On The Valuation Of Cashflow CDOs
Without Monte Carlo Simulation

by

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Abstract

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The most common method of pricing a cashflow collateralized debt obligation (cashflow CDO) is to use Monte Carlo integration. However, Monte Carlo integration is computationally intensive and often faster methods of pricing are required. Gallagher et al. (2009) proposed a semi-analytic approximation that allows fast pricing of cashflow CDOs. This thesis has two goals: (i) A self contained description of the mathematical background necessary for practical implementations of cashflow CDO pricing and (ii) a critical examination of the semi-analytic cashflow CDO pricing method proposed by Gallagher et al. (2009). We examine one of the main arguments in their paper surrounding the modality of the underlying probability distribution and suggest an alternative explanation for the accuracy of their method. With this new understanding, we describe the conditions under which the approximation will be most accurate.
Declaration

I hereby declare that this thesis is entirely my own work, and has not been submitted for any other awards at this or any other university. Where use has been made of the work of other people it has been fully acknowledged and referenced.

Signed: 

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Date: 

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3.3.5 Actual versus Risk-Neutral Default Probabilities .......................... 39
3.3.6 Default Dependence and Default Correlation ............................. 41
3.3.7 Copula Functions .......................................................................... 43
3.3.8 Factor Copula Model ................................................................. 46

4 Pricing Cashflow CDO Tranches Without Monte Carlo Simulation 49
4.1 Probability Bucketing ...................................................................... 50
4.2 The Difficulty in Evaluating the Expected Tranche Coupon ............... 53
4.3 The Approximation of Gallagher et al. (2009) ................................. 55
4.4 Modality of the X-Y Distribution ..................................................... 58
   4.4.1 Unimodal X-Y Distributions ..................................................... 59
   4.4.2 Multimodal X-Y Distributions ................................................ 61
4.5 Integration Along the Q-Q Curve .................................................... 64
4.6 Conditions Under which the Tranche Coupon Function Approximates a
   Linear Function ................................................................................. 67
   4.6.1 X-Y Distributions in a Single Region of \( g_k(x, y) \) .................... 68
   4.6.2 X-Y Distributions Across Multiple Regions ............................. 75
4.7 Incorporating IC and OC Tests ......................................................... 79

5 Conclusion ......................................................................................... 83
5.1 Concluding Remarks ........................................................................ 83
5.2 Directions for Future Work ............................................................ 84

A Default Correlation ........................................................................... 85
B Tranche Coupon Function .................................................................. 89
C Fisher’s Non-Central Hypergeometric Distribution .......................... 93

Bibliography ......................................................................................... 95
## List of Figures

2.1 Creation of a cashflow CDO. ................................. 9
2.2 Example of a cashflow CDO. ................................. 11
2.3 CDO after 3 years. ........................................ 12

4.1 Available Interest at $t_i$ versus Pool Redemption at $t_{i-1}$ for a unimodal distribution. ........................................ 59
4.2 Available Interest at $t_2$ versus Pool Redemption at $t_1$ for a unimodal distribution. ........................................ 60
4.3 Available Interest at $t_7$ versus Pool Redemption at $t_6$ showing a bimodal distribution. ........................................ 62
4.4 Available Interest at $t_{11}$ versus Pool Redemption at $t_{10}$ showing a multimodal distribution. ........................................ 62
4.5 Available Interest at $t_7$ versus Pool Redemption at $t_6$ showing a bimodal distribution and the Q-Q RFD plot. ........................................ 63
4.6 Available Interest at $t_2$ versus Pool Redemption at $t_1$ for an underlying pool that contains an outlier with respect to recovery rate. ........................................ 64
4.7 Tranche coupon function, $g_1(x,y)$, for Tranche 1. ................................. 69
4.8 Tranche coupon function, $g_1(x,y)$, for Tranche 1 for a given $x < N_1$. ........................................ 69
4.9 Tranche coupon function, $g_1(x,y)$, for Tranche 1 for a given $y < K_1N_1$. ........................................ 70
4.10 Tranche coupon function, $g_2(x,y)$, for Tranche 2. ................................. 70
4.11 Tranche coupon function, $g_3(x,y)$, for Tranche 3. ................................. 71
4.12 Tranche coupon function, $g_4(x,y)$, for Tranche 4. ................................. 71
4.13 General tranche coupon function, $g_k(x,y)$, for Tranche $k$. ................................. 72
4.14 Tranche coupon function, $g_4(x,y)$, where Tranche 4 is the equity tranche. ................................. 72
4.15 The regions for the most senior tranche coupon function, $g_1(x,y)$, drawn over the $X$-$Y$ distribution. ........................................ 74
4.16 The regions for the second most senior tranche coupon function, $g_2(x,y)$, drawn over the $X$-$Y$ distribution. ........................................ 74
4.17 A bimodal normal distribution centered on $\langle 2,2 \rangle$. ................................. 75
4.18 The difference between the Q-Q integral and the Monte Carlo expected value plotted against the value $m$ in the function $x + my$. ................................. 76
4.19 Tranche coupon function, $g_2(x,y)$, for Tranche 2 for a given $y < y_1$ in Figure 4.16. ................................. 77
4.20 The regions that define the third most senior tranche, $g_3(x,y)$, drawn over the $X$-$Y$ distribution for Deal B. ................................. 78
4.21 The tranche coupon function, $g_3(x,y)$, for the third most senior tranche for a given value of $y$. ................................. 78

C.1 The probability density function of $R_{Total}$. ................................. 94
## List of Tables

4.1 Deal A price comparisons without IC and OC tests. .......................... 58
4.2 Deal B price comparisons without IC and OC tests. ......................... 58
4.3 Run time to price each cashflow CDO without IC and OC tests for Monte Carlo and Q-Q RFD methods. ........................................ 58
4.4 Values of $C_k(t_2)$ for Monte Carlo and Q-Q RFD method for the unimodal X-Y distribution in Figure 4.2. ........................................... 64
4.5 Values of $C_k(t_7)$ for Monte Carlo and Q-Q RFD method for the bimodal X-Y distribution in Figure 4.3. ........................................... 65
4.6 Deal A price comparisons with IC and OC tests. ............................. 82
4.7 Deal B price comparisons with IC and OC tests. ............................. 82
Chapter 1

Introduction

Cashflow collateralized debt obligations (cashflow CDOs) are complex financial instruments used by financial institutions to convert a portfolio of bonds or loans into a number of products that can be sold on to investors. They are used, for example, by banks to remove loans, such as mortgages, from their balance sheets. Before we describe the exact structure of a cashflow CDO in Section 3.2, we will begin in this chapter by discussing the economic climate that led to the popularity of cashflow CDOs in recent years. Specifically, we will give a brief synopsis of the events that led to the U.S. subprime mortgage crisis in 2007 since this will provide a good overall context for the study of cashflow CDO pricing. Although we deal only with the U.S. case, the general sequence of events is similar to the growth of the CDO market in Europe and Asia.

1.1 The Growth of the Cashflow CDO Market and the U.S. Subprime Mortgage Crisis

The U.S. and global economic crisis of 2007 and 2008 has been attributed to, amongst other things, the failure of financial institutions to adequately model credit risk, specifically in relation to the securitization\(^1\) of mortgage loans in the form of collateralized debt obligations. Brunnermeier (2009) details how, during the peak of the U.S. housing boom in 2006, banks strayed from traditional banking practices of lending and holding loans for their full life, in favour of “originate and distribute” policies in which banks

\(^1\)Securitization is the process in which the interest and principal repayments from pools of mortgage loans, car loans, credit card loans, etc., are grouped together and repackaged in order to be sold to investors. A security is a legal claim for future cash payment.
created loans in order to sell them to investors through different forms of securitization. The situation arose out of a climate of low interest rates in the U.S. in which banks were able to borrow money at a low cost. Since the banks could obtain such cheap credit they could afford to offer inexpensive loans to home buyers and so the housing market began to strengthen. At the same time, the low returns being offered by the U.S. Federal Reserve on Treasury Bonds was not sufficient for many investors, especially the many investors from emerging Asian economies who sought to hedge currency depreciation by buying U.S. securities. In order to match the supply and demand of the two markets, a new innovation allowing investors the opportunity to indirectly fund the residential mortgage market became widespread. This was achieved by banks selling a portfolio of loans to a special purpose vehicle (SPV), which pooled the loans together and repackaged them into a number of products that were then sold to investors. The investors would then receive a return on their investments that was funded by the mortgage payments of the underlying loans. This is the basic structure of a cashflow CDO. The products that are issued by the SPV and sold to the investors are known as tranches. A detailed description of the terminology surrounding cashflow CDOs will follow in Chapter 2.

A CDO that is backed by mortgage loans, corporate bonds, credit card loans or similar cashflow assets is known as a cashflow CDO to distinguish it from synthetic CDOs, which are backed by a pool of credit default swaps.¹

Eventually, the popularity of cashflow CDOs backed by residential mortgages led to a decline in lending practices by the mortgage lenders, see Keys, Mukherjee, Seru, and Vig (2008). Since the demand for the tranches of cashflow CDOs was so high the banks could offload the risk of default quickly. There was therefore less reason for them to be as stringent as normal in approving loan applications. There was also a belief that since property prices were rising any mortgage holders that could not make payments could simply sell their houses to close out the loans. These practices resulted in the now infamous NINJA loans in which loan applicants were no longer required to prove the amount of their income, whether or not they had a job or whether they owned any assets (No Income, No Jobs, No Assets). At the time, many investors who purchased CDO tranches would have relied on the credit rating agencies to determine the risk associated with their investments. However, Benmelech and Dlugosz (2009), having studied the data for nearly 4000 CDOs between 2004-2008, observed a “mismatch between the credit

¹A credit default swap (CDS) is a contract that provides insurance against the risk of a default by particular company. The company is known as the reference entity and a default by the company is known as a credit event. The buyer of the insurance obtains the right to sell a particular bond issued by the company for its par value when a credit event occurs.
ratings of CLO\textsuperscript{1} securities and the credit quality of the underlying collateral.” Many 
CDO tranches that were given AAA ratings by the rating agencies were backed entirely 
by subprime mortgages.

Once these subprime mortgages began defaulting and the housing bubble burst, the 
spread of risk created by the CDOs meant that not only the mortgage lenders were 
exposed to large losses but so were many other investors and financial institutions that 
bought tranches of CDOs backed by residential mortgages. From early 2007 onwards, 
following significant losses for a number of hedge funds due to subprime mortgage de-
faults, the rating agencies began to downgrade the tranches of many mortgage backed 
CDOs. A sudden lack of confidence in both the valuation of CDO tranches and the 
tranche ratings assigned by agencies led to a liquidity crisis due to the difficulty of institu-
tions in selling these toxic assets. As the book value of CDO tranches began to decline 
the balance sheets of many institutions took substantial write-downs. The increasing 
delinquency rates\textsuperscript{2} for residential mortgage loans culminated in the largest bankruptcy 
in U.S. history in September 2008 when the investment bank Lehman Brothers failed in 
an attempt to sell the company after making massive losses on their large portfolio of 
mortgage backed CDO tranches. The scale of Lehman’s business and its large number of 
counter-parties meant that the knock-on effects were felt all across the U.S. and through-
out the world. Stock markets took severe losses as uncertainty spread surrounding the 
level of contagion that would result. Shortly after the collapse of Lehman’s the U.S. 
government was forced to provide a bailout to the insurance company AIG after their 
stock price fell by more than 90%. In October 2008, the U.S. government passed the 
Emergency Economic Stabilization Act, a $700 billion rescue package that implemented 
the Troubled Asset Relief Program (TARP), whereby the U.S. government would pur-
chase distressed mortgage-backed securities from financial institutions. A requirement 
of the act was that the institutions selling assets would have to issue securities in the 
form of equity, equity warrants\textsuperscript{3} or senior bonds to the U.S. Treasury Department with 
the intention that they would purchase them back once they had recovered. TARP 
has since acquired assets from many institutions, including Citigroup, Bank of America, 
AIG, JP Morgan Chase and more. Currently, many of the largest TARP recipients 
have purchased back the securities issued and the program is not expected to cost the 
American taxpayer as much as initially feared.

\textsuperscript{1}A collateralized loan obligation (CLO) is a CDO made up primarily of loans.

\textsuperscript{2}The delinquency rate is the percentage of loans in a portfolio that have delinquent payments. A 
delinquent payment is a failure to make payments on a loan. Sometimes a loan may be regarded 
delinquent if say no scheduled payment has been made in 60 days.

\textsuperscript{3}An equity warrant is a note that gives the holder the right to purchase equity from the issuers at a 
certain price with a certain time frame.
1.2 Criticisms of CDOs and the Need for Fast CDO Pricing Methods

One of the main criticisms of CDOs over the last number of years has been that rather than controlling or transferring risk, they in fact contributed to amplifying and spreading the risk associated with the U.S. housing market. Moral hazard has been blamed for increasing the risk, with many CDO originators criticized for not continuing the usual practice of retaining the riskiest tranches of the CDO (see Brunnermeier (2009)). By not holding any part of the CDO, originators clearly had little incentive to ensure that the underlying assets of the CDO were of high credit quality. Credit rating agencies and financial modelers have also been blamed for not fully understanding the risk associated with CDO tranches while the use of the Gaussian copula function, which will be discussed in Section 3.3.7, has been heavily criticized for its failure to adequately model the likelihood that companies may default in clusters. Though not the main focus of this thesis, for completeness, we will briefly discuss the shortcomings of the Gaussian copula in Section 3.3.8.

Due to the failings of the CDO market in recent years, the pricing of CDOs has come under much more scrutiny than previously. Academics and practitioners alike are attempting to make CDO pricing more robust and much more effort is being made to understand the consequences for CDO tranches under extreme conditions. Using Monte Carlo simulation, an average sized cashflow CDO (about 150 underlying assets) may take 15 to 20 minutes to be priced, which we will see in Chapter 4. For large financial institutions, with potentially hundreds of CDOs that need to be priced every day, each with different pricing parameters and with different scenarios, this quickly becomes unwieldy. Fast methods for pricing synthetic CDOs were introduced by Hull and White (2004) but these methods have limitations when it comes to pricing a cashflow CDO. Gallagher et al. (2009) extended the Hull and White “probability bucketing” method for use with cashflow CDOs using an analytic approximation. One of the goals of this thesis will be gaining an understanding of how this method works and under which conditions it is most accurate. However, before we deal with this method our first objective will be to review the theory behind cashflow CDO pricing. Alongside this, we will also deal with practical methods such as stripping hazard rates from bond prices, bootstrapping a credit curve, sampling from a probability distribution and generating a factor copula model. All this should provide a strong background for practical implementations of cashflow CDO pricing used by financial institutions and companies such as our industry...
partner Eudaemon Consulting. We have worked closely with Donal Gallagher of Eudaemon Consulting on their C++ cashflow CDO pricing routines, including Monte Carlo simulation and semi-analytic methods.

In Chapter 3 we will provide an introduction to pricing simple financial instruments such as loans and bonds and we will explicitly detail the structure of a cashflow CDO. This will be followed by a review of the background theory and practical methods used in pricing a cashflow CDO, focusing on the use of Monte Carlo integration. This also includes a review of the theory and practice of modelling credit risk, constructing a credit curve and modelling default dependence using copulas, which will also form the basis of the theory behind the semi-analytic pricing used by Gallagher et al. (2009).

Chapter 4 deals with pricing a cashflow CDO without Monte Carlo integration. We describe the difficulty in pricing a cashflow CDO without Monte Carlo integration and introduce the analytic approximation proposed by Gallagher et al. (2009). This will lead on to the discussion of the modality of the underlying probability distribution and to our discussion of the accuracy of the method.

We will now move on to Chapter 2, which will give an introduction to some of the terminology and basic principles surrounding bonds, loans and credit risk. Chapter 2 will also give an introduction to cashflow CDOs with an example that demonstrates the mechanics behind their structure.
Chapter 2

Background Theory and Cashflow

CDO Example

As discussed in Chapter 1, the underlying building blocks of a cashflow CDO are a pool of cashflow assets such as bonds or loans. Since bonds and loans are similar in structure we will often use the terms interchangeably. In Sections 3.3.3 - 3.3.4 (on building a credit curve and stripping hazard rates) we will specifically be dealing with corporate loans and bonds, of which market data is readily available.

We will now familiarize the reader with the terminology surrounding bonds and loans before describing the concepts of credit risk and credit derivatives. This will pave the way for a more detailed definition of a cashflow CDO in Section 3.2 and we will end the chapter with an example.

2.1 Bonds, Loans and Derivatives

A bond is a debt security, in which the issuer agrees to repay a sum of currency (principal) to the holder along with periodic interest payments (coupons) at a future date, called maturity, e.g., a 10 year bond with par value\(^1\) of €100 and a fixed rate of 5% will pay a coupon of €5 each year for 10 years along with a final principal payment of €100 on the maturity date. Governments sell bonds to investors in order to finance expenditure and companies might issue bonds to raise money for long term investment. Loans are similar to bonds except that a loan is usually an over-the-counter negotiated trade between two

\(^1\)The stated value on the bond. Also called face value.
organizations. So whereas bonds of fixed denominations are sold publicly by the borrower to raise capital, loans (either private or corporate) are applied for by the borrower on an individual basis and an amount and price is negotiated. In this thesis bonds and loans are considered structurally identical.

A derivative is a financial instrument whose value depends on the value of other underlying variables. At its most basic, it is a financial contract whose price is dependent on the price of an underlying asset. For example, a forward contract is an agreement to buy or sell an asset at a certain future time for a certain price. So the value of a forward contract obliging the holder to sell a barrel of oil for $75 in 6 months is dependent on the price of oil 6 months in the future.

2.2 Credit Risk and Credit Derivatives

Credit risk is the risk that a debtor may default on their obligations. For example, a company may go bankrupt and be unable to pay back its outstanding loans. In this situation any amount of the loan that the company can pay back (as a result of liquidation of the company’s assets) would be known as the recovery amount. The difference between the expected return and the recovery amount is called the loss-given-default.

In trying to evaluate the amount of loss suffered by a pool of bonds or loans, one of the most important factors to consider is default dependence. Consider a pool of 100 mortgage loans where the individual probability of default of each loan is known. We are interested in calculating the expected loss on the entire portfolio. Considering each loan individually and getting an expected loss does not take into account the dependencies between the loans and their likelihood to default together. For example, in the case of mortgage loans there are certain economic factors which are likely to affect a large number of mortgage holders at once (such as a property crash) thus leading to a very strong default dependence. One of the most widely used methods to incorporate default dependence into a pricing model is to use a factor copula method. Copulas will be discussed further in Chapter 3.

A credit derivative is a derivative whose value depends on the credit risk of an underlying asset such as a loan or a bond. In Section 3.3.4 we will discuss credit default swaps which are credit derivatives that act like insurance on bonds in the case of default. We will also

\[1\text{Fail to make payment.}\]
see that a tranche of a cashflow CDO is a credit derivative since its price is dependent on the credit risk of an underlying portfolio of bonds or loans.

The other main form of risk besides credit risk is market risk. Market risk is the risk associated with fluctuations in the market price of an instrument. For example, there is no credit risk involved with purchasing shares of a company but there is market risk since the price of a share may decrease.

### 2.3 Credit Rating

A credit rating is an indicator, set by credit rating agencies, to demonstrate the credit worthiness of a corporation. Credit rating agencies use the financial records of the company, such as its assets and current liabilities to estimate its ability to pay back its debt. This information is then used to assign a rating to the company. The different rating agencies, such as Moody’s, Standard and Poor’s and Fitch use different measures to rate the companies but most use a system which rates the companies using letters from A - D. Typically, the companies with the highest standard of credit worthiness (least possibility of defaulting) are given a AAA rating and the lowest rated companies are given a rating of C or D, with many layers in between. As mentioned in Chapter 1, credit rating agencies having been heavily criticized in recent years for failing to adequately quantify the risk associated with tranches of CDOs. In their study of the ratings of nearly 4000 CDOs, Benmelech and Dlugosz (2009) concluded that “there is a striking difference between the credit rating structure of CDOs and the credit quality of the collateral.”

### 2.4 Cashflow CDOs

A collateralized debt obligation is a credit derivative that creates a number of different securities from a pool of risky assets. Essentially a cashflow CDO is a structure that allows an originator (e.g. a bank) to group together a number of bonds or loans into a package so that they may be resold to investors. There are a number of motivations for creating a cashflow CDO, the main being that it allows the originator to remove risky assets from their balance sheet while it affords the investor the ability to invest in a diversified portfolio of assets that they would otherwise not easily be able to invest in.

In Chapter 1 we differentiated between the two main types of CDOs, namely cashflow
CDOs and synthetic CDOs. A cashflow CDO can further be broadly classified as a *balance sheet* or an *arbitrage* CDO depending on the motivation of the originators. Balance sheet CDOs would be created by a bank with the purpose of removing assets from their balance sheet. This may be in order to reduce their exposure to a particular market or simply to free up funds for other investments. An arbitrage CDO would be set up with the aim of earning a profit on the spread between the yield on the underlying portfolio of assets and the resulting CDO structure.

As discussed, the creation of a CDO involves the setting up of a trust called a *special purpose vehicle* (SPV) by the originator. The bonds or loans are then sold to the SPV and tranches are issued to investors. The SPV pays coupons and principal to the tranche holders from the coupons and principals it receives from the underlying assets. In the event of default of an underlying asset, the recovery amount is used to fund principal payment to the tranche holders. Principal payment, whether it be scheduled notional repayment or recovery amounts due to default, will often be referred to as *redemption* (since it is used to redeem the tranche holders). Since the tranches income payments are derived from the underlying assets, they are known as *asset-backed-securities*.

