr>
<td></td>
<td>Understanding Linear Functions</td>
</tr>
<tr>
<td></td>
<td>Understanding Mathematics</td>
</tr>
<tr>
<td></td>
<td>Understanding Over time</td>
</tr>
<tr>
<td>Modelling</td>
<td>Leaving Cert</td>
</tr>
<tr>
<td></td>
<td>Report Writing</td>
</tr>
<tr>
<td>GeoGebra</td>
<td>GeoGebra understanding</td>
</tr>
</tbody>
</table>

**Teacher Reflective Journals**

The Reflective Journal was designed to allow opportunities for the participating teachers to reflect on their overall experience of each lesson and provide an insight to the intervention from their perspective. Each journal in the Implementation Phase was one page per lesson. They were expected to extend their comments based on specific titles such as:

- Accompanying material
- Real-life problems
- GeoGebra applets.
While all three teachers completed the journals, to varying degrees of success, the responses were often descriptive in their nature as opposed to reflective. The reflections of significance provided in the Reflective Journal are included in the relevant nodes as outlined in Section 7.3.2.

7.3.2 Findings

Following analysis of the nodes a number of relationships were established between the nodes as can be seen in Figure 7.2. These relationships are shown diagrammatically and are discussed in the following paragraphs.

Node: Applications

A dominant theme expressed by the ten students with regard to the applications was one of relevance. Almost all expressed a satisfaction with the applications when they could relate it either to their everyday experiences or experiences in other school subjects.

(Jack) ‘Like the river one. We’ve done a lot of that in Geography and stuff.’

(Anne) ‘Yeah they were everyday things that we have, like iPods & temperature and stuff. Things you’d have every day.’

(Lisa) ‘Yeah.. yeah because all the pictures and the question like the hot-air balloon and other things like we use them in Science as well. So we’d be able to use them at home as well.’
Figure 7.2: The relationship between the nodes
(Amy) ‘Yeah I could, like when we did the part on, am, the air-balloon, like I
could understand that and it helped to understand how the pressure works
in an air-balloon.’

(Tommy) ‘Em, yeah, the one about the iPod was alright. Like the price going up
for the value of it. So that was realistic.’

(Teacher3) ‘They were delighted with the Shannon problem as were they with the
hot air balloon as it was relevant and not like the abstract maths they had
become accustomed to.’ (Reflective Journal)

Within the sample of students interviewed there was much agreement where the
students were content that they could easily see the relevance of the applications in
the world around them. However, one of the applications caused concern amongst
the students was that of ‘Pelican Eggs’ (See Appendix P, Lesson 3). The participating teachers provided an insight into the students’ problem with that particular
application.

(Teacher3) ‘There was one I think on pelican eggs, they had a problem with that
because it made you use some Science at the start and some just switched
off.’

(Teacher3) ‘They couldn’t relate to it because I think maybe I have a lot of
weaker students and they saw that this really wasn’t something they knew a
lot about and there was chemicals and stuff in it. They were kind of asso-
ciating it back to the maths they had done before and immediately they just
weren’t as interested at the start of it. I suppose it wasn’t as simple a story
as had it been something basic. That’s what I felt anyway.’
(Teacher3) ‘The problem with Example 3 (pelican Eggs) was that it was not as relevant to students as the previous two had been. The mention of PCB at the start made it seem more complicated that it was. My weaker students tended to switch off.’ (Reflective Journal)

(Teacher2) ‘The one they didn’t like or disliked the most was the one to do with the pelican eggs. They had no knowledge or background and couldn’t relate to anything, so therefore they were wondering where that would come up.’

(Teacher2) ‘Yeah they struggled and couldn’t understand. They could understand what it was about, but putting that function together, why or the concentration and that they just didn’t like it. They didn’t like that one and it stuck out most in their minds.’

Node: Syllabus

There was a mixture of feelings regarding the position of applications in the mathematics syllabus expressed by the students interviewed.

(Ron) ‘Em, I way prefer the ordinary things about where there is a method kind of.’

(Helen) ‘Once you get to think about them they are a lot easier than the other (traditional) course.’

(Hugh) ‘Yeah may be a few like. But I found it harder to do those questions. Just me as a person like.’

(Anne) ‘It would probably be easier than if it was just kind of numbers thrown at you. Something that you can relate to.’
The need for students to learn mathematics through the medium of applications and modelling is widely accepted and its merits for inclusion are widely recognised (Burkhardt, 2006; Ferrucci & Carter, 2003; James, 1985). It seems that the sample of students interviewed are unsure as to the position of applications within the mathematics syllabus here in Ireland. However, with the introduction of Project Maths in all schools in Ireland since September 2010, the participating teachers are aware of the role that applications will play in future mathematics courses at second-level.

(Teacher3) ‘Em I think it’s good because you kind of seeing the whole process in maths rather than just a sum and get an answer. You’re seeing where it’s coming from. You understand it from start to finish. And I think that’s important, because that’s what’s missing in the Leaving Cert at the minute. You open a text book and the problems are there. There’s no background to them, you learn a formula for the sum and you get full marks. Without really having any real knowledge I would feel anyway about a lot of the concepts behind them. So I would think it would be a good idea for kids to maybe understand it better what’s going on.’

Node: Modelling

While the students were unsure as to the position of applications in the mathematics syllabus, all expressed positive reactions to the provision of a modelling aspect as part of the assessment process for the Leaving Certificate here in Ireland.

(Helen) ‘Yeah I think it would be good as a project.’
(Ger) ‘Yeah, it’d be good because it gives different kinds of people chances to get more points and stuff.’

(Anne) ‘Yeah other subjects do it and its better to have something going in to the Leaving Cert. Like in the practical subjects we have projects so there’s no reason why this can’t be part of mathematics either.’

Two of the participating teachers’ echoed these sentiments where they felt that Project Maths could have incorporated this approach in the assessment process:

(Teacher2) ‘I would like some type of written project element, where students could work on something like we worked on here. Like taking functions and putting a project together based on a function problem.’

(Teacher2) ‘Well it works in other subjects so there no reason why it can’t work in Maths. I can’t understand why it hasn’t been introduced with regards to Project Maths.’

(Teacher3) ‘If for a Leaving Cert project that would be a good idea as well, because a lot of other subjects have it and again it would make it a lot more relevant to them. I think with Leaving Cert maths you can get away with a lot of formulas and doing sums you don’t really have to understand a lot of the questions. I think a report would be a good idea. You would see then who understood what they were talking about.’

(Teacher3) ‘Yeah I think it would be a good idea. Maybe worth 10 or 20 % or something like that. I think it would be very fair and it would really see who knows their stuff as opposed to those that just plug in numbers and learning formulas. They would have a better understanding too.’
One of the teachers felt that the assessment procedures/marking of reports could prove to be too much of a stumbling block for the inclusion of modelling as part of the assessment process.

(Teacher1) ‘Who assesses it? If it’s project based I am assuming then it would be more than one individual working on it. What way do you divvy out the work load? You know there are loads of areas, huge bones of contention: Plagiarism, equal distribution of marks. Supposing you had a very capable student originally and they produce a very good report, have they really grown through the process? Or I might have a student who is maybe a ‘C’ student and they produce a report that is an ‘A’ standard, should they get more marks than the students who is always an ‘A’ and still at an ‘A’. Huge level of difficulty there. So no I wouldn’t like it. Too complicated.’

Another teacher expresses a contrasting view to this where she states:

(Teacher2) ‘I would like it to be group work, but in terms of marking, being realistic at the end of the day, it will have to be individual. But if you take what goes on in geography we all go off and do a project and we gather the data together, but the answers then are individual write-ups. Something similar to that.’

Node: Modelling/Report Writing

Report writing was a novel aspect of the modelling process for both the teachers and students. It is an element of mathematics that is not widely practised in
second-level mathematics education. It is important that sufficient time is provided to allow the students and teachers to familiarise themselves with the techniques necessary to complete a comprehensive and informative report of a mathematics problem. Nevertheless, the responses showed that both the students and teachers embraced the report writing. In general, positive attitudes were expressed by the majority of interviewees.

(Ron) ‘Yeah I did like that actually. Yeah that was one of the easiest parts. It was that there was steps to it. And that was kind of a bit of a method to follow.’

(Jack) ‘It was ok. It was kind of hard to understand where to put the useful solutions and that but it was ok.’

(Ger) ‘Ah it was different. It was alright though. It was good. Like you got to do something different than just algebra or whatever you normally do in class.’

(Helen) ‘Eh yeah it was ok. I don’t think anyone likes writing reports but they weren’t too bad you know. They were something different.’

(Teacher1) ‘I think it’s very relevant, because, ultimately this day and age, everything is paper-driven. There has to be a paper trail and students need to know that they actually have to be able to stand over their own work. You can’t just say ‘I did this’, there has to be accountability. So from that point of view, I think it’s very good. I also like that report writing as opposed to, ah, say essay writing on an imaginary topic, ah, or say whatever; you know maybe it can go into a stream of consciousness. But report writing is extremely structured and I like that. It’s trying to instil a sense of logic with students and I think that’s important. You know and clarity and structure is
very succinct, you know, very brief, one or two sentences under that bullet point.’

(Teacher2) ‘Am I think it’s necessary. I think it’s important to get my own understanding and the students understanding both on the one page at the end of each lesson. I think it’s valuable. The students probably didn’t like it, but at the same time it’s important. It’s valuable.’

(Teacher3) ‘Am I think it’s good because you are kind of seeing the whole process in maths rather than just a sum and get an answer. You’re seeing where it’s coming from. You understand it from start to finish. And I think that’s important, because that’s what’s missing in the Leaving Cert at the minute. You open a text book and the problems are there. There’s no background to them, you learn a formula for the sum and you get full marks. Without really having any real knowledge, I would feel anyway, about a lot of the concepts behind them. So I would think it would be a good idea for kids to be maybe understand it better what’s going on.’

(Teacher3) ‘I think with Leaving Cert maths you can get away with a lot of formulas and doing sums you don’t really have to understand a lot of the questions. I think a report would be a good idea. You would see then who understood what they were talking about.’

**Node: Understanding**

Understanding is closely linked to the applications that the students were exposed to in this intervention, as previously discussed in the applications node, and this is easily recognised when questioned about their understanding of the mathematical
concepts. On numerous occasions the students refer to the fact that understanding was made easier for them as a result of their ability to relate the problems to the world around them.

(Lisa) ‘Yeah I have an iPod at home and I found that if I was stuck I could look at it and question it.’

(Ger) ‘Yeah because it gave you problems that you would need maths for. So that’s what you need.’

(Helen) ‘Yeah definitely. Because before we could never see that maths had a valid point to it and to everyday life, whereas, we can see it now.’

(Declan) ‘Yeah it did because it was showing you how maths is important in your life and it just made things clearer and easier.’

(Helen) ‘Em they were very easy to relate to. Well with real-life there basically in your own words not just in numbers and letters.’

(Teacher3) ‘Because what they kept saying to me was they did a certain amount of functions at Junior Cert level and you know when I was trying to link it back to the functions they were doing at Junior Cert they couldn’t believe that this was the same topic at all. They were saying to me “God you know we had no idea at all last year” and now they felt, especially the weaker students, that this was relevant.’

(Teacher3) ‘Students in my class were surprised to realise that the real life problems were in fact the functions that they had been working with in Junior Cycle.’ (Reflective Journal)
(Teacher3) ‘By the end of the lesson, even the weaker student had a good grasp of what a function was.’ (Reflective Journal)

(Teacher3) ‘I was surprised by how quickly the weaker students grasped the concepts’. (Reflective Journal)

(Teacher3) ‘The slope formula, which at Junior-Cycle was just a series of letters with no real meaning, became relevant. They got the idea of increasing and decreasing functions through the (real-life) problems’. (Reflective Journal)

(Teacher3) ‘It has to be said that they are much better working on the real-life problems’. (Reflective Journal)

(Teacher3) ‘Had this been a standard mathematical sum from a text, I don’t think this would have happened (correct answers). They were working harder as it captured their interest.’ (Reflective Journal)

However, Teacher 1 noted that her class was more comfortable with the traditional approach to teaching mathematics. In fact the teacher questions the approach and the affect it is having on her students ability to reason mathematically.

(Teacher1) ‘As a higher level class, they were very comfortable with the familiar rhetoric of \( y = f(x) \).’ (Reflective Journal)

(Teacher1) ‘One or two students realised that the speed = 0 but still could not align it to solving a quadratic. However, when I triggered “how do we solve a quadratic?” they replied as expected.. factorise or formula, and one word flashed through my mind – institutionalised! What have we done?’ (Reflective Journal)
(Teacher1) ‘Commenting on the class efforts I asked why somebody might draw a $-x^2$ graph and the response coincided with my thoughts on the reasoning behind it.. purely because we had just revised the negative $t^2$ graph relating to the car-crash and the student assumed the next example followed suit.’ (Reflective Journal)

(Teacher1) ‘In terms of the real-life examples used, students who had successfully derived solutions in Example 1 faltered at Example 2. Lack of active thought perhaps accounting for $-t^2$ being perceived as $(−t)^2$ (illogical!) by very capable students academically.’ (Reflective Journal)

**Node: Understanding/Time**

Loucks-Horsley et al. (1998) argues that deep learning of mathematical concepts takes time, and takes place over time. This view is supported by many other researchers (Cockcroft, 1982; Carpenter & Lehrer, 1999) where they acknowledge that the development of understanding in mathematics classrooms takes place over time. The sample of students interviewed in this intervention share common views on this approach providing an insight into their understanding of the way they learned throughout the intervention.

(Ron) ‘Yeah as the weeks went on it was kind of easier to understand.’

(Lisa) ‘It was a bit hard at the beginning. As we got used to it was a lot easier.’

(Jack) ‘Yeah if we had more of them I’d say we’d get a bit better at them.’

(Ger) ‘Yeah well we got a problem the other day in class and it would have been easier to solve it now than if we got it at the start because of the way we
were learning.’

(Hugh) ‘It takes time and we are not used to that way.’

(Amy) ‘If I was always taught this way it would be easier.’

(Tommy) ‘Am, yeah you did have to flick back and see what they meant by certain questions and stuff, but otherwise it was a lot easier then. It was difficult at the start but the more we did the easier it became.’

(Teacher1) ‘Am sure even when the first time you show someone something it’s a disaster, and then eventually improves. There’s no reason why we couldn’t then do the same thing when it comes to teaching applications. Right the first time it doesn’t work but by a process with re-iteration it will improve their skill. They quite enjoy it though! And I think that will then lend itself to maybe a faster development of that skill.’

(Teacher1) ‘Most students appear to be gaining in confidence (by Lesson 4). More in touch with what is expected of them.’ (Reflective Journal)

Node: Understanding/ Group Work

The positive benefits of collaborative learning have been widely recognised in mathematics education for some time now (Davidson, 1990; Brown, 1994; Webb & Palincsar, 1996). In a review of collaborative learning in secondary mathematics, Suri (1997) reported an overall positive effect both in the cognitive domain and in the social and affective domain. Furthermore, small group work is widely recognised as an important tool and worthy of inclusion in successful modelling
approaches (Bajpai et al., 1975; Harper, 1975; Ormell, 1984; Niss 1987; Clatworthy, 1989).

(Lisa) ‘We worked well in groups than we would have on our own. There was more communication going on and with the teacher as well.’

(Amy) ‘Am some of them were a bit hard but we did them in groups so it was ok.’

(Anne) ‘Ah no we worked through them all in group work. So if you had any problems we worked them out.’

(Tommy) ‘Yeah, that was like a lot better, because everyone had their own opinion for each way and the way to do it and stuff. And we worked together.’

(Teacher1) ‘They don’t tend to think of it as something you can take a trained skill and not put it into something which is a process. They are not quite thinking laterally like that. They are not quite sure what their skill-set is or how they can import different skills at certain times for certain applications. That is key, but then again that is a very high-level skill and something that not all students have the ability to do. We’re hoping that more will be able to do this through maybe pair-work or team-work. I think its something that we should maybe pursue though.’

Node: Understanding/GeoGebra

Daugherty (2007) identified that we have not seen widespread adoption of visualisation techniques in mathematics teaching and learning. Eisenberg and Dreyfus (1991) further this debate, where they analysed why there is such widespread reluctance on the part of both teachers and students to choose visual methods in
problem solving and in establishing a basic understanding of fundamental notions. They concluded that visualisation techniques are cognitively more demanding of the learner than analytical techniques which are more algorithmic in nature, thus providing negative attitudes (fear, anxiety etc.) for both teachers and students towards using visualisation methods.

(Hugh) ‘Probably (the computer/GeoGebra) was good for my learning alright. But I didn’t understand it too much’

(Teacher3) ‘I would not be a fan of using a computer in the classroom. I never have been. But I have to say when I got into that it’s very easy to use’

Snir (1995) argues that computers can make a unique contribution to the clarification and correction of commonly held misconceptions of phenomena by visualising those ideas. Instead of spending considerable time learning by hand routines, West (1995) argues that the computer can alleviate such time constraints, allowing the students to move more swiftly to higher-level conceptual matters and a variety of practice problems. In addition, West (1995) acknowledges that students who learn mathematics through the use of visual techniques have shown that, in comparison with traditional courses, they understand the basic concepts better, can remember the information longer and can apply the concepts to practical uses more effectively.

(Jack) ‘Yeah they made you understand what was happening’.

(Ger) ‘Yeah, because you got to see what you were learning and it made it easier.’

