Effect of elastic support on the linear buckling response of quasi-isotropic cylindrical shells under axial compression

Quaiyum M. Ansari *, Luan C. Trinh, Giovanni Zucco, Paul M. Weaver

School of Engineering and Bernal Institute, University of Limerick, Ireland

ARTICLE INFO

Keywords:
Cylindrical shell
Linear buckling
Boundary effects
Elastic foundation
Modal analysis

ABSTRACT

Cylindrical shells under compressive loading are highly sensitive to boundary conditions. Considering that these structures are connected by surrounding structural components with finite stiffness, an accurate evaluation of the effects of their boundary stiffness is crucial in their design. As such, this work investigates the effect of elastic support on the linear buckling behaviour of cylindrical shells under compressive loading. To achieve this goal, a virtual testing investigation on the effect of translational and rotational constraints to the linear buckling response of a quasi-isotropic cylinder subjected to axial compression is performed. Therefore, based on these similarities, seven groups of classical boundary conditions are introduced to classify all possible linear buckling behaviours exhibited by the cylinder under consideration. Finally, these findings can support the development of theoretical models for cascade, or flange, designs of multiple connecting cylinders.

1. Introduction

Thin-walled cylindrical shells are highly efficient structures used in many practical applications for aerospace, mechanical and civil engineering. However, when these structures are loaded in compression, their design is often driven by buckling behaviour. Since, most of the time, buckling behaviour is catastrophic, an effective prediction of the buckling load capacity of thin-cylindrical shells is of much interest.

To understand their buckling behaviour, extensive experiments on cylindrical shells were carried out during the 1920s to 1980s. During these tests, it was observed that the theoretical buckling load is higher than the tested buckling load. For example, in 1933, Lundquist [1] reported the buckling loads of 45 duralumin circular shells with clamped ends. In 1934, Donnell [2] carried out almost a hundred axial and bending buckling tests of steel/brass cylinders and, based on the obtained experimental results, proposed an empirical expression for the failure stress of cylindrical shells under axial load. Moreover, he suggested that the equivalent bending critical load is about 1.4 times the compressive stress of the material discussed previously.

deviation was due to geometrical imperfections which was later quantified by Koiter [3]. In 1965 Weingarten et al. [4,5] shared extensive experimental data on buckling of thin-walled cylinders and cones under axial compression and external pressure and found that buckling load capacity is highly sensitive to initial imperfections. In 1971, Guist [6] tested seven radially expandable cylinders with different thicknesses and found that the ratio of buckling loads in expanded cylinders to the classical buckling load varies from 0.50 to 0.71, which is higher than those in purely geometrically imperfect cases presented in the literature discussed previously.

However, the reduction of the buckling load of thin-walled cylindrical shells is not only due to geometrical imperfections [7–10]. In fact, imperfections can be in the form of material [11], loading [12] and boundary conditions [13–16]. The main source of these imperfections is due to manufacturing processes, material property variations, surrounding structures and complex loading conditions, respectively. Chryssanthopoulos and Poggi [17] developed a probabilistic based methodology for axially compressed composite cylindrical shells to perform geometric imperfection sensitivity analysis. Alternatively to static loading, Xu and co-authors [18] studied the effect of axially compressed impact loading on buckling of cylindrical shells. This
buckling load was grouped into two classes based on local and global buckling. Some influential work regarding the influence of imperfections in the boundary conditions was carried out by Kriegesmann and Hilburger [19], Singer and Rosen [20], and Hilburger and Nemeth [21]. Hence, one or more types of imperfection can be present in real structures and can significantly reduce their critical buckling load. Broggi and Schueller [22] discussed the large scatter in buckling load in tested data of composite cylinders before buckling. They proposed an averaging technique to capture variation in material properties due to thickness variations in manufacturing and employed Monte Carlo simulations for buckling analysis of cylinders under compression and torsion. Later, Kepple et al. [23] developed an improved stochastic method to model geometric imperfections of composite cylindrical shells instead of using experimental data as in [22].