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**Figure 2.1:** Creation of a cashflow CDO. A pool of loans is sold to a special purpose vehicle which then issues tranches of varying risk qualities.

The tranches have a strict ordering of seniority over the cashflow payments from the underlying assets. That is, if a certain amount of cash is available from the pool to make payments to the tranche holders, the most senior tranche is paid first, followed by the next senior and so on until finally the least senior tranche receives whatever is remaining. The top tranches have the highest credit rating and are known as *senior* tranches, with the top tranche known as the super-senior tranche. The middle tranches
are known as the *mezzanine* tranches and the least senior tranche is known as the *equity* tranche.

So any loss incurred by the portfolio of underlying assets (due to defaults) is first felt by the equity tranche, then the mezzanine tranches and finally the senior tranches. The tranches receive coupons relative to the level of risk being undertaken. So the most senior tranches (least level of risk) would have the lowest rate of return while the least senior tranches (greatest level of risk) receive the highest rate.

The principal and interest proceeds from the portfolio of underlying assets (bonds or loans) are paid to the tranches in a method known as a *waterfall payment*. There are two separate waterfall payments in a cashflow CDO - the interest waterfall and the principal waterfall. In the interest waterfall, any interest received from the underlying pool of assets will be used to first pay the senior tranches, then the mezzanine tranches and finally whatever residual interest is left over will be paid to the equity tranche. A certain tranche holder will only receive any interest payment once the the more senior tranche holders have received their interest payments in full. So the least senior tranche (equity tranche) will be the first to feel any loss whereas the most senior tranche will be the last affected by loss. The principal waterfall is similar except that it uses principal payment from the pool to make principal payments to the tranche holders.

### 2.5 Example of a cashflow CDO

In preparation for the formal mathematical description of a cashflow CDO in Section 3.2 and the pricing theory and practice reviewed in Section 3.3, we provide here a simple example of a cashflow CDO, as shown in Figure 2.2. We have a portfolio of 10 loans, each with a notional of €100, that pay interest annually at 5% and that all mature in 5 years (in practice the portfolio could be made up of a large variety of loans with different notionals, interest rates and maturities). The loans are sold to a special purpose vehicle and five tranches are issued. Tranche 1 is the most senior tranche and Tranche 5 is the least senior tranche. We see that Tranche 1 has a notional of €500 and pays 3% annually, Tranche 2 has a notional of €275 and pays 5% annually, Tranche 3 has a notional of €100 and pays 7% annually, Tranche 4 has a notional of €75 and pays 10% annually and finally Tranche 5 has a notional of €50 and receives any residual interest only after the first four tranches have been paid interest in full. We will assume that if a loan defaults the recovery rate is 40%, i.e., if one of the loans default we get back €40.
Let us assume that in the first year no loan has defaulted and all make coupon payments. Each loan pays €5 and so we have €50 available from the pool to make interest payments to the tranche holders. Tranche 1 is owed €15 (this will also be referred to as the interest claim of Tranche 1), Tranche 2 is owed €13.75, Tranche 3 is owed €7 and Tranche 4 is owed €7.5. This adds up to €43.25 and so we have €6.75 left over to pay to Tranche 5.

Let say that in the second year, two out of the ten loans default. We now have 8 loans paying coupons of €5 and so we have €40 available to make payments to the tranche holders. The first 3 tranches are paid their full due amount, i.e., Tranche 1 is paid €15, Tranche 2 is paid €13.75 and Tranche 3 is paid €7. This only leaves €4.25 left over from our original €40. This is paid to Tranche 4 and Tranche 5 receives nothing.

Since two loans have defaulted we have €80 (€40 from each) available for principal payments to the tranche holders. This is used to pay back Tranche 1 and the notional of Tranche 1 is written-down to €420.

If, in the third year, there is one more default we now have €35 available to pay interest and €40 available to pay principal. Tranche 1 receives 3% of its current notional value, which is €420 and so receives €12.60, Tranche 2 receives €13.75 and Tranche 3 receives €7. This leaves €1.65 and is paid to Tranche 4 while again Tranche 5 receives no
payment. Tranche 1 is paid €40 and its notional is written-down to €380. We can see the situation after 3 years in Figure 2.3.

If there are no further defaults, at the end of the fifth year there will be €700 worth of principal repayments available to pay €380 to Tranche 1, €275 to Tranche 2, and €45 left over to pay to Tranche 3. Tranche 4 and Tranche 5 will not receive any principal payment.

This is the basic structure of the waterfall payment system over the life of a cashflow CDO. At the beginning of the trade the value of each tranche is dependent on the probability of default of the underlying loans. If none of the loans default then every tranches will receive their full interest payment each year and their full principal payment at the end. In our example we see that if two or more loans are very likely to default then Tranche 4 and Tranche 5 become worthless since they will not receive any interest or principal payments. Tranche 1 on the other hand, would need 6 or more loans to default before it would feel any loss. So pricing a cashflow CDO involves determining the likelihood of default of the underlying assets.

It should be noted in our example that the amount of coupons received by a tranche holder at a certain date is dependent on the number of defaults at two different times. The coupons paid to a certain tranche is the minimum of the total available interest (from the pool) and that tranches interest claim. However, the available interest at a scheduled
payment date is dependent on all defaults to that date, whereas the interest claim is dependent on the number of defaults at the previous payment date (since the interest claim is calculated using the notional at the last payment date, which is dependent on the number of defaults at the last payment date). We will see in Chapter 4 that this adds difficulty to any semi-analytic methods for cashflow CDO pricing.

In Chapter 3 we will take the basic structure above and explicitly describe the cashflow CDO pricing equations. Two new features, interest coverage and over-collateralisation tests, will be introduced and we review existing methods for pricing a cashflow CDO.
Chapter 3

Cashflow CDO Structure and Pricing

This chapter will review the practice and theory behind pricing a tranche of a cashflow CDO. Even though we will be specifically dealing with Monte Carlo pricing, Sections 3.3.3 - 3.3.7 are general enough that they will also serve as the background theory for pricing using a semi-analytic approach in Chapter 4. We will start by showing in Section 3.1 how to price a simple instrument like a bond. We will then describe in Section 3.2 the main pricing equations of a cashflow CDO and will introduce two new features, namely interest coverage and overcollateralisation tests. Finally, in Section 3.3 we will review pricing methods, including how to model credit risk, constructing a credit curve and how to model default dependence. By the end of the chapter the reader should have a complete understanding of the practicalities of pricing a cashflow CDO tranche using Monte Carlo integration, including how to strip hazard rates from market data and the use of copula functions, thus setting the scene for Chapter 4 on pricing without Monte Carlo simulation.

3.1 Pricing a Bond

One of the most fundamental aspects of the pricing any financial instrument involves evaluating the current price of a future cashflow. Intuitively, it can be seen that money is always worth more now than in the future. For example, given the choice, one would naturally prefer to receive a sum of money today over receiving the same sum ten years from now. This assumes that money can always be invested at a risk-free interest rate.
for any period of time, i.e., assumes it is always possible to put money on deposit in a bank and receive interest for an indefinite amount of time. Clearly then, if money always accumulates over time, €100 today will always be more valuable than €100 at any time in the future since €100 today can be put on deposit immediately.

This leads us to define the risk-free discount factor, \( D(t,T) \), as the amount by which a future cashflow at time \( T \) needs to be multiplied by in order to give an “equivalent” amount at time \( t \):

\[
D(t,T) = e^{-r\delta(t,T)},
\]

where \( r \) is the constant risk-free interest rate and \( \delta(t,T) \) is the \textit{year fraction}\(^1\) between time \( t \) and \( T \). Assuming a risk-free interest rate of 3.5%, a note, which entitles the holder to €100 two years from now would be worth €100 \( \times D(0,2) = 100 \times e^{(-0.035 \times 2)} = €93.24 \) today.

When \( r \) changes over time the following expression represents the discount factor

\[
D(t,T) = e^{-\int_t^T r(s)ds}.
\]

Consider a bond with notional \( N \), that pays coupons at a fixed rate \( k \) at times \( t_1, t_2, t_3, \ldots, t_n \) and pays back the principal \( N \) at \( t_n \). Calculating the price of this bond at time \( t \) (assuming no default) simply involves discounting all future cashflows and getting their sum:

\[
\Pi(t) = \sum_{i=1}^n N k \delta(t_{i-1}, t_i) D(t, t_i) + ND(t, t_n), \tag{3.1}
\]

where \( \Pi(t) \) is the price or value of the bond at time \( t \).

Say we want to incorporate credit risk (probability of default) into the above pricing. Let \( \tau \) be a random variable representing the \textit{time of default} (also known as \textit{survival time}) of the bond. Assume the probability density function of \( \tau \) is known; call this \( \rho(t) \). The probability that a bond defaults within a certain time interval \([a,b]\) is then

\(1\)The \textit{year fraction} \( \delta(t,T) \), is the time difference in years between \( t \) and \( T \). A \textit{day-count convention} is the method used to measure the year fraction between two dates. When measuring the number of years (or fraction of a year) between two dates the day-count convention decides how many days represent a year, whether or not to count weekends and how many days in a month, etc. For example, a “30/360” convention assumes that each month has 30 days and that a year is 360 days long.
The probability of default before time \( t \) is

\[
P_d(t) = P_d[0 \leq \tau \leq t] = \int_0^t \rho(s)ds.
\]

The survival probability, i.e., the probability that the bond does not default before time \( t \) is therefore

\[
S(t) = 1 - P_d(t).
\]

The price of the risky fixed rate bond is the expected value of the discounted future cashflows under the probability measure introduced. Since we are trying to include the possibility of default into our pricing, a recovery amount \( RN \) (letting \( R \) be the recovery rate) is now a possible cashflow. Unlike the other cashflows (interest and principal payments), which have discrete payment dates, a recovery amount can be paid at any time before maturity and so it is necessary to integrate over the continuous time range up to maturity:

\[
\Pi(t) = \sum_{i=1}^{n} Nk\delta(t_{i-1}, t_i)D(t, t_i)S(t_i) + ND(t, t_n)S(t_n) + RN \int_t^{t_n} D(t, s)\rho(s)ds. \tag{3.2}
\]

Whether a bond is priced as risky or not depends on the issuer. For example US Treasury bonds or German Bunds would be priced using Equation 3.1, i.e., assuming the probability of default is 0. This is almost certainly the case with US Treasury bonds since the US can always print more money in order to pay back its debt. This is not strictly true for German debt but the rate on a German Bund is still used as a benchmark for the risk-free interest rate in Europe, as the probability of default is so low. Corporate bonds\(^1\) are priced as risky bonds since even the highest rated companies have at least some probability of default. It can clearly be seen by comparing Equation 3.1, for a risk free bond, and Equation 3.2, for a risky bond, that a risky bond with identical notional and coupon payments to a risk free bond will always be cheaper than

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\(^1\)Corporate bonds are bonds issued by corporations in order to raise money. For example, a company might sell bonds to raise money needed for capital expenditure.
the risk free bond. The reason for this is that investors must be compensated for the fact that the bond they are investing in may not pay back in full. So investors demand a higher yield (lower price) on the risky bonds.

Unlike a bond, a CDO tranche is dependent on the risk associated with many different companies and therefore pricing is not as straightforward. Before we move on to pricing a CDO tranche in Section 3.3, we will first describe in detail the mechanics of how a cashflow CDO is structured.

### 3.2 Cashflow CDO Structure

Consider a portfolio of \( m \) cashflow assets (loans or bonds). The \( j \)th asset has a notional value of \( N^j \), makes interest payments \( C^j(t) \) at times \( t^j_1, \ldots, t^j_{p_j} \), and has a maturity \( T^j = t^j_{p_j} \). We will assume that the underlying assets are bullet structures, i.e., they only make one single principal repayment at maturity. They can be fixed or floating rate notes. So, assuming no default, each asset makes a principal repayment of \( N^j \) at \( T^j \).

On default, a recovery rate of \( r^j \) is assumed and so the \( j \)th asset will have a recovery amount of \( r^j N^j \). Assuming no credit risk, the price of the \( j \)th asset at time \( t \) is

\[
\Pi^j(t) = \sum_{i=1}^{p^j} C^j(t^j_i) D(t, t^j_i) + N^j D(t, T^j).
\]

We consider a cashflow CDO with \( n \) tranches. At time \( t \), the notional of tranche \( k \) is \( N_k(t) \) and the interest rate on tranche \( k \) is \( K_k(t) \). The amount of interest due to the holder of tranche \( k \) at any time is equal to the interest rate times the value of the notional at the previous payment date. Therefore tranche \( k \) has scheduled interest payments \( K_k(t_i) N_k(t_{i-1}) \) at times \( t_1, \ldots, t_p \) (sometimes time \( t_i \) will be referred to as the \( i^{th} \) period). The actual amount of interest (coupon) paid at time \( t_i \) is \( C_k(t_i) \). This can be 0 if no interest is available for tranche \( k \) and any amount up to a max of \( K_k(t_i) N_k(t_{i-1}) \) depending on how much pool interest is available to the tranche. We will assume that interest that is not paid to a tranche in a certain period is not made up in subsequent periods.

The notional is written-down at each period as tranche \( k \) receives redemption, \( R_k(t_i) \), from either scheduled repayments or recovery amounts:
A tranche can receive payments from either coupons, \( C_k(t_i) \), or redemptions, \( R_k(t_i) \), and these are dependent on the number and types of defaults in the pool at each time step. If \( C_k(t_i) \) and \( R_k(t_i) \) are treated as random variables on the sample space of all possible defaults, then using a suitable probability measure the price of tranche \( k \) is:

\[
\Pi_k(t) = \sum_{i=1}^{p} \mathbb{E}[C_k(t_i)] D(t,t_i) + \sum_{i=1}^{p} \mathbb{E}[R_k(t_i)] D(t,t_i),
\]

(3.4)

where \( \mathbb{E} \) is the expected value under a credit measure.

Before considering a probability/credit measure on which to evaluate the above expected values in Section 3.3 we will first look at the structure of \( C_k(t_i) \) and \( R_k(t_i) \) and determine how they are affected by the coupons and redemptions from the underlying pool.

### 3.2.1 Evaluating Tranche Notionals

First we will examine how to determine the tranche notional \( N_k(t) \) (and hence \( R_k(t) \) by Equation 3.3), given a total amount of portfolio redemption, \( Red(t) \), up to time \( t \).

Now \( Red(t) \) is just the sum of the individual redemptions, \( Red^j(t) \), (either from scheduled repayments of principal or recovery amounts in the event of default) of all \( m \) underlying assets:

\[
Red(t) = \sum_{j=1}^{m} Red^j(t).
\]

(3.5)

The redemption values \( Red^j(t) \) are one of three values; 0 if time \( t \) is less than the maturity time \( T^j \) of the asset and the asset has not defaulted; \( r^j N^j \) if the asset has defaulted at any time before \( t \); \( N^j \) if the asset has not defaulted at \( t \) and \( t \) is past the maturity of the asset. Letting \( \tau^j \) be the time of default of the \( j^{th} \) asset, this can be written as,

\[
Red^j(t) = \begin{cases} 
  r^j N^j \mathbf{1}_{\{\tau^j < t\}}, & \text{for } t < T^j, \\
  r^j N^j \mathbf{1}_{\{\tau^j < T^j\}} + N^j \mathbf{1}_{\{\tau^j \geq T^j\}}, & \text{for } t \geq T^j,
\end{cases}
\]

(3.6)
and the indicator function, $1_A$, is defined as

$$1_A = \begin{cases} 
1 & \text{if } A \text{ is true}, \\
0 & \text{otherwise}.
\end{cases}$$

This can be written completely in terms of indicator functions as

$$Red^j(t) = r^j N^j \mathbf{1}_{\{\tau^j < t\}} \mathbf{1}_{\{t < T^j\}} + N^j \left[ r^j \mathbf{1}_{\{\tau^j < T^j\}} + \mathbf{1}_{\{\tau^j \geq T^j\}} \right] \mathbf{1}_{\{t \geq T^j\}}.$$

Assume that the total redemption from the pool at time $t$ is $Red(t)$. We note that this is the total redemption from all assets in all previous time steps, i.e., it is the sum of all the redemptions in the time periods $t_1, t_2, \ldots$ up until time $t$. The notional of the first tranche at the start of the trade is $N_1(0)$. So, at this point, the first notional can either be completely paid back (if $Red(t) \geq N_1(0)$) or have a portion of its notional paid back (if $Red(t) < N_1(0)$). Similarly, if $Red(t)$ is greater than the notional of the first tranche then the second tranche can have a portion of its notional paid back or it can be completely paid back if $Red(t)$ is greater than the combined first two notional. Continuing, it can be seen that the notional of a given tranche at time $t$ is dependent on whether or not the more senior tranches have been paid back in full. This gives us the following set of recursive equations for the tranche notionals:

$$N_1(t) = \max \left( N_1(0) - Red(t), 0 \right), \quad (3.7)$$

$$\vdots$$

$$N_k(t) = \max \left( N_k(0) + \min \left( \sum_{j=1}^{k-1} N_j(0) - Red(t), 0 \right), 0 \right), \quad (3.8)$$

$$\vdots$$

$$N_n(t) = \max \left( N_n(0) + \min \left( \sum_{j=1}^{n-1} N_j(0) - Red(t), 0 \right), 0 \right). \quad (3.9)$$

Assuming a probability measure is known, the expected value of the tranche notionals, $\mathbb{E}[N_k(t)]$, can be evaluated and hence by Equation 3.3 the expected value of the redemptions, $\mathbb{E}[R_k(t)]$, can also be evaluated. This gives the second term of Equation 3.4 for
the price of the tranche.

In the next section we will see how calculate the remaining term of Equation 3.4, the coupons, $C_k(t_i)$, of the tranche.

### 3.2.2 Evaluating Tranche Coupons

The coupons that will be paid to the tranche holders are dependent on the amount of interest that has been received from the underlying pool. We will call this the *available interest*. At time $t_i$ it is the sum of the interest received from all the underlying assets at $t_i$:

$$\text{Available Interest} = \sum_{j=1}^{m} C_j(t_i) 1_{\{\tau_j \geq t_i\}}. \quad (3.10)$$

The interest claim of the $k^{th}$ tranche is $K_k(t_i)N_k(t_{i-1})$. The amount of interest paid to the most senior tranche at $t_i$ will thus be the minimum of the tranche interest claim and the available interest:

$$C_1(t_i) = \min \left( K_1(t_i)N_1(t_{i-1}), \sum_{j=1}^{m} C_j(t_i) 1_{\{\tau_j \geq t_i\}} \right).$$

From the above section we have $N_1(t_{i-1}) = \max(N_1(0) - \text{Red}(t_{i-1}), 0)$ and so

$$C_1(t_i) = \min \left( K_1(t_i) \max \left( N_1(0) - \text{Red}(t_{i-1}), 0 \right), \sum_{j=1}^{m} C_j(t_i) 1_{\{\tau_j \geq t_i\}} \right).$$

Replacing with the general term for the tranche notional and noting that the available interest will be decreased as each tranche receives payment, we get
Chapter 3. Cashflow CDO Structure and Pricing

\[ C_k(t_i) = \min \left( K_k(t_i) N_k(t_{i-1}), \max \left( \sum_{j=1}^{m} C^j(t_i) \mathbf{1}_{\{r_j \geq t_i\}} - \sum_{l=1}^{k-1} K_l(t_i) N_l(t_{i-1}), 0 \right) \right), \]

\[ = \min \left( K_k(t_i) \max \left( N_k(0) + \min \left( \sum_{j=1}^{k-1} N_j(0) - \text{Red}(t_{i-1}), 0 \right), 0 \right), 0 \right), \]

\[ \max \left( \sum_{j=1}^{m} C^j(t_i) \mathbf{1}_{\{r_j \geq t_i\}} - \sum_{l=1}^{k-1} K_l(t_i) \max \left( N_l(0), 0 \right) \right) \]

\[ + \min \left( \sum_{j=1}^{l-1} N_j(0) - \text{Red}(t_{i-1}), 0 \right), 0 \right), \quad (3.11) \]

The equity tranche, tranche \( n \), receives whatever residual interest is left over after all the other tranches have been paid:

\[ C_n(t_i) = \max \left( \sum_{j=1}^{m} C^j(t_i) \mathbf{1}_{\{r_j \geq t_i\}} - \sum_{l=1}^{n-1} K_l(t_i) \max \left( N_l(0), 0 \right) \right) \]

\[ + \min \left( \sum_{j=1}^{l-1} N_j(0) - \text{Red}(t_{i-1}), 0 \right), 0 \right), \quad (3.12) \]

The above equations characterize a cashflow CDO in its simplest form. Most CDOs though, have additional procedures that add more complexity to the above equations.