(Teacher1) ‘It was very, very useful. I’ve already done it (used GeoGebra in my teaching outside the intervention) and I intend to use it in the future. Yes
**Node: Impact of the intervention on Teachers**

One of the significant questions posed to the teachers was about the impact the intervention had on their teaching of mathematics outside of the intervention. They were asked ‘would you approach teaching applications in the real world differently now especially with the introduction of Project Maths next year?’ All of the teachers concluded that the intervention would have a positive impact on their teaching of mathematics in the future.

*(Teacher1)* ‘Yes I would. Am, I felt that I am not in a position that I am fully equipped to do it at this point. I need to think about the way the students
think and relate to them and then relate it back to the concepts mathematically. Am I need to do a bit of homework for my teaching but not just because of Project Maths but after this project (intervention) it makes me want to make Maths more real I think.’

(Teacher1) ‘But I think to be a pro-active teacher I have to constantly want to engage with the students and to do that you have to be able to relate to them on their terms. And you have to be aware of what’s happening in the modern society. So it does mean that I need to re-think where I’m getting my resources from, am I suppose I need to get resources which are pertinent to improving my capability of teaching applications. Definitely.’

(Teacher2) ‘Absolutely. Its an issue that I had myself over the last few years. We weren’t preparing our students for third level. And a lot of my students would have come back to me at different stages with a lot of the problems with the manual we worked through for help and understanding. So with Project Maths I think it will big time in the next couple of years.’

(Teacher2) ‘So it would definitely be something I would use again I would like to see more changes coming into the Leaving Cert rather than what Project Maths are introducing. I think going on what we worked on this intervention is the way to go for the Leaving Cert.’

(Teacher3) ‘I would, especially with a weaker class. My students were much more interested and they understood it much faster. As I said especially the first lesson struck me and I could spend three lessons trying to explain dependent and independent variables with f(x) and they probably wouldn’t have understood it. Whereas immediately they were getting it with real life
7.3.3 Delayed Reflection

It was decided that a follow-up reflection was necessary given the nature of the responses of the reflective journals. The teachers were asked two reflective questions (see questions below), via email, approximately five months after the completion of the intervention. The time-frame of five months was used so as to ensure sufficient time had elapsed since the intervention and as a result the teachers could form a more objective view, thus resulting in more valid responses.

1. In the intervention, given that you used the 6-stage process when teaching modelling, do you think a similar approach (diagrammatic outline) would work for teaching applications? (Why?)

2. After completion of the intervention, what is your overall reflection of the experience? Would it influence your teaching in the future?

Responses to Question 1

(Teacher1) If by "teaching applications" you mean mathematical problems relating to real-life (i.e. comprehension style questions) then yes. Comprehension style questions that are language laden require students to deploy a systematic approach in attempting a solution process. Some students are naturally inclined to be systematic in their behaviour; others however, need to be guided in order to develop such high-order skills. Also by including a visual representation of the solution process it extends the appeal of such
an area (i.e. applications) to a broader audience; those visually triggered as well as those non-visual, and in terms of learning, reinforces a student’s individual assimilation process.

(Teacher2) Yes, by using stages students appreciate what they are learning as they have better understanding from beginning to end- being able to relate to learning stages benefits students, leads to clarity of thinking and a clear understanding of process

(Teacher3) Regarding this approach for teaching applications, I do feel it would work for teaching applications. It is a very student centred process. It is a process which enables the student to process small pieces of information at a time. This was useful for weaker students in particular. Visual aids as well as the software involved help keep the student interested and was certainly more effective than the traditional "talk and chalk" method. Students saw the relevance of the topic concerned and discovered maths in a way in which many probably hadn’t before. It is a very thorough way of discussing a topic and my class certainly benefited hugely from the process. It is something that would benefit all students hugely!

Responses to Question 2

(Teacher1) I feel it was a highly beneficial experience encouraging all of us, students and teachers, to step outside of our traditional comfort zone (or non-comfort zone in some cases!) and harness skills currently underused due to the existing restraints and format of the syllabus. It was a wonderful opportunity to gain some insight into the potential of Project Maths. I
would like to think that I would allow this experience to influence some of my approaches to teaching independently in spite of Project Maths. The opportunity to "discuss" mathematics with students and to have this pooling of ideas facilitated by the content of Project Maths excites me further and hopefully, by adopting a fresh approach to teaching and learning we can encourage our young students to become stimulated and excited in our wonderful subject!

(Teacher2) Timing of research was excellent with the introduction of Project Maths, great opportunity for teacher to get head around new approach to teaching maths- would love to had more time (my own fault!!) to work with the students, evident that a lot of work involved in putting material together- would love to be involved in further research.

(Teacher3) At the beginning of the process, I was quite apprehensive as to what it would be like. As I was not used to technology in my classroom, this was a cause of concern for me. I also had quite a weak group of Transition Years who struggled with a lot of the basic concepts. They had all heard of the word "function" from their Junior Cert but did not really have any understanding as to what it meant. By the end of lesson one though, they had a very clear idea in their minds as to what it was. GeoGebra was an invaluable asset to me through the course of these lessons. The students absolutely loved it! They had a great interest in seeing what happened to the various functions when they moved a point etc. It also helped them to understand the concept of a slope. What really happened was that the maths in the textbooks came to life for them. They suddenly saw the relevance with the real life applications and were enthusiastic about coming up with their
own ideas. This has certainly opened my eyes as to what can be achieved with a weak group of students in the teaching of maths. I would now try to keep things as real as possible for these students and to use the aids that are available. I would imagine the same is true for a more academic group. Teaching maths in this way creates a deeper understanding among students about the concepts involved. It gives them a more positive view of a subject they struggle with, and they begin to see how relevant it is to them. Hopefully Project Maths will do the same.

7.3.4 Discussion of Findings

This qualitative analysis of the teaching intervention begins with the identification of the initial nodes and re-categorised nodes formed after careful analysis of the data. The findings related to each node are presented and discussed. From the outset it is evident that the findings of the interviews are consistent with the quantitative analysis of the student questionnaires. The students certainly find mathematics more interesting when taught through applications. Consequently, when the students can relate to the problems and identify mathematics in the real-world around them it makes understanding more accessible. Learning through the medium of applications takes time and this was evident in the views of the participating students. The students found that as the weeks progressed the teaching approach was easier to identify with and thus made understanding easier. Group work is a learning approach that is favored by almost all students that participated in the intervention where the opportunities to discuss and identify problem areas is present. This was an approach not widely utilised in the students’ prior mathematical learning experiences. The objective of integrating applications into the
mathematics syllabus was positively met from the students’ viewpoint, yet the participating teachers are aware of the upcoming integration of Project Maths to all schools in Ireland in September 2010. Project Maths, the new revised mathematics syllabus, has an increased use of contexts and applications that will enable students to relate mathematics to everyday experience. Mathematical modelling was favoured by almost all student interviewees where they would like to see some project-based assessment available as part of the Leaving Certificate here in Ireland. As previously discussed, the need for suitable applications is paramount in providing a positive mathematics experience for the students within the school setting. Researchers such as Burkhardt (2006) and Niss (1987) have highlighted the need to provide applications which the students can relate to. They call for non-trivial applications in which mathematics plays a crucial role in their solution. The qualitative data suggests that the students must be provided with applications which they can relate to in order to ensure understanding occurs. Exposure to excessively specialised problems would result in over-complication of material and marginalised understanding by the students. Research has shown that the current assessment process widely used, which is exclusively end-of-term examination here in Ireland, merely requires regurgitation of standard techniques applied to standard examples (Bajpai et al., 1975; Clatworthy, 1989; Burton, 1997). Quite clearly, an effective evaluative strategy for the inclusion of modelling techniques and skills is unlikely to be an end-of-term written examination (Burton, 1997; Clatworthy, 1989). Undoubtedly, there will be numerous deliberations as to the desirable forms of evaluation considered most useful in assessing modelling and applications within mathematics education. However, there exists almost uniform agreement that current practises are outdated and need to be
revitalised (Burkhardt, 2006).

7.4 Evaluation of the Intervention

Central to the success of any intervention is the quality of data on which to base the particular intervention. This provides background information to the intervention and opportunities to evaluate the effectiveness of the program (Regan, 2005; Sandoval, 1993). Thus, to conclude this chapter the author will use Shapiro’s model for evaluating an intervention (Shapiro, 1987), where the following four components are discussed:

- Treatment effectiveness
- Treatment integrity
- Social validity
- Treatment acceptability

Treatment effectiveness

The effectiveness of the intervention is an important measure in the evaluation of any strategy employed. This was ensured through Section A of the questionnaire where it consisted of 7 statements assessing students’ views and reactions to the intervention. The breakdown of these results is discussed previously in Section 7.2. However, encouragingly the results show that the students found that mathematics was more interesting when taught through applications to real-life. Furthermore, the students displayed a strong sense of the importance of mathematics in their everyday lives.
Treatment integrity

To ensure the intervention can be implemented with replicable results, the integrity of the intervention strategy is of utmost importance. That is the extent to which the specified scheme is actually executed in the manner prescribed in the intervention documentation. To ensure this, the theoretical analysis and subsequent instructional materials (including all applications, applets, exercises etc.) are provided for the participating teachers in this research study. Teachers’ need both knowledge of the mathematical concepts concerned and the ‘know how’ to teach these concepts. In addition, the teachers will need knowledge of the applications area. Thus, the author provides an inclusive package that in a sense is ‘self-contained’ as regards teacher knowledge for applications. Such a self-contained approach ensured the participating teachers could focus on the quality of their instruction, rather than the acquisition of knowledge of other subject areas needed to teach the applications. Thus, all three participating teachers were provided with a teacher’s pack which included:

- Teacher Manual
- Teacher Guidelines
- Teacher Journal
- Solutions to Exercises
- CD
- Modelling Problem & Solution
Social validity

Social validity was defined by Shapiro (1987, p. 293) as the ‘evaluation of the intervention by the clients or consumers’. Consequently, the question ‘to what extent did the project meet its overall goals?’ needs to be answered by analysing the participant’s views. This was ensured through follow-up interviews with the participating teachers and a sample of the participating students (n=10). This is discussed in more detail in Section 7.3 of this Chapter. The reviews from the students and teachers were positive overall indicating that the applications and instructional materials used in the intervention were appropriate for the age range and mathematical ability of the students.

Treatment acceptability

Kazdin (1981, p. 494) defined treatment acceptability as: “judgements by laypersons, clients and others on whether treatment procedures are appropriate, fair and reasonable for the problem or client”. Again, this was ensured through follow-up interviews with the participating teachers and a sample of the participating students (n=10). The qualitative data suggests that the students must be provided with applications which they can relate to in order to ensure understanding occurs. Furthermore, learning through the medium of applications takes time and this was evident in the views of both the participating teachers and students. The delayed reactions of the participating teachers, in particular, highlighted the positive attitudes towards the intervention and thus, overall, the intervention proved to be acceptable to both teachers and students.
7.5 Conclusion

The study focuses on the problem of improving the teaching and learning of mathematics, particularly upper secondary level in Ireland, by making a provision for the effective teaching of applications. The purpose of this chapter was to provide an insight into the teaching intervention employed at the Implementation Phase in this research study and thus to provide a more in-depth knowledge of the teaching and learning of applications in Senior-Cycle mathematics.

The ‘Reflective Phase’ of the research was documented, which consisted of quantitative analysis of the student questionnaire to examine the views of students in Senior-Cycle mathematics who participated in the intervention regarding the appropriateness and success of the intervention, and qualitative analysis to promote a more in-depth knowledge of its impact on both the students and the co-operating teachers who participated in the intervention. This analysis showed that students find mathematics more interesting when taught through applications. Consequently, when the students can relate to the problems and identify mathematics in the real-world around them it makes understanding more accessible. In addition, this chapter highlighted the issue of assessment in the teaching of applications and modelling, where the participating students and teachers called for a review of assessment procedures in Senior-Cycle mathematics here in Ireland.

The next chapter will present some conclusions and recommendations for future work in this area of research in Ireland generated from the key findings of the research.
Chapter 8

Thesis Contributions, Recommendations, and Future Work

This study focuses on the problem of improving the teaching and learning of mathematics, particularly upper secondary level in Ireland, by making a provision for the effective teaching of applications. While approaches have been developed to cater for applications in mathematics to date, none have been developed for Senior-Cycle students in Ireland. Thus, this is the first investigation of its type to be undertaken in Ireland and provides significant findings and insights into the context that exists in Ireland.

The key findings from this thesis were presented and discussed in Chapter 7, where the author created a synthesis on the various themes and understandings that emerged during the previous two phases. The quantitative and qualitative analysis of the data collected at the Implementation Phase was discussed at this
stage, including the dissemination of relevant findings. This chapter concludes the thesis by summarising the conclusions of the study, discussing the contribution of the work to the field of mathematics education, making recommendations and putting forward suggestions for future research based on the findings of this investigation. This chapter also provides insights and reflections on the teaching and learning of applications in Senior-Cycle mathematics by providing a ‘systematic approach’ to be adopted by teachers when teaching applications of mathematics.

8.1 Conclusions

The author was concerned with a number of significant aspects which were highlighted through the implementation of a small-scale intervention designed to test the appropriateness of the research design. These aspects included:

Theoretical Analysis

The theoretical analysis put forward by Dubinsky and his colleagues at RUME is validated in both the Exploratory Phase and Implementation Phase of this research study. The author has demonstrated how a genetic decomposition can be used not only to predict how the mathematical constructs of the learner develops, but has taken the process further and used the genetic decomposition as a tool for highlighting the role of applications in the mathematical knowledge of the learners. The author also demonstrated that performing a genetic decomposition of a mathematical concept enabled the author to examine his understanding of the concept of ‘linear functions’ and its place within the domain of functions. It provided a trajectory of learning that a learner may undertake when learning a
new mathematical concept. Such practise by teachers would have the potential to ensure enhanced teacher knowledge of the concepts taught, not just Senior-Cycle mathematics but at all levels of mathematics education.

ACE Teaching Cycle

The *ACE Teaching Cycle* is a good model for classroom practise when teaching mathematics. It provides the teachers with an overall model from which to base their typical mathematics lesson. While in essence it is not a strict model, it provides the teachers with an easy to remember ideal to follow when teaching mathematics, particularly applications. The *activities* provide an excellent opportunity for the applications to be introduced and a vehicle for introducing and exploring mathematical modelling; the *classroom discussion* ensures applications of mathematics can be used to make connections with the mathematical concepts; while the *exercises* allow the students to practise and perform routine problems.

Group Work

It appears that the students from the intervention enjoyed working in groups or pairs. It led to a more enjoyable experience in the mathematics classroom and fuelled better discussion of mathematical concepts and problems. One of the key aspects of group work is proper implementation within the classroom setting. It is important to assign places in a group where weak students are given the opportunity to work with strong students, thus creating opportunities for peer learning. Opportunities for peer discussion, teacher-student and student-teacher discussion are provided in the implementation of appropriate group work. Discussion is vital, and its importance should be stressed at all times by the teacher.
Visualisation

Allowing opportunities for visualisation of mathematical concepts provides opportunities for enhanced understanding. Utilising the dynamic capabilities of GeoGebra ensures this approach can be integrated seamlessly into lessons. While the author acknowledges the cost of technology (i.e. laptop & projector) is a deterrent in such an approach, GeoGebra is free to download. It is widely acknowledged that visualisation is one of the most common techniques employed in effective mathematics teaching (Zarzyki, 2004; Jonassen, 2000; Snir, 1993). When visualising the concepts, it provides an opportunity to maximise the potential benefits of ICT in a classroom situation. Snir (1995) argues that computers can make a unique contribution to the clarification and correction of commonly held misconceptions of phenomena by visualising those ideas. Instead of spending considerable time learning routines by hand, West (1995) argues that the computer can alleviate such time constraints, allowing the students to move more swiftly to higher-level conceptual matters and a variety of practice problems.

Relevance of Applications

The importance of relevant applications cannot be understated. As has been stressed by many researchers (Burkhardt, 2006; Niss 1987; Bajpai, 1976) success in teaching applications depends on the students’ grasp of the applications used in the classroom. In order to create and maintain interest in mathematics set in real-life context the applications/problems used must create a link with the lives of the students. Teachers and educators must explore the backgrounds that are of interest to the students they are teaching, including: sports/music/art/tv/film/subjects at school etc. Without such an insight, the applications presented will be out-dated
and may lead to negative attitudes.

**Time Factor**

Even an experienced teacher will need time to become familiar with the new pedagogical approaches and techniques employed in teaching applications in the mathematics classroom. Furthermore, an extra subject demand on teachers will be experienced in terms of contextual knowledge and knowledge of applications. Thus, it goes without saying that it will take the students time to become familiar with new ways of learning this mathematics and it will also take the teachers time to become familiar with new ways of teaching applications. An introductory phase should be expected when first introducing the students to the teaching and learning of mathematical concepts through the medium of applications.

**Assessment Procedures**

Traditional assessment procedures are not the best measure of understanding for this type of learning. Almost all participants in the intervention expressed an interest in some sort of project based assessment as part of the overall assessment procedure. Similarities in the assessment procedures to other subjects examined in school were expressed such as: geography, science, woodwork etc. where up to 40% of the overall assessment marks are awarded for some sort of project based work. While there are no changes to the assessment procedures implemented in ‘Project Maths’ in terms of project based assessment, assessment for learning is stressed and there will be changes in the terminal assessment. In addition, individual teachers and schools can still explore different types of in-class assessment and mathematical modelling can provide an approach to exploring larger real-life
problems. Mathematical modelling can be assessed through various approaches such as: posters, reports, video reports.