To consider the discrepancy between the theoretical buckling load and the real buckling load, researchers introduced a design factor also known as the Knockdown factor (KDF), which is defined as the ratio of the experimental buckling load to the theoretical buckling load [24–28]. The initial design guidelines and design factors developed by NASA SP-8007 [24] in 1968 and NASA TN D-5561 in 1969 [29] are currently in use to design shell structures. The empirical relations were developed using lower bound experimental data which give highly conservative design factors. In fact, in many cases, a design factor with value equal to 0.65 is frequently used to design a structure [25]. Moreover, these relations do not include important data such as ply orientations, stacking sequences and boundary conditions. On the other hand, the use of these relations for designing cylindrical shells made with new technologies and manufacturing processes is most likely conservative and leads to excessive mass and cost of these structures. This consideration opens up the possibility for more robust KDF [26], which can take into account also those aspects not considered in the KDF developed by NASA.

To further understand the discrepancies between theoretical and classical buckling loads Tahir and Mandal [30] used an artificial neural network to predict the buckling load of thin cylindrical shells under axial compression. They trained, tested, and validated 390 test data using two networks with eight and ten neurons. Recently, Sadowsky and Kriváček [31] published their work on the choice of influential eigenmode imperfections for their use in nonlinear finite element (FE) analysis with imperfections for lower buckling strength calculation of thin-walled cylindrical shells.

Fig. 1 shows the major source of boundary imperfections due to surrounding structures and test boundaries, which lead to significant reductions of the critical buckling load [20,21,32,33]. To understand the boundary effect, vibration correlation techniques were adopted [34,35] to predict the real buckling load. However, the time and resource required for testing can be prohibitive and analytical or empirical models, which take into account the effect of boundary conditions, are needed. Furthermore, a better prediction of boundary effects can help mitigate over-conservative design issues.

The main objective of this study is to perform detailed analysis on the linear buckling behaviour of cylindrical shells, supported on different boundary conditions to understand the effect of stiffness that may come from surrounding structures or cascade along the cylinder as shown in Fig. 1.

Some earlier major contributions in this topic for isotropic cylindrical shells are mentioned hereafter. Nachbar and Hoff [36] investigated free-edge effects on the buckling behaviour of semi-infinite isotropic cylindrical shells. Based on a closed-form solution, they expressed radial and tangential displacements in an exponential form and solved for buckling load and mode shape in the case that both translations are free. Hoff and Rehfeld [37] extended the solutions to compare four simply supported conditions and highlighted the contribution of constraining circumferential translation to almost double the buckling load. It is worth mentioning that they assumed that all the displacements including circumferential translation decay in a short distance from the loaded edge. Hoff and Soong [38] then performed a study on thin-walled circular cylinders finite in lengths considering nine different boundary conditions. By expressing the axial and tangential displacements in terms of radial displacement (i.e. Donnell’s expressions), they investigated the symmetric and antisymmetric buckling modes of four simply supported, four fixed ends and a free edge shell. The solutions for the simply supported ends with free circumferential translation and the

<table>
<thead>
<tr>
<th>Nomenclature</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
</tr>
<tr>
<td>( R )</td>
</tr>
<tr>
<td>( H )</td>
</tr>
<tr>
<td>( \nu )</td>
</tr>
<tr>
<td>( E )</td>
</tr>
<tr>
<td>( A_{ij} )</td>
</tr>
<tr>
<td>( D_{ij} )</td>
</tr>
<tr>
<td>( P_{cl} )</td>
</tr>
<tr>
<td>( P_{cr} )</td>
</tr>
<tr>
<td>( K_t )</td>
</tr>
<tr>
<td>( K_\phi )</td>
</tr>
<tr>
<td>( u, w, v )</td>
</tr>
<tr>
<td>( \phi_u, \