### 3.2.3 Interest Coverage and Overcollateralisation Tests

We will now introduce two important structural features of most cashflow CDOs that have not been mentioned so far, namely interest coverage (IC) tests and overcollateralisation (OC) tests. These are tests that are performed for each tranche (in order of seniority) that determine whether more interest needs to be diverted to the tranche to pay off principal, if certain requirements (on the amount of available interest and portfolio notional size) are not met. First we will introduce the indicators used to perform these tests. The structure of IC/OC tests introduced below follows from the same terminology and notation used by Papadopoulos and Tan (2007).
Supported Debt Level and Overcollateralisation Ratio

The supported debt level for tranche $k$ at time $t$, $\sigma_k(t)$, is defined as the sum of the notionals of all the tranches more senior to, and including tranche $k$:

$$\sigma_k(t) = \sum_{j=1}^{k} N_j(t).$$

The outstanding notional on the pool at time $t$ is the sum of the current notionals of all the assets that have not defaulted:

$$\sum_{j=1}^{m} N^j(t) 1_{\{\tau^j \geq t\}}.$$

The overcollateralisation (OC) ratio for tranche $k$ at time $t$, $OC_k(t)$, is the ratio of the outstanding notional on the pool to the supported debt level:

$$OC_k(t) = \frac{\sum_{j=1}^{m} N^j(t) 1_{\{\tau^j \geq t\}}}{\sigma_k(t)}.$$

An OC test measures the extent to which the outstanding notional of the pool is greater than the debt required to pay back the notional of the tranches. Tranche holders will not want the level of pool notional to drop below a certain size relative to the amount of debt required for their tranche to be paid back. So an OC test for a certain tranche essentially involves checking whether the calculated OC ratio has fallen below a contractually prespecified trigger, $OC_k$. If an OC test fails, any available interest is diverted to be used as redemption to bring the tranche notionals to a level that passes the test. This is known as a cure payment. If $OC_k(t) < OC_k$ then the tranche notionals $N_k(t)$ are reduced (in order of decreasing seniority) and hence $\sigma_k(t)$ is reduced until $OC_k(t)$ is equal to $OC_k$.

Supported Debt Service Level and Interest Coverage Tests

The supported debt service level for tranche $k$ at time $t_i$, $c_k(t_i)$, is the sum of the interest claims of all the tranches more senior to, and including tranche $k$:
\[ c_k(t_i) = \sum_{j=1}^{k} K_j(t_i)N_j(t_{i-1}). \]

The interest coverage (IC) ratio for tranche \( k \) at time \( t \), \( IC_k(t) \), is the ratio of the amount of available interest from the pool to the interest due on tranche \( k \) and all tranches senior to it, i.e., it is the ratio of the available interest to the supported debt service level:

\[ IC_k(t_i) = \frac{\sum_{j=1}^{m} C_j(t_i)1_{\{r_j \geq t\}}}{c_k(t_i)}. \]

An IC test measures the amount of interest available on the pool relative to the size of the interest due to the tranches. As in the case of an OC test, a tranche holder will not want the amount of available interest fall below a certain fraction of the supported debt service level. So, similar to OC tests, there will be prespecified IC levels, \( IC_k \), for the \( k \)th tranche that will trigger cure payments. Again any available interest will be diverted to reduce the tranche notionals until \( c_k(t_i) \) is the required amount.

**Cure Payments and Notional Reduction**

We now introduce the equations that calculate the amount of cure payments necessary to be paid to each tranche for each test. Assume a test is performed for every tranche except the equity tranche. That means for an \( n \) tranche CDO there are \( n-1 \) tests at each payment date. The \( k \)th test refers to the testing of the \( k \)th tranche. So for the \( k \)th test cure payments can be made to at most \( k \) tranches. Similar to the interest waterfall, the cure payments are made in order of seniority and the \( k \)th tranche will only receive a cure amount if the notional of tranche \( k-1 \) has been fully redeemed (\( N_{k-1} = 0 \)). Interest payments for a given tranche have priority over IC/OC cure amounts. So a test will only be performed (and any cure payments made) after an attempt has been made to pay interest due to that tranche. We use the word “attempt” because it could be possible that there is no available interest to pay to the tranche at all (in this case obviously there will be no interest available to make any cure payments either). So the interest waterfall is ordered such that the \( k \)th tranche will be paid interest first, then IC/OC tests for tranche \( k \) and any cure payments, then tranche \( k+1 \) will be paid interest, then IC/OC tests for tranche \( k+1 \) and any cure payments, etc., from tranche 1 down to tranche \( n-1 \), and finally tranche \( n \) will receive any left over interest but will not have any tests performed.
Suppose we are performing an OC test on the most senior tranche \((k = 1)\). Since the interest payments receive priority over the OC test the amount of available interest for any cure payments will be the available pool interest minus what has been paid in interest to tranche 1. Therefore the available interest is now

\[
\max \left( \sum_{j=1}^{m} C^j(t_i) \mathbf{1}_{\{\tau_j \geq t_i\}} - K_1(t_i) N_1(t_{i-1}), 0 \right).
\]  

(3.13)

We take the max with 0 since obviously available interest can only be a nonnegative value. Assuming the OC test has failed \((OC_1(t_i) < OC_1)\) the desired OC ratio for tranche 1 to pass the test is \(OC_1\). So the desired supported debt level, \(\tilde{\sigma}_1(t_i)\), is:

\[
\tilde{\sigma}_1(t_i) = \sum_{j=1}^{m} \frac{N_j(t_i) \mathbf{1}_{\{\tau_j \geq t_i\}}}{OC_1}.
\]

The amount of interest to be diverted to redeem tranche 1 in order for it to pass the OC test will be the minimum of the size of the tranche, \(N_1(t_i)\), and the difference between the calculated supported debt level and the required supported debt level \((\sigma_1(t_i) - \tilde{\sigma}_1(t_i))\):

\[
k_{OC}^1(t_i) = \max \left( \min \left( N_1(t_i), \sigma_1(t_i) - \tilde{\sigma}_1(t_i) \right), 0 \right).
\]

(3.14)

The cure amount to be paid for the curing of the first OC test at time \(t_i\), \(Cure_1^{OC}(t_i)\), is equal to the minimum of the required notional reduction (Equation 3.14) and the available interest (Equation 3.13):

\[
Cure_1^{OC}(t_i) = \max \left( \min \left( k_{OC}^1(t_i), \sum_{j} C^j(t_i) \mathbf{1}_{\{\tau_j \geq t_i\}} - K_1(t_i) N_1(t_{i-1}) \right), 0 \right).
\]

(3.15)

Note that \(Cure_1^{OC}(t_i) = 0\) corresponds to the case where either the OC test has passed or there is not enough available interest to make any cure payment.

The equations for determining the amount of cure payment to be paid for the curing of the first IC test are similar to the case of the OC test. The amount of available interest for the curing of tranche 1 is still given by Equation 3.13. Assuming the test fails when \(IC_1(t_i) < IC_1\), then the desired supported debt service level, \(\tilde{c}_1(t_i)\), is
\[ \tilde{c}_1(t_i) = \sum_{j=1}^{m} C_j(t_i) \mathbf{1}_{\{\tau_j \geq t_i\}} \frac{1}{IC_1}, \]

The amount of interest to be diverted in order for tranche 1 to pass the IC test is the minimum of the tranche notional and the amount needed to make \( c_k(t_i) = \tilde{c}_k(t_i) \).

Assuming the test fails, the amount required to be paid to tranche 1 to make \( c_k(t_i) = \tilde{c}_k(t_i) \) is

\[ c_k(t_i) - \tilde{c}_k(t_i) \]

\[ \frac{1}{K_1(t_i)}. \]

So the equation for calculating \( Cure_1^{IC}(t_i) \) for an IC test is similar to Equation 3.15 but with

\[ Cure_1^{IC}(t_i) = \max \left( \min(\kappa_1^{IC}(t_i), \sum_j C_j(t_i) \mathbf{1}_{\{\tau_j \geq t_i\}} - K_1(t_i) N_1(t_i-1)), 0 \right) \] (3.16)

\[ \kappa_1^{IC}(t_i) = \max \left( \min \left( N_1(t_i), \frac{c_k(t_i) - \tilde{c}_k(t_i)}{K_1(t_i)} \right), 0 \right). \] (3.17)

In practice, IC and OC tests are done at the same time for a given tranche and the maximum of the two values, \( Cure_1(t_i) = \max \left( Cure_1^{OC}(t_i), Cure_1^{IC}(t_i) \right) \), is paid.

To extend these equations to the \( k^{th} \) test, the amount of cure payments already paid for the curing of higher tests \( (1, \ldots, k-1) \) must be taken into account, along with the amount paid to each tranche for the curing of a \( k^{th} \) test. The available interest for the curing of the \( k^{th} \) test is simply the amount of interest received from interest payments minus what has been paid out in tranche interest payments and cure payments:

\[ \max \left( \sum_{j=1}^{m} C_j(t_i) \mathbf{1}_{\{\tau_j \geq t_i\}} - \sum_{j=1}^{k} K_j(t_i) N_j(t_i-1) - \sum_{j=1}^{k-1} Cure_j(t_i), 0 \right), \] (3.18)

where \( Cure_j(t_i) \) is the total amount of cure payments (over all tranches) paid for the curing of the \( j^{th} \) test (this includes both IC and OC tests):

\[ Cure_k(t_i) = \max \left( \min \left( \kappa_k(t_i), \sum_{j=1}^{m} C_j(t_i) \mathbf{1}_{\{\tau_j \geq t_i\}} - \sum_{j=1}^{k} K_j(t_i) N_j(t_i-1) - \sum_{j=1}^{k-1} Cure_j(t_i) \right), 0 \right), \]
with

\[ \kappa_k(t_i) = \max \left( \sum_{j=1}^{k} \Delta N_{IC}^{jk}(t_i), \sum_{j=1}^{k} \Delta N_{OC}^{jk}(t_i) \right). \]

Here \( \Delta N_{IC}^{jk}(t_i) \) and \( \Delta N_{OC}^{jk}(t_i) \) are the cure payments required for the \( j^{th} \) tranche, for curing the \( k^{th} \) IC and OC tests, respectively. For each tranche, both \( \Delta N_{IC}^{jk}(t_i) \) and \( \Delta N_{OC}^{jk}(t_i) \) cannot be greater than \( N_j(t_{i-1}) \).

The cure payments \( \Delta N_{IC/OC}^{jk}(t_i) \) will be equal to 0 unless \( \Delta N_{IC/OC}^{j-1k}(t_i) = N_{j-1}(t_{i-1}) \), i.e., a tranche will only be redeemed if the more senior tranche is fully redeemed. The following equation for \( \Delta N_{OC}^{jk}(t_i) \) is similar to Equation 3.14 but where the cures to the higher tranches have been taken into consideration:

\[ \Delta N_{OC}^{jk}(t_i) = \max \left( \min \left( N_j(t_{i-1}), \sigma_k(t_i) - \tilde{\sigma}_k(t_i) - \sum_{q=1}^{j-1} \Delta N_{OC}^{qk}(t_i) \right), 0 \right), \]

The equation for \( \Delta N_{IC}^{jk}(t_i) \) is similar to Equation 3.17 but also takes into consideration the cures paid to higher tranches and the effects of each tranches weighting (due to different \( K_k(t_i) \)) in \( c_k(t_i) \):

\[ \Delta N_{IC}^{jk}(t_i) = \frac{\Delta (KN)^{jk}(t_i)}{K_j(t_i)}, \]

where

\[ \Delta (KN)^{jk}(t_i) = \max \left( \min \left( K_k(t_i)N_j(t_{i-1}), c_k(t_i) - \tilde{c}_k(t_i) - \sum_{m=1}^{j-1} \Delta (KN)^{mk}(t_i) \right), 0 \right). \]

Effects of Cure Payments on the Pricing Equation

It was seen in Section 3.2 that the cashflow CDO pricing equation (Equation 3.4) is determined by \( C_k(t_i) \) and \( R_k(t_i) \). Recalling that \( C_k(t_i) \) is the minimum of the interest claim and the available interest, it can be shown that the cure payments affect Equation 3.11 in two ways. Firstly, the available interest is decreased by any cure amounts paid to the higher tranches and, secondly, the interest claim of tranche \( k \) will be decreased.
due to notional reduction of the tranche on receipt of any cure amounts:

\[ N_k(t_i) = \max \left( N_k(0) + \min \left( \sum_{j=1}^{k-1} N_j(0) - \text{Red}(t_i) - \sum_{b \leq i} \sum_{l=1}^{n-1} \text{Cure}_l(t_b), 0 \right), 0 \right). \] (3.19)

Thus,

\[ C_k(t_i) = \min \left( K_k(t_i) \max \left( N_k(0) + \min \left( \sum_{j=1}^{k-1} N_j(0) - \text{Red}(t_{i-1}) - \sum_{b \leq i} \sum_{l=1}^{n-1} \text{Cure}_l(t_b), 0 \right), 0 \right), \right. \]

\[ \left. \max \left( \sum_{j=1}^{m} C_j(t_i) \mathbf{1}_{\{\tau \geq t_i\}} - \sum_{l=1}^{k-1} K_l(t_i) N_l(t_{i-1}) - \sum_{b \leq i} \sum_{l=1}^{n-1} \text{Cure}_l(t_b), 0 \right) \right) \). \] (3.20)

The equity tranche will receive whatever interest is available after all higher claims and higher cures have been paid:

\[ C_n(t_i) = \max \left( \sum_{j=1}^{m} C_j(t_i) \mathbf{1}_{\{\tau \geq t_i\}} - \sum_{l=1}^{n-1} K_l(t_i) N_l(t_{i-1}) - \sum_{b \leq i} \sum_{l=1}^{n-1} \text{Cure}_l(t_b), 0 \right) \).

Now that it has been explained how IC and OC tests are performed and how the cure amounts are calculated, we will explicitly review how the interest and principal waterfall payment works.

### 3.2.4 Interest and Principal Waterfalls

Consider a cashflow CDO in which the principal and interest are paid in separate waterfalls. The principal waterfall payments take place after the interest waterfall so that any notional reduction in the interest waterfall (due to cure payments) is already accounted for in the principal waterfall.

#### Interest Waterfall

- The available interest for the interest waterfall comes from the interest payments of the underlying portfolio of assets. On a specific payment date \((t_i)\) we will have a pool of interest that has been received from the underlying assets.
This pool will first be used to pay the coupons of the most senior tranche. Using the same notation as above, the most senior tranche (tranche 1) will have an interest claim of $K_1(t_i)N_1(t_{i-1})$. If the available interest is only large enough to pay all or a portion of the most senior tranche’s interest claim then the interest waterfall is complete. If, however, there is any available interest left over, we move on to the IC/OC tests for the senior tranche.

- If either or both of the IC and OC tests fail we use the available interest to redeem (make principal payments to) tranche 1 until either the test has been cured (has the desired IC/OC ratios using Equations 3.15 and 3.16 ) or the available interest runs out.

- If the tests for tranche 1 have passed or there is still interest left over in the pool after the curing of tranche 1, we move on to the payment of the coupons to tranche 2. As above we only progress to the IC/OC tests for tranche 2 if tranche 2 has received its full interest claim ($K_2(t_i)N_2(t_{i-1})$).

- We continue like this until we reach the last tranche (equity tranche). Any available interest that is left over from the steps above is paid to the equity tranche as a coupon.

**Principal Waterfall**

- The available pool of principal proceeds (redemption) for the principal waterfall comes from the scheduled principal repayments of the underlying assets and from recovery amounts in the case of default (see Equation 3.6).

- Any available redemption is used to pay back the notional of the tranches in order of seniority. A tranche only receives any principal payments once all the tranches senior to it have been paid back in full.

- The principal waterfall is complete once the available redemption in a period runs out or if all the tranches have been paid back in full.

Having described the payoff functions for the coupons and redemptions of a cashflow CDO tranche, we will next consider the models used to price a tranche using these functions. We will first explicitly define the concept of an expected value under a credit measure and introduce Monte Carlo pricing. Along with a study of credit risk and default dependence, this will form the basis of the factor copula model which we use to price cashflow CDO tranches.
3.3 Cashflow CDO Pricing

Now that we have detailed the exact structure of a cashflow CDO we can begin to discuss pricing methods. Recall the main pricing equation, namely Equation 3.4, for a cashflow CDO:

\[
\Pi_k(t) = \sum_{i=1}^{p} \mathbb{E}[C_k(t_i)]D(t, t_i) + \sum_{i=1}^{p} \mathbb{E}[R_k(t_i)]D(t, t_i).
\]

By Equations 3.3 and 3.8 we see that \( R_k(t_i) \) can be written in terms of the pool redemption at time \( t_i \), \( \text{Red}(t_i) \). Similarly by Equation 3.11 we see that \( C_k(t_i) \) can be expressed in terms of the available interest at time \( t_i \) and the pool redemptions up until time \( t_{i-1} \), \( \text{Red}(t_{i-1}) \). It can also be seen by Equations 3.5, 3.6 and 3.10 that the pool redemptions and available interest at any time can be written in terms of the survival times of the underlying assets of the pool. We can therefore write our pricing equation in terms of the survival times of the assets:

\[
\Pi_k(t) = \mathbb{E}[\Pi_k(\tau_1, ..., \tau_m|t)],
\]

where

\[
\Pi_k(\tau_1, ..., \tau_m|t) = \sum_{i=1}^{p} \left( C_k(t_i) + R_k(t_i) \right) D(t, t_i),
\]

using Equations 3.3, 3.5, 3.6, 3.8, 3.10 and 3.11 to calculate \( C_k(t_i) \) and \( R_k(t_i) \), given a survival time for each of the \( m \) assets. In this section we will see how we can model the survival times of the assets and use Monte Carlo integration to evaluate the expectation in Equation 3.21. In order to simulate the survival times we will need to model the credit risk of the assets so that we can determine the probability of default for each asset at a certain time. We will discuss one of the most prominent credit risk models, namely the reduced form model in Section 3.3.3. For our practical implementation we will use a reduced form model. As we will see, to calculate the probability of default for an asset using the reduced form model, we require a term structure of hazard rates for each asset. In Section 3.3.4 we will see how we can “strip” these hazard rates from market data, such as the price of a bond or a CDS spread. We will then briefly discuss in Section 3.3.5 the difference between default probabilities stripped from market data and actual observed historical default probabilities, and how the former is most relevant for pricing. In Section 3.3.6 and 3.3.7 we will discuss default dependence and how we can use a copula function to model correlation between the survival times of the assets.
Finally, in Section 3.3.8 we will bring together the theory and methods from the previous sections and show how we can use a factor copula model to simulate correlated survival times for the pool of underlying assets. Using Equation 3.22, we can then calculate the price of the tranche for each sample, thus allowing us to perform a Monte Carlo integration over many samples.

Before we begin our discussion of the reduced form model, stripping hazard rates and default correlation as described above, we will first discuss the Monte Carlo method and sampling from a probability distribution in Sections 3.3.1 and 3.3.2, respectively.

### 3.3.1 The Monte Carlo Method

The Monte Carlo method is a method of computing the expected value of a random variable (or a function of a random variable) by repeated random sampling. In real world applications, it is the most common method used to price complex derivatives with high dimensions of uncertainty such as a cashflow CDO.

Consider a random variable $X$ on $(-\infty, \infty)$. If $f$ is a function of $X$ then the expected value of $f(X)$, $\mathbb{E}[f(X)]$, is defined as follows:

$$
\mathbb{E}[f(X)] := \int_{-\infty}^{\infty} f(x) dP(x),
$$

where $P$ is the cumulative distribution function of $X$. The Monte Carlo method involves evaluating the expected value by repeated sampling of $X$ from the distribution $P$, calculating $f(X)$ for each sample and getting the sample mean,$^{1}$ $\hat{f}(X)_N$. The strong law of large numbers tells us that $\hat{f}(X)_N$ is almost surely equal to the expected value of $f(X)$ as the sample size $N$ goes to infinity:

$$
P \left( \lim_{N \to \infty} \frac{f(X)}{N} = \mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x) dP(x) \right) = 1,
$$

which naturally leads to the approximation that for large $N$,

$^{1}$Say an experiment is performed and $N$ samples of $X$, $(X_1, X_2, \ldots, X_N)$ are taken. The sample mean, $\bar{X}_N$, is then defined as the average of these values:

$$
\bar{X}_N = \frac{1}{N} \sum_{i=1}^{N} X_i.
$$
\[ \bar{f}(X)_N \approx \int_{-\infty}^{\infty} f(x) dP(x). \]

So the sample mean for large \( N \) is a good approximation to the expected value. In the case of cashflow CDOs we sample survival times from an underlying probability distribution. A sample price is computed using Equation 3.22 the mean price over all samples is calculated. Note that in order to perform Monte Carlo pricing of a cashflow CDO it is first necessary to be able to sample random survival times for each of the underlying assets. A more detailed discussion of Monte Carlo methods in financial modelling can be seen in Glasserman (2003). Before we eventually discuss using a factor copula model to sample survival times in Section 3.3.8 we will first see how we can sample from a general probability distribution.

### 3.3.2 Sampling from a Probability Distribution

Suppose we can generate a random variable, \( U \), with uniform distribution\(^1\) on the interval \([0, 1]\) (standard uniform distribution). It can easily be shown that the random variable \( X = F^{-1}(U) \) has a cumulative probability distribution \( F \). For a single asset we suppose the cumulative probability of default, \( P_d(t) \), is known. Recall that \( P_d(t) \) is the probability that the asset will default before time \( t \). To generate a random survival time from this distribution we use the inverse transformation method as follows:

- Generate a random variable \( u \) from the standard uniform distribution.
- Let \( \tau = P_d^{-1}(u) \).
- \( \tau \) is now a random survival time sampled from the cumulative default probability distribution \( P_d(t) \).

So in order to be able to simulate defaults for a single asset we need to know the cumulative probability of default, \( P_d(t) \), for that asset.