It is also important to ensure the students interpret, justify and validate answers, and teachers do not just accept them without consideration of the application at hand. The students should be asked to explain their answers in words or graphs, which is an approach not traditionally employed in Senior-Cycle mathematics in Ireland.

**Systematic Approach to Teaching Applications of Mathematics**

Following careful analysis of the pedagogical approaches used in teaching applications throughout the intervention it was decided that a systematic approach to the teaching and learning of applications was needed. The six-stage modelling process was favoured by the teachers when used in teaching the modelling process and as a result was the basis from which to begin the development of the systematic approach to the teaching and learning of applications. Thus, the author designed an eight-stage approach to teaching applications of mathematics, where Figure 8.1 provides a diagrammatic overview of the approach.
Figure 8.1: Systematic Approach to Teaching Real-Life Applications of Mathematics

**Stage 1: Application**

Stage 1 begins with the introduction of an appropriate application. The application is chosen based on the age range and mathematical ability of the students. It should also take into account the gender, personal backgrounds, and interests of the students. The application may consist of a word problem, diagram or table of values and without mathematical intervention, the problem may not be solved or necessary conclusions made.
Stage 2: Collateral Knowledge

‘Collateral knowledge’ refers to the basic (non-mathematical) knowledge that is required to understand the application. Each application, as it is presented, requires a basic understanding of the subject/field to which it is relevant to. While the application is presented in a mathematics classroom, to be solved mathematically, the students and teacher require ‘collateral knowledge’ to establish a proper sense of the problem and its context in the real world. It may be necessary to make assumptions regarding the application, taking into account external factors that may have a bearing on the application in the real world, but are not necessary for the purpose of the exercise. ‘Collateral knowledge’ is required by both the teacher and student so that a successful transition can be made to the next stage of the process. It is important that sufficient time (but not too much) is spent at this stage to understand the application and make sense of the problem as it occurs in the real world.

Suggested classroom techniques include:

The teacher should divide the class into small groups/pairs and allow them to discuss the problem. Some classes may need the structure of focusing questions for their discussion. A good technique is to get the students to list key words or to restate the problem. Following discussion in the groups, it is important that the teacher talks to the class as a whole. At this stage the teacher can prompt the students to identify the necessary ‘collateral knowledge’ for the application. Not alone is this approach student-led but also ensures that all groups will become aware of the necessary ‘collateral knowledge’ for the application.
Stage 3: Identify Mathematical Content

This consists of the identification and listing all the variables/equations/values involved in the application. This is a vital stage in the process and it is important that the students are comfortable identifying the mathematical content from the given application. Opportunities should be provided to the students to practise this stage.

*Suggested classroom techniques include:*

Individually, or in groups, have the students brainstorm the problem to uncover the variables/equations/values involved. It is necessary to return to the ‘collateral knowledge’ to determine the mathematical content. No teacher intervention is required here as it is important that the students identify the mathematical content and attempt the next stage by themselves.

Stage 4: Draw a Diagram

At this stage of the process it is useful to draw a diagram/sketch a graph and identify, if possible, the given and required quantities on the diagram. This allows the learner to visualise the problem. Again, opportunities should be provided for the students to practise this stage.

*Suggested classroom techniques include:*

Individually, or in groups, have the students practise drawing a diagram based on the information at hand. Following the students’ opportunity at drawing a diagram, it is important that the teacher shows a correct and accurate diagram to the whole class. Not alone does this provide the students with correct material but it allows an opportunity for the students to correct their thinking of the problem.
if their diagram is not accurate. It may be necessary to establish why you have
provided such a diagram. This will allow the students to re-focus on the problem
and guide them to the next stage.

Stage 5: Determine the Required Mathematical Content

Based on the diagram it may be necessary to reassess the list of variables/equations/values
involved, then to try to simplify or modify the list. This differentiates between the
material that is provided and the material that you require to solve the problem
mathematically. It is important that the mathematical content that is necessary to
obtain the mathematical solution is identified at this stage.

Suggested classroom techniques include:

Individually, or in groups, have the students reassess the problem, based on the
diagram, to uncover the variables/equations/ variables that are necessary to obtain
a mathematical solution to the application. Following the students’ opportunity
to identify the required mathematical content to solve the problem, it is important
that the teacher shows the correct mathematical content to the whole class, so that
groups can re-assess their own choice. It may be necessary to establish why you
have chosen such variables/equations/values over others provided in the original
problem. This will allow the students to re-focus on the problem and obtain the
mathematical solution.

Stage 6: Obtain the Mathematical Solution

At this stage the teacher/student may use the diagram to obtain the solution. How-
ever, it is necessary to check the solution as often the diagram may not be an
accurate or precise drawing. Therefore, in addition the teacher/student should utilise the given variables/equations/values involved and, using algebraic methods, obtain the solution.

Suggested classroom techniques include:

The teacher might obtain the solution on the board using directions from the students. Alternatively, the students, either individually or in groups, might obtain the solution. If the students obtain the solution themselves, it may be necessary for the teacher to provide a correct solution on the board and the method used to obtain the solution. This allows the students to assess their methods. If correct, the students then can reinforce the correct procedures in obtaining the mathematical solution. If incorrect, the students can then identify where they made mistakes and address these mistakes accordingly.

Stage 7: Interpret the Mathematical Solution

After obtaining their solutions, the students are directed back to the problem. They must check to ensure that they have answered the problem within the assumptions they have made. This is an important step in helping students realise that solutions to real-life problems are constrained by the context of the given problem and are not easily transferable to other situations.

Suggested classroom techniques include:

Again the teacher might interpret the solutions using directions from the students. Or alternatively, the students might interpret the solution. It is necessary for the teacher to discuss with the class the validity of the interpretations. In order to determine the validity of the interpretations it is necessary to return to the original
application (Stage 1) and assess the solutions based on the given application. If the solutions are not sufficient or inappropriate it may be necessary to repeat the process in its entirety.

**Stage 8: Make connections to the mathematical concepts involved**

At this stage, the teacher should make the students aware of the key concepts included in the application. This provides opportunities for the teacher to: provide definitions, visualise concepts through diagrams/applets/data tables and make connections between different mathematical concepts. It is widely acknowledged that visualisation is one of the most common techniques employed in effective mathematics teaching (Zarzyki, 2004; Jonassen, 2000; Snir, 1993). When visualising the concepts, it provides an opportunity to maximise the potential benefits of ICT in a classroom situation. Snir (1995) argues that computers can make a unique contribution to the clarification and correction of commonly held misconceptions of phenomena by visualising those ideas.

*Suggested classroom techniques include:*

This is a very appropriate time, if indeed the optimal time, in the students’ learning of a mathematical concept, for the teacher to provide a general definition for a concept based on the application that has been used in the lesson. This allows the students to make connections between what has just been learned through self and guided discovery and a general definition for a given mathematical concept. For quick and accurate graphical representations, the teacher could use dynamic mathematics software such as GeoGebra or Cabri. For statistical and data representations, teachers could use Excel or SPSS.
Conclusion:

This approach requires students to do much more than just give answers (as is prevalent in current assessment and evaluation methods worldwide), instead looking for students to provide reasoning and justification for their answers, thus encouraging student understanding of concepts as opposed to mastery of concepts void of understanding. It approaches the teaching and learning of mathematics from an alternative direction than is traditionally practised in mathematics classrooms throughout the world. Rather than introducing a new mathematical concept, learning the techniques and skills, and then applying these skills to real life-problems, this approach begins with applications and uses these as basis for the learner to make connections between real-life situations and mathematics. It allows the learners to see and appreciate the relevance of mathematics to their everyday lives, while ensuring one of the key objectives of the Leaving Certificate mathematics Syllabus is catered for – that of ‘relational understanding’. It is intended that the diagram (Figure 8.1) visualises the approach, thereby providing the teachers with an easily accessible approach to teaching applications, while staying consistent with the philosophy of the approach.

8.2 Recommendations for Mathematics Teachers

In this section of the thesis the author would like to take the opportunity to discuss further these insights and outline recommendations for mathematics teachers. These recommendations are suggestions based on the author’s opinion and insights into this area of research, generated from undertaking this research project. The focus of this investigation has been on the teaching of applications of math-
ematics in Senior-Cycle schools in Ireland. Clearly the mathematics teacher is going to play a significant role in facilitating approaches to the teaching of applications. The following are a number of suggestions for teachers that can be incorporated into their pedagogic practises:

- **Establish a distinct theoretical analysis for each mathematical topic taught** i.e. functions, trigonometry, co-ordinate geometry etc. By doing so the teacher can gain insights into the model of cognition that a learner may undertake when learning a new mathematical concept. Furthermore it visualises the process, thus making it easier for the teacher to plan lessons and schemes of work.

- **Introduce concepts through real-life examples.** The author feels that there is a need to present the students with an environment which contains as much as possible about the concept at the beginning, as opposed to being sequentially organised, as is traditionally practised within mathematics classrooms. Such an approach ensures the teaching and learning of mathematics is approached from an alternative direction than is traditionally practised in mathematics classrooms throughout the world. Rather than introducing a new mathematical concept, learning the techniques and skills, and then applying these skills to a real life-problem, this approach begins with applications and uses these as a basis for the learner to make connections between real-life situations and mathematics. It allows the learners to see and appreciate the relevance of mathematics to their everyday lives, while ensuring one of the key objectives of the Leaving Certificate Mathematics Syllabus is catered for. As a result, it is intended that the learner will then be better equipped to advance their mathematical knowledge.
• **Ensure the applications are (where possible) relevant to the students’ interests and backgrounds.** As has been stressed by many researchers (Burkhardt, 2006; Niss 1987; Bajpai et al., 1976) success in teaching applications depends on the student’s grasp of the applications used in the classroom. Thus, the teacher should use applications which are culturally dependent and specific to the environment in which they are taught and the personal backgrounds/interests of the students involved.

• **Each application, as it is presented, requires a basic understanding of the subject/field to which it is relevant to.** While the application is presented in a mathematics classroom, to be solved mathematically, the students and teacher require ‘collateral knowledge’ to establish a proper sense of the problem and its context in the real world. It may be necessary to make assumptions regarding the application, taking into account external factors that may have a bearing on the application in the real world, but are not necessary for the purpose of the exercise.

• **Make connections to the mathematical concepts involved** - the teacher should make the students aware of the key concepts included in the application. This provides opportunities for the teacher to: provide definitions, visualise concepts through diagrams/applets/data tables and make connections between different mathematical concepts.

• **There is an urgent need for training and professional development of current teachers in Ireland with respect to the teaching of applications.** The author strongly believes that current mathematics teachers need to develop pedagogic skills in order to develop appropriate problem solving
skills in students so as to facilitate the transition to third-level mathematics courses.

- **Appropriate applications are needed so as to provide a platform for teachers from which they can implement them in their own teaching of mathematics.** However, it must be stressed that teachers need to be aware of the importance of updating and revising applications to suit the needs of their students with regard to their mathematical ability and background interests.

### 8.3 Contributions

A explained in the introduction chapter, this study focused on the problem of improving the teaching and learning of mathematics, particularly upper secondary level in Ireland, by making a provision for the effective teaching of applications. While approaches have been developed to cater for applications in mathematics to date, none have been developed for Senior-Cycle students in Ireland. The author field tested the resultant approach in a small scale intervention in Irish Senior-Cycle schools. Both quantitative and qualitative analysis were used to develop an insight into the appropriateness of the approach. Thus, this is the first investigation of its type to be undertaken in Ireland and provides significant findings and insights into the context that exists in Ireland.

With the introduction of ‘Project Maths’ to all secondary schools in Ireland in 2010 this research study is timely and necessary because of the emphasis on applications in ‘Project Maths’. The approach outlined by the author in Section 5.2 allows the learners to see and appreciate the relevance of mathematics to their ev-
everyday lives, while ensuring one of the key objectives of the Leaving Certificate Mathematics Syllabus is catered for. Such an approach ensures the teaching and learning of mathematics is approached from an alternative direction than is traditionally practised in mathematics classrooms throughout the world and in Ireland. Thus, this research is a significant national contribution. It adds to our knowledge and understanding of the teaching and learning of applications of mathematics from an Irish context and contributes in a positive way to improving the transition from second-level to third-level service mathematics courses.

The approaches developed by Bajpai (1975), the Harvard Calculus Consortium (1991) and others (Ormell, 1972; Meyer & Ludwig, 1991) are practise-led and lack a wider pedagogical perspective suited to secondary school mathematics teaching. APOS Theory offers just such a perspective and theoretical pedagogical framework for mathematics teaching and learning based on constructivist principles. It is a purpose built theory for mathematics teaching at third-level and was developed by Dubinsky (1996) and his colleagues in the Research for Undergraduate Mathematics Education Community (RUMEC). However, APOS Theory offers a perspective on how learning a mathematical concept might take place, it offers no specific foundation for the role of applications in the learning of a mathematical concept. Thus, the author has adapted APOS Theory so as to include a provision for applications.

In his research the author demonstrated how a genetic decomposition can be used not only to predict how the mathematical constructs of the learner develops, but has taken the process further and used the genetic decomposition as a tool for highlighting the role of applications in the mathematical knowledge of the learners. Such an approach provides a trajectory of learning that a learner may under-
take when learning a new mathematical concept. Such practise by teachers would have the potential to ensure enhanced teacher knowledge of the concepts taught, not just Senior-Cycle mathematics but at all levels of mathematics education.

Much of the research carried out by RUMEC was primarily focused on students who were studying mathematics as mathematical specialists. The author has adapted APOS Theory so as to provide for mainstream mathematics students. As a result, this approach ensures it can be used at both second-level mathematics of all levels (foundation, ordinary and higher) and third-level service mathematics, as opposed to the current limitation of mathematics specialists.

Cooperative learning and computer programming were activities that were promoted so as to foster the mental constructions called for by the theoretical analysis and implemented in the ACE Teaching Cycle. This was subsequently adapted to remain consistent with the overall aims of the research study. The Activities were centred around students working in small groups in the computer lab on computer tasks in GeoGebra designed to foster specific mental constructions suggested by the theoretical analysis. Furthermore, the activities were designed so as to ensure the students became familiar with mathematical modelling. In the Classroom Discussion the teacher could avail of the opportunity to provide definitions, explanations and overviews of the concepts being discussed and worked on through the medium of applications. GeoGebra was used, where appropriate, in the classroom discussion to visualise the concepts. Exercises were presented in relatively traditional fashion for students to work on. However, often the students are asked to explain their answers in words or graphs, an approach not traditionally employed in Senior-Cycle mathematics in Ireland.

Finally, the author outlines future research that may be needed to be undertaken
so as to establish a deeper understanding of the teaching of applications of mathematics in Senior-Cycle schools in Ireland.

8.4 Future Work

This investigation is a positive step towards providing support for teachers in Ireland with the teaching of applications of mathematics. The author has demonstrated that there is a need to continue with work in the area of the teaching of applications and modelling in an Irish context. Some suggestions for future research are outlined below:

- Further research is needed to test the potential of the ‘systematic approach to teaching applications’ with teachers of mathematics in second-level both in Ireland and abroad.
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Appendix A
A Chara,

My name is Brian Carroll and I am currently undertaking a PhD in mathematics education at the University of Limerick, under the guidance of Prof. John O Donoghue. The purpose of the study is to investigate the role of applications as they help facilitate the transition from second-level to university mathematics. There is a growing need for more students to make a successful transition from second-level mathematics education to third-level service mathematics education so as improvements in retention and completion rates can be achieved in higher education courses involving service mathematics.

Therefore I am writing to you to invite you and your school to participate in this project. The key proceedings of the project that I would like emphasise are:

**Students Participation:** Pre and post questionnaires designed to test their attitudes towards mathematics, as well as their knowledge and perceptions of applicable mathematics. They will participate in follow-up interviews/focus groups.

**Teachers Participation:** Implement the intervention programme, complete a personal teachers journal, observations of their lessons will be undertaken and they will be required to participate in a follow-up interview.

In addition, throughout this research project:
- All participation is on a voluntary basis and participants can withdraw from the study at any time.
- The tests will remain anonymous.
- The school will not be identified in the write up of the findings.
- All information gathered will be stored confidentially and will be used only for the purpose of this project.

I have enclosed a copy of the Teacher Information Sheet accompanied by the Parent/Carer Information Sheet and the Consent Form for the students, should your school wish to participate in this research project. Please ensure that the students return the consent form to you.

I would like to thank you for taking time to read this letter and if you require any further information please do not hesitate in contacting me at 087- 9063627 or by email at brian.carroll@ul.ie. I look forward to hearing from you.

Le meás,

_______________________
Brian Carroll
Appendix B
Principal Consent Form

Thank you for indicating an interest in participating in my research study. I will be carrying out all research as part of the requirements for completing a PhD by research at the University of Limerick. This form will be signed by you and the researcher and both parties will keep a copy.

Project Title:
An Investigation into Using Applications to Facilitate the Transition from Second-Level to University Mathematics

Researcher: Brian Carroll
Supervisor: Prof. John O’Donoghue

- I have read the Principal Information Sheet and the purpose of the study has been explained to me. I understand the purpose of the study and agree to allow it take part.
- I understand that I may withdraw the school from the study at any stage and if I do so all data relating to the school’s participation will be destroyed immediately.
- I understand that the school will not be identified through the school’s participation in the study and through the supply of information relating to me or any of the school’s students/employees.
- I understand that the data will be stored for the duration of the study with access only by the researcher and the supervisor. I understand that all computer files containing teacher and/or student data and information...
will be kept password protected, while all other data relating to participants will be secured in a locked cabinet.

- I understand that I may contact the researcher or supervisor if I require any further information about the study.
- I understand that a copy of the interview transcript will be made available to you should you wish to check it for accuracy.

Research Consent:
I have read and understood the conditions under which I will participate in this study and give my consent, on behalf of the school, to be a participant.

Do you agree to participate in the CBSAM intervention?
Yes______ No______

Principal
Name: ____________________
Date: ____________________

Researcher
Name: ____________________
Date: ____________________

Contact Name and Number:

Researcher:
Brian Carroll
PhD Student
Dept. of Mathematics & Statistics
University of Limerick
Mobile: 087 9063627
Email: Brian.Carroll@ul.ie

Supervisor:
Prof. John O’Donoghue
Dept. of Mathematics & Statistics
University of Limerick
Phone: 061 202481
Email: John.ODonoghue@ul.ie
Appendix C
A n Investigation into Using Applications to Facilitate the Transition from Second-Level to University Mathematics

Parent/Guardian Information Sheet

The Study:

This project is an investigation into how the introduction of real-life everyday contexts to students at senior-cycle level impacts on mathematics teaching and learning. In particular the project wants to identify:

- If the use and/or exploration of mathematical applications can further enhance this experience for the students.
- The students’ views on what particular mathematical topics are applicable to real-life contexts.
- Teaching strategies to improve the understanding of real-life contexts for these students.

Participation Information:

Your son/daughter will be required to complete the following:

- A Mathematical Report based on a real-life problem
- A Mathematical Attitude Questionnaire

Both of these procedures will take place in school, during school hours, and at a fixed time and date, as arranged with the mathematics teacher. Furthermore, they will be required to attend mathematics class as normal where the following will be explored:

- Applications of real-life contexts
- Approaches to mathematical modelling

There are no risks involved in this study. All information gathered will remain confidential and used only for the purpose of this study. Students will not be required to sign their name to the test instruments. The information gathered will be stored safely with access only available to the investigator.

Your son/daughter is under no obligation to participate in this study. Should you have any questions or do not understand something, please contact me and I will clarify any issues that you are concerned about.

Contact Details: Brian Carroll,
Department of Mathematics and Statistics,
University of Limerick,
Limerick.
087-9063627
Appendix D
Consent Form

**Title of Project:** An Investigation into Using Applications to Facilitate the Transition from Second-Level to University Mathematics

Your child is under no obligation to participate in this study. If they agree to participate, but at a later stage feel the need to withdraw, they are free to do so. It will not affect them in any way.

Please answer all of the following (tick the appropriate box):

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I have read and understood the subject information sheet.

I understand what the project is about, and what the results will be used for.

I am fully aware of all of the procedures involving my child and of any risks and benefits associated with the study.

I know that my child’s participation is voluntary and that they can withdraw from the project at any stage without giving any reason.

I am aware that my child’s results will be kept confidential.

**I agree to participate in the above study**

__________________________  ____________

Signature of Participant

Date

__________________________  ____________

Signature of Parent/Guardian
Appendix E
A n Investigation into Using A pplications to Facilitate the Transition from Second-Level to University M athematics

Student Information Sheet

The Study:

This project is an investigation into how the introduction of real-life everyday contexts to students at senior-cycle level impacts on mathematics learning and teaching. In particular the project wants to identify:

• If the use and/or exploration of mathematical computer software can further enhance this experience for the students.
• The students’ views on what particular mathematical topics are applicable to real-life contexts.
• Teaching strategies to improve the understanding of real-life contexts for these students.

Participation Information:

You will be required to complete the following:

• A Mathematical Attitude Questionnaire
• Geogebra (Computer Software) Training

Both of these procedures will take place in school, during school hours, and at a fixed time and date, as arranged with the mathematics teacher Miss Paula Mangan. Both procedures will last approx. 40 minutes in duration. Furthermore, you will be required to attend mathematics class as normal where the following will be explored:

• Applications of real-life contexts
• Approaches to mathematical modelling

There are no risks involved in this study. All information gathered will remain confidential and used only for the purpose of this study. Students will not be required to sign their name to the test instruments. The information gathered will be stored safely with access only available to the investigator.

You are under no obligation to participate in this study. Should you have any questions or do not understand something, please contact me and I will clarify any issues that you are concerned about.

Contact Details:  
Brian Carroll,  
Department of Mathematics and Statistics,  
University of Limerick,  
Limerick.  
061 - 202207
Appendix F
An Investigation into Using Applications to Facilitate the Transition from Second-Level to University Mathematics

Teacher Information Sheet

This form outlines the purpose of the study and provides a description of your involvement and rights as a participant.

1. What is the purpose of the study?

The purpose of the study is to investigate the role of applications as they help facilitate the transition from second-level to university mathematics. The study aims to design, develop, implement and evaluate a teaching intervention. The intervention is concerned with the teaching and learning of upper second-level mathematics which will primarily involve the use of ICT within a distinct modelling approach specifically aimed at highlighting the potential of mathematics in relation to real-life contexts. In addition, it is intended that an insight into the factors determining the level of understanding and attitudes towards learning school mathematics at upper-secondary level will be incorporated into the research.

2. What is the rationale for the study?

The significance of the ‘Mathematics Problem’ combined with the growing concerns that higher education graduates of engineering, science, business and computing are lacking required levels of mathematical ability for economic development are issues of concern both in Ireland (O’Donoghue, 1996; Hourigan & O’Donoghue, 2006) and worldwide (Croft, 2001; Evensky et al., 1997; Barry & Davis, 1999). There is a growing need for more students to make a successful transition from second-level mathematics education to third-level service mathematics education so as improvements in retention and completion rates can be achieved in higher education courses involving service mathematics. Additionally, it must be noted that the relatively poor take-up of Higher level mathematics (17% of students sitting their Leaving Certificate in 2009 completed the Higher level mathematics paper) rightly gives cause for concern, since it has implications for the follow-on study of mathematics at undergraduate level (State Examinations Commission, 2008).
3. **What will I have to do?**

As a teacher participant in this study you will be required to:

- Take part in an intervention entitled “Support for Applicable Mathematics”.
- Keep a Personal Teacher’s Journal of your experiences teaching the intervention throughout the course of this study.
- Partake in an interview regarding your experiences teaching the intervention.

4. **What are the guiding principles of the intervention?**

The primary focus of this intervention is on conceptual understanding and it implies that where appropriate, topics should be presented graphically, numerically, and symbolically. Ultimately, the order in which these approaches are used varies, and all are regarded as important (O’Keeffe, 1994). This particular approach provides an environment within which applications to everyday real-life contexts are prevalent. Key to this approach is the teacher routinely makes use of technology as they develop the concepts. Furthermore, this approach ensures the students are often unable to appreciate the fact that often a combination of techniques are used when solving problems arising in real-life contexts, both in industry and everyday experiences.

5. **What are the teaching requirements involved?**

The **ACE Teaching Cycle** proposed by Dubinsky and his colleagues in the Research in Undergraduate Mathematics Education Community (RUMEC) has been employed in this intervention and is expected to be undertaken where possible. The three components of the ACE Cycle are: Activities, Class Discussion, and Exercises.

**Activities** are centred around students working in small groups in the computer lab on computer tasks designed to foster specific mental constructions suggested by the theoretical analysis.

**Class Discussion** provides an opportunity for these same groups to work on paper and pencil tasks based on the computer activities. The teacher can also avail of the opportunity to provide definitions, explanations and overviews of the concepts being discussed and worked on.

**Exercises** are presented in relatively traditional fashion for students to work on in teams. They are generally expected to be completed as homework.
6. What if I do not want to take part?

Your participation in this research is voluntary. If at any point during the course of the study you wish to withdraw, you have the right to do so for any reason, and without any prejudice.

7. What happens to the information?

The information gathered will be treated with the utmost confidence and anonymity. Once analysis has been completed all data records will be destroyed. The results from the analysis will be reported in my thesis or in any research papers which disseminate from this. Your real name will not be used at any point in the collection of information, or in the thesis. Instead, you and any other person mentioned during the course of the interviews will be given pseudonyms that will be used in all verbal or written records.

8. What if I have more questions or do not understand something?

You are encouraged to ask any questions at any time about the nature of the study and the methods that I am using. Your suggestions and concerns are important to me; please contact me at any time at the address/phone number/email address below. If you have any further questions or concerns regarding this study, you may also contact my supervisor Professor John O’Donoghue, whose address is also provided below.

9. Contact name and number

**Researcher:**
Brian Carroll  
PhD Student  
Dept. of Mathematics & Statistics  
University of Limerick  
Mobile: 087 9063627  
Email: Brian.Carroll@ul.ie

**Supervisor:**
Prof. John O’Donoghue  
Dept. of Mathematics & Statistics  
Phone: 061 202481  
Email: John.ODonoghue@ul.ie  
University of Limerick
Appendix G
Teacher Consent Form

Thank you for indicating an interest in participating in my research study. I will be carrying out all research as part of the requirements for completing a PhD by research at the University of Limerick. This form will be signed by you and the researcher and both parties will keep a copy.

Project Title:
An Investigation into Using Applications to Facilitate the Transition from Second- Level to University Mathematics

Researcher: Brian Carroll
Supervisor: Prof. John O’Donoghue

• I have read the Teacher Information Sheet and the purpose of the study has been explained to me. I understand the purpose of the study and agree to take part.
• I understand that I may withdraw from the study at any stage and if I do so all data relating to my participation will be destroyed immediately.
• I understand that I will not be identified through my participation in the study and through the supply of information relating to me.
• I understand that the data will be stored for the duration of the study with access only by the researcher and the supervisor. I understand that all computer files containing teacher and/or student data and information
will be kept password protected, while all other data relating to participants will be secured in a locked cabinet.

- I understand that I may contact the researcher or supervisor if I require any further information about the study.
- I understand that a copy of the interview transcript will be made available to you should you wish to check it for accuracy.

Research Consent:
I have read and understood the conditions under which I will participate in this study and give my consent to be a participant.

Do you agree to participate in the intervention?
Yes______ No______

Do you grant permission to be quoted directly?
Yes______ No______

Do you grant permission to be audio taped?
Yes______ No______

<table>
<thead>
<tr>
<th>Participant</th>
<th>Researcher</th>
</tr>
</thead>
<tbody>
<tr>
<td>Name: ___________</td>
<td>Name: ___________</td>
</tr>
<tr>
<td>Date: ___________</td>
<td>Date: ___________</td>
</tr>
</tbody>
</table>

Contact Name and Number:

**Researcher:**
Brian Carroll
PhD Student
Dept. of Mathematics & Statistics
University of Limerick

**Mobile:** 087 9063627
**Email:** Brian.Carroll@ul.ie

**Supervisor:**
Prof. John O’Donoghue
Dept. of Mathematics & Statistics
John.ODonoghue@ul.ie
University of Limerick

**Phone:** 061 202481
**Email:**
Appendix H
# Teacher’s Journal

<table>
<thead>
<tr>
<th>Date:</th>
<th>Class Duration:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Venue:</td>
<td>Topic:</td>
</tr>
</tbody>
</table>

## Class Aims:

## Learning Outcomes:
Summary of Class:

Please comment on the following:

1) Student’s Work
2) Accompanying Material

3) Real-life Problems

4) Geogebra Applets
Overall Reflection:
Appendix I
Teacher’s Journal

Lesson No:

Please comment on your thoughts regarding the appropriateness and effectiveness of the following:

1) Accompanying Material (i.e. Teacher Manual/PowerPoint)

2) Real-life Problems

3) Geogebra Applets

Overall Reflection:

Code: __________
Appendix J
An Investigation into Using Applications to Facilitate the Transition from Second-Level to University Mathematics

Please answer **All** questions honestly and to the fullest extent. Please **tick/circle** the appropriate answer to each question. All information will be kept confidential and only used for this study.

**Student Profile**

Please indicate your answer clearly:

1. What is your age? ___________

2. Male □
   Female □

3. Junior Cert Level:  
   Higher □
   Ordinary □

4. Junior Cert Grade:

   A  B  C  D  E  F  NG
Section A

Section A is aimed at discovering which mathematical topics you (the student) consider applicable to real-life contexts.

1. What mathematical topics would you consider applicable to real-life contexts? i.e. mathematics that can be used in everyday situations

_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
_______________________________________________

2. What other subject areas (that you study in school) would you consider that provides the opportunity to use mathematics?

_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
3. Please tick the following mathematical topics according to what you consider applicable to real-life contexts. You may tick more than one answer.

<table>
<thead>
<tr>
<th>Topic</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic</td>
<td></td>
</tr>
<tr>
<td>Algebra</td>
<td></td>
</tr>
<tr>
<td>Linear equations</td>
<td></td>
</tr>
<tr>
<td>Quadratic equations</td>
<td></td>
</tr>
<tr>
<td>Inequalities</td>
<td></td>
</tr>
<tr>
<td>Logarithms</td>
<td></td>
</tr>
<tr>
<td>Complex numbers</td>
<td></td>
</tr>
<tr>
<td>Matrices</td>
<td></td>
</tr>
<tr>
<td>Geometry</td>
<td></td>
</tr>
<tr>
<td>Vectors</td>
<td></td>
</tr>
<tr>
<td>Trigonometry</td>
<td></td>
</tr>
<tr>
<td>Sequences and Series</td>
<td></td>
</tr>
<tr>
<td>Functions</td>
<td></td>
</tr>
<tr>
<td>Curve Sketching</td>
<td></td>
</tr>
<tr>
<td>Differentiation</td>
<td></td>
</tr>
<tr>
<td>Integration</td>
<td></td>
</tr>
<tr>
<td>Statistics</td>
<td></td>
</tr>
<tr>
<td>Probability</td>
<td></td>
</tr>
</tbody>
</table>
4. Please tick the following mathematical topics according to what you consider **least** applicable to real-life contexts. You may tick more than one answer.

<table>
<thead>
<tr>
<th>Arithmetic</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebra</td>
<td></td>
</tr>
<tr>
<td>Linear equations</td>
<td></td>
</tr>
<tr>
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<td></td>
</tr>
<tr>
<td>Inequalities</td>
<td></td>
</tr>
<tr>
<td>Logarithms</td>
<td></td>
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<tr>
<td>Complex numbers</td>
<td></td>
</tr>
<tr>
<td>Matrices</td>
<td></td>
</tr>
<tr>
<td>Geometry</td>
<td></td>
</tr>
<tr>
<td>Vectors</td>
<td></td>
</tr>
<tr>
<td>Trigonometry</td>
<td></td>
</tr>
<tr>
<td>Sequences and Series</td>
<td></td>
</tr>
<tr>
<td>Functions</td>
<td></td>
</tr>
<tr>
<td>Curve Sketching</td>
<td></td>
</tr>
<tr>
<td>Differentiation</td>
<td></td>
</tr>
<tr>
<td>Integration</td>
<td></td>
</tr>
<tr>
<td>Statistics</td>
<td></td>
</tr>
<tr>
<td>Probability</td>
<td></td>
</tr>
</tbody>
</table>

Code: _______
Appendix K
An Investigation into Using Applications to Facilitate the Transition from Second-Level to University Mathematics

Please answer ALL questions honestly and to the fullest extent.
Please tick/circle the appropriate answer to each question.
All information will be kept confidential and only used for this study.

Student Profile

Please indicate your answer clearly:
1. What is your age? __________

2. Male □
   Female □

3. Junior Cert Level: Higher □
   Ordinary □

4. Junior Cert Grade:
   A    B    C    D    E    F    NG

Section A
Section A is concerned with determining your attitudes and reactions to the teaching intervention that you have just undertaken.

**Instructions:** Draw a circle around the letter(s) that show(s) how closely you agree or disagree with each statement:
SD = Strongly Disagree, D = Disagree, U = Undecided, A = Agree, SA = Strongly Agree.

<table>
<thead>
<tr>
<th>Circle the appropriate response for each statement below.</th>
<th>Strongly disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. I believe studying mathematics helps me with problem solving in other areas/subjects.</td>
<td>SD</td>
<td>D</td>
<td>U</td>
<td>A</td>
<td>SA</td>
</tr>
<tr>
<td>2. I do not like writing reports for mathematics activities e.g. a report on a problem.</td>
<td>SD</td>
<td>D</td>
<td>U</td>
<td>A</td>
<td>SA</td>
</tr>
<tr>
<td>3. Mathematics is more interesting when taught through applications to real-life situations.</td>
<td>SD</td>
<td>D</td>
<td>U</td>
<td>A</td>
<td>SA</td>
</tr>
<tr>
<td>4. Mathematics is not important in everyday life.</td>
<td>SD</td>
<td>D</td>
<td>U</td>
<td>A</td>
<td>SA</td>
</tr>
<tr>
<td>5. Working in pairs or groups makes mathematics class more enjoyable.</td>
<td>SD</td>
<td>D</td>
<td>U</td>
<td>A</td>
<td>SA</td>
</tr>
<tr>
<td>6. I do not understand why we need to study mathematics in school.</td>
<td>SD</td>
<td>D</td>
<td>U</td>
<td>A</td>
<td>SA</td>
</tr>
<tr>
<td>7. I believe studying mathematics will help me when I go to college or when I am working.</td>
<td>SD</td>
<td>D</td>
<td>U</td>
<td>A</td>
<td>SA</td>
</tr>
</tbody>
</table>
Section B

Section C is aimed at discovering which mathematical topics you (the student) consider applicable to real-life contexts.