Recall Section 3.1 where we defined \( P_d(t) \) in terms of the probability density function, \( \rho(t) \), of the survival time of the asset:

\(^1\)A continuous random variable, \( X \), has uniform distribution if the probability of \( X \) being on any interval in its range is equal to the probability of \( X \) being on any other interval of the same length. On a interval \([a, b]\) the probability density function of the uniform distribution is:

\[ p(x) = \begin{cases} \frac{1}{b-a}, & \text{for } a \leq x \leq b, \\ 0, & \text{for } x < a \text{ or } x > b. \end{cases} \]
\[ P_d(t) = \int_0^t \rho(s) \, ds. \]

We also saw that the survival probability was then defined as

\[ S(t) = 1 - P_d(t). \]

We can also represent the survival time in terms of a conditional probability density, \( h(t) \), i.e., a probability density function for the survival time conditional on survival to time \( t \). Let us say that \( h(t) \Delta t \) approximates the probability of default between \( t \) and \( \Delta t \), conditional on survival to \( t \). So

\[
\begin{align*}
    h(t) \Delta t &= \frac{P_d(t + \Delta t) - P_d(t)}{S(t)} \\
    &= \frac{(1 - S(t + \Delta t)) - (1 - S(t))}{S(t)} \\
    h(t) \Delta t &= -\frac{S(t + \Delta t) - S(t)}{S(t)}
\end{align*}
\]

Letting \( \Delta t \to 0 \), we have

\[ h(t)dt = -\frac{dS(t)}{S(t)}, \]

and solving the resulting differential equation for \( S(t) \) gives

\[ S(t) = e^{-\int_0^t h(s) \, ds}. \]

The function \( h(t) \) is commonly called the *hazard rate*. It can be thought of as the instantaneous rate of default conditional on survival to time \( t \). The four parameters \( P_d(t) \), \( S(t) \), \( \rho(t) \) and \( h(t) \) are interchangeable and a graph of any one of these terms against time is generally called a *credit curve* or a *default probability term structure*.

For our Monte Carlo pricing we require a credit curve in order to be able to evaluate \( P_d^{-1}(u) \). We will be using a reduced form credit risk model to construct a credit curve and we will introduce the model in the next section.
### 3.3.3 Modelling Credit Risk

To simplify our study of modelling credit risk we will discuss pricing a zero-coupon bond. A zero-coupon bond is a bond that makes one single payment at maturity, i.e., no coupons. We consider a bond that pays one euro at maturity, $T$. We will assume a risk-free interest rate, $r(t)$. In the absence of default the price of the bond at time $t$ would therefore be:

$$
\Pi(t) = e^{-\int_t^T r(s) \, ds}.
$$

If a constant risk-free interest rate is assumed, the price of the bond is:

$$
\Pi(t) = e^{-r(T-t)}.
$$

The basis of reduced form models involves treating the survival time as the arrival of the first event in a Poisson process. Assume we have a Poisson process with intensity $\lambda$. The number of events that arrive in the time interval $(t, t + \Delta t]$ is given by the following probability distribution:

$$
\mathbb{P}[(M(t + \Delta t) - M(t)) = k] = e^{-\lambda \Delta t} \frac{(\lambda \Delta t)^k}{k!},
$$

where $M(t)$ is the number of events that arrive by time $t$ and so $M(t + \Delta t) - M(t)$ is the number of events that occur in the interval $(t, t + \Delta t]$.

Letting survival time be the first arrival, the survival probability for $t$ years is the probability that no event occurs between the time 0 and $t$:

$$
S(t) = \mathbb{P}[(M(t) - M(0)) = 0] = e^{-\lambda t} = e^{-\lambda t}.
$$

In this instance $\lambda$ is usually referred to as the default intensity. The expected survival time (time to default) is $1/\lambda$. So a default intensity of 0.04 would mean that the probability of default in 1 year, $P_d(1) = 1 - e^{-0.04} = 3.92\%$ and that the expected time to default is 25 years.

If the default intensity is a deterministic function of time then the survival probability becomes
\begin{equation}
S(t) = e^{-\int_0^t \lambda(s) ds},
\end{equation}

and the default probability is given by

\begin{equation}
P_d(t) = 1 - e^{-\int_0^t \lambda(s) ds}.
\end{equation}

Assuming a constant risk-free interest rate that is independent of default intensity and a recovery rate $R$, the bond price is given by:

\begin{equation}
\Pi(t) = Re^{-r(T-t)}(1 - S(T)) + e^{-r(T-t)} S(T),
\end{equation}

\begin{equation}
= Re^{-r(T-t)} \left( 1 - e^{-\int_t^T \lambda(s) ds} \right) + e^{-r(T-t)} e^{-\int_t^T \lambda(s) ds}.
\end{equation}

Again, for clarity, it is assumed, even though default can occur at any time, that the recovery amount will be paid at time $T$.

The default intensity, $\lambda(t)$, is of course very similar to the hazard rate, $h(t)$ discussed above (Section 3.3.2). The difference between the two is that by definition the hazard rate at time $t$ is conditional only on survival to $t$ whereas the default intensity is conditional on all available information to time $t$. For the remainder of this thesis we will assume survival is the only information available and so the two terms can be considered interchangeable. Under our model, the cumulative probability of default for an asset at time $t$ is known provided that the hazard rate for all times up to $t$ is known. A credit curve or a term structure of hazard rates is a simply a number of values of $\lambda(t)$ for various values of $t$ (various terms). We will now see how we can “strip” a term structure of hazard rates from market data.

### 3.3.4 Stripping Hazard Rates from Market Data and Constructing a Credit Curve

Two methods of stripping hazard rates from market data will be discussed. The first involves stripping the hazard rates from bond prices and the second from the spreads on credit default swaps.
First, assume that the prices of a number of bonds of different terms (1 year, 2 years, ..., 10 years), issued by the same issuer, is available from the market. Recall Equation 3.2 for the price of a risky bond with principal $N$, coupon rate $k$, maturity $t_n$ and $n$ payment dates at $t_1, t_2, ..., t_n$:

$$
\Pi(t) = \sum_{i=1}^{n} Nk\delta(t_{i-1}, t_i)D(t, t_i)S(t_i) + ND(t, t_n)S(t_n) + RN \int_{t}^{t_n} D(t, s)\rho(s) ds,
$$

where $S(t)$ is the survival probability, $\rho(t)$ is the probability density function of the survival time $\tau$, $D(t, t_i)$ is the risk-free discount factor at time $t$ and $\delta(t_{i-1}, t_i)$ is the year fraction between time $t_{i-1}$ and $t_i$. Let us assume a constant hazard rate over the term of the bond of lowest maturity (the 1 year bond). By this we mean that the 1 year bond is priced using a constant hazard rate, $\lambda_{0,1}$, between $t = 0$ and $t = 1$. That is, the survival probability for $0 \leq t \leq 1$ will be given by $S(t) = e^{-\lambda_{0,1}t}$.

The price at time $t = 0$ of the 1 year bond in terms of a constant hazard rate, $\lambda_{0,1}$, is given by:

$$
\Pi_1(0) = \sum_{i=1}^{n_1} Nk_1\delta(t_{i-1}, t_i)D(0, t_i)e^{-\lambda_{0,1}t_i} + ND(0, 1)e^{-\lambda_{0,1}} + RN \int_{0}^{1} D(0, s)\rho(s) ds,
$$

(3.27)

where there are $n_1$ coupon payment dates, $t_{n_1} = 1$ and $k_1$ is the coupon rate of the 1 year bond. Using the fact that the probability density function of the survival time can be written as

$$
\rho(t) = -\frac{\partial S(t)}{\partial t},
$$

(3.28)

and that

$$
S(t) = e^{-\lambda_{0,1}t},
$$

the integral over $\rho(s)$ can be estimated at the coupon payment dates and we get
\[ \Pi_1(0) = \sum_{i=1}^{n_1} N k_1 \delta(t_{i-1}, t_i)D(0, t_i)e^{-\lambda_{0,1}t_i} + ND(0, 1)e^{-\lambda_{0,1}} + RN \sum_{i=1}^{n_1} D(0, t_i)(e^{-\lambda_{0,1}t_{i-1}} - e^{-\lambda_{0,1}t_i}). \]  
\[ (3.29) \]

Again we have assumed that the discount factors are independent of \( \rho(t) \). So now, given the market price of the 1 year bond at time 0, \( \Pi_1(0) \), a recovery value \( R \) and risk-free discount factors we can solve for an implied 1 year hazard rate \( \lambda_{0,1} \).

We then move on to the next bond, in this case the 2 year bond, and, using the 1 year hazard rate previously calculated, \( \lambda_{0,1} \), and the current market price of the bond, \( \Pi_2(0) \) we strip the implied hazard rate, \( \lambda_{1,2} \) between year 1 and year 2:

\[ \Pi_2(0) = \sum_{i=1}^{n_1} N k_2 \delta(t_{i-1}, t_i)D(0, t_i)e^{-\lambda_{0,1}t_i} + \sum_{i=n_1+1}^{n_2} N k_2 \delta(t_{i-1}, t_i)D(0, t_i)e^{-\lambda_{0,1}t_i}e^{-\lambda_{1,2}(t_i-1)} + ND(0, 2)e^{-e^{-\lambda_{0,1}}e^{-\lambda_{1,2}}} + RN \sum_{i=1}^{n_1} D(0, t_i)(e^{-\lambda_{0,1}t_{i-1}} - e^{-\lambda_{0,1}t_i}) + RN \sum_{i=n_1+1}^{n_2} D(0, t_i)(e^{-\lambda_{0,1}e^{-\lambda_{1,2}(t_i-1)}} - e^{-\lambda_{0,1}e^{-\lambda_{1,2}(t_i-1)}}), \]  
\[ (3.30) \]

where the two year bond has \( n_2 \) coupon payment dates, \( t_{n_1} = 1 \) and \( t_{n_2} = 2 \). We then use \( \lambda_{0,1}, \lambda_{1,2} \) and \( \Pi_3(0) \) to calculate \( \lambda_{2,3} \) and continue for all available bonds. This method of iteratively building up hazard rates by forward substitution is known as bootstrapping. The bootstrapped term structure of hazard rates gives us the following structure of survival probabilities:
Chapter 3. *Cashflow CDO Structure and Pricing*

\[ S(t) = \begin{cases} 
  e^{-\lambda_{0,1} t} & 0 \leq t \leq 1 \\
  e^{-\lambda_{0,1} - \lambda_{1,2}(t-1)} & 1 \leq t \leq 2 \\
  e^{-\lambda_{0,1} - \lambda_{1,2} - \lambda_{2,3}(t-2)} & 2 \leq t \leq 3 \\
  \vdots & \vdots \\
  e^{-\lambda_{0,1} - \lambda_{1,2} - \lambda_{2,3} \cdots - \lambda_{8,9}(t-8)} & 8 \leq t \leq 9 \\
  e^{-\lambda_{0,1} - \lambda_{1,2} - \lambda_{2,3} \cdots - \lambda_{9,10}(t-9)} & t > 9 
\end{cases} \] (3.31)

and hence this implies the following term structure of default probabilities:

\[ P_d(t) = \begin{cases} 
  1 - e^{-\lambda_{0,1} t} & 0 \leq t \leq 1 \\
  1 - e^{-\lambda_{0,1} - \lambda_{1,2}(t-1)} & 1 \leq t \leq 2 \\
  1 - e^{-\lambda_{0,1} - \lambda_{1,2} - \lambda_{2,3}(t-2)} & 2 \leq t \leq 3 \\
  \vdots & \vdots \\
  1 - e^{-\lambda_{0,1} - \lambda_{1,2} - \lambda_{2,3} \cdots - \lambda_{8,9}(t-8)} & 8 \leq t \leq 9 \\
  1 - e^{-\lambda_{0,1} - \lambda_{1,2} - \lambda_{2,3} \cdots - \lambda_{9,10}(t-9)} & t > 9 
\end{cases} \] (3.32)

Of course, in practice it may not be the case that we have bond prices for every year, in which case interpolation over periods longer than a year is required.

The other method of stripping implied hazard rates from market data is to strip the rates from the spreads on credit default swaps. A credit default swap (CDS) is an insurance contract between two parties against credit default. The buyer of the protection may wish to insure against the risk of default from an debt issuer. So they will pay a regular premium (spread) to a protection seller and in the event that the issuer in question defaults, the protection seller will reimburse the protection buyer.

The two legs of a CDS trade are known as the *premium leg* and the *protection leg*. The premium leg is the series of regular payments made by the protection buyer to the protection seller. The protection leg is the contingent payment made to the buyer in the event of default.

Say the buyer wants to buy protection on a bond with notional value $N$. They will make regular payments at a rate $s$ on the notional $N$ to the protection seller, where $s$ is known as the CDS *spread* or *premium*. In the event that there is a default on the bond, the protection seller will pay the protection buyer $(1 - R)N$, where $R$ is the recovery.
rate of the bond. Say there are \( n \) payment dates, \( t_1, \ldots, t_n \), where \( t_n \) is the maturity date of the CDS contract. The value of the premium leg at time \( t \) is:

\[
\text{PremiumLeg}(t) = sN \sum_{i=1}^{n} \delta(t_{i-1}, t_i)D(t_i)S(t_i) + sN \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \delta(t_{i-1}, \tau)D(t, \tau)\rho(\tau)d\tau.
\]

The first term is the expected value of the discounted scheduled payments where \( S(t) \) is the survival probability of the underlying asset, i.e., the bond, at time \( t \). The second term is the expected value of the amount of premium accrued if default occurs at time \( \tau \). For example, if default were to occur at time \( \tau \), where \( t_{i-1} < \tau < t_i \), then the first term would represent the scheduled payments from \( t_1 \) up to \( t_{i-1} \) and the second term would represent the accrued premium from \( t_{i-1} \) to \( \tau \), i.e., \( sN\delta(t_{i-1}, \tau)D(t, \tau) \). The value of the protection leg at time \( t \) is the expected value of the amount paid in the event of default:

\[
\text{ProtectionLeg}(t) = (1-R)N \int_{t}^{t_n} D(t, \tau)\rho(\tau)d\tau.
\]

A fair CDS trade is one which has a spread, \( s \), such that

\[
\text{ProtectionLeg}(t) = \text{PremiumLeg}(t).
\]

This gives us the following equation:

\[
(1-R) \int_{t}^{t_n} D(t, \tau)\rho(\tau)d\tau \quad = \quad s \sum_{i=1}^{n} \delta(t_{i-1}, t_i)D(t_i)S(t_i) \\
+ \quad s \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \delta(t_{i-1}, \tau)D(t, \tau)\rho(\tau)d\tau,
\]

(3.33)

Using Equation 3.28 again, we obtain the following equation which can be used, given a CDS spread \( s \), to solve for the implied hazard rate, \( \lambda \):
(1 − R) \sum_{i=1}^{n} D(t, t_i)(e^{-\lambda_{t_i-1}} − e^{-\lambda_{t_i}}) = s \sum_{i=1}^{n} \delta(t_{i-1}, t_i)D(t, t_i)e^{\lambda_{t_i}}
+ s \sum_{i=1}^{n} \delta(t_{i-1}, t_i)D(t, t_i)(e^{-\lambda_{t_i-1}} − e^{-\lambda_{t_i}}).

Using the same method as with the bond pricing equation, we can use this equation to bootstrap a term structure of implied hazard rates from a term structure of market CDS spreads.

Using Equation 3.24, an implied hazard rate at a certain time (from bond prices or CDS spreads) can be used to calculate a default probability. However, these default probabilities do not coincide with actual real world probabilities but rather, are known as risk-neutral default probabilities. We discuss the difference between actual and risk-neutral probabilities in the next section.

### 3.3.5 Actual versus Risk-Neutral Default Probabilities

By data comparison, it can be seen that default probabilities backed out from bond prices or CDS spreads do not match actual historical default probabilities on debt from similarly rated issuers. For example, it is noted in Hull, Predescu and White (2005) that for an A-rated corporate bond, the implied default probability from the bonds price can be up to 10 times that of the historical default probability for an A-rated bond. One of the earliest comparisons in the research literature of historical default data versus implied default probability was undertaken by Altman (1989). In his paper, Altman finds that corporate bond prices between 1971 and 1987 were much lower than could be accounted for by pricing using historical rates of default. This implies that the bonds are being valued by investors assuming a much greater risk than can be accounted for by actual default probabilities. It suggested that this extra risk premium\(^1\) could be due to other forms of risk such as liquidity risk\(^2\) or recovery risk\(^3\).

---

\(^1\)The risk premium is the extra expected return on a risky investment above the return from a risk free investment. In our risky bond pricing model the higher return (lower bond price) was to compensate for the possibility of default on the bond.

\(^2\)Liquidity risk is the risk that the holder of the bond might not be able to sell the bond in order to prevent a loss.

\(^3\)Recovery risk reflects the uncertainty as to how much will be recovered in the event that the bond does default.
Since default probabilities stripped from market prices do not represent actual default probabilities they are known as risk-neutral\(^1\) default probabilities. Intuitively, it might seem that actual default probabilities would be the appropriate probability measure with which to sample survival times in a Monte Carlo pricing of the tranches of a cashflow CDO. However, the pricing of the tranches of a cashflow CDO using actual default probabilities would lead to arbitrage opportunities in the market. An arbitrage opportunity exists in a market when an investor can make a risk-less profit at zero cost. In this case the arbitrage would exist since the price of the portfolio of underlying assets of the cashflow CDO would be less than the total price of all the tranches of the cashflow CDO (due to the extra risk premium on the individual bonds). So it would be possible to short sell all the tranches of the CDO, buy the underlying portfolio and then use this portfolio to create a cashflow CDO to close out the short position. The investor would make the positive difference between the two prices at no cost and with zero risk. When pricing derivatives, the correct price is naturally considered to be the price that does not create any arbitrage opportunities.

Harrison and Pliska (1981) showed that a market is free of arbitrage if and only if there exists a risk-neutral probability measure that is equivalent to the actual probability measure. They also showed that in an arbitrage-free market the price of a derivative is given by the expectation of future discounted cashflows under this equivalent risk-neutral probability measure. This result is sometimes known as “The Fundamental Theorem of Asset Pricing”. See Delbaen and Schachermayer (1994) and Chapter 2 of Brigo and Mercurio (2006) for a detailed discussion of the subject of risk-neutral pricing.

Recall Equation 3.21 for the price of a cashflow CDO:

\[
\Pi_k(t) = \mathbb{E}[\Pi_k(t|\tau_1, ..., \tau_m)].
\]

We see that the appropriate probability measure under which to calculate the above expected value is the risk neutral probability measure, i.e., using probabilities stripped from bond prices or CDS spreads as detailed in Section 3.3.4. So when evaluating the expected value using the Monte Carlo method described in Section 3.3.1, we will sample from the risk-neutral probability distribution using risk-neutral default probabilities.

From now on when we use \(P_d(t)\) we will be referring to risk-neutral default probabilities.

\(^1\)They are known as risk-neutral probabilities since they represent the probability of default in a risk-neutral world, i.e., a world in which investors do not require extra compensation to buy risky debt.
Having the default probability for each individual asset though does not give us a complete model of the risk of default for the CDO since it does not take into account the relationships between the different companies and their likelihood to default at about the same time. In the next section we will describe attempts to deal with this issue by introducing default dependence and default correlation. This will lay the foundation for the introduction of copula functions and the one factor copula model in Section 3.3.7 3.3.8, respectively.

### 3.3.6 Default Dependence and Default Correlation

Assuming that we know the cumulative probability of default, \( P_d(t) \), for each asset, we could perform the inverse transformation method described in Section 3.3.2 for each of the \( m \) assets in the portfolio to get a vector of survival times. We could do this multiple times, using Equation 3.21 to get a price each time and eventually getting an average result as our price. However, sampling survival times independently of each other would assume that the default probabilities of the assets are independent.

Lucas (1995) discusses the reasons why a collection of companies cannot be considered as having independent probabilities of default. The most obvious is the situation where one company is a creditor of the other. Clearly the default of the debtor significantly increases the probability of default of the creditor. There are also more general causes of default dependence such as the relationship between companies within specific industries. Companies which rely on the strength of the same market clearly have linked probabilities of default. For example, as the market for the purchase of CDs and DVDs is diminished in favor of downloadable products, the fortunes of media retailers that sell such products are clearly connected. It is also noted that the occurrence of defaults is related to the overall health of the economy and, as such, even companies not directly connected by credit or a specific industry can be considered to have a default dependence.

Note that Equation 3.21 can also be written in the form

\[
E[\Pi_k(t)] = \int \Pi_k(\tau_1, ..., \tau_m|t) dP_d(\tau_1, ..., \tau_m),
\]

where \( P_d(\tau_1, ..., \tau_m) \) is the joint cumulative distribution function of the survival times. The survival times are independent if and only if this distribution factors:
\[ P_d(\tau_1, \tau_2, ..., \tau_m) = P_{d1}(\tau_1)P_{d2}(\tau_2)...P_{dm}(\tau_m). \] (3.35)

Clearly, in order to perform the Monte Carlo integration of Equation 3.34 we need to sample from the joint distribution, \( P_d(\tau_1, \tau_2, ..., \tau_m) \). However, instead of having the joint distribution we only have the marginal risk-neutral probability distributions:

\[
egin{align*}
P_{d1}(\tau_1) &= \int_{\tau_2} \int_{\tau_3} ... \int_{\tau_m} P_d(\tau_1, \tau_2, ..., \tau_m) d\tau_2 d\tau_3 ... d\tau_m, \\
P_{d2}(\tau_2) &= \int_{\tau_1} \int_{\tau_3} ... \int_{\tau_m} P_d(\tau_1, \tau_2, ..., \tau_m) d\tau_1 d\tau_3 ... d\tau_m, \\
&\vdots \\
P_{dm}(\tau_m) &= \int_{\tau_1} \int_{\tau_2} ... \int_{\tau_{m-1}} P_d(\tau_1, \tau_2, ..., \tau_m) d\tau_1 d\tau_2 ... d\tau_{m-1},
\end{align*}
\]

where \( P_{di}(\tau_i) \) is the risk-neutral probability that asset \( i \) will default before time \( \tau_i \). These are the default probabilities stripped from either bond prices or CDS spreads as described in Section 3.3.4.