1. What mathematical topics would you consider applicable to real-life contexts? i.e. mathematics that can be used in everyday situations

_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
_________________________________________________________________

2. What other subject areas (that you study in school) would you consider that provides the opportunity to use mathematics?

_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
_________________________________________________________________
3. Please tick the following mathematical topics according to what you consider applicable to real-life contexts. You may tick more than one answer.

<table>
<thead>
<tr>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic</td>
</tr>
<tr>
<td>Algebra</td>
</tr>
<tr>
<td>Sets</td>
</tr>
<tr>
<td>Number Systems</td>
</tr>
<tr>
<td>Measure (Area &amp; Volume etc.)</td>
</tr>
<tr>
<td>Statistics</td>
</tr>
<tr>
<td>Functions</td>
</tr>
<tr>
<td>Trigonometry</td>
</tr>
<tr>
<td>Geometry</td>
</tr>
<tr>
<td>Graphs</td>
</tr>
</tbody>
</table>

4. Please tick the following mathematical topics according to what you consider **least** applicable to real-life contexts. You may tick more than one answer.

<table>
<thead>
<tr>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic</td>
</tr>
<tr>
<td>Algebra</td>
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<tr>
<td>Sets</td>
</tr>
<tr>
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</tr>
<tr>
<td>Measure (Area &amp; Volume etc.)</td>
</tr>
<tr>
<td>Statistics</td>
</tr>
<tr>
<td>Functions</td>
</tr>
<tr>
<td>Trigonometry</td>
</tr>
<tr>
<td>Geometry</td>
</tr>
<tr>
<td>Graphs</td>
</tr>
</tbody>
</table>
Appendix L
The Study

The Applications:

• Did you find the problems to be realistic? In other words could you relate to them? (If yes, which ones?)

• Would you like to see more of these problems on the mathematics syllabus for all topics?

• Did this approach help you understand why you study mathematics (or at least this topic – Linear Functions)?

Modelling:

• Is the modelling process easy to follow?

• Did you like to do report writing for a real life problem? Why?

• Would you like to see this as part of the assessment for the Leaving Certificate? (If yes - In what way?)

Geogebra:

• Did you think that the Geogebra Applets were beneficial to your learning?

• Would you use Geogebra in your own time?
Appendix M
Teacher Interview Questions

The Study

The Applications:

• Did you find the applications to be realistic? Which ones?

• Do you think it allowed the students to see the relevance of mathematics to the real world?

• Would you approach teaching applications in the real world differently now especially with the introduction of Project Maths next year?

Modelling:

• What did you think of mathematical modelling from a teaching perspective?

• Is the modelling process easy to follow? Do you think it is feasible in a classroom setting?

• What are your views on report writing?

• Would you like to see modelling as part of the assessment for the Leaving Certificate? (If yes - In what way?)

Geogebra:

• Were you comfortable using Geogebra?

• Would you use it in your teaching of mathematics in the future?

• Do you think it would benefit the students understanding of mathematical concepts?
Appendix N
Objectives:
- Examine the intercepts of a graph
- Investigate linear functions and their applications
- Investigate the graph of a linear function in slope-intercept form
- Investigate quadratic functions and their applications
- Evaluate functions at a given point

The Graph of an Equation

In 1637 René Descartes, a French mathematician and philosopher, transformed the study of mathematics by connecting the two major fields of mathematics - algebra and geometry. His book *La Géométrie* gave birth to analytical geometry, better known as Cartesian geometry.

The same approach can be followed in your study of mathematics, by viewing mathematics from numerous perspectives - graphically, numerically, and symbolically (algebraically); you will increase your understanding of core concepts.

Example 1:

Consider the equation: \(-x + y = -5\).

To find solutions for this equation, solve the equation for \(y\).

\[ y = x - 5 \quad \text{Algebraic approach} \]

A table of values can also be constructed by substituting several values of \(x\) into \(y\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>-5</td>
<td>-4</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
</tr>
</tbody>
</table>

*Table 1: Numerical approach*

From the table, you can see that (0, -5), (1, -4), (2, -3), (3, -2) and (4, -1) are solutions of the original equation \(-x + y = -5\). However, these are not the only solutions, as this equation has an infinite number of solutions. The set of all solution points is the graph of the equation as shown in Figure 1.
Descartes made many contributions to philosophy, science and mathematics. The idea of representing points in the plane by pairs of real numbers and representing curves in the plane by equations was described Descartes in his book *La Géométrie* in 1637. The work established Descartes as the founder of analytic geometry.
Using the Straight Line Applet:

**Figure 2: “Straight Line” Applet**

Open the Geogebra applet entitled “The Straight Line” from the webpage: http://www.ul.ie/cemtl/applets.htm

The page above should be displayed:

There are two sliders in this applet, labelled m and c, that represent the slope (m) and the y-axis intercept (c). You can use your mouse to catch each of these sliders and increase or decrease the values of the slope (m) and the y-intercept (c). We use the Greek letter, \( \theta \), to name the angle that the line makes with the positive direction of the x-axis. The slope of the line is then given by \( m = \tan \theta \).

As you move the slider marked m you will see that the slope of the line changes. Note that the point where the line meets the y-axis is unchanged regardless of the value of m selected. This is because when a line intersects the y-axis its x value is 0 so this gives

\[
\begin{align*}
y &= mx + c \\
y &= m \times 0 + c \\
y &= 0 + c \\
y &= c
\end{align*}
\]
so the point \((0, c)\) is always on the line regardless of the \(m\) value selected.

Now move the second slider, marked \(c\), and note what happens to the line. Remember that the variable \(c\) controls the \(y\)-intercept of the line. This means that as \(c\) is varied the location where the line crosses the \(y\)-axis varies. Whatever value the slider \(c\) is fixed at represents the value where the \(y\)-axis and the straight line intersect.

**Intercepts of a Graph**

Two types of solutions that are especially useful when graphing an equation are those having zero as their \(x\) or \(y\) co-ordinates. Such points are called **intercepts** because they are the points at which the graph cuts the \(x\) or \(y\) axis.

<table>
<thead>
<tr>
<th>Intercepts of a Graph:</th>
</tr>
</thead>
<tbody>
<tr>
<td>• an (x)-intercept is a point on the graph where (y) is zero, and</td>
</tr>
<tr>
<td>• a (y)-intercept is a point on the graph where (x) is zero.</td>
</tr>
</tbody>
</table>

In addition to the above considerations, you should think of the following terms interchangeably:

"\(x\)-intercepts" = "roots" = "solutions"

In other words, the following exercises are equivalent:

- Find the \(x\)-intercept(s) of \(y = x^3 + 2x^2 - x - 3\)
- Solve \(x^3 + 2x^2 - x - 3 = 0\),
- Find the roots of \(f(x) = x^3 + 2x^2 - x - 3\)

It is possible for a graph to have no intercepts, or it might have several:
Figure 3: No x-intercepts, One y-intercept

Figure 4: Three x-intercepts, One y-intercept
Example 2:

Find the x and y intercepts of the graph of \( y = x^3 - x \)

Solution

To find the x-intercepts, let y equal to zero and solve for x.

\[
\begin{align*}
x^3 - x &= 0 \\
x(x^2 - 1) &= 0 \\
x^2 - 1 &= 0 \quad \text{and} \quad x = 0
\end{align*}
\]

Factorise

\[
\begin{align*}
x &= \pm 1
\end{align*}
\]

As the equation has three solutions, we can now conclude that the graph has three x-intercepts:

\((0,0), (-1, 0) \text{ and } (1, 0)\)  \text{x-intercepts}

To find the y-intercepts, let x equal to zero and solve for y.

Doing this produces \( y = 0 \). So the y-intercept is:

\((0,0)\)  \text{y-intercept}
Exercise 1.1

Find the x and y intercepts (if any) of the following:

1). \( y = x^2 - 4x + 9 \)  
2). \( y = -2x^2 + x^3 \) 
3). \( y = 4 - 8x - 2x^2 \)  
4). \( x^2 + y^2 = 5 \) 
5). \( y = 8 - 2x^2 \)  
6). \( y = x^3 - 6x^2 + 9x \)

Functions

Olympic and World Records

During the early years of the Olympics, the height of the men’s winning pole vault increased approximately 8 inches every four years. Table 2 shows that the height started at 130 inches in 1900, and increased by the equivalent of 2 inches per year. So the height was a linear function of time from 1900 to 1912. If \( y \) is the winning height and \( t \) is the number of years since 1900, we can write

\[
y = f(t) = 2t + 130
\]
Since \( y = f(t) \) increases with \( t \), we say that \( f \) is an increasing function. The coefficient 2 tells us the rate, in inches per year, at which the height increases.

<table>
<thead>
<tr>
<th>Year</th>
<th>1900</th>
<th>1904</th>
<th>1908</th>
<th>1912</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (inches)</td>
<td>130</td>
<td>138</td>
<td>146</td>
<td>154</td>
</tr>
</tbody>
</table>

**Table 2: Men’s Olympic pole vault winning height (approximately)**

This rate of increase is the slope of the line in Figure 7. The slope is given by the ratio:

\[
\frac{\text{rise}}{\text{run}} = \frac{146 - 138}{8 - 4} = \frac{8}{4} = 2 \text{ inches/year}
\]

![Figure 7: Graph of Men’s Olympic pole vault winning height](image)

Calculating the slope using any two points on the line gives the same value.

What about the constant 130? This represents the initial height in 1900, when \( t = 0 \). Geometrically, 130 is the intercept of the vertical (y) axis.

Does the linear trend continue beyond 1912? Not surprisingly it doesn’t. The formula \( y = f(t) = 2t + 130 \) predicts that the height in the 2012 London games would be 354 inches!!

As the pole vault heights have increased over the years, the time to run the mile has decreased. If \( y \) is the world record time to run the mile, in seconds, and \( t \) is the number of years since 1900, then the record shows that, approximately,

\[
y = g(t) = 260 - 0.4t
\]

The 260 tells us that the world record was 260 seconds in 1900 (at \( t = 0 \)). The slope -0.4 tells us that the world record decreased by about 0.4 seconds per year. We say \( g \) is a decreasing function.
Definition of a function

In mathematics, a **function** is used to represent the dependence of one quantity upon another. We can define a function as follows:

Let $X$ and $Y$ be sets of real numbers. A **function** $f$ is a rule that assigns to each number $x$ in $X$ a single number $y$ in $Y$. $X$ is called the **domain** of $f$. The number $y$ is the **image** of $x$ under $f$ and is denoted by $f(x)$. The set of images of elements of $X$ is called the **range** of $f$.

Or more simply put:

A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

The input variable ($x$) is called the **independent variable**, and the output variable ($y$ or $f(x)$) is called the **dependent variable**.

Functions are often depicted pictorially as in Figure 7. The diagram illustrates, for example, that the function $f$ takes a value, $a$, in the domain and assigns it to a unique value $f(a)$ in its range.

A function could, for example, indicate the take-home pay of an employee, ($y$) given the employee earns the minimum wage of €8.65 for each hour worked ($t$). This would allow us to express the function as:

$$f(t) = 8.65 \times t$$
Figure 8: Pictorial representation of $f(t) = 8.65 \, t$

We can represent any given function pictorially; however, it is not always appropriate to do so!
For example in the function $f(x) = x^2$, we can easily represent $-2 \leq x \leq 3$ on a Venn Diagram.

Figure 9: Pictorial representation of $f(x) = x^2$,

However, it would not be appropriate to illustrate the function, $f$, for $-250 \leq x \leq 300$.

The modern notation for a function is derived from the efforts of many 17th and 18th Century mathematicians, and in particular the work of Gottfried Wilhelm Leibniz and Leonhard Euler
Function Notation

The word function was first used by Gottfried Wilhelm Leibniz in 1694 as a term to denote any quantity connected with a curve, such as the co-ordinates of a point on a curve or the slope of a curve. Forty years later, Leonhard Euler used the word function to describe any expression made up of a variable and some constants. He introduced the notation

\[ y = f(x) \]

Gottfried Wilhelm Leibniz (1646 – 1716)

Gottfried Leibniz was a German mathematician who developed the present day notation for the differential and integral calculus (independently of Sir Isaac Newton). In his correspondence with the leading intellectual and political figures of his era, he discussed mathematics, logic, science, history, law, and theology. Furthermore, Leibniz is known among philosophers for his wide range of thought about fundamental philosophical ideas and principles.
Leonhard Euler (1707 – 1783)

Leonhard Euler was a Swiss mathematician who made enormous contributions to a wide range of mathematics and physics including analytic geometry, trigonometry, geometry, calculus and number theory. Furthermore, Euler was one of the first to apply calculus to real-life problems in physics. His extensive published writings include such topics as shipbuilding, acoustics, optics, astronomy, mechanics and magnetism.

By the end of the 18th Century, mathematicians and scientists had concluded that many real-world phenomena could be represented by mathematical models taken from a collection of functions called elementary functions.

Elementary functions fall into three categories:
1. Algebraic functions (polynomial, radical, rational)
2. Trigonometric functions (sine, cosine, tangent etc.)
3. Exponential and logarithmic functions

The most common type of algebraic function is a polynomial function

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \quad a_n \neq 0 \]

where the positive integer \( n \) is the degree of the polynomial. The numbers \( a_i \) are coefficients, with \( a_n \) the leading coefficient and \( a_0 \) the constant term of the polynomial function.

For polynomial functions of low degree the following simpler forms are often used:

- Zeroth Degree: \( f(x) = a \) — Constant function
- First Degree: \( f(x) = ax + b \) — Linear Function
- Second Degree: \( f(x) = ax^2 + bx + c \) — Quadratic Function
- Third Degree: \( f(x) = ax^3 + bx^2 + cx + d \) — Cubic Function
**Exercise 1.2**

Classify each of the following functions.

1). \( f(x) = 2x^2 + 5x + 10 \) 
2). \( f(x) = 12x^2 + 5x^3 + 1 \)
3). \( f(x) = 15x + 7 \)
4). \( f(x) = 2x^2(x - 4) \)
5). \( f(x) = \frac{1}{2}x^3 + 2 \)
6). \( f(x) = \cos(x - 4) \)
7). \( f(x) = \ln(x) \)
8). \( f(x) = e^{2x^2 + 4} \)

**Linear Functions**

A function is linear if its slope, or rate of change, is the same at every point. Linear functions are used to model situations that show a constant rate of change between 2 variables.

For example, linear functions are used in such ways to explore mobile phone plans, hire rates for taxis, consultation fees for hospitals and salary arrangements for companies. Each issue carries their own significance and it is important that we as consumers, clients, and employees etc. can appreciate the influence of mathematics behind real-life decisions.

**Mobile Phone Bill Tariff**

In the example below a mobile phone bill tariff is examined where there is a fixed charge of €50 for 300 minutes. By examining the graph we can deduce that for each 100 minutes used thereafter there is a €12.60 charge.
Example 3: Engineering Design

In a daredevil car stunt, the ramp rises to a height of 6 feet, with the ramp length being 21 feet long, as shown in Figure 7. The slope of the ramp is the ratio of its height (the rise) to the length of its base (the run).

\[
\text{Slope of ramp} = \frac{\text{rise}}{\text{run}} = \frac{6 \text{ feet}}{21 \text{ feet}} = \frac{2}{7}
\]

Rise is vertical change, Run is horizontal change

Figure 10: Real-life Application - Phone Bill Tariff

Figure 11: Dimensions of a stunt ramp
Slope of a linear function:

We use the symbol ∆ (the Greek letter capital delta) to mean “change in”. Therefore, ∆x means change in x and ∆y means change in y.

We can then calculate the slope of a linear function $y = f(x)$ as follows: As you move from left to right along the line, a vertical change of

$\Delta y = y_2 - y_1$ Change in y

corresponds to a horizontal change of

$\Delta x = x_2 - x_1$ Change in x

The definition of the slope of a line or linear function can be expressed as:

$$ m = \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta y} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1} $$

![Figure 12: Slope of a linear function](image)

In general, the greater the value of the slope of a line, the steeper the line is. For instance in Figure 13c the line with a slope of -2.5 is steeper than the line in Figure 13a with a slope of 1/5.
Figure 13a: If \( m \) is positive, then the line rises from left to right.

Figure 13b: If \( m \) is zero, then the line is horizontal.

Figure 13c: If \( m \) is negative, then the line falls from left to right.
Figure 13d: If \( m \) is undefined (i.e. \( \Delta x = 0 \)), the line is vertical.

**Example 5:**

Estimate the slope of the line, \( l \), from its graph:

Figure 14: Graph of the line \( l \)

Solution
From the graph above we can see that $\Delta y = 2$, i.e. from the point $(0,0)$ to $(0, 2)$ there is a change of +2 for the y-value.
The $\Delta x = 3$ i.e. from the point (-3,0) to (0, 0) there is an increase of +3 for the x-value.
From the formula $m = \frac{\Delta y}{\Delta x}$, we can conclude that $m = \frac{2}{3}$.