The only way to exactly capture the full dependence between the survival times is to have the joint distribution \( P_d(\vec{\tau}) \). However, there are many measures of dependence (also known as measures of association) that can be used to quantify the relationship between dependent random variables. These measures, if calibrated to market data, can be used to fill in some of the blanks in the joint distribution and provide a more accurate model of default dependence. In Appendix A we introduce three measures of dependence, Pearson’s correlation coefficient, Spearman’s rho and Kendall’s tau, as well as a method to estimate Pearson’s correlation using historical default data.

Suppose we now have the marginal probabilities of default for each asset plus a correlation structure between each of the assets. Clearly there is no unique joint distribution that has these properties. But how do we at least define an arbitrary joint distribution function that does have these properties? We need a method of constructing a joint distribution from marginal distributions and a default correlation structure. Li (2000) was the first to use a copula function to achieve this.
3.3.7 Copula Functions

Copulas were first introduced to the statistics literature by Sklar (1973). They were called copulas due to the fact that they “coupled” 1-dimensional marginal distributions to their full multi-dimensional joint distributions. Sklar defines a copula as follows:

**Definition (Copula)** A copula is an n-dimensional function \( C \) with \( \text{Dom} \ C = [0, 1]^n \) and \( \text{Ran} \ C = [0, 1] \), which satisfies the following conditions:

(a) \( C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i \) for each \( i \leq j \) and all \( u_i \in [0, 1] \),

(b) \( C(u_1, \ldots, u_n) = 0 \) if \( u_i = 0 \) for any \( i \leq n \),

(c) \( C \) is \( n \)-increasing.\(^1\)

Sklar proved the following theorem which states that for any multi-dimensional joint cumulative distribution there exists a copula function that links the joint distribution to its marginals:

**Theorem 3.1.** For any \( n \)-dimensional joint cumulative distribution function \( F \) \((n \geq 2)\) with 1-dimensional marginal cumulative distributions \( F_1, \ldots, F_n \) there exists a copula \( C \) such that

\[
F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))
\]

If the \( n \) marginal distributions \( F_1, \ldots, F_n \) are continuous then the copula \( C \) is unique.

Li proposed using the converse of this theorem so that given a set of marginals \( F_1, \ldots, F_n \) and a copula function \( C \) a joint distribution can be produced.

So given a copula function \( C \) such that \( F_{XY}(x, y) = C(F_X(x), F_Y(y)) \), the expected value of a function \( g(x, y) \) with respect to the joint distribution \( F_{XY} \) can also be written in terms of the copula as follows:

\[
\mathbb{E}[g(x, y)] = \int \int g(x, y) dF_{XY}(x, y) = \int \int g(F_X^{-1}(u), F_Y^{-1}(v)) dC(u, v)
\]

\(^1\) \( C \) is \( n \)-increasing if for all \((a_1, \ldots, a_d), (b_1, \ldots, b_d) \in [0, 1]^d \) with \( a_i \leq b_i \) we have

\[
\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1 + \cdots + i_d} C(u_{1i_1}, \ldots, u_{di_d}) \geq 0,
\]

where \( u_{ij} = a_j \) and \( u_{ij} = b_j \) for all \( j \in \{1, \ldots, d\} \).
With respect to Monte Carlo simulation this means that sampling $x$ and $y$ from the distribution $F_{XY}(x, y)$ and computing $g(x, y)$ is identical to sampling $u$ and $v$ from $C(u, v)$ and computing $g(F^{-1}_X(u), F^{-1}_Y(v))$. Returning to Equation 3.34 for the price of tranche $k$, we get the following expectation if we use a copula $C$ such that $P(\tau_1, ..., \tau_m) = C(P_{d1}(\tau_1), ..., P_{dm}(\tau_m))$:

$$
E[\Pi_k(t_i)] = \int \Pi_k(\tau_1, ..., \tau_m|t_i) dP_d(\tau_1, ..., \tau_m)
= \int_0^1 \Pi_k(P^{-1}_{d1}(u_1), ..., P^{-1}_{dm}(u_m)|t_i) dC(u_1, ..., u_m),
$$

(3.36)

letting $\tau_i = P^{-1}_{di}(u_i)$. So to sample $m$ survival times from the portfolio we can sample $u_1, ..., u_m \in [0, 1]$ from our copula $C$ and set $\tau_i = P^{-1}_{di}(u_i)$. The choice of the copula $C$ for our pricing will determine the default dependence between the survival times of the underlying assets. Applying Li’s method to the definitions of Pearson’s correlation, Spearman’s rho and Kendall’s tau gives us:

$$
r_{XY} = \frac{1}{\sqrt{Var(X)Var(Y)}} \int_0^1 \int_0^1 \left[ C(u, v) - uv \right] dF^{-1}_X(u) dF^{-1}_Y(v),
$$

$$
\rho_{XY} = 12 \int_0^1 \int_0^1 \left[ C(u, v) - uv \right] dudv,
$$

$$
\tau_{XY} = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1.
$$

From these equations it can be seen that, unlike Pearson’s correlation coefficient, $r_{XY}$, Spearman’s rho, $\rho_{XY}$, and Kendall’s tau, $\tau_{XY}$, can both be specified completely in terms of a copula function. However, Pearson’s correlation coefficient is dependent on the marginal distributions of the two random variables. It is therefore not necessarily invariant under a copula transformation. This makes $r_{XY}$ generally less suitable for use with copula functions than $\rho_{XY}$ or $\tau_{XY}$.

The multivariate Gaussian copula is defined as

$$
C^G_{R}(u_1, ..., u_n) := \Phi^g_R(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_m)),
$$

(3.37)
where $R$ is the Pearson’s correlation matrix for the $n$ variables and $\Phi_R^n$ is the joint multivariate standard normal distribution function that has Pearson’s correlation $R_{ij}$ between $U_i$ and $U_j$. The bivariate Gaussian copula has the following form

$$C^{Ga}_{r_{ij}}(u_i, u_j) = \int_{-\infty}^{\Phi^{-1}(u_i)} \int_{-\infty}^{\Phi^{-1}(u_j)} \frac{1}{2\pi \sqrt{1 - r_{ij}^2}} \exp \left\{ - \frac{s^2 - 2r_{ij}st + t^2}{2(1 - r_{ij}^2)} \right\} ds dt, \quad (3.38)$$

where $r_{ij}$ is the Pearson’s correlation between the two Gaussian variates. Since Pearson’s correlation is not invariant under the copula transformation the following theorem (whose proof can be found in McNeil et al. (2008)) is important:

**Theorem 3.2.** If $X$ and $Y$ follow a bivariate Gaussian distribution then the following is true:

$$\tau_{XY} = \frac{2}{\pi} \arcsin (r_{XY}),$$

$$\rho_{XY} = \frac{6}{\pi} \arcsin \left( \frac{1}{2} r_{XY} \right).$$

The multivariate $t$-copula is defined as

$$C^{t}_{R,\nu}(u_1, ..., u_n) := t^n_R(t^{-1}_\nu(u_1), ..., t^{-1}_\nu(u_n)), \quad (3.39)$$

where $t^{-1}_\nu$ is the inverse of the univariate $t$-distribution with $\nu$ degrees of freedom and $t^n_R$ is the $n$-dimensional $t$-distribution with mean 0 and Pearson’s correlation matrix $R$.

The bivariate $t$-copula can be written as

1. Let two random variables $Z$ and $Y$ be sampled independently from a standard normal distribution and a chi-squared distribution with $\nu$ degrees of freedom, respectively. Then

$$t := Z \sqrt{\frac{\nu}{Y}}$$

is distributed as a $t$-distribution with $\nu$ degrees of freedom. A random variable $Y$ follows a chi-squared distribution with $\nu$ degrees of freedom if

$$Y = \sum_{i=1}^{\nu} Z_i^2,$$

where $Z_1, ..., Z_\nu$ are independent, standard normal random variables.
\[
C^t_{r_{ij},\nu}(u_i, u_j) = \int_{-\infty}^{t^{\nu-1}(u_i)} \int_{-\infty}^{t^{\nu-1}(u_j)} \frac{1}{2\pi \sqrt{1 - r_{ij}^2}} \left\{ 1 + \frac{s^2 - 2r_{ij}st + t^2}{\nu(1 - r_{ij})} \right\}^{(\nu+2)/2} ds dt,
\]

where again \( r_{ij} \) is the Pearson’s correlation coefficient between bivariate \( t \)-distributed variates.

We will now discuss how, given the implied default probabilities of a number of assets and a copula structure, we can generate a factor copula model.

### 3.3.8 Factor Copula Model

Say our portfolio consists of \( m \) cashflow assets where the implied default probability of the \( i^{th} \) asset at time \( t \) is \( P_{di}(t) \). First we will consider a one-factor copula model. We sample a common factor \( M \) and \( m \) independent variables, \( Z_1, ..., Z_m \), such that both \( M \) and all \( Z_i \) have zero mean and unit variance. Next, we define \( m \) random variables \( x_i \) such that

\[
x_i = a_i M + \sqrt{1 - a_i^2} Z_i,
\]

where \(-1 \leq a_i < 1\). This definition of \( x_i \) ensures that the Pearson’s correlation coefficient between \( x_i \) and \( x_j \) is \( a_i a_j \). Now suppose the distribution of the \( x_i \) is \( F_i \), the distribution of the \( Z_i \) is \( H_i \), and the distribution of the common factor \( M \) is \( F_M \). If we let

\[
u_i = F_i(x_i),
\]

then the \( u_i \) will be correlated random variables on the interval \([0, 1]\) sampled from a copula function. The type of one-factor copula model will be dependent on the distributions \( H_i, F_i \) and \( F_M \). If \( H_i = F_i = F_M = \Phi \), where \( \Phi \) is the cumulative Gaussian distribution, then the \( u_i \)'s will be sampled from a multivariate Gaussian copula (Equation 3.37). Different distributions will result in different copulas, so long as \( F_i \) and \( F_M \) have zero mean and unit variance.

We can extend the model to an \( n \)-factor copula model by letting
\[ x_i = a_{i1}M_1 + a_{i2}M_2 + ... + a_{in}M_n + \sqrt{1 - a_{i1}^2 - a_{i2}^2 - ... - a_{in}^2}Z_i, \]

where \( a_{i1}^2 + a_{i2}^2 + ... + a_{in}^2 < 1 \) and each of the \( n \) \( M_j \) are distributed independently with zero mean and unit variance. The Pearson’s correlation coefficient between \( x_i \) and \( x_j \) will then be \( a_{i1}a_{j1} + a_{i2}a_{j2} + ... + a_{in}a_{jn} \).

The one-factor Gaussian copula model has been heavily criticized in recent years, especially following the subprime mortgage crisis. These criticisms have been discussed by Donnelly and Embrechts (2010). They note three main drawbacks to the one-factor Gaussian copula model. Firstly, they note the “inadequate modelling of default clustering”. They explain this by describing how in reality, in extreme situations with lots of defaults, a company defaulting increases the likelihood that other companies will default within a short time space. However, in the factor Gaussian model, defaults become less clustered as the total number of defaults increase. Other copulas, such as the \( t \)-distribution copula are generally better suited to modelling extreme conditions. Secondly, they note the inconsistency between the implied default correlations of tranches of the same CDO. If we are using a one-factor Gaussian copula model with a common default correlation \( \rho \) i.e., letting \( x_i = \rho M + \sqrt{1 - \rho^2}Z_i \) in our model, then the implied default correlation for a specific tranche is the value of \( \rho \) that gives the market price of that tranche. One would expect that this should be the same for all tranches of a CDO but in fact the Gaussian model gives different implied default correlations for each tranche. The third drawback is that a copula model does not model dependence by using actual economic relationships between companies. This makes the model unsuitable for stress-testing economic factors.

Now, given our copula model we can proceed with our Monte Carlo integration. Recall Equation 3.24 for the cumulative default probability of the \( i^{th} \) asset:

\[ P_{di}(t) = 1 - e^{-\int_0^t \lambda(s)ds}. \]

Given our \( u_i \) sampled from a certain copula function, we can then use the inverse transformation method to generate a survival time for each asset:

\[ \tau_i = P_{di}^{-1}(u_i). \] (3.41)
Equations 3.5 and 3.10, for the pool redemption and available interest can then be used with Equations 3.11 and 3.6, to get the tranche coupon and tranche redemption cashflows at each payment date, given a certain set of survival times. These cashflows can then be discounted to the pricing date and totaled using Equation 3.22 to give the price of each tranche, given these survival times. This process is repeated many times, sampling many different survival times and an average price is calculated. This is then our Monte Carlo price for each tranche of the cashflow CDO.

Getting an accurate price for a tranche requires averaging over many different simulations. Cashflow CDO tranches often require 10,000 - 20,000 samples, which may take 15-20 minutes to price and this quickly becomes very unwieldy if one has to price hundreds of CDOs for many different scenarios, every business day. It is for this reason, that developing fast methods to price cashflow CDOs without Monte Carlo integration is necessary and is of particular interest to our industrial partner Eudaemon Consulting. Fast pricing methods allow practitioners more flexibility with regards to the particular correlation values, copula functions and factor model being used. For example, calibrating the correlation value in a one-factor model to market prices becomes much more manageable when one has fast pricing methods available. Gallagher et al. (2009) proposed a semi-analytic approximation to price cashflow CDOs using the probability bucketing method introduced by Hull and White (2004). In the next chapter, we will review the Hull and White probability bucketing method and introduce the Gallagher et al. approach. We will critically examine this method and the theory behind it using real world C++ implementations. We will then present a new understanding of the method and attempt to describe the conditions under which it produces accurate tranche prices.
Chapter 4

Pricing Cashflow CDO Tranches
Without Monte Carlo Simulation

Having provided the mathematical background necessary to price a cashflow CDO using Monte Carlo simulation in Section 3.3, we now move on to the second goal of this thesis, a critical look at the semi-analytic approximation introduced by Gallagher et al. (2009). This approximation allows for fast pricing of cashflow CDOs without Monte Carlo simulation using the Hull and White (2004) probability bucketing method. First, in Section 4.1 we will introduce the Hull and White probability bucketing method, which is used to price synthetic CDOs. In Section 4.2 we will show that this method cannot easily be extended to the pricing of cashflow CDOs due to the joint dependence of the tranche coupon on the available interest in the current payment date and the total pool redemption at the previous payment date. Following this, in Section 4.3 we will describe the method proposed by Gallagher et al. (2009) to overcome this difficulty. This will lead to a discussion of the modality of the underlying joint distribution, which Gallagher et al. claimed would “be dominated by a single peak”. In Section 4.4 we will show that this is generally not the case. In Section 4.5 we will present a new explanation for the accuracy of the Gallagher et al. approximation and given this, in Section 4.6, we will attempt to describe the conditions under which the approximation will be most accurate. Finally, in Section 4.7 we will see how IC and OC tests can be included into the semi-analytic pricing method.

In the following sections, we quote numerical results from C++ implementations that were initially developed in conjunction with our industry partner Eudaemon Consulting.
The code is built upon the QuantLib\(^1\) open-source C++ financial library. The Monte Carlo and semi-analytic implementations share the same underlying structures and setup and differ only by the actual pricing routines. Default probabilities for the assets are calculated using a reduced form model with the hazard rates being stripped from the asset prices (see Section 3.3.4). Both the Monte Carlo and semi-analytic implementations use the same market data and underlying portfolios supplied by Eudaemon Consulting and both use a one-factor Gaussian copula model. We will begin discussing the Hull and White (2004) probability bucketing method.

\section{Probability Bucketing}

In order to demonstrate the use of the probability bucketing method we will first show how it can be used to evaluate the expected tranche redemptions at a given date. We will then see that this method cannot easily be applied to evaluating the expected tranche coupons due to the joint dependence of the tranche coupon on the available pool coupons at a given payment date and the pool redemptions at the previous payment date.

Recall Equation 3.4 for evaluating the price of the \(k\)\(^{th}\) tranche of a CDO at time \(t\):

\[
\Pi_k(t) = \sum_{i=1}^{p} E[C_k(t_i)]D(t, t_i) + \sum_{i=1}^{p} E[R_k(t_i)]D(t, t_i),
\]

where \(C_k(t_i)\) are the coupons paid to tranche \(k\) at time \(t_i\) and \(R_k(t_i)\) is the notional redemption paid to tranche \(k\) at time \(t_i\). The price of tranche \(k\) is simply the sum of its discounted expected future cashflows.

We want to evaluate the expected value of the tranche redemption \(R_k(t_i)\) at each time step. Using Equation 3.3 we get:

\[
E[R_k(t_i)] = E[N_k(t_{i-1})] - E[N_k(t_i)].
\]

So clearly knowledge of the tranche notional at each time step is equivalent to knowledge of the tranche redemption at each step. If we consider the case without IC and OC tests then by using Equation 3.8, we get

\(^1\)QuantLib is a free/open-source library for quantitative finance. See http://www.quantlib.org.
\[ \mathbb{E}[N_k(t_i)] = \mathbb{E}\left[ \max \left( N_k(0) + \min \left( \sum_{j=1}^{k-1} N_j(0) - \text{Red}(t_i), 0 \right) \right) \right], \]

where \( \text{Red}(t) \) is the total redemption received from the pool up until time \( t \) and \( N_k(0) \) is the notional of the \( k^{th} \) tranche at the beginning of the life of the cashflow CDO. By Equation 3.5 and 3.6 we see that \( \text{Red}(t) \) is a function of the survival times of the pool. If we let \( X_t := \text{Red}(t) \) be a real-valued random variable then we have

\[ \mathbb{E}[N_k(t_i)] = \int_{-\infty}^{\infty} n(x_t) f_{X_t}(x_t) dx_t, \] (4.1)

where \( f_{X_t} \) is the probability density function of \( X_t \) and where

\[ n(x_t) = \max \left( N_k(0) + \min \left( \sum_{j=1}^{k-1} N_j(0) - x_t, 0 \right) \right). \]

In order to perform a fast numerical integration of Equation 4.1 we need a method of approximating \( f_{X_t}(x_t) \). We will approximate \( f_{X_t}(x_t) \) using the Hull and White probability bucketing method, following the description of the algorithm in Hull and White (2004).

Recall the copula model from Section 3.3.7 which mapped \( x_i \) to \( \tau_i \) by letting \( \tau_i = P^{-1}_{d_i}(F_i(x_i)) \) where \( P_{d_i}(t) \) is the probability that company \( i \) will default by time \( t \) and \( F_i(x_i) \) is dependent on the specific copula function we are using. By Equation 3.40 we have

\[ \mathbb{P}(x_i < x|M) = H_i \left[ \frac{x - a_i M}{\sqrt{1 - a_i^2}} \right], \]

where \( H_i \) is dependent on the copula function being used. The copula model allows us to write \( x = F_i^{-1}(P_{d_i}(t)) \) and this gives us

\[ \mathbb{P}(\tau_i < t|M) = H_i \left[ \frac{F_i^{-1}(P_{d_i}(t)) - a_i M}{\sqrt{1 - a_i^2}} \right], \] (4.2)

which is the probability that the \( i^{th} \) asset will default before time \( t \) conditional on \( M \). For simplicity, we will write \( q_i^M(t) := \mathbb{P}(\tau_i < t|M) \). Given \( q_i^M \) for each asset, we will
construct \( f_{X_i} \) conditional on \( M \) and finally integrate over \( M \), as follows,

\[
E[N_k(t_i)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n(x_t)f_{X_i|M}(x_t|M)f_M(M)dx_tdM,
\]

where \( f_M \) is the probability density function of \( M \) and is dependent on the copula. Again, assume we have a portfolio of \( m \) assets and that we can calculate \( q_i^M(t) \) for each asset. For simplicity, let us drop the \( M \) and \( t \) and refer to \( q_i \) as the conditional probability that asset \( i \) will default at the given time step, i.e., at the time step that we want to evaluate \( f_{X_i} \). The probability bucketing algorithm works by building up a discrete distribution for the redemption by adding each asset one at a time. So if we divide the possible values for portfolio redemption into the following \( I \) intervals or buckets \{0, b_0\}, \{b_0, b_1\}, ..., \{b_{I-1}, \infty\} we work out the probability that the redemption will be within any given bucket. We will refer to \{b_{j-1}, b_j\} as the \( j^{th} \) bucket.

Let \( p_j \) be the probability that the redemption is in the \( j^{th} \) bucket and let \( A_j \) be the mean redemption conditional on the redemption being in the \( j^{th} \) bucket. Since we build up the distribution of the buckets one asset at a time, initially we set \( p_0 = 1 \) and \( p_i = 0 \) for all \( i > 0 \). We also set \( A_0 = 0 \) and \( A_j = 0.5(b_{j-1} + b_j) \) for \( 1 \leq j \leq I - 1 \) and \( A_I = b_{I-1} \).