**Exercises 1.3**
Estimate the slope of the line from the following graphs:
1). 
2). 
3). 
4).
Families of Linear Functions

A linear function has the form

\[ y = f(x) = mx + b \]

Its graph is a line such that
- \( m \) is the slope, or rate of change of \( y \) with respect to \( x \)
- \( b \) is the y intercept at \((0, b)\)

Formulas such as \( f(x) = mx + b \), in which the constants \( m \) and \( b \) can take on various values, give a family of functions. All the functions in a family share certain properties - in this case all the graphs are straight lines. The constants \( m \) and \( b \) are called parameters; their meaning is shown in Figure 15 and 16.
Non-Linear Functions

The Quadratic Function

Using the “Quadratic Graph” Applet:
Expressions of the form \( f(x) = ax^2 + bx + c \), \( a, b, c \in \mathbb{Z} \) are called quadratic expressions.

The name comes from the Latin word, quadratus, which means square. When we graph functions of the form \( y = f(x) = ax^2 + bx + c \) we get a curve. There are two possibilities for the general shape of the curve: U-shaped or \( \cap \)-shaped. A quadratic curve will be U-shaped whenever the parameter \( a \) (i.e. the \( x^2 \) coefficient) is positive and \( \cap \)-shaped whenever \( a \) is negative.

There are three sliders in this applet to allow you change the values of \( a \), \( b \) and \( c \). Move the slider, marked \( a \), first. See the way the shape of the graph changes as the sign of \( a \) changes from + to −.

Next move the slider marked \( c \). Note the way the graph moves vertically up and down. The reason is the same as in the linear case.

If we put \( x = 0 \) then

\[
\begin{align*}
y &= a \times 0^2 + b \times 0 + c \\
y &= 0 + 0 + c \\
y &= c
\end{align*}
\]

which means that the point \((0, c)\) is always on the graph.

Now move the slider marked \( b \). Note that the point \((0, c)\) stays fixed and the graph moves around it.
The rate of change of a function that is not linear may vary from point to point. A **non-linear function** is defined as a polynomial function of degree 2 or higher, while trigonometric, exponential or logarithmic functions can also be defined as non-linear functions.

One very important application is to find the stopping distance of a car travelling at a given velocity. Suppose that a car is travelling at a certain speed, $s$, and you apply the brakes, how long will it take to stop? Everybody should be interested in this question, especially if it means avoiding an accident!

If $s$ is the speed of the car in km/ph, and $t$ is the time in seconds, then from the time the car starts moving to the time it stops can be expressed as

$$f(s) = -t^2 + 5t$$

We can see from the graph that from the moment the brakes are applied ($t = 2.5$) the speed of the car deceases until the cars stops ($s = 0$).

Solving the quadratic equation correctly here could, quite literally, save your, or someone else's, life!

Non-linear functions can be found in numerous aspects of our real lives including such situations as: areas, tax, architecture, sundials, stopping, electronics, micro-chips, fridges, acceleration, paper, planets, shooting.

**Evaluating a Function**
Function notation provides the advantage of clearly identifying the function \( f, x \) as the independent variable and the dependant variable as \( f(x) \). Therefore, instead of asking: “What value of \( y \) corresponds to \( x = 5 \)?” you can ask “What is \( f(5) \)?”.

**Example 3:**

For the function \( f \) defined by \( f(x) = x^2 - 10 \), evaluate each of the following.

- \( f(5a) \)
- \( f(b-2) \)
- \( f(x + \Delta x) \)
- \( f(x + \Delta x) - f(x) \)

**Solution**

\[
\begin{align*}
\text{Evaluate } f(5a) & \quad = (5a)^2 - 10 \quad \text{Substitute } 5a \text{ for } x \\
& \quad = 25a^2 - 10 \quad \text{Simplify} \\

\text{Evaluate } f(b-2) & \quad = (b-2)^2 - 10 \quad \text{Substitute } b - 2 \text{ for } x \\
& \quad = b^2 - 4b + 4 - 10 \quad \text{Simplify} \\
& \quad = b^2 - 4b - 6 \\

\text{Evaluate } f(x + \Delta x) & \quad = (x + \Delta x)^2 - 10 \\
& \quad = x^2 + 2x\Delta x + (\Delta x)^2 - 10 \\

\text{Evaluate } f(x + \Delta x) - f(x) & \quad = x^2 + 2x\Delta x + (\Delta x)^2 - 10 - (x^2 - 10) \\
& \quad = 2x\Delta x + (\Delta x)^2 \\

\end{align*}
\]

**Exercises 1.4**

Evaluate each of the following:

1. \( f(x) = 2x - 7 \)
   - a) \( f(0) \)
   - b) \( f(-3x) \)
   - c) \( f(ab) \)
   - d) \( f(x-5) \)

2. \( f(x) = 3 + x - x^2 \)
   - a) \( f(10) \)
   - b) \( f(c) \)
   - c) \( f(\sqrt{3}) \)
   - d) \( f(x + \Delta x) \)
Exercises and Problems for Section 1

Exercises
1. The population of a city, \( P \), in millions, is a function of \( t \), the number of years since 1970, so \( P = f(t) \). Explain the meaning of the statement \( f(35) = 12 \) in terms of the population of the city.

2. The value of a car, \( V = f(a) \), in thousands of euros, is a function of the age of the car, \( a \), in years.
   a). Interpret the statement \( f(7) = 8 \)
   b). Sketch a possible graph of \( V \) against \( a \). Is \( f \) an increasing or decreasing function. Explain.
   c). Explain the significance of the horizontal and vertical intercepts in terms of the value of the car.

For Exercises 3-6, find the equation of the line that passes through the given points using slope-intercept equation of a line.

3. (0, 0) and (1, 5)
4. (0, 2) and (2, 9)
5. (0,-5) and (4, 7)
6. (0, 1.5) and (-2.5, 7.5)

For exercises 7-10, determine the slope and the y-intercept of the line whose equation is given.

7. \( 7y + 12x - 8 = 0 \)
8. \( -4y + 2x + 13 = 0 \)
9. \( 12x = 6y + 17 \)
10. \( -9x -9y = 18 \)

11. Match the graphs in Figure 18 with the following equations
   a) \( y = 2x - 5 \)  
   b) \( -7x + 3 = y \)  
   c) \( -2 = y \)  
   d) \( y = 4x + 5 \)  
   e) \( y = x - 9 \)  
   f) \( y = \frac{2x}{2} \)

12. Match the graphs in Figure 19 with the following equations
a) \( y = -2.86x \)  

b) \( y = 0.02x + 0.05 \)  

c) \( y = 12.8 + 0.5x \)  

d) \( y = 12.8x + 15 \)  

e) \( y = -8.9 -2x \)  

f) \( y = \frac{x}{-3.14} \)  

In exercises 13-20, evaluate (if possible) the function at the given value(s) of the independent variable. Simplify the results.

13. \( f(x) = 2x - 7 \)  
a) \( f(0) \)  
b) \( f(-5) \)  
c) \( f(x -1) \)  
d) \( f(b) \)  

14. \( f(x) = \sqrt{x +13} \)  
a) \( f(-2) \)  
b) \( f(6) \)  
c) \( f(c) \)  
d) \( f(x + \Delta x) \)  

15. \( f(x) = 9 -x^2 \)  
a) \( f(0) \)  
b) \( f(\sqrt{3}) \)  
c) \( f(-2) \)  
d) \( f(t- 1) \)  

16. \( g(x) = x^2(x +5) \)  
a) \( g(4) \)  
b) \( g(\frac{5}{2}) \)  
c) \( g(c) \)  
d) \( g(m + 4) \)  

17. \( h(x) = x^3 \)  

18. \( f(x) = 3x - 1 \)  

19. \( g(x) = \frac{1}{\sqrt{x -1}} \)  

20. \( f(x) = x^3 -x \)  

In exercises 21-24, match the data with a function from the following list

i) \( f(x) = dx \)  

ii) \( g(x) = dx^2 \)  

iii) \( h(x) = d\sqrt{|x|} \)  

iv) \( r(x) = \frac{d}{x} \)  

21.  

<table>
<thead>
<tr>
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<th>-4</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>4</th>
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<td>-2</td>
<td>0</td>
<td>-2</td>
<td>-32</td>
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</table>

22.  

<table>
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<th>-1</th>
<th>0</th>
<th>1</th>
<th>4</th>
</tr>
</thead>
<tbody>
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<td>( y )</td>
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<td>-( \frac{1}{4} )</td>
<td>0</td>
<td>( \frac{1}{4} )</td>
<td>1</td>
</tr>
</tbody>
</table>

23.  

<table>
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<tr>
<th>( x )</th>
<th>-4</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-8</td>
<td>-32</td>
<td>undefined</td>
<td>32</td>
<td>8</td>
</tr>
</tbody>
</table>
24.

\[
\begin{array}{c|c|c|c|c|c}
 x & -4 & -1 & 0 & 1 & 4 \\
 y & 6 & 3 & 0 & 3 & 6 \\
\end{array}
\]

Determine the value of the constant, \(d\), for each function such that the function fits the data shown in the table.

**Problems**

25. Match each story to the most appropriate graph in Figure 20. Write a story for the remaining graph.
   a). On the way out of town I had to stick to the speed limit, when I then realised I forgot something so had to turn around and go back.
   b). My car broke down so I had to wait until the AA arrived with assistance.
   c). I drove as far as the train station and then got a train the rest of the journey.

26. A car starts out slowly and then goes faster and faster until a tire blows out. Sketch a possible graph of the distance the car travels as a function of time.

27. A flight from Dublin to Shannon has to circle Shannon several times before being allowed to land. Sketch a graph of the distance of the plane from Dublin to Shannon against time, from the moment of takeoff until landing.

28. The monthly charge for a waste collection service is €42 for 100kg of waste and is €58 for 180kg of waste.
   a) Find a linear formula for the cost, \(C\), of waste collection as a function of the number of kilograms of waste, \(w\).
   b) What is the slope of the line found in part (a)? Give units and interpret your answers in terms of the cost of waste collection.
   c) What is the vertical intercept of the line found in part (a)? Give units with your answer and interpret it in terms of the cost of waste collection.

29. Residents of the town Maple Grove who are connected to the town water supply are billed a fixed amount yearly plus a charge for each litre of water used. A household using 1000 litres was billed €120, while one using 1600 litres was billed €155.
   a) What is the charge per litre?
   b) Write an equation for the total cost of a resident’s water as a function of cubic feet of water used.
   c) How many cubic feet of water used would lead to a bill of €105.

30. The table gives an average weight, \(w\), in kilograms of Irish men in their sixties for various heights, \(h\), in inches.

\[
\begin{array}{cccccccc}
 h \text{ (inches)} & 68 & 69 & 70 & 71 & 72 & 73 & 74 & 75 \\
 w \text{ (in pounds)} & 83 & 85 & 87 & 89 & 91 & 93 & 95 & 97 \\
\end{array}
\]

   a) Does the data in this table could represent a linear function? How do you know?
   b) Find weight, \(w\), as a linear function of height, \(h\). What is the slope of the line? What are the units for the slope?
c) Find height $h$, as a linear function of weight, $f$. What is the slope of the line? What are the units for the slope?

**Figure 18**

i)  

ii)  

iii)  

iv)
Figure 19

i) ii)

iii) iv)
Figure 20

i)

![Graph i)

ii)

![Graph ii)

iii)

![Graph iii)

iv)

![Graph iv)
Appendix O
A Report On:
The Possibility of a Temperature Being the Same Number Using Both Fahrenheit & Centigrade Scales of Temperature

By
B. Carroll

University of Limerick
Contents

1. Preliminary Sections
   1.1 Summary
   1.2 Glossary of Terms

2. Main Sections
   2.1 Problem Statement
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   2.3 Mathematical Analysis
      2.3.1 Formulate a mathematical model
      2.3.2 Obtain the mathematical solution
      2.3.3 Interpret the mathematical solution
   2.4 Conclusions
1. Preliminary Sections

1.1 Summary
The problem considered is to assess whether it is possible for a temperature to be the same number using both Fahrenheit and Centigrade scales of temperature. In order to obtain a viable solution to the problem a two-stage procedure was used; a linear equation with supporting graph, and algebraic methods were used to obtain exact solutions.

1.2 Glossary of Terms

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>Centigrade temperature</td>
</tr>
<tr>
<td>F</td>
<td>Fahrenheit temperature</td>
</tr>
<tr>
<td>T</td>
<td>Temperature</td>
</tr>
</tbody>
</table>

2. Main Sections

2.1 Problem Statement
We recognise that in the Fahrenheit temperature scale water freezes at 32 degrees and boils at 212 degrees. Furthermore, we are aware that on the Centigrade scale freezes at 0 degrees and boils at 100 degrees.

2.2 Assumptions
For the remainder of this report we shall assume the following:

- there is a temperature that is the same in both temperature scales
2.3 Mathematical Analysis
2.3.1 Formulate a mathematical model

In order to formulate a mathematical model for the problem it is necessary to make a graph of the data. When it is 0 degrees Centigrade it is 32 degrees Fahrenheit. This gives one data point:

\[(C, F) = (0, 32),\text{ where } C \text{ is the temperature in Centigrade and } F \text{ is the temperature in Fahrenheit.}\]

Another data point is \((C, F) = (100, 212)\). This allows us to draw a graph of the function as seen in the diagram below.

Diagram: Graph representing the Centigrade and Fahrenheit Scales of Temperature
2.3.2 Obtain the mathematical solution

In order to obtain a mathematical solution for the problem it is necessary to find an equation for the line we have just drawn. Therefore:

Using the two data points we can compute the slope of the graph.

\[
\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{F_2 - F_1}{C_2 - C_1}
\]

\[
= \frac{212 - 32}{100 - 0} = \frac{180}{100} = 1.8
\]

We can now calculate the equation of the line, where:

**Equation of a line** = \( y - y_1 = m(x - x_1) \)

\[
\therefore F - 32 = 1.8(C - 0)
\]

\[
F = 1.8 \, C + 32
\]

Therefore, this allows us to express the equation: \( F = 1.8 \, C + 32 \)

In order to obtain the solution we must assume that there is a temperature that is the same in both temperature units. We will call that temperature \( T \).

Therefore, the point we are looking for is \( (T, T) \).

Using the equation \( F = 1.8 \, C + 32 \), that shows us that:

\[
T = 1.8T + 32
\]

\[
-0.8T = 32
\]

\[
T = \frac{32}{-0.8}
\]

\[
T = -40
\]
2.3.3 Interpret the mathematical solution

If \( T = -40 \) the temperature is -40 degrees Centigrade, it will also be -40 degrees Fahrenheit. In order to further confirm this mathematical solution, we can utilise the graph and interpret that indeed if the temperature is 40 degrees below Centigrade, it will also be 40 degrees below zero Fahrenheit. See diagram below.

![Diagram 2: Temperature at -40 degrees in both Fahrenheit and Centigrade units.](image)

2.4 Conclusions

With respect to the procedures examined we can conclude that if the temperature is -40 degrees Centigrade, it will also be -40 degrees Fahrenheit.
Appendix P
1. Writing Functions

The modern notation for a function is derived from the efforts of many 17th and 18th Century mathematicians, and in particular the work of Gottfried Wilhelm Leibniz and Leonhard Euler.

**Function Notation**

The word function was first used by Gottfried Wilhelm Leibniz in 1694 as a term to denote any quantity connected with a curve, such as the coordinates of a point on a curve or the slope of a curve. Forty years later, Leonhard Euler used the word function to describe any expression made up of a variable and some constants. He introduced the notation

\[ y = f(x) \]

The input variable \( x \) is called the *independent variable*, and the output variable \( y \) or \( f(x) \) is called the *dependent variable* as it depends on whatever value \( x \) holds.

**Example: Hot Air Balloon/Cruise Boat**

For each situation, express the given function in your own words. Write two variables as an ordered pair and write the function with the notation

\[ y = f(x) \]

1. You are riding in a hot air balloon. As the balloon rises, the surrounding atmospheric pressure decreases (causing your ears to pop!).
2. You on a cruiser heading down the Shannon River. You notice that the width of the river changes as you travel southward with the current.

Solution 1
- The pressure depends on your altitude, so we say that the pressure changes with respect to altitude.
- Pressure \( P \) is the dependant variable and altitude \( A \) is the independent variable: \((\text{altitude}, \text{pressure})\).
- We can write the function as:
  \[ P = f(A) \]

Solution 2
- The river width depends on your distance from the river's source, so we say that the river width changes with respect to your distance from the river's source.
- River width \( w \) is the dependant variable and distance from the river's source \( d \) is the independent variable: \((\text{River width}, \text{distance from the river's source})\).
- We can write the function as:
  \[ w = f(d) \]

2. Representing Functions
In 1637 René Descartes, a French mathematician and philosopher, transformed the study of mathematics by connecting the two major fields of mathematics – algebra and geometry. His book *La Géométrie* gave birth to analytical geometry, better known as Cartesian geometry. The same approach can be followed in your study of mathematics, by viewing mathematics from numerous perspectives - graphically, numerically, and symbolically (algebraically); you will increase your understanding of core concepts.

**Example: Hot Air Balloon**

Imagine measuring the atmospheric pressure as you rise upward in a hot-air balloon. Table 1 shows typical values you might find for the pressure at different altitudes, with the pressure given in units of inches of mercury. (This pressure unit is used with barometers, which measure pressure by height of a column of mercury.)

Use these data to graph a function showing how atmospheric pressure depends on altitude. Use the graph to predict the atmospheric pressure at an altitude of 15,000 feet, and discuss the validity of your prediction.