Now consider adding the \( j^{th} \) asset to our distribution. Recall from Equation 3.6 that Red\( j^j(t) \) for the \( j^{th} \) asset can have two possible non zero values, i.e., \( r_jN_j \) if the asset has defaulted and \( N_j \) if the asset has not defaulted and the maturity has passed. If we are dealing in a time step before the maturity date of the asset then the \( j^{th} \) asset can add an amount \( r_jN_j \) to the redemption with a probability \( q_j \). We iterate over each of the \( I \) buckets in turn. Suppose we are considering the \( i^{th} \) bucket. Let \( u(i) \) be the bucket containing \( A_i + r_jN_j \). The probability bucketing algorithm essentially just moves an amount of probability equal to \( p_iq_j \) from the \( i^{th} \) bucket to bucket \( u(i) \) and updates the \( A_i \) and \( A_{u(i)} \) correspondingly:

\[
p_i = p_i - p_iq_j,
\]

\[
p_{u(i)} = p_{u(i)} + p_{u(i)}q_j,
\]

\[
A_i = A_i,
\]
A_{u(i)} = \frac{p_i^* A^*_i u(i) + p_i^* q_i (A^*_i + r^j N^j)}{p_i^* + p_i^* q_i},$

where $p_i^*, p_{u(i)}^*, A^*_i$ and $A^*_{u(i)}$ are the values of $p_i, p_{u(i)}, A_i$ and $A_{u(i)}$ before the iteration. If the addition of $r^j N^j$ does not shift the redemption from the current bucket, i.e., if $u(i) = i$ then we set

$$p_i = p_i^*,$$

$$A_i = A_i^* + q_i r^j N^j.$$  

We repeat this for all buckets. If we are considering a time period after the maturity of the $j^{th}$ asset then there are two possible non zero values for the redemption, i.e., $r^j N^j$ with probability $q_i$ and $N_j$ with probability $1 - q_i$. We must then complete the above procedure for both possibilities, once in the same way as above and a second time where $u(i)$ is the bucket containing $A_i + N^j$ and we replace $q_j$ in the above formulas with $1 - q_i$. Once we have iterated over all of the $m$ assets for each possible value of redemption we have built up a discrete approximation for $f_{X_i|M}$. This can then be used to get the expected value of $n(x_i)$ conditional on $M$ and once we integrate over all possible values of $M$ we have the value $E[N_k(t_i)]$.

So we have shown now that the second term on the right-hand side of Equation 3.4 for the price of a cashflow CDO can be evaluated using the Hull and White probability bucketing method. The first part on the right-hand side of Equation 3.4 involves evaluating the expected tranche coupon, $E[C_k(t_i)]$. We will now see that this is a more complex task than the above method for evaluating the tranche redemptions and requires a new approach for semi-analytic pricing.

### 4.2 The Difficulty in Evaluating the Expected Tranche Coupon

Recall Equation 3.11 for the tranche coupon (without IC and OC tests):
Chapter 4. Pricing Cashflow CDO Tranches Without Monte Carlo Simulation

\[ C_k(t_i) = \min \left( K_k(t_i) \max \left( N_k(0) + \min \left( \sum_{j=1}^{k-1} N_j(0) - Red(t_{i-1}), 0 \right), 0 \right), 0 \right), \]
\[ \max \left( \sum_{j=1}^{m} C_j(t_i) \mathbf{1}_{\{\tau_j \geq t_i\}} - \sum_{l=1}^{k-1} K_l(t_i) \max \left( N_l(0), 0 \right) \right) + \min \left( \sum_{j=1}^{l-1} N_j(0) - Red(t_{i-1}), 0 \right), 0 \right) \right), \]

where \( K_k(t_i) \) is the tranche coupon rate at time \( t_i \) and \( \sum_{j=1}^{m} C_j(t_i) \mathbf{1}_{\{\tau_j \geq t_i\}} \) is the sum of the coupons from the pool of assets. Note that unlike the case with the tranche redemption, \( R_k(t_i) \), the tranche coupon is dependent on the number of defaults at two different times. It is dependent on the available interest at time \( t_i \) and the pool redemption at time \( t_{i-1} \). Now, if we define the random variables \( X \) and \( Y \) as the following two-dimensional integral

\[ X := Red(t_{i-1}) \]
\[ Y := \sum_{j=1}^{m} C_j(t_i) \mathbf{1}_{\{\tau_j \geq t_i\}} \]

then we can write the expected tranche coupon in terms of \( X \) and \( Y \) as follows

\[ E[C_k(t_i)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_k(x, y) f_{XY}(x, y) dx dy, \quad (4.3) \]

where \( f_{XY} \) is the joint probability density function of \( X \) and \( Y \), and the tranche coupon function \( g_k(x, y) \) is given by Equation 3.11:

\[ g_k(x, y) = \min \left( K_k(t_i) \max \left( N_k(0) + \min \left( \sum_{j=1}^{k-1} N_j(0) - x, 0 \right), 0 \right), 0 \right), \]
\[ \max \left( y - \sum_{l=1}^{k-1} K_l(t_i) \max \left( N_l(0) + \min \left( \sum_{j=1}^{l-1} N_j(0) - x, 0 \right), 0 \right), 0 \right) \right). \]

For the equity tranche we get
\[ g_n(x, y) = \max \left( y - \sum_{l=1}^{n-1} K_l(t_i) \max \left( N_l(0) + \min \left( \sum_{j=1}^{l-1} N_j(0) - x, 0 \right), 0 \right), 0 \right). \]

The pool redemption at \( t_{i-1} \), \( X \), is dependent on the number of defaults at time \( t_{i-1} \) while the available interest at time \( t_i \), \( Y \), is dependent on the number of defaults at time \( t_i \). Clearly, these are not independent. If a large number of defaults is very likely at time \( t_{i-1} \), then a large number of defaults will still have to be very likely at time \( t_i \). The Hull and White probability bucketing method can be used to establish the marginal densities \( f_X(x) \) and \( f_Y(y) \). However, due to the dependence of \( X \) and \( Y \) we require the joint density \( f_{XY}(x, y) \) in order to evaluate Equation 4.3 (and so complete the pricing of the \( k^{th} \) tranche as per Equation 3.4). We cannot calculate this two dimensional distribution using the Hull and White method. Gallagher et al. (2009) proposed a solution to the problem which allows the marginals \( f_X(x) \) and \( f_Y(y) \) to be used to generate an approximation to the true joint distribution \( f_{XY} \), which enables the integration in Equation 4.3 to be performed very efficiently.

4.3 The Approximation of Gallagher et al. (2009)

The method proposed by Gallagher et al. (2009) to evaluate the expected tranche coupon, using the marginals \( f_X(x) \) and \( f_Y(y) \), is to reduce the integration to one dimension by matching \( X \) to \( Y \) along a specific curve. Gallagher et al. note that the total pool redemption, \( X \), and the available interest, \( Y \), are heavily dependent on each other since they are both dependent on the number of defaults up until time \( t_{i-1} \). Therefore when \( X \) is far from its risk-free value then \( Y \) is likely to be far from its risk-free value also. Motivated by this fact, they associate a certain \( x \) to a certain \( y \) if the cumulative probability of the deviation of \( x \) from its risk-free value is equal to the cumulative probability of the deviation of \( y \) from its risk-free value. They note that this matching of random variables by cumulative probability is known as the Q-Q plot in the statistics literature.

Formally, the method can be described as follows. First, define risk-free values of \( X \) and \( Y \). We shall call these values \( X_{rf} \) and \( Y_{rf} \) and they are simply the values of \( X \) and \( Y \) in the absence of defaults. They can be written as,
Let $\hat{X}$ and $\hat{Y}$ be the deviations of $X$ and $Y$ from their risk-free values. They are defined as follows,

$$\hat{X} := X_{rf} - X,$$  \hspace{1cm} (4.4)
$$\hat{Y} := Y_{rf} - Y.$$  \hspace{1cm} (4.5)

The approximation of Gallagher et al. is to integrate along a curve in the $x$-$y$ plane by associating $\hat{x}^*$ with $\hat{y}^*$ if

$$F_{\hat{X}}(\hat{x}^*) = F_{\hat{Y}}(\hat{y}^*),$$

where $F_{\hat{X}}$ and $F_{\hat{Y}}$ are the cumulative probability distributions of $\hat{X}$ and $\hat{Y}$. By using Equations 4.4 and 4.5 they implicitly define a relationship $x^*(y^*)$ where

$$x^* = X_{rf} - \hat{x}^*,$$
$$y^* = Y_{rf} - \hat{y}^*.$$ 

Along this curve the joint distribution reduces to

$$f_{XY}(x, y) = f_X(x^*) \delta(y - y^*) = f_Y(y^*) \delta(x - x^*),$$

and the expected tranche coupon can now be written as a one dimensional integral in terms of either $x$ or $y$:

$$\mathbb{E}(C_k(t_i)) = \int_{-\infty}^{\infty} g_k(x^*, y^*) f_X(x^*) dx^* = \int_{-\infty}^{\infty} g_k(x^*, y^*) f_Y(y^*) dy^*. \hspace{1cm} (4.6)$$
So, using the probability bucketing method, \( f_Y(y) \) can be evaluated and we can then compute the expected tranche coupon using the Gallagher et al. approximation to match a value of \( x \) to each \( y \). We shall refer to the Gallagher et al. method as the Q-Q Risk-Free Deviation (Q-Q RFD) method since it constructs a curve matching the Q-Q plot of the deviation of the distributions from their risk-free values.

We have priced two cashflow CDOs using the same deals as Gallagher et al. (2009). The underlying pool of assets in both CDOs are corporate bonds. Deal A has 8 tranches and is backed by 223 assets. The mean notional of the pool underlying Deal A is \( €2.55m \) with standard deviation \( €1.71m \). Deal B has 7 tranches and is backed by 158 assets. The mean notional of the pool underlying Deal B is \( €2.22m \) with standard deviation \( €1.16m \). Both pools consist of fixed and floating rate bonds that pay coupons quarterly, i.e., \( t_1 = 3 \) months, \( t_2 = 6 \) months, etc. We priced each deal using Monte Carlo (MC) integration and using the Q-Q RFD method. Both pricing methods use the C++ code built on QuantLib as we discussed in the introduction. The Monte Carlo pricing routine uses 20,000\(^1\) samples to get a result. The \texttt{QuantLib::GaussianRandomDefaultModel}\(^2\) class along with the \texttt{QuantLib::OneFactorGaussianCopula}\(^3\) class are used to generate the correlated unit variates for the Monte Carlo model. These unit variates are then transformed to survival times using the inverse transformation method and the default probabilities calculated by stripping hazard rates from the bond prices (Equation 3.41). Equation 3.22 is then used to calculate the sample price of the bond given these survival times. After computing 20,000 prices a sample mean is calculated and we have our Monte Carlo price.

The Q-Q RFD method also uses the \texttt{QuantLib::OneFactorGaussianCopula} class to generate the factor conditional default probabilities as per Equation 4.2. Using these factor conditional default probabilities, the probability bucketing algorithm uses 200 buckets to build up discrete approximations for \( f_X(x) \) and \( f_Y(y) \). We use these to evaluate an expected value for \( C_k(t_i) \) at each time step by matching \( x \) to \( y \) using the Q-Q RFD method and getting an expected value with respect to \( y \) as in Equation 4.6.

Table 4.1 shows the results for Deal A and Table 4.2 shows the results for Deal B. The prices are shown as a percentage of the initial notional value of the tranches. Note that these results are without IC and OC tests.

\(^1\)We found by trial and error that after 20,000 samples the tranche prices did not significantly change with respect to two decimal places.

\(^2\)http://quantlib.org/reference/class_quant_lib_1_1_gaussian_random_default_model.html

\(^3\)http://quantlib.org/reference/class_quant_lib_1_1_one_factor_gaussian_copula.html
As can be seen by Tables 4.1 and 4.2, the two methods provide good agreement on the tranche prices, with the errors (which are shown as a percentage of the Monte Carlo price) less than 1% for both deals. In Table 4.3 we show the run times for each deal for both methods. We see that the Q-Q RFD method produces similar results up to 50 times faster than Monte Carlo.

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![Table 4.1: Deal A price comparisons without IC and OC tests.](image1)

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Tranche Size (%)</th>
<th>Π MC (%)</th>
<th>Π Q-Q RFD (%)</th>
<th>Error (as % of MC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>74.32</td>
<td>74.92</td>
<td>74.78</td>
<td>-0.19</td>
</tr>
<tr>
<td>2</td>
<td>6.21</td>
<td>34.97</td>
<td>34.97</td>
<td>0.00</td>
</tr>
<tr>
<td>3</td>
<td>3.19</td>
<td>29.29</td>
<td>29.16</td>
<td>-0.44</td>
</tr>
<tr>
<td>4</td>
<td>0.67</td>
<td>28.37</td>
<td>28.21</td>
<td>-0.56</td>
</tr>
<tr>
<td>5</td>
<td>4.03</td>
<td>29.87</td>
<td>29.72</td>
<td>-0.50</td>
</tr>
<tr>
<td>6</td>
<td>3.86</td>
<td>22.53</td>
<td>22.54</td>
<td>0.44</td>
</tr>
<tr>
<td>7</td>
<td>0.84</td>
<td>22.87</td>
<td>22.92</td>
<td>0.22</td>
</tr>
<tr>
<td>8</td>
<td>6.88</td>
<td>41.39</td>
<td>41.54</td>
<td>0.36</td>
</tr>
</tbody>
</table>

![Table 4.2: Deal B price comparisons without IC and OC tests.](image2)

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Tranche Size (%)</th>
<th>Π MC (%)</th>
<th>Π Q-Q RFD (%)</th>
<th>Error (as % of MC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6</td>
<td>102.78</td>
<td>102.28</td>
<td>-0.48</td>
</tr>
<tr>
<td>2</td>
<td>71.78</td>
<td>73.20</td>
<td>73.16</td>
<td>-0.05</td>
</tr>
<tr>
<td>3</td>
<td>5.11</td>
<td>36.37</td>
<td>36.31</td>
<td>-0.16</td>
</tr>
<tr>
<td>4</td>
<td>7.95</td>
<td>30.03</td>
<td>29.96</td>
<td>-0.23</td>
</tr>
<tr>
<td>5</td>
<td>3.62</td>
<td>24.95</td>
<td>25.02</td>
<td>0.28</td>
</tr>
<tr>
<td>6</td>
<td>3.41</td>
<td>26.71</td>
<td>26.72</td>
<td>0.04</td>
</tr>
<tr>
<td>7</td>
<td>7.53</td>
<td>48.75</td>
<td>49.02</td>
<td>0.55</td>
</tr>
</tbody>
</table>

![Table 4.3: Run time to price each cashflow CDO without IC and OC tests for Monte Carlo and Q-Q RFD methods.](image3)

<table>
<thead>
<tr>
<th>Deal</th>
<th>Monte Carlo Time</th>
<th>Q-Q RFD Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>39m22s</td>
<td>00m56s</td>
</tr>
<tr>
<td>B</td>
<td>60m28s</td>
<td>01m55s</td>
</tr>
</tbody>
</table>

We now move on to discuss one of the main theoretical aspects of the Gallagher et al. paper which is the modality of the $X$-$Y$ distribution.

### 4.4 Modality of the $X$-$Y$ Distribution

In their error analysis of the Q-Q RFD method, Gallagher et al. argue that the main contributing factor to the accuracy of the method is the unimodality of the joint $X$-$Y$ distribution on cross sections of the $x$-$y$ plane. They argue that this unimodality on cross sections will result in the $X$-$Y$ distribution being dominated by a single peak, such
as in Figure 4.1. In Section 4.4.1 we will show an example of an $X$-$Y$ distribution that is unimodal on cross sections and that does have a single peak that is clearly matched by the Q-Q RFD curve. However, contrary to the claim by Gallagher et al., we will show in Section 4.4.2 that the $X$-$Y$ distribution can be multimodal\(^1\) on cross-sections in certain instances. Despite this, we will present results that show that the accuracy of the Q-Q RFD approximation (compared to Monte Carlo pricing) is the same regardless of whether the distribution is unimodal or multimodal.

### 4.4.1 Unimodal $X$-$Y$ Distributions

Figure 4.2 shows a scatter plot of $X$ against $Y$ for the pool in Deal B from the Monte Carlo run at time $t_2$, i.e., each point corresponds to $X$ at time $t_1$ (3 months) and $Y$ at time $t_2$ (6 months). The recovery rate used for these particular bonds is 0.01 and so the average recovery amount from a bond in this pool in the event of default is about €22,000. The maximum pool redemption if all 158 bonds default is therefore just less than €3.5m. The average coupon payment from a bond in the pool at this period is about €33,000, giving a maximum total available interest of €5.2m if none of the 158 bonds default.

\(^1\)By multimodal we mean that the distribution will have more than one local mode.
The shape of the distribution in Figure 4.2 is driven by the fact that the pool redemption, $X$, increases with the number of defaults at time $t_1$, whereas the available interest, $Y$, decreases as the number of defaults at time $t_1$ increases. There is an added “thickness” to the general line given by the variability of the asset recovery amounts and coupon size and the fact that defaults may occur between time $t_1$ and time $t_2$. The point $(x = 0, y = 5.14m)$ in Figure 4.2 corresponds to the case where none of the companies default and so there is available interest from every company but no redemptions (this is assuming that there are no scheduled repayments from any of the bonds at or before $t_2$). The point $(x = 3.43m, y = 0)$ corresponds to the situation where every company has defaulted and so there is no interest available but there is an amount of redemption equal to the sum of all the recovery amounts. For this pool of bonds, the most likely outcome at this time period is that only a few of the bonds will have defaulted. Hence, most of the Monte Carlo points are located around $(x = 0, y = 5.14m)$, whereas the points become sparse in the area around $(x = 3.43m, y = 0)$ since points in this area correspond to the unlikely outcome that nearly all bonds will have defaulted.

We see in Figure 4.2 that the joint distribution of $X$ and $Y$ is unimodal and that the Q-Q RFD curve matches the peak of the Monte Carlo distribution well. In the next section we will show that the $X$-$Y$ distribution can contain multiple peaks and that when it does the Q-Q RFD curve does not match the $X$-$Y$ distribution well. However, we will
see that even when the distribution is multimodal, the Q-Q RFD method still provides similarly accurate results to the unimodal case, when compared to Monte Carlo.

### 4.4.2 Multimodal $X$-$Y$ Distributions

We will show two situations that will lead the $X$-$Y$ distribution to have multiple peaks. The first is when one of the assets has a scheduled principal repayment and the second is when the underlying pool contains outliers with respect to notional or recovery amount.

First, if one of the assets has a significant scheduled principal repayment at a certain time period then it can be seen that the $X$-$Y$ distribution becomes dichotomized around the notional amount of this asset. In Figure 4.3 we again show the $X$-$Y$ distribution for the pool in Deal B but at time $t_7$ for $Y$ and $t_6$ for $X$. We see in Figure 4.3 that the distribution becomes split into two regions due to the fact that one asset has a scheduled principal repayment of €3.7m at some time between $t_5$ and $t_6$. The shape of both regions will be driven by the same factors as the unimodal case in Figure 4.2 but will split according to the probability that this asset defaults or not. The fact that the line on the left-hand side of Figure 4.3 has a greater density of Monte Carlo points indicates that it is more likely that this bond will have defaulted by this time period than not. If we plot the Q-Q RFD curve alongside this distribution, such as in Figure 4.5, we see that the two do not match.

If more than one bond has a scheduled repayment before the time period that $X$ is being considered then multiple lines will appear in the distribution, with each line corresponding to the different linear combinations of redemption that can result. In Figure 4.4 we again show the $X$-$Y$ distribution for Deal B but for $Y$ at time $t_{11}$ and $X$ at time $t_{10}$. Here there are 4 different bonds that have scheduled principal repayments before time $t_{10}$ and so we get multiple lines appearing in the distribution.

There are also other situations that can cause the $X$-$Y$ distribution to be multimodal besides scheduled principal repayments. In our examples above the underlying pools have been fairly homogeneous with respect to notional amount, recovery amount and coupon size. However, the addition of outliers to the pool can cause the distribution to be multimodal. For example, if we take the pool in Figure 4.2 and change just one of the recovery amounts to be 0.4 then we get a second line appearing in the distribution, similar to when there is a scheduled repayment. This can be seen in Figure 4.6.

Gallagher et al. (2009) show that the underlying distribution of defaults in a pool of assets can be expressed in terms of Fisher’s Non-Central Hypergeometric Distribution.
Chapter 4. Pricing Cashflow CDO Tranches Without Monte Carlo Simulation

Figure 4.3: Available Interest at $t_7$ versus Pool Redemption at $t_6$ showing a bimodal distribution. One bond has a scheduled repayment of €3.7m at some time between $t_5$ and $t_6$. The line that connects the points $(x = 0, y = 5.14m)$ and $(x = 3.43m, y = 0)$ corresponds to the situation where this asset does default, whereas the second line beginning at $(x = 3.7m, y = 5.12m)$ corresponds to the situation that this bond does not default and pays back in full.

Figure 4.4: Available Interest at $t_{11}$ versus Pool Redemption at $t_{10}$ for a multimodal distribution. Since more than one bond has a schedule repayment before time $t_{10}$ we get multiple lines corresponding to the possibility that each bond will pay back in full, along with their linear combinations.
Given this, they show that under certain rather restrictive conditions the pool redemption and available interest are unimodal. However, in Appendix C we provide an example using Fisher’s Non-Central Hypergeometric Distribution that shows that the inclusion of outliers to the pool (such as above) can result in the pool redemption and available interest being multimodal.

Despite these examples that show that the $X-Y$ distribution is not always unimodal, the results from the Q-Q RFD method when the distribution is not unimodal are equally as accurate (when compared to Monte Carlo) as the results when the distribution is unimodal. Table 4.4 shows the mean value of $C_k(t_2)$ using the Monte Carlo pricing (Mean MC) and the values of $C_k(t_2)$ using the Q-Q RFD method for the unimodal distribution in Figure 4.2. We see that the errors as a percentage of the Monte Carlo value are on average less than 0.2%.

Table 4.5 shows the same results as Table 4.4 but for the bimodal distribution in Figure 4.5. The errors are in the same range as the errors seen in the unimodal distribution in Table 4.4. It seems clear that the modality of the $X-Y$ distribution does not affect the accuracy of the Q-Q RFD method. In the next section we will see that the Q-Q RFD curve is in fact equivalent to a general Q-Q curve ($F_X(x) = F_Y(y)$) and that the
properties of a general Q-Q curve can be used to explain the accuracy of the Q-Q RFD method, without relying on unimodality of the underlying distribution.