<table>
<thead>
<tr>
<th>Altitude (ft.)</th>
<th>Pressure (inches of mercury)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>5,000</td>
<td>25</td>
</tr>
<tr>
<td>10,000</td>
<td>22</td>
</tr>
<tr>
<td>20,000</td>
<td>16</td>
</tr>
<tr>
<td>30,000</td>
<td>10</td>
</tr>
</tbody>
</table>

**Table 1: Typical Values of Pressure at Different Altitudes**

**Solution:**

In solution 1 we identified *Pressure* (*P*) is the dependent variable and *altitude* (*A*) is the independent variable. We can write the function as:

\[ P = f(A) \]

Graph of the Function:
Using the graph, we can predict that the atmospheric pressure at 15,000 feet is about _____ inches of mercury.

How valid is your prediction?
(Hint: How accurate is your prediction?)

_________

**René Descartes (1596 – 1650)**

Descartes made many contributions to philosophy, science and mathematics. The idea of representing points in the plane by pairs of real numbers and representing curves in the plane by equations was described Descartes in his book *La Géométrie* in 1637. The work established Descartes as the founder of analytic geometry.

### 3. Definition of a Function

**Application of a Function – Wages**

Functions are often depicted pictorially as in Figure 1. The diagram illustrates, for example, that the function \( f \) takes a value, \( a \), in the domain and assigns it to a unique value \( f(a) \) in its range.

A function could, for example, indicate the take-home pay of an employee, \( y \) given the employee earns the minimum wage of €8.65 for each hour worked \( t \). This would allow us to express the function as:
We can represent any given function pictorially; however, it is not always appropriate to do so!
For example in the function \( f(x) = x^2 \), we can easily represent \(-2 \leq x \leq 3\) on a Venn Diagram.

In mathematics, a **function** is used to represent the dependence of one quantity upon another. We can define a function as follows:

Let \( X \) and \( Y \) be sets of real numbers. A **function** \( f \) is a rule that assigns to each number \( x \) in \( X \) a single number \( y \) in \( Y \). 
\( X \) is called the **domain** of \( f \). The number \( y \) is the **image** of \( x \) under \( f \) and is denoted by \( f(x) \). The set of images of elements of \( X \) is called the **range** of \( f \).

Or more simply put:
A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

Can you define what a function is in your own words?

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________

Can you think of two other examples of functions? Is it appropriate to provide a diagram representing these (if so, sketch the resulting diagram)?
Gottfried Wilhelm Leibniz (1646 – 1716)

Gottfried Leibniz was a German mathematician who developed the present day notation for the differential and integral calculus (independently of Sir Isaac Newton). In his correspondence with the leading intellectual and political figures of his era, he discussed mathematics, logic, science, history, law, and theology. Furthermore, Leibniz is known among philosophers for his wide range of thought about fundamental philosophical ideas and principles.

Leonhard Euler (1707 – 1783)

Leonhard Euler was a Swiss mathematician who made enormous contributions to a wide range of mathematics and physics including analytic geometry, trigonometry, geometry, calculus and number theory. Furthermore, Euler was one of the first to apply calculus to real-life problems in physics. His extensive published writings include such topics as shipbuilding, acoustics, optics, astronomy, mechanics and magnetism.

Question: Can you think of any other famous mathematicians? What are they famous for?
Exercises 1

For exercises 1-4, write a short statement that expresses a possible relationship between the given variables.

Example: (age, shoe size)

Solution: As a child grows, shoe size increases. Once the child is fully grown, shoe size remains constant.

1. (volume of diesel tank, cost to fill the tank)
2. (time in second s after jumping from an airplane, speed of skydiver)
3. (travel time from Dublin to Limerick, average speed of car)
4. (distance from speaker, intensity of sound)

For each of the function in exercises 9-12 make a rough sketch of a graph of the function.

5. A car starts out slowly and then goes faster and faster until a tire blows out. Sketch a possible graph of the distance the car travels as a function of time.

6. A flight from Dublin to Shannon has to circle Shannon several times before being allowed to land. Sketch a graph of the distance of the plane from Dublin to Shannon against time, from the moment of takeoff until landing.
1. Elementary Functions

By the end of the 18th Century, mathematicians and scientists had concluded that many real-world phenomena could be represented by mathematical models taken from a collection of functions called elementary functions.

Elementary functions fall into three categories:
4. Algebraic functions (polynomial)
5. Trigonometric functions (sine, cosine, tangent etc.)
6. Exponential and logarithmic functions (e.g. $\log_{10}$; $e^{-3x}$)

For polynomial functions of low degree the following simpler forms are often used:

<table>
<thead>
<tr>
<th>Degree</th>
<th>Function Form</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zeroth</td>
<td>$f(x) = a$</td>
<td>$f(x) = 2$</td>
</tr>
<tr>
<td>First</td>
<td>$f(x) = ax + b$</td>
<td>$f(x) = 3x + 4$</td>
</tr>
<tr>
<td>Second</td>
<td>$f(x) = ax^2 + bx + c$</td>
<td>$f(x) = 2x^2 - 5x + 3$</td>
</tr>
<tr>
<td>Third</td>
<td>$f(x) = ax^3 + bx^2 + cx + d$</td>
<td>$f(x) = x^3 - 2x^2 + x + 1$</td>
</tr>
</tbody>
</table>

By definition, trigonometric, exponential and logarithmic functions are not polynomial functions.

The general rule of a polynomial function states:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \quad a_n \neq 0$$

where the positive integer $n$ is the degree of the polynomial. The numbers $a_i$ are coefficients, with $a_n$ the leading coefficient and $a_0$ the constant term of the polynomial function.
2. Evaluating Functions

Function notation provides the advantage of clearly identifying the function $f$, $x$ as the independent variable and the dependent variable as $f(x)$.

Therefore, instead of asking:
“What value of $y$ corresponds to $x = 5$?” you can ask “What is $f(5)$?”.

Example 1: Temperature Scales

If $x$ represents the temperature of an object in degrees Celsius, the temperature in degrees Fahrenheit is a function of $x$, given by:

$$f(x) = 1.8 \times +32$$

a) Water freezes at $0^\circ$ C (Celsius) and boils at $100^\circ$ C. What are the corresponding temperatures in degrees Fahrenheit?
b) Aluminium melts at $660^\circ$ C. What is the melting point in degrees Fahrenheit?

Solution

a)

b)
Example 2: Preventing a Car- Crash

Suppose that a car is travelling at a certain speed, $s$, and you apply the brakes, how long will it take to stop?

If $s$ is the speed of the car in km/ph, and $t$ is the time in seconds, then from the time the car starts moving to the time it stops can be expressed as:

$$s = f(t) = -t^2 + 5t$$

a) What is the speed of the car after 3 seconds?
b) After how long does it take the car to stop?
Example 3: Pelican Eggs

The pollutant PCB (polychlorinated biphenyl) affects the thickness of pelican eggs.

Thinking of the thickness, $T$, of the eggs, in mm, as a function of the concentration, $P$, of PCB’s in ppm (parts per million), we have the function:

$$T = f(P)$$

a) Explain the meaning of the statement $f(200)$ in terms of the thickness of pelican eggs and concentration of PCB’s.

b) Sketch a possible graph of $T$ against $P$. Is $f$ an increasing or decreasing function? Explain.

Solution

a)

b)
Example 4:

For the function \( f \) defined by \( f(x) = x^2 - 10 \), evaluate each of the following.

- \( f(5a) \)
- \( f(b-2) \)
- \( f(x + \Delta x) \)
- \( f(x + \Delta x) - f(x) \)

Solution

\[
\begin{align*}
f(5a) & = (5a)^2 - 10 \\Rightarrow 25a^2 - 10 \\
f(b-2) & \\
f(x + \Delta x) & = (x + \Delta x)^2 - 10 \
&= x^2 + 2x\Delta x + (\Delta x)^2 - 10 \\
f(x + \Delta x) - f(x) & = 
\end{align*}
\]
For exercises 1-8 classify each of the following functions.

1). \[ f(x) = 2x^2 + 5x + 10 \]  
2). \[ f(x) = 12x^2 + 5x^3 + 1 \]

3). \[ f(x) = 15x + 7 \]  
4). \[ f(x) = 2x^2(x - 4) \]

5). \[ f(x) = \frac{1}{2}x^3 + 2 \]  
6). \[ f(x) = \cos(x - 4) \]

7). \[ f(x) = \ln(x) \]  
8). \[ f(x) = e^{2x^2 + 4} \]

9. The population of a city, \( P \), in millions, is a function of \( t \), the number of years since 1970, so \( P = f(t) \).

c) Explain the meaning of the statement \( f(35) = 12 \) in terms of the population of the city.

d) Sketch a possible graph of \( P \) against \( t \). Is \( f \) an increasing or decreasing function. Explain.

10. The value of a car, \( V = f(a) \), in thousands of euros, is a function of the age of the car, \( a \), in years.

a). Interpret the statement \( f(7) = 8 \)

b). Sketch a possible graph of \( V \) against \( a \). Is \( f \) an increasing or decreasing function. Explain.

c). Explain the significance of the horizontal and vertical intercepts in terms of the value of the car.
In exercises 11-18, evaluate (if possible) the function at the given value(s) of the independent variable. Simplify the results.

11. \( f(x) = 2x - 7 \)
   a) \( f(0) \)
   b) \( f(-5) \)
   c) \( f(x - 1) \)
   d) \( f(b) \)

12. \( f(x) = \sqrt{x + 13} \)
   a) \( f(-2) \)
   b) \( f(6) \)
   c) \( f(c) \)
   d) \( f(x + \Delta x) \)

13. \( f(x) = 9 - x^2 \)
   a) \( f(0) \)
   b) \( f(\sqrt{3}) \)
   c) \( f(-2) \)
   d) \( f(t - 1) \)

14. \( g(x) = x^2(x + 5) \)
   a) \( g(4) \)
   b) \( g\left(\frac{5}{2}\right) \)
   c) \( g(c) \)
   d) \( g(m + 4) \)

15. \( h(x) = x^3 \)

16. \( f(x) = 3x - 1 \)
   \( \frac{h(x + \Delta x) - h(x)}{\Delta x} \)
   \( \frac{f(x) - f(1)}{x - 1} \)

17. \( g(x) = \frac{1}{\sqrt{x - 1}} \)
   \( \frac{g(x) - g(10)}{x - 2} \)
   \( \frac{f(x) - f(2)}{x - 1} \)

18. \( f(x) = x^3 - x \)
1. Applications of Linear Functions

Example: Pole Vault/Mile Run

During the early years of the Olympics, the height of the men’s winning pole vault increased approximately 8 inches every four years. Table 2 shows that the height started at 130 inches in 1900, and increased by the equivalent of 2 inches per year. So the height was a linear function of time from 1900 to 1912. If \( y \) is the winning height and \( t \) is the number of years since 1900, we can write

\[
y = f(t) = 2t + 130
\]

Since \( y = f(t) \) increases with \( t \), we say that \( f \) is an \textit{increasing function}. The coefficient 2 tells us the rate, in inches per year, at which the height increases.

<table>
<thead>
<tr>
<th>Year</th>
<th>1900</th>
<th>1904</th>
<th>1908</th>
<th>1912</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (inches)</td>
<td>130</td>
<td>138</td>
<td>146</td>
<td>154</td>
</tr>
</tbody>
</table>

\textbf{Table 2}: Men’s Olympic pole vault winning height (approximately)
Sketch a Graph of the Function:

The slope is given by the ratio:

\[ m = \frac{\text{rise}}{\text{run}} = \frac{146 - 138}{8 - 4} = \frac{8}{4} = 2 \text{ inches/} \text{year} \]

Calculating the slope using any two points on the line gives the same value.

What does the constant 130 tell us?

________________________________________________________
________________________________________________________
________________________________________________________
________________________________________________________
________________________________________________________

Does the linear trend continue beyond 1912?

________________________________________________________
________________________________________________________
________________________________________________________
________________________________________________________
________________________________________________________

Did you know?
In 1985 Sergey Bubka (Ukraine) became the first pole vaulter to clear 6 metres; he also holds the current outdoor world record at 6.14 metres, set on 31 July 1994 in Sestriere.
As the pole vault heights have increased over the years, the time to run the mile has decreased. If $y$ is the world record time to run the mile, in seconds, and $t$ is the number of years since 1900, then the record shows that, approximately,

$$y = g(t) = 260 - 0.4t$$

The 260 tells us that the world record was 260 seconds in 1900 (at $t = 0$). The slope -0.4 tells us that the world record decreased by about 0.4 seconds per year. We say $g$ is a *decreasing function*.

Sketch a Graph of the Function:

**Did you know?**  
Sir Roger Bannister (UK) was the first man to break the four minute mile barrier. In 1954 he ran a time of 3:59.4 at avenue in Oxford. By the end of the 20th century, the record had been lowered to 3:43.13, by Hicham El Guerrouj of Morocco in 1999 at race meeting in Rome.
2. Definition of a Linear Function

A linear function has the form
\[ y = f(x) = mx + c \]

Its graph is a line such that
- \( m \) is the slope, or rate of change of \( y \) with respect to \( x \)
- \( b \) is the \( y \) intercept at \((0, c)\)

A function is linear if its slope, or rate of change, is the same at every point. Linear functions are used to model situations that show a constant rate of change between 2 variables.

3. Uses of Linear Functions

Linear functions are used in such ways to explore:
Mobile phone plans, hire rates for taxis, consultation fees for hospitals and salary arrangements for companies.

Each issue carries their own significance and it is important that we as consumers, clients, and employees etc. can appreciate the influence of mathematics behind real-life decisions.
Example: Mobile Phone Bill Tariff

In the example below the graph represents a particular mobile phone bill tariff.

a) Identify the independent and dependent variables.

b) By examining the graph what is the charge for each 100 minutes used thereafter?

c) What is the charge for using 600 minutes in any given month?

d) Describe the function in words.
For each of the following graphs shown do the following:

a) Identify the independent and dependent variables.

b) In your own words, describe the function shown on the graph.

c) Briefly discuss the conditions under which a linear function is a realistic model for the given situation

Exercises 3
1. Application of Intercepts of a Linear Function

Example: Demand Function

The demand function is given by the equation \( P = 100 - 0.5 \, Q \), where \( Q \) is the number of ipod shuffles demanded daily throughout Ireland; \( P \) is the price per ipod shuffles, in Euros.

Complete the following table which plots the quantity demanded over the range \( 0 \leq Q \leq 200 \)

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Price (in Euros)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>40</td>
<td>80</td>
</tr>
<tr>
<td>80</td>
<td>60</td>
</tr>
<tr>
<td>120</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td></td>
</tr>
</tbody>
</table>

Describe the table in your own words?

Sketch a graph of the function:
Based on the function $P = 100 - 0.5Q$, what is the y-intercept?

________________________________________________________

________________________________________________________

What does this mean?

________________________________________________________

________________________________________________________

Based on the table above, what is the x-intercept?

________________________________________________________

________________________________________________________

If you were the managing director of the company, based on the information at hand, what price would you set the iPod shuffles at? Why?

________________________________________________________

________________________________________________________

2. Intercepts of a Graph

Two types of solutions that are especially useful when graphing an equation are those having zero as their x or y co-ordinates. Such points are called intercepts because they are the points at which the graph cuts the x or y axis.

**Intercepts of a Graph:**

- an $x$-intercept is a point on the graph where $y$ is zero, and
- a $y$-intercept is a point on the graph where $x$ is zero.

In addition to the above considerations, you should think of the following terms interchangeably:
"x-intercepts" = "roots" = "solutions"

In other words, the following exercises are equivalent:

- Find the $x$-intercept(s) of $y = x^3 + 2x^2 - x - 3$
- Solve $x^3 + 2x^2 - x - 3 = 0$.
- Find the roots of $f(x) = x^3 + 2x^2 - x - 3$

It is possible for a graph to have no intercepts, or it might have several:

**Figure 3:** No $x$-intercepts, One $y$-intercept
Example: Linear Function

Find the x and y intercepts of the graph of \( y = 5x - 15 \)

Solution

To find the x-intercepts, let y equal to zero and solve for x.

\[
5x - 15 = 0
\]

\[
5x = 15
\]

\[
x = 3 \quad \text{(3, 0) } \text{x-intercept}
\]

To find the y-intercepts, let x equal to zero and solve for y.

\[
y = 5(0) - 15
\]

\[
y = -15 \quad \text{(0, -15) } \text{y-intercept}
\]

Sketch a graph of \( f(x) \) and label the x and y intercepts on the diagram.
**Example: Quadratic Function**

Find the x and y intercepts of the graph of \( y = 3x^2 - 14x + 8 \)

**Solution**

To find the x-intercepts, let \( y \) equal to zero and solve for \( x \)

\[
3x^2 - 14x + 8 = 0
\]

\[
(3x - 2)(x - 4) = 0
\]

\[
x = \frac{2}{3} \quad \text{and} \quad x = 4
\]

As the equation has two solutions, we can now conclude that the graph has two x-intercepts:

\[
\left(\frac{2}{3}, 0\right) \quad \text{and} \quad (4, 0) \quad \text{x- intercepts}
\]

Find the y-intercepts, by letting \( x \) equal to zero and solve for \( y \).