### 4.5 Integration Along the Q-Q Curve

We will now attempt to explain the accuracy of the Q-Q RFD method in a way that is independent of the modality of the \( X-Y \) distribution. First we will show that the Q-Q RFD curve, i.e., the set of \((\hat{x}, \hat{y})\) points satisfying \( F_X(\hat{x}) = F_Y(\hat{y}) \), is equivalent to the general Q-Q curve, i.e., the set of \((x, y)\) points satisfying \( F_X(x) = F_Y(y) \). We will then
<table>
<thead>
<tr>
<th>Tranche</th>
<th>$C_k(t_7)$ (€) (Mean MC)</th>
<th>$C_k(t_7)$ (€) (Q-Q RFD)</th>
<th>Error (as % of MC )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14,088.70</td>
<td>14,105.40</td>
<td>0.12</td>
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<tr>
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<td>2,265,820.00</td>
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<td>152,495.00</td>
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<td>0.18</td>
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<td>4</td>
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<td>252,717.00</td>
<td>0.04</td>
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<tr>
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<td>0.04</td>
</tr>
<tr>
<td>6</td>
<td>168,001.00</td>
<td>168,184.00</td>
<td>0.11</td>
</tr>
<tr>
<td>7</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 4.5: Values of $C_k(t_7)$ for Monte Carlo and Q-Q RFD method for the bimodal X-Y distribution in Figure 4.3.

show that integrating along the general Q-Q curve, with respect to certain functions like the tranche coupon function, can approximate the expected value of these functions well.

First we see that $F_X(\hat{x})$ and $F_Y(\hat{y})$ can be expressed in terms of $F_X(x)$ and $F_Y(y)$:

$$F_X(\hat{x}) = \mathbb{P}(\hat{X} \leq \hat{x}) \text{ by definition of } F_X(\hat{x})$$
$$= \mathbb{P}(X_{rf} - X \leq X_{rf} - x) \text{ by definition of } \hat{X}$$
$$= \mathbb{P}(X \geq x)$$
$$= 1 - F_X(x).$$

Here we have assumed that $F_X(x)$ is continuous, implying that $\mathbb{P}(X = a) = 0$ for any $a$ and so $F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X < x)$, which is necessary for the last line above.

Using the same argument for $F_Y(\hat{y})$ we get:

$$F_X(\hat{x}) = F_Y(\hat{y})$$
$$\implies 1 - F_X(x) = 1 - F_Y(y)$$
$$\implies F_X(x) = F_Y(y).$$

Thus, integration along the Q-Q RFD curve is identical to integration along the Q-Q curve. Parametrically, the curve $F_X(x) = F_Y(y)$ can be defined as
\[
x(t) = F_X^{-1}(t),
\]
\[
y(t) = F_Y^{-1}(t).
\]

where \( t \in [0, 1] \). This can also be written as

\[
F_X(x) = t,
\]
\[
F_Y(y) = t.
\]

Now, if \( f_X(x) \) is continuous at \( x \) and \( f_Y(y) \) is continuous at \( y \) we have, by definition,\( f_X(x) = \frac{d}{dx}F_X(x) \) and \( f_Y(y) = \frac{d}{dy}F_Y(y) \). This gives us

\[
f_X(x) = \frac{dt}{dx},
\]
\[
f_Y(y) = \frac{dt}{dy}.
\]

Suppose we have an arbitrary function \( g(x, y) \), for example, one of the tranche coupon functions of Equation 3.11. Define integration along the Q-Q curve as the integral along the parametric curve \( t \in [0, 1] \):

\[
\langle g(x, y) \rangle_{QQ} := \int_0^1 g(x(t), y(t))dt.
\]

Using the fact that \( dt = f_X(x)dx \) from Equation 4.7, we can change variable from \( t \) to \( x(t) \) and we get

\[
\langle g(x, y) \rangle_{QQ} = \int_{-\infty}^\infty g\left(x, F_Y^{-1}(F_X(x))\right)f_X(x)dx.
\]

Similarly, using Equation 4.8, we change variable in Equation 4.9 from \( t \) to \( y(t) \) to get

\[
\langle g(x, y) \rangle_{QQ} = \int_{-\infty}^\infty g\left(F_X^{-1}(F_Y(y)), y\right)f_Y(y)dy.
\]
Suppose now that \( g(x, y) \) is a linear function of \( x \) and \( y \) of the form

\[
g(x, y) = ax + by + c, \tag{4.12}
\]

where \( a, b \) and \( c \) are constants. Suppose we want to evaluate the expected value of \( g(x, y) \) with respect to a joint probability density function \( f_{XY}(x, y) \)

\[
E[g(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (ax + by + c) f_{XY}(x, y) dy dx.
\]

It can be shown that integration along a Q-Q curve for any linear function of the form of Equation 4.12 is equal to the above expected value, as follows,

\[
\langle g(x, y) \rangle_{QQ} = \int_0^1 g(x(t), y(t)) dt = \int_0^1 (ax(t) + by(t) + c) dt
\]

\[
= a \int_0^1 x(t) dt + b \int_0^1 y(t) dt + c
\]

\[
= a \int_{-\infty}^{\infty} x f_X(x) dx + b \int_{-\infty}^{\infty} y f_Y(y) dy + c \quad \text{(by Equations 4.10 and 4.11)}
\]

\[
= a \mathbb{E}[x] + b \mathbb{E}[y] + c
\]

\[
= E[g(x, y)].
\]

So, by the equivalence of the Q-Q RFD curve and the Q-Q curve, the Gallagher et al. approximation will be exact for tranche coupon functions \( g(x, y) \) that are linear functions of \( x \) and \( y \). In general however, the tranche coupon function, \( g_k(x, y) \), is not a linear function - in fact it is a continuous piecewise linear function over the \( xy \) plane. However, will see that under certain conditions, the result shown above can explain why the Gallagher et al. approximation gives accurate prices.

### 4.6 Conditions Under which the Tranche Coupon Function Approximates a Linear Function

In the last section we saw that the accuracy of the Q-Q method (and hence the Q-Q RFD method) is dependent on how well the tranche coupon function, \( g_k(x, y) \), approximates a linear function. Let us now look at the structure of this function. From Equation 4.4
it can be seen that $g_k(x, y)$ is a piecewise linear function in $x$ and $y$. The tranche coupon function for the most senior tranche, $g_1(x, y)$, is defined in 3 distinct regions, as follows:

$$g_1(x, y) = \begin{cases} 
  y & \text{if } y \leq K_1(N_1 - x) \text{ and } x < N_1, \\
  K_1(N_1 - x) & \text{if } K_1(N_1 - x) < y \text{ and } x < N_1, \\
  0 & \text{otherwise.}
\end{cases} \quad (4.13)$$

These regions can be seen in Figure 4.7. Note that there are no jumps in this function, i.e., it is continuous. This can be seen in Figures 4.8 and 4.9 where we show how $g_1(x, y)$ changes with $x$ and $y$ respectively. Figures 4.10 - 4.14 show the structure of the tranche function for the next few tranches as well as the general structure of the $k^{th}$ tranche and the equity tranche. Their full equations can be found in Appendix B. From these diagrams it can be seen that the $k^{th}$ tranche has $k + 4$ regions in the $x$-$y$ plane that define $g_k(x, y)$. The equity tranche has $n + 1$ regions that define $g_n(x, y)$ (for a cashflow CDO with $n$ tranches). The accuracy of the Q-Q integration in computing the expected value of the tranche coupon function will be dependent on the position of the $X$-$Y$ distribution (i.e., the location of the probability mass) with respect to these defining regions. In Section 4.6.1 we will show that when the distribution is completely contained within one of these regions then the Q-Q approximation will be at its most accurate. In Section 4.6.2 we will deal with the situation where the distribution is spread over more than one region. We will see that because $g_k(x, y)$ is a continuous function and since the slopes of the lines defining the regions are small, $g_k(x, y)$ tends to approximate a linear function well, even when the distribution is spread across more than one region.

### 4.6.1 $X$-$Y$ Distributions in a Single Region of $g_k(x, y)$

If the $X$-$Y$ distribution is completely contained within any one of the regions, i.e., if $f_{XY}(x, y) = 0$ in all but one region then the Q-Q integration reduces to the expected value of a linear function and the approximation is exact\(^1\). This can be shown as follows: assume $f_{XY}(x, y) = 0$ everywhere outside a certain region $\Omega$ in the $x$-$y$ plane and that the tranche coupon function, $g_k(x, y)$, is equal to a linear function of the form $ax + by + c$ inside $\Omega$. The expected value of $g_k(x, y)$ reduces to

\[^1\text{Note that practically, it will only be exact to the extent that the loss bucketing algorithm approximates } f_X(x) \text{ and } f_Y(y).\]
Figure 4.7: Tranche coupon function, $g_1(x,y)$, for Tranche 1. There are 3 possible regions in which the coupon function of the first tranche is defined. The blue region corresponds to the tranche getting all available interest. In the green region the tranche gets its coupon rate, $K_1$ times the current notional value of the tranche, $N_1 - x$, provided the total redemption to that point, $x$, is less than the initial notional, $N_1$, of the tranche. If $x$ is greater than or equal $N_1$ then the first tranche has received its full principal repayment and so receives no coupon. See Equation 4.13 for full equation.

Figure 4.8: Tranche coupon function, $g_1(x,y)$, for Tranche 1 for a given $x < N_1$. The tranche coupon function is equal to the available interest $y$ until it reaches its maximum value and saturates when $y$ is equal to the scheduled coupon, $K_1(N_1 - x)$. 
Chapter 4. Pricing Cashflow CDO Tranches Without Monte Carlo Simulation

Figure 4.9: Tranche coupon function, $g_1(x, y)$, for Tranche 1 for a given $y < K_1N_1$. The tranche coupon function is equal to the available interest $y$ until the scheduled interest claim is less than $y$, at which point the tranche coupon decreases linearly with slope $-K_1$ until the redemption is greater than $N_1$ and the scheduled coupon claim is 0.

Figure 4.10: Tranche coupon function, $g_2(x, y)$, for Tranche 2. See Equation B.1 in Appendix B for full equation.
Figure 4.11: Tranche coupon function, $g_3(x, y)$, for Tranche 3. See Equation B.2 in Appendix B for full equation.

Figure 4.12: Tranche coupon function, $g_4(x, y)$, for Tranche 4.
Figure 4.13: General tranche coupon function, $g_k(x, y)$, for Tranche $k$. See Equation B.3 in Appendix B for full equation.

Figure 4.14: Tranche coupon function, $g_4(x, y)$, where Tranche 4 is the equity tranche. Since an equity tranche receives whatever interest is left over from the pool, there is no upper bound on the tranche coupon function like there is on the senior and mezzanine tranches. See Equation B.4 in Appendix B for the general equation for an equity tranche.
\[ E[g_k(x, y)] = \int \int_{\Omega} (ax + by + c)f_{XY}dxdy, \]

which is the expected value of a linear function and from Section 4.5 we know that this is exactly equal to the Q-Q integration in this region.

In Figure 4.15 we have drawn in the regions for \( g_1(x, y) \) for Deal B and we see that a large majority of the distribution (> 99%) is contained within one of the regions. In this region \( g_1(x, y) = K_1(N_1 - x) \) is a linear function and so the Q-Q integration will be close to exact. In Figure 4.16 we see the same distribution but we have drawn the regions for \( g_2(x, y) \). Again, a large majority of the weight of the distribution (> 97%) is in one of the regions. Note that here \( g_2(x, y) = K_2N_2 \) and is independent of \( x \) and \( y \).

Let \( y_1 \) be the value of \( y \) where the dividing lines of this region intersect the \( y \)-axis (see Figure 4.16). This allows us to write

\[
E[g_2(x, y)] = \int_{-\infty}^{y_1} \int_{-\infty}^{\infty} g_2(x, y)f_{XY}(x, y)dxdy + \int_{y_1}^{\infty} \int_{-\infty}^{\infty} g_2(x, y)f_{XY}(x, y)dxdy,
\]

\[ = \int_{-\infty}^{y_1} \int_{-\infty}^{\infty} g_2(x, y)f_{XY}(x, y)dxdy + \int_{y_1}^{\infty} K_2N_2f_Y(y)dy, \]

(4.14)

since \( f_{XY}(x, y) = 0 \) outside of the \( K_2N_2 \) region for \( y > y_1 \). Similarly using Equation 4.11 we can write

\[
\langle g_2(x, y) \rangle = \int_{-\infty}^{y_1} g_2 \left( F_X^{-1}(F_Y(y)), y \right) f_Y(y)dy + \int_{y_1}^{\infty} g_2 \left( F_X^{-1}(F_Y(y)), y \right) f_Y(y)dy,
\]

\[ = \int_{-\infty}^{y_1} g_2 \left( F_X^{-1}(F_Y(y)), y \right) f_Y(y)dy + \int_{y_1}^{\infty} K_2N_2f_Y(y)dy, \]

(4.15)

as long as \( F_X^{-1}(F_Y(y)) \) stays within the \( K_2N_2 \) region. So the value of Q-Q integration of \( g_2(x, y) \) above \( y_1 \) will be exactly equal to the expected value of \( g_2(x, y) \) above \( y_1 \). In the section below \( y_1 \) the distribution crosses over more than one region and so the Q-Q approximation will not be exact. In the next section we will deal with the case where the distribution is spread over more than one region.
Figure 4.15: The regions for the most senior tranche coupon function, $g_1(x, y)$, drawn over the $X$-$Y$ distribution. Here, more than 99% of the weight of $f_{XY}$ is contained with the $K_1(N_1 - x)$ region. So, to a good approximate, the Q-Q integration reduces to $\langle K_1(N_1 - x) \rangle_{QQ}$ and hence $E[g_1(x, y)] \approx \langle K_1(N_1 - x) \rangle_{QQ}$.

Figure 4.16: The regions for the second most senior tranche coupon function, $g_2(x, y)$, drawn over the $X$-$Y$ distribution. Here the distribution crosses over three different regions. However, most of the weight of $f_{XY}$ is contained within a region that is independent of $x$ and $y$, leading to Q-Q integration that approximates the expected value well.
Figure 4.17: A bimodal normal distribution centered on (2,2). The function $g(x, y)$ is piecewise linear function defined on either side of $x = 2$.

4.6.2 $X$-$Y$ Distributions Across Multiple Regions

Before we deal with specific examples of the $X$-$Y$ distribution crossing over more than one region of the tranche coupon function, let us consider a more general function $g(x, y)$ defined as follows:

$$g(x, y) = \begin{cases} x + y & \text{if } x \leq 2, \\ x + my & \text{otherwise}, \end{cases}$$  \quad (4.16)$$

where $m \in \mathbb{R}$. Suppose we want to get the expected value of $g(x, y)$ with respect to a bivariate normal distribution, where $X$ and $Y$ are independent and both have mean 2 and standard deviation 2. A scatter plot of the Monte Carlo points for the distribution is shown in Figure 4.17. We have calculated both the expected value of $g(x, y)$, $\mathbb{E}[g(x, y)]$, and the Q-Q integral of $g(x, y)$, $\langle g(x, y) \rangle_{QQ}$, for a range of values of $m$ spaced equally between -1 and 1. In Figure 4.18 we show how the difference between these methods $(\langle g(x, y) \rangle_{QQ} - \mathbb{E}[g(x, y)])$ changes with $m$. We see that the difference is linear and approaches 0 as $m$ approaches 1, i.e., the error is small if $g(x, y)$ is well approximated by a linear function.

In practice we find that because the tranche coupon function is continuous by definition, and since the slopes of the lines dividing the regions are generally relatively small, the
tranche coupon function tends to approximate a linear function well and so the errors introduced by the Q-Q RFD method are small. Recall Figure 4.16 from Section 4.6.1. Here we saw that below $y_1$ the X-Y distribution crosses over two regions. In Figure 4.19 we see how the tranche coupon function changes as it crosses between these two regions. We see that, given a fixed $y < y_1$ and $x < N_1$, the slope of $g_2(x, y)$ is $K_1$. The error between the Q-Q integral and the expected value in this instance will be smallest as $K_1$ approaches 0. Since $K_1$ is the coupon rate for the most senior tranche it will in general be small, e.g., 3-5%, and so the errors introduced by the Q-Q method will be small in this instance.

Figure 4.20 shows the regions that define $g_3(x, y)$ for Deal B. Here we see that there are two large sections above and below the main band in which $g_3(x, y)$ is completely independent of $x$ and $y$. So, similar to the argument above, the Q-Q integration in these intervals on the $y$-axis (above and below the band) will be equal the expected value on the intervals. The error in the method will be due to the probability mass contained within the thin band that runs through the middle of the distribution. The height of this band is determined by the size of $K_3N_3$, as can be seen in Figure 4.11. Since $K_3N_3$ is small compared to the maximum value of $y$, this band will only contain a small proportion of the probability mass $f_{XY}$. So the errors introduced by this band
Figure 4.19: Tranche coupon function, \( g_2(x, y) \), for Tranche 2 for a given \( y < y_1 \) in Figure 4.16. The slope of \( g_2(x, y) \) for \( x < N_1 \) is \( K_1 \). The Q-Q method will be most accurate the closer \( K_1 \) is to 0.

will be small. In this band, for a given value of \( y \), the function can pass through all four regions. This would look similar to Figure 4.21. Here \( g_3(x, y) \) is made up of 4 lines. The slope of the first line is 0, the second is \( K_1 \), the third is \( K_2 \) and the last is 0. Typical values for \( K_1 \) would be 0.03-0.05 and \( K_2 \) would be slightly greater. So here, \( g_3(x, y) \) will approximate a linear function well.

From the examples above it is clear that when the distribution crosses over more than one region of the tranche coupon function the conditions that will make the Q-Q method most accurate will be that \( K_k \) is small for all tranches and that the initial tranche coupon claim, \( K_k N_k \), is only a small fraction of the maximum value of the available interest, \( y \). This is due to the fact that these conditions will lead to \( g_k(x, y) \) approximating a linear function well and we showed in Section 4.5 that this is sufficient for the Q-Q integration to approximate the expected value. Tranche coupon rates, \( K_k \), for even the least senior tranches are generally less than 8-10% for cashflow CDOs. The initial tranche claims, \( K_k N_k \), will initially be much less than the maximum interest available. However, due to many of the assets either reaching maturity or defaulting, the maximum available interest will decrease as the life of the CDO progresses. So there will be greater errors in evaluating the tranche coupon function in later periods. However, in later periods, as the tranche nationals decrease, the tranche coupons decrease and these values add
Figure 4.20: The regions that define the third most senior tranche, \( g_3(x, y) \), drawn over the X-Y distribution for Deal B. Above and below a horizontal band the function is independent of \( x \) and \( y \). Within this band the function can pass between 2, 3 or 4 regions.

Figure 4.21: The tranche coupon function, \( g_3(x, y) \), for the third most senior tranche for a given value of \( y \). Here \( g_3(x, y) \) crosses over 4 regions. The slope of the line in the second region is \( K_1 \) and the slope of the third line is \( K_2 \). Since both \( K_1 \) and \( K_2 \) are small \( g_3(x, y) \) will be well approximated by a linear function in this instance.

less to the total price of the tranche anyway. So, for most regular cashflow CDOs that satisfy these conditions, such as Deal A and Deal B, the Q-Q method provides good approximations for tranche prices overall.

So far we have been discussing the Q-Q method with respect to cashflow CDOs without IC and OC tests. In the next section we will see that using the Q-Q method for pricing tranches with IC and OC tests becomes more difficult.
4.7 Incorporating IC and OC Tests

We will end this Chapter by discussing how IC and OC tests can be incorporated into our semi-analytic pricing framework. In Section 3.2.3 we discussed how IC and OC tests are performed for a cashflow CDO trade. In this section we will see that when IC and OC tests are considered, the tranche coupon function will no longer be expressed simply in terms of the pool redemption, $x$, and the available interest, $y$. This will make the two-dimensional Q-Q method redundant and will require us to extend the method to more than two dimensions. We will also see that once IC and OC tests are included, the tranche redemption, $R_k(t_i)$, will also no longer be expressed simply in terms of $x$ and will require a multidimensional Q-Q RFD method also. First, let us recall Equation 3.4 for the price of the $k^{th}$ tranche at time $t$:

$$
\Pi_k(t) = \sum_{i=1}^{p} E[C_k(t_i)]D(t, t_i) + \sum_{i=1}^{p} E[R_k(t_i)]D(t, t_i).
$$

We saw in Section 4.1 that evaluating the expected value of $R_k(t_i)$ at each time step is equivalent to evaluating the expected value of the tranche notional, $N_k(t_i)$, at each time step. When IC and OC tests are included, the tranche notional $N_k(t_i)$ is given by Equation 3.19:

$$
N_k(t_i) = \max \left( N_k(0) + \min \left( \sum_{j=1}^{k-1} N_j(0) - \text{Red}(t_i) - \sum_{b \leq i} \sum_{l=1}^{n-1} \text{Cure}_l(t_b), 0 \right), 0 \right),
$$

where $\text{Cure}_l(t_i)$ is the cure amount paid to the $l^{th}$ tranche at time $t_i$. Similarly, the tranche coupon function, $C_k(t_i)$, when IC and OC tests are included is given by Equation 3.20:

$$
C_k(t_i) = \min \left( K_k(t_i) \max \left( N_k(0) + \min \left( \sum_{j=1}^{k-1} N_j(0) - \text{Red}(t_{i-1}) - \sum_{b \leq i} \sum_{l=1}^{n-1} \text{Cure}_l(t_b), 0 \right), 0 \right), 0 \right),
$$

\[
\max \left( \sum_{j=1}^{m} C^j(t_i)1_{\{r_j \geq t_i\}} - \sum_{l=1}^{k-1} K_l(t_i)N_l(t_{i-1}) - \sum_{b \leq i} \sum_{l=1}^{n-1} \text{Cure}_l(t_b), 0 \right) \right).}
\]
From these two equations and from Equations 3.15 and 3.16 it can be seen that \( C_k(t_i) \) and \( N_k(t_i) \) are now dependent on the available interest, \( \sum_{j=1}^{m} C_j(t)1_{\{\tau_j \geq t\}} \), and total pool redemption, \( Red(t) \), at every time step before and including \( t_i \). They are also dependent on the outstanding pool notional, \( \sum_{j=1}^{m} N_j(t)1_{\{\tau_j \geq t\}} \), at every time step before and including \( t_i \). Similar to Section 4.2 we can define the following random variables:

\[
X_i := Red(t_i),
\]
\[
Y_i := \sum_{j=1}^{m} C_j(t_i)1_{\{\tau_j \geq t_i\}},
\]
\[
Z_i := \sum_{j=1}^{m} N_j(t_i)1_{\{\tau_j \geq t_i\}}.
\]

Using these random variables, the expected tranche coupon can be expressed as follows:

\[
E(C_k(t_i)) = \int \cdots \int \cdots \int \int \int g_k(x_{i-1}, \ldots, x_1, y_1, \ldots, y_1, z_1, \ldots, z_1) f_{X_i-1 \ldots X_1 Y_i \ldots Y_1 Z_i \ldots Z_1}(x_{i-1}, \ldots, x_1, y_1, \ldots, y_1, z_1, \ldots, z_1) \, dx_{i-1} \cdots dx_1 \, dy_{i-1} \cdots dy_1 \, dz_{i-1} \cdots dz_1,
\]

where \( g_k(x_{i-1}, \ldots, x_1, y_1, \ldots, y_1, z_1, \ldots, z_1) \) is the tranche coupon function in terms of these random variables and \( f_{X_i-1 \ldots X_1 Y_i \ldots Y_1 Z_i \ldots Z_1} \) is the joint probability density function for the random variables. Similarly the expected tranche redemption can be expressed as

\[
E(R_k(t_i)) = \int \cdots \int \cdots \int \int \int r_k(x_i, \ldots, x_1, y_1, \ldots, y_1, z_i, \ldots, z_1) f_{X_i X_1 Y_i \ldots Y_1 Z_i \ldots Z_1}(x_i, \ldots, x_1, y_1, \ldots, y_1, z_i, \ldots, z_1) \, dx_i \cdots dx_1 \, dy_i \cdots dy_1 \, dz_i \cdots dz_1,
\]

where \( r_k(x_i, \ldots, x_1, y_1, \ldots, y_1, z_i, \ldots, z_1) \) is the tranche redemption function, and \( f_{X_i X_1 Y_i \ldots Y_1 Z_i \ldots Z_1} \) is the joint probability density. We can then extend the Q-Q integration method to any number of dimensions by letting
\[
x_i(t) = F_{X_i}^{-1}(t),
y_i(t) = F_{Y_i}^{-1}(t),
z_i(t) = F_{Z_i}^{-1}(t),
\]

and defining integration along the multidimensional Q-Q curve as the integral along the parametric curve \( t \in [0, 1] \):

\[
\langle g_k(x_{i-1}, \ldots, x_1, y_1, \ldots, z_1) \rangle_{QQ} := \int_0^1 g_k(x_{i-1}(t), \ldots, x_1(t), y_1(t), \ldots, z_1(t)) \, dt,
\]

\[
\langle r_k(x_1, \ldots, x_1, y_1, \ldots, z_1) \rangle_{QQ} := \int_0^1 r_k(x_1(t), \ldots, x_1(t), y_1(t), \ldots, z_1(t)) \, dt.
\]

Using the same argument as in Section 4.5 for the two dimensional case we can show that for a linear function in \( x_i, y_i \) and \( z_i \), the multidimensional Q-Q integration is equal to the expected value of the function. So, the accuracy of the multidimensional Q-Q method will be dependent on how well the multidimensional tranche coupon and tranche redemption functions approximate linear functions. A detailed study of both these functions is beyond the scope of this thesis. However, we have, in conjunction with Eudaemon Consulting, developed a practical implementation of this method that allows us to price both Deal A and Deal B with IC and OC tests included. The results are shown in Tables 4.6 and 4.7. We see that the errors introduced by this method are greater than the two dimensional Q-Q method when no IC and OC tests are performed. However, we hope this work will provide a basis for further understanding, and improvement, of the method, so that it may be used in practical situations where IC and OC tests are important.
Table 4.6: Deal A price comparisons with IC and OC tests.

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<th>II Q-Q RFD (%)</th>
<th>Error (as % of MC)</th>
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</tr>
<tr>
<td>6</td>
<td>17.23</td>
<td>17.12</td>
<td>-0.64</td>
</tr>
<tr>
<td>7</td>
<td>14.87</td>
<td>14.78</td>
<td>-0.60</td>
</tr>
<tr>
<td>8</td>
<td>13.35</td>
<td>12.70</td>
<td>-4.87</td>
</tr>
</tbody>
</table>

Table 4.7: Deal B price comparisons with IC and OC tests.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>II MC (%)</th>
<th>II Q-Q RFD (%)</th>
<th>Error (as % of MC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>101.40</td>
<td>100.56</td>
<td>-0.83</td>
</tr>
<tr>
<td>2</td>
<td>75.95</td>
<td>76.07</td>
<td>0.16</td>
</tr>
<tr>
<td>3</td>
<td>42.87</td>
<td>43.36</td>
<td>1.14</td>
</tr>
<tr>
<td>4</td>
<td>28.87</td>
<td>28.05</td>
<td>-2.84</td>
</tr>
<tr>
<td>5</td>
<td>21.53</td>
<td>20.17</td>
<td>-6.32</td>
</tr>
<tr>
<td>6</td>
<td>17.94</td>
<td>17.29</td>
<td>-3.62</td>
</tr>
<tr>
<td>7</td>
<td>25.10</td>
<td>25.41</td>
<td>1.24</td>
</tr>
</tbody>
</table>
Chapter 5

Conclusion

5.1 Concluding Remarks

The first goal of this thesis was to review the existing theory that would allow for practical implementations of cashflow CDO pricing. Chapters 1-3 provide all the background theory and practical techniques necessary to perform a Monte Carlo pricing of a cashflow CDO. This includes how to strip hazard rates from bond prices and CDS spreads and how to bootstrap a credit curve in Section 3.3.4; how to model default dependence between assets in a pool using a copula function in Sections 3.3.6 and 3.3.7; and how to sample survival times from a factor copula model in Section 3.3.8.

In Chapter 4 we dealt with the second goal of this thesis, a critical examination of the method proposed by Gallagher et al. (2009) to price cashflow CDOs without Monte Carlo simulation. First, we saw that the tranche coupon function could be expressed in terms of two random variables: the total pool redemption to the previous time period, $X$, and the available interest in the current time step, $Y$. We introduced the Gallagher et al. method in Section 4.3, which we called the Q-Q RFD method, and we showed that it involves evaluating the expected value of the tranche coupon function by integrating along a specific two dimensional curve in $X$ and $Y$. In Section 4.4 we showed that, contrary to the claim by Gallagher et al., the accuracy of the Q-Q RFD method does not rely on the unimodality of the underlying $X$-$Y$ distribution. We showed that the $X$-$Y$ distribution could be multimodal and, in fact, we presented results that showed that the method was equally as accurate regardless of the modality of the distribution. This led us to explore the Q-Q RFD method in more detail in Section 4.5 where we showed that the Q-Q RFD curve is, in fact, equivalent to a general Q-Q curve, i.e., a
curve that matched \( X \) to \( Y \) by their quantiles. In this section we concluded that the accuracy of the Q-Q RFD method is due to the fact that integration along the Q-Q curve for a linear function of \( X \) and \( Y \) could be shown to be equal to the expected value of this function. We stated that the accuracy of the Q-Q RFD method would be dependent on how well the tranche coupon function approximated a linear function. Following this, in Section 4.6 we examined the conditions under which the tranche coupon function would approximate a linear function well and stated that most cashflow CDOs would satisfy these conditions in general.

In the final section of Chapter 4 we saw that the addition of IC and OC tests to the pricing equations meant that the tranche coupon function was no longer simply a two dimensional function \( X \) and \( Y \). The function was now dependent on the values of \( X \) and \( Y \) at every time step up to that point, as well as the outstanding pool notional at all times before that point. We then showed that the Q-Q method could be extended naturally to more than two variables and presented results from a practical implementation of this method. We found this method to be less accurate than the two dimensional case without IC and OC tests.

### 5.2 Directions for Future Work

IC and OC tests are generally contractual requirements of most cashflow CDOs. We saw in Tables 4.6 and 4.7 that when the Q-Q method is extended to include IC and OC tests the results become less accurate. Any future work should focus on this and especially on developing an understanding of Equations 3.19 and 3.20 for the tranche coupon and tranche redemption when IC and OC tests are included. A study of how the accuracy of the general Q-Q method changes as the number of dimensions is increased would also further the understanding of the possible accuracy of this method for pricing with IC and OC tests. The ultimate goal would be to develop techniques to reduce the errors of pricing with IC and OC tests to the same level as the errors when IC and OC tests are not included.
Appendix A

Default Correlation

A discrete approach to defining linear default correlation was explored by Lucas (1995) as follows. First recall the definition of Pearson’s correlation coefficient:

**Definition (Pearson’s correlation coefficient)** Let $A$ and $B$ be two random variables. Pearson’s correlation coefficient, $r$, measures the linear dependence between $A$ and $B$ and is defined as:

$$r_{AB} = \frac{Cov(A, B)}{\sqrt{Var(A)Var(B)}},$$  \hspace{1cm} (A.1)

where the covariance of $A$ and $B$, $Cov(A, B)$, is defined as:

$$Cov(A, B) = E[AB] - E[A]E[B],$$  \hspace{1cm} (A.2)

and the variance of $A$, $Var(A)$, is defined as:

$$Var(A) = E[(A - E[A])^2].$$  \hspace{1cm} (A.3)

Now say we have two companies, **Company A** and **Company B**, and the random variables $A$ and $B$ are indicator functions that take the value 1 if the company has defaulted and 0 otherwise:

$$A = \begin{cases} 
1 & \text{if Company A has defaulted,} \\
0 & \text{otherwise,} 
\end{cases}$$
Appendix A. Default Correlation

\[ B = \begin{cases} 
1 & \text{if Company B has defaulted,} \\
0 & \text{otherwise.} 
\end{cases} \]

Let \( P(A) \) be the probability that Company A defaults within a certain time frame, e.g. 1 year, and let \( P(B) \) be the probability that Company B defaults within the same time. We can now evaluate the covariance and variances of the random variables A and B:

\[ \mathbb{E}[A] = 1 \times P(A) + 0 \times (1 - P(A)) = P(A), \quad (A.4) \]

\[
Var(A) = P(A)(1 - \mathbb{E}[A])^2 + (1 - P(A))(0 - \mathbb{E}[A])^2
= P(A)(1 - P(A))^2 + (1 - P(A))(0 - P(A))^2
= P(A)(1 - P(A)), \quad (A.5)
\]

\[ Cov(A, B) = P(AB) - P(A)P(B). \quad (A.6) \]

Combining the result gives us a discrete default correlation parameter

\[ r_{AB} = \frac{P(AB) - P(A)P(B)}{\sqrt{P(A)(1 - P(A))P(B)(1 - P(B))}}. \quad (A.7) \]

Lucas uses Equation A.7 to calculate default correlations between companies by their ratings. So, for example, to calculate the default correlation between two B-rated companies \( P(A) \) and \( P(B) \) are taken from the historical default rate for 1 year. The joint probability of A and B defaulting, \( P(AB) \), is evaluated from historical data also. Suppose, for example, we are considering a pool of \( N \) B-rated companies. Suppose that \( X \) of these B-rated companies defaulted in a given year. The joint probability of two B-rated companies defaulting is evaluated by dividing the number of possible pairs of defaulted companies, \( [X(X - 1)]/2 \), by all the possible pairs of B-rated companies, \( [N(N - 1)]/2 \):

\[ P(AB) = \frac{X(X - 1)}{N(N - 1)}. \quad (A.8) \]
This is summed over all the years for which data is available. It is then possible to calculate a value for the correlation between two B-rated companies. This can be repeated for all ratings, for time periods greater than a year to build up a term structure of correlation values.

The discrete default correlation above is calculated over a particular time period, i.e., $P(A)$, $P(B)$ and $P(AB)$ are taken over 1 year or 2 years etc. To get a more general form of default correlation (using Pearson’s formula for correlation), Li (2000) defines it as the Pearson’s correlation between the survival times, $\tau$, of any two companies. If we have two companies A and B with survival times $\tau_A$ and $\tau_B$ then the default correlation, $r_{AB}$, between A and B is defined as:

$$
 r_{AB} = \frac{\text{Cov}(\tau_A, \tau_B)}{\sqrt{\text{Var}(\tau_A)\text{Var}(\tau_B)}} 
$$

(A.9)

Given the joint density $p(s,t)$ of the times to default and the marginal densities $p_A$ and $p_B$, the covariance and variance in Li’s general definition above can be evaluated over any time period to give Lucas’ discrete default correlation.

If we have two random variables $X$ and $Y$ with joint distribution $F_{XY}(x,y)$ and marginal distributions $F_X(x)$ and $F_Y(y)$, then Pearson’s correlation coefficient can also be written as:

$$
 r_{XY} = \frac{1}{\sqrt{\text{Var}(X)\text{Var}(Y)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F_{XY}(x,y) - F_X(x)F_Y(y)]dydx. 
$$

(A.10)

Other measures of association besides Pearson’s correlation can be used to model the dependence of two random variables. These include Spearman’s rho, $\rho$, and Kendall’s tau, $\tau$:

$$
 \rho_{XY} = 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F_{XY}(x,y) - F_X(x)F_Y(y)]dF(y)dF(x), 
$$

(A.11)

$$
 \tau_{XY} = 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{XY}(x,y)dF(x,y) - 1. 
$$

(A.12)
Detailed discussions of measures of dependence can be found in Wolff and Schweizer (1981) and Lehman (1966).
Appendix B

Tranche Coupon Function

Here we give the full expressions for tranche coupon functions (the second and third most senior tranches as well as the general function for the $k^{th}$ and the equity tranche) discussed in Section 4.6. Here $y$ is the available interest at time $t_i$, $x$ is the total pool redemption up until time $t_{i-1}$, $N_k$ is the initial notional of the $k^{th}$ tranche and $K_k$ is the coupon rate of the $k^{th}$ tranche.

\[
g_2(x,y) = \begin{cases} 
  y & \text{if } y < K_2(N_1 + N_2 - x) \text{ and } N_1 \leq x < N_1 + N_2, \\
  K_2(N_1 + N_2 - x) & \text{if } y \geq K_2(N_1 + N_2 - x) \text{ and } N_1 \leq x \leq N_1 + N_2, \\
  y - K_1(N_1 - x) & \text{if } K_1(N_1 - x) < y < K_1(N_1 - x) + K_2N_2 \text{ and } 0 \leq x < N_1, \\
  K_2N_2 & \text{if } y \geq K_1(N_1 - x) + K_2N_2 \text{ and } 0 \leq x < N_1, \\
  0 & \text{otherwise.}
\end{cases}
\]

(B.1)
Appendix B. Tranche Coupon Function

\[ g_3(x, y) = \begin{cases} 
  y & \text{if } y < K_3(N_1 + N_2 + N_3 - x) \\
  K_3(N_1 + N_2 + N_3 - x) & \text{if } y \geq K_3(N_1 + N_2 + N_3 - x) \\
  y - \left( K_1(N_1 - x) + K_2N_2 \right) & \text{if } K_1(N_1 - x) + K_2N_2 < y, \\
  K_3N_3 & \text{if } y \geq K_1(N_1 - x) + K_2N_2 + K_3N_3 \text{ and } 0 \leq x < N_1, \\
  y - K_2(N_1 + N_2 - x) & \text{if } K_2(N_1 + N_2 - x) < y, \\
  K_3N_3 & \text{if } y \geq K_2(N_1 + N_2 - x) + K_3N_3 \text{ and } 0 \leq x < N_1, \\
  0 & \text{otherwise.} 
\]
### Appendix B. Tranche Coupon Function

The tranch coupon function is defined as:

\[
g_k(x, y) = \begin{cases} 
  y & \text{if } y < K_k \left( \sum_{i=1}^{k} N_i - x \right) \\
  K_k \left( \sum_{i=1}^{k} N_i - x \right) & \text{if } y \geq K_k \left( \sum_{i=1}^{k} N_i - x \right) \\
  y - \left( K_1 (N_1 - x) + \sum_{i=2}^{k-1} K_i N_i \right) & \text{if } y < K_1 (N_1 - x) + \sum_{i=2}^{k-1} K_i N_i < y, \\
  K_k N_k & \text{if } y \geq K_1 (N_1 - x) + \sum_{i=2}^{k-1} K_i N_i \\
  y - \left( K_2 (N_1 + N_2 - x) + \sum_{i=3}^{k-1} K_i N_i \right) & \text{if } y < K_2 (N_1 + N_2 - x) + \sum_{i=3}^{k-1} K_i N_i < y, \\
  K_k N_k & \text{if } y \geq K_2 (N_1 + N_2 - x) + \sum_{i=3}^{k-1} K_i N_i \\
  \vdots & \text{if } y - K_{k-1} \left( \sum_{i=1}^{k-1} N_i - x \right) \text{ and } \sum_{i=1}^{k-2} N_i \leq x < \sum_{i=1}^{k-1} N_i, \\
  K_k N_k & \text{if } y \geq K_{k-1} \left( \sum_{i=1}^{k-1} N_i - x \right) + K_k N_k \\
  0 & \text{otherwise.} 
\end{cases}
\]

(B.3)
Tranche Coupon Function

\[
g_n(x, y) = \begin{cases} 
y & \text{if } x \geq \sum_{i=1}^{k-1} N_i, \\
y - \left( K_1(N_1 - x) + \sum_{i=2}^{k-1} K_i N_i \right) & \text{if } y > K_1(N_1 - x) + \sum_{i=2}^{k-1} K_i N_i \\
y - \left( K_2(N_1 + N_2 - x) + \sum_{i=3}^{k-1} K_i N_i \right) & \text{if } y > K_2(N_1 + N_2 - x) + \sum_{i=3}^{k-1} K_i N_i \\
\vdots & \text{if } \sum_{i=1}^{k-2} N_i \leq x < \sum_{i=1}^{k-1} N_i, \\
y - K_{k-1} \left( \sum_{i=1}^{k-1} N_i - x \right) & \text{if } y > K_{k-1} \left( \sum_{i=1}^{k-1} N_i - x \right) \\
0 & \text{otherwise.} 
\end{cases}
\]

(B.4)
Appendix C

Fisher’s Non-Central Hypergeometric Distribution

Gallagher et al. (2009) show that, conditional on the number of defaults and conditional on the copula factor $M$ in Equation 3.40, the distribution of defaults in the underlying pool of a cashflow CDO is given by Fisher’s Non-Central Hypergeometric Distribution. They do this by comparing the pool of assets to a bag-of-balls example. Consider a bag that is filled with balls of a number of different colours. A certain ball being picked from the bag is analogous to a certain asset defaulting. Each colour has a weight that dictates the likelihood of a ball of that colour being picked from the bag. This corresponds to the likelihood that a company with a certain default probability will default in a given time period. Conditional on the total number of balls picked, the number of balls of each colour picked is given by Fisher’s Non-Central Hypergeometric Distribution. Gallagher et al. show, in the case where there are only two different colours of balls, that since Fisher’s Non-Central Hypergeometric Distribution is unimodal with respect to the number of balls of each colour picked and since the redemption is just a linear combination of this distribution, the redemption will be unimodal. However, it can be shown that this result does not extend to the general case and a bag-of-balls example with more than two colours can make the distribution multimodal.

Consider the following example: We have a bag with 20 balls in total consisting of:

- 10 green balls with weight 4,
- 9 yellow balls with weight 6,
- 1 red ball with weight 20.
We assign a value of redemption to each colour. Let these be $R_g$, $R_y$ and $R_r$ for the green, yellow and red balls respectively. The total redemption, $R_{\text{Total}}$ is then given by the linear combination of $R_g$, $R_y$ and $R_r$:

$$R_{\text{Total}} = n_g R_g + n_y R_y + n_r R_r,$$

where $n_g$, $n_y$, and $n_r$ are the number of green, yellow and red balls picked, respectively, conditional on $n_{\text{Total}}$ balls picked in total, i.e., $n_g + n_y + n_r = n_{\text{Total}}$. Letting $R_g = 1$, $R_y = 2$ and $R_r = 10$ and using the methods in Fog (2008b), we sample from Fisher’s Non-Central Hypergeometric Distribution and use Monte Carlo simulation to pick 10 balls each time ($n_{\text{Total}} = 10$), to build up a discrete probability distribution for $R_{\text{Total}}$. The resulting distribution can be seen in Figure C.1. Clearly, this distribution is bimodal.

![Figure C.1: The probability density function of $R_{\text{Total}}$. Here we sampled 10 balls from Fisher’s Non-Central Hypergeometric Distribution 1000 times and calculated $R_{\text{Total}}$ each time. We then used these values to estimate a discrete probability density within equally spaced intervals.](image-url)
Bibliography