Sketch a graph of \( f(x) \) and label the x and y intercepts on the diagram.
**Example: Cubic Function**

Find the x and y intercepts of the graph of \( y = x^3 - x \)

**Solution**

To find the x-intercepts, let \( y \) equal to zero and solve for \( x \).

\[
x^3 - x = 0
\]

Let \( y \) equal to zero

\[
x(x^2 - 1) = 0
\]

Factorise

\[
x^2 - 1 = 0 \quad \text{and} \quad x = 0
\]

Solve for \( x \)

\[
x^2 = 1
\]

\[
x = \pm 1
\]

As the equation has three solutions, we can now conclude that the graph has three x-intercepts:

\[(0,0), (-1, 0) \text{ and } (1, 0)\]  x-intercepts

To find the y-intercepts, let \( x \) equal to zero and solve for \( y \).

\[
y = (0)^3 - 0
\]

Doing this produces \( y = 0 \). So the y-intercept is:

\[(0,0)\]  y-intercept

Sketch a graph of \( f(x) \) and label the x and y intercepts on the diagram.
Exercises 4

Find the x and y intercepts (if any) of the following:

1). \( y = x^2 - 4x + 9 \)  
2). \( y = -2x^2 + x^3 \)  
3). \( y = 4 - 8x - 2x^2 \)  
4). \( x^2 + y^2 = 5 \)  
5). \( y = 8 - 2x^2 \)  
6). \( y = x^3 - 6x^2 + 9x \)

7. The equation of the demand function is given by \( Q = 64 - 4P \), where \( Q \) is the number of helicopter flights demanded daily and \( P \) is the price per flight in Euros.
   a) Plot the demand function with \( Q \) on the vertical axis
   b) What is the demand when \( P = 0 \)?
   c) What is the price when \( Q = 0 \)?
   d) Describe the function in your own words.

8. Given the supply function, \( P = 10 + 2Q \), where \( P \) is the price of a bottle of exclusive French wine (2000 Châteauneuf-du-Pape, Domaine Pierre Andre) in Euros, and \( Q \) is the number of litres supplied.
   • Graph the supply function for \( 0 \leq Q \leq 50 \).
   • What is the meaning of the value at the vertical intercept?
1. Application of the Slope of a Linear Function

Example: Rain Gauge

Imagine that we measure the depth of rain accumulating in a rain gauge as a steady rain falls. The rain stops after 6 hours, and we want to describe how the rain depth varied with time during the storm.

What is the dependent variable?

______________________________

What is the independent variable?

______________________________

Based on the measurements with the rain gauge, we find the rain depth function as the following graph:

![Graph showing rain depth vs. time](image-url)
**Rate of Change**
The graph shows that during the storm, the rain depth increased by ____ inch per _____.

Thus we can say that the **rate of change** of the rain depth with respect to time was ___ inch per _____.

This rate of change was constant throughout the storm.
The following graphs show results from three other steady rainstorms.

a) ![Graph a](image)

The constant rate of change in graph a is: _____________________

b) ![Graph b](image)

The constant rate of change in graph b is: _____________________

c)
The constant rate of change in graph c is: _____________________

Comparing these three graphs leads us to a crucial observation:

The greater the ____________, the __________ the graph.

2. Definition of the Slope of a Linear Function

We use the symbol Δ (the Greek letter capital delta) to mean “change in”. Therefore, Δ x means change in x and Δ y means change in y.

The definition of the slope of a line or linear function can be expressed as:

$$ m = \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta y} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1} $$
Example: Slope given two points

Find the slope of the line, $l$, through the points $a(-1, 2)$ and $b(3, -4)$

Solution

\[
\text{Slope of } ab = \frac{y_2 - y_1}{x_2 - x_1}
\]

\[
a(-1, 2) \quad \downarrow \downarrow \quad b(3, -4) \\
(x_1, y_1) \quad \downarrow \downarrow \quad (x_2, y_2)
\]

\[
\text{Slope of } ab = \frac{-4 - 2}{3 + 1} = \frac{-6}{4} = -\frac{3}{2}
\]
In general, the greater the value of the slope of a line, the steeper the line is. For instance in Figure 3c the line with a slope of -2.5 is steeper than the line in Figure 3a with a slope of $\frac{1}{5}$.

**Figure 3a:** If $m$ is positive, then the line rises from left to right

**Figure 3b:** If $m$ is zero, then the line is horizontal
Figure 3c: If $m$ is negative, then the line falls from left to right

Figure 3d: If $m$ is undefined (i.e. $\Delta x = 0$), the line is vertical
Example 5: Determining slope from a graph

Estimate the slope of the line, $l$, from its graph:

![Graph of the line $l$](image)

**Figure 14:** Graph of the line $l$

**Solution**

From the graph above we can see that $\Delta y = 2$, i.e. from the point $(0,0)$ to $(0, 2)$ there is an change of $+2$ for the $y$-value.

The $\Delta x = 3$ i.e. from the point $(-3,0)$ to $(0, 0)$ there is an increase of $+3$ for the $x$-value.

From the formula $m = \frac{\Delta y}{\Delta x}$, we can conclude that $m = \frac{2}{3}$. 
3. Linear functions not written in standard form

Finally, let us consider what happens when we have linear functions that are not in the form \( f(x) = mx + c \).

For simplicity, we shall put \( y = f(x) \) here.

Example:

Suppose that we have the equation \( 4x - 3y = 2 \).

To get this into the required form we need to make \( y \) the subject:

\[
\begin{align*}
-3y &= 2 - 4x \\
3y &= 4x - 2 \\
y &= \frac{4}{3}x - \frac{2}{3}
\end{align*}
\]

Then, since \( y = f(x) \), this equation represents the function \( f(x) = \frac{4}{3}x - \frac{2}{3} \).

The graph of the function will have a gradient of \( \frac{4}{3} \) (i.e. \( m = \frac{4}{3} \)) and a y-intercept of \((0, -\frac{2}{3})\).
Exercises 5

For exercises 1-4, determine the slope and the y-intercept of the line whose equation is given by writing the function the form \( y = mx + c \).

1. \( 7y + 12x - 8 = 0 \)
2. \(-4y + 2x + 13 = 0\)
3. \( 12x = 6y + 17 \)
4. \(-9x -9y = 18 \)

For Exercises 5 -8, find the equation of the line that passes through the given points using slope-intercept equation of a line.

5. \((0, 0) \) and \((1, 5)\)
6. \((0, 2) \) and \((2, 9)\)
7. \((0,-5) \) and \((4, 7)\)
8. \((0, 1.5) \) and \((-2.5, 7.5)\)

11. Match the graphs in Figure 18 with the following equations
   a) \( y = 2x - 5 \)  
   b) \(-7x + 3 = y \)  
   c) \(-2 = y \)  
   d) \( y = 4x + 5 \)  
   e) \( y = x - 9 \)  
   f) \( y = \frac{2x}{2} \)

12. Match the graphs in Figure 19 with the following equations
   a) \( y = -2.86x \)  
   b) \( y = 0.02x + 0.05 \)  
   c) \( y = 12.8 + 0.5x \)  
   d) \( y = 12.8x + 15 \)  
   e) \( y = -8.9 -2x \)  
   f) \( y = \frac{x}{-3.14} \)
Exercises
Estimate the slope of the line from the following graphs:
1).  
2). 
3).  
4).
Figure 18

i)  

ii)  

iii)  

iv)
Figure 19

i) 

ii) 

iii) 

iv)
Appendix Q
Teacher Guidelines

The key aspects of this teaching intervention include:

- The intervention aims to show the students the different representations of a function (i.e. algebraic, graphical and numerical)
- All concepts, rules and definitions are introduced through the medium of applications.
- The applications are deemed to be appropriate for the age range of the students.
- The mathematical content is based on prior mathematical experiences of the students and the expected current level of senior-cycle students.
- GeoGebra Applets are provided to allow the teacher to demonstrate the visual aspect of a particular concept.
- A PowerPoint presentation is provided for the teacher so as to enable the teachers to incorporate the use of ICT into the lessons. In particular it allows the teacher to provide clear and concise diagrams easily.
- The modelling session is provided to allow the students to readily see the usefulness and importance of mathematics, while affording them an opportunity to work on a given real-life problem that utilises mathematics to solve the problem.

Teacher Outcomes

After participation in this intervention, it is intended that the teacher will have the following pedagogical outcomes:

- An enhanced understanding of the use of mathematics in other subject areas and in the real world.
- An extended range of teaching methodologies employed in the teaching of functions to Senior-Cycle level, which includes both the exposure to applications and the different representations of a function (i.e. algebraic, graphical and numerical)
- An extended range of ICT skills through the use of PowerPoint presentation and GeoGebra Applets.
- An opportunity to prepare for the introduction of ‘Project Maths’.

Student Outcomes

After participation in this intervention, it is intended that the student will have the following learning outcomes:

- An enhanced understanding of the use of mathematics in other subject areas and in the real world.
- An enhanced understanding of the different representations of functions (i.e. algebraic, graphical and numerical)
- An exposure to applications and problems in the real-world which use mathematics in solving them.
• An opportunity to appreciate the role of mathematical modelling and in particular writing reports.
• An opportunity to appreciate the role of mathematics in one’s future education and/or work-life.

**Linear Functions – Outline of Lessons**

**Lesson 1**

This lesson is designed to teach the basic representations of a function, while provide the learner with a general definition of a function. The lesson begins by introducing the learner to writing functions, while identifying the general method of writing functions i.e. \( y = f(x) \). The learner is then shown the different representations of a function (i.e. algebraic, graphical and numerical). This is one of the key aims of the intervention. Finally, the learner is provided with a general definition of a function and afforded opportunities to write a definition in their own words. Each concept covered in this lesson is introduced through application and these applications include:

1. **Writing Functions**
   - **Example:** Hot Air Balloon/ Cruise Boat

2. **Representing Functions**
   - **Example:** Hot Air Balloon

3. **Definition of a Function**
   - **Example:** Wages

**Lesson 2**

This lesson is designed to introduce the learner to Elementary functions, while providing a visual stimulus in the form of Geogebra Applets. It is not intended that the learner is able to comprehend each type of function at this stage in the learning process, but should be able to identify the different types of polynomial functions both algebraically and graphically. The learner is then introduced to evaluating functions. Both applications and standard mathematical examples are used at this stage of the lesson. The applications used in this lesson include:

1. **Elementary Functions**
   - **Geogebra Applets**

2. **Evaluating Functions**
   - **Example 1:** Temperature Scales
   - **Example 2:** Preventing a Car-Crash
   - **Example 3:** Pelican Eggs

**Lesson 3**

In this lesson the learner is introduced to linear functions. Two applications looking at Olympic Events allow the learner to form their own
understanding and mathematical knowledge of the idea of a linear function and the mathematical concepts involved with linear functions i.e. slope, increasing/decreasing functions, slope and intercepts.

Building on the initial introduction to a linear function, the learner is provided with a definition of a linear function.

The applications used in this lesson include:

1. Applications of Linear Functions  
   Example: Pole Vault / Mile Run

2. Definition of a Linear Function  
   Geogebra Applet

3. Uses of Linear Functions  
   Example: Mobile Phone Bill Tariff

Lesson 4

To further establish the concepts learned in the previous lesson, this lesson focuses specifically on the intercepts of a linear function. Again a visual stimulus in the form of Geogebra Applets is provided.

So as to ensure the learner is aware of the mathematical capacities of x and y intercepts the learner is introduced to finding intercepts not only in linear functions, but also in quadratic and cubic functions.

The applications used in this lesson include:

1. Application of Intercepts of a Graph  
   Example: Demand Function Graph

2. Intercepts of a Graph  
   Geogebra Applet  
   Example: Linear/Quadratic/Cubic Functions

Lesson 5

This lesson focuses specifically on the slope concept of a linear function. The lesson begins by examining a real-life application of a rain gauge, thereby allowing the learner to form their own mental constructions of the slope of a linear function. Building on the initial introduction to the slope of a linear function, the learner is provided with a definition of the slope of a linear function using two data points. Again a visual stimulus in the form of Geogebra Applets is provided.

Finally, the lesson focuses on functions not written in standard form and the mathematical procedures needed in such cases.

The applications used in this lesson include:

1. Application of the Slope of a Linear Function  
   Example: Rain Gauge

2. Definition of the Slope of a Linear Function  
   Geogebra Applet  
   Example: Determining slope from a graph
Lesson 6

This lesson focuses specifically on mathematical modelling and is a basic introductory lesson to mathematical modelling within the school setting. The learner is introduced to the benefits of mathematical modelling and report writing. Each of the 6 stages of the modelling process is outlined through examining a sample report provided.

1. Outline of mathematical modelling
2. Example of mathematical modelling

Lesson 7

In this lesson the learner works in small groups to solve a real-life mathematical problem. It is intended that each group begins the report of the procedures that they undertook in solving the problem. The report is to be provided as part of the assessment procedure for the next lesson.

Group modelling problem
Using the Straight Line Applet:

**Straight Line**

Use your mouse to change the values of m and c. The slope of the line is given by m. The intercept on the y-axis is given by c.

Open the Geogebra applet entitled “The Straight Line” from the webpage:  
http://www.ul.ie/cemtl/applets.htm

The page above should be displayed:

There are two sliders in this applet, labelled m and c, that represent the slope (m) and the y-axis intercept (c). You can use your mouse to catch each of these sliders and increase or decrease the values of the slope (m) and the y-intercept (c). We use the Greek letter, θ, to name the angle that the line makes with the positive direction of the x-axis. The slope of the line is then given by $m = \tan \theta$.

As you move the slider marked $m$ you will see that the slope of the line changes. Note that the point where the line meets the y-axis is unchanged regardless of the value of $m$ selected. This is because when a line intersects the y-axis its x value is 0 so this gives

\[ y = mx + c \]
\[ y = m \times \theta + c \]
\[ y = \theta + c \]
\[ y = c \]

so the point (0, c) is always on the line regardless of the m value selected.

Now move the second slider, marked c, and note what happens to the line. Remember that the variable c controls the y-intercept of the line. This means that as c is varied the location where the line crosses the y-axis varies. Whatever value the slider c is fixed at represents the value where the y-axis and the straight line intersect.
Appendix R
Mathematical Modelling Within A School Setting

Introduction:

The increased use of mathematics in the non-physical sciences such as geography, economics and biology, together with the existing practices in physics, technology and engineering, allows the students to readily see the usefulness and importance of mathematic to succeed in these disciplines. Mathematical modelling activities, within a school setting, permit the student to effortlessly gain an insight into these practices and procedures employed by not only mathematicians but by employees of numerous other disciplines. Furthermore, mathematical modelling at this stage in a student's education can become a vehicle through which the student can facilitate the transition from second-level to university undergraduate mathematics where mathematical modelling is widely practised.

The modelling process

The modelling process is best illustrated in diagrammatic form (see Figure 1). Although there are many variations of the process, most of the modelling processes are, in essence, presenting comparable features. Despite the relative diversity in approaches to the teaching of modelling worldwide, the focal perspective of modelling courses has consistently remained within the structure of the existing mathematics curricula. The conventional approaches to instruction and course work are almost entirely altered within a modelling approach, yet the mathematical substance of models remains largely unchanged (1).

Why include modelling in the teaching and learning of mathematics?

The benefits of including modelling within mathematics programs are widely recognised (2, 3, 4), with many countries calling for increased instructional emphasis on mathematical modelling techniques (2). Mathematics classrooms are by and large dominated by students who can be described as ‘non-specialists’, in that they are not fully committed to mathematical study, or entirely convinced of the value of mathematics to their overall education and its place in their future lives (5). Consequently, Ormell deduces that we need courses which are Capable of Appealing to the Non-Specialist (CANS). CANS courses aim to provide opportunities for the ‘non-specialist’ to perceive value in mathematics and mathematical thinking alleviating their alienation from mathematics. Burghes (6) identifies modelling as an activity that has the ability to cater for the ‘non-specialist’, whereby it encourages the use of discussion, investigation and application in the teaching of mathematics and for this reason merits inclusion within a mathematics program.
The modelling process in the classroom

Stage 1: Real-world problem
The problem statement should be very general and free of as much data as possible, as later stages of the modelling process will consider and gather what is needed.

Suggested classroom techniques include:
Divide the class into groups and allow them to discuss the problem. Some classes may need the structure of focusing questions for their discussion. A good technique is to get the students to list key words, to restate the problem, and to determine what units will be used to express the answer.

Stage 2: Make assumptions
This consists of listing all the variables involved and then trying to simplify or modify the list. In this process, it becomes obvious that there is a need to obtain certain information that will constitute the initial conditions of the problem.

Suggested classroom techniques include:
Individually, or in groups, have the students brainstorm the problem to uncover the variables involved. The teacher could collect and write the variables on the board.

Stage 3: Formulate mathematical problem
The teacher has the option of algebraically constructing the model with the class or providing a spreadsheet containing the model.

Suggested classroom techniques include:
• Groups discussing the mathematics they can use to make sense of the problem and reporting back to the class. For example, if the situation or problem includes a collection of data, then the use of
various graphs may be an adequate model.
• Listing the calculations that need to be made.

Stage 4: Obtain the mathematical solution

This stage describes the process used by the students when applying a procedure to given data. Using the modelling process may mean a return to the initial assumptions in order to modify the problem being considered.

Stage 5: Interpret the solution

After obtaining their solutions, the students are directed back to the problem. They must check to ensure that they have answered the problem within the assumptions they have made. Interpretations made should take into account the assumptions and initial conditions. This is an important step in helping students realise that solutions to problems are constrained by the context and are not easily transferable to other situations.

Stage 6: Report

This is a valuable part of the process, as students need experience in using language to express mathematical ideas. It is advised to use the sample report that is provided (see Appendix).

Bibliography


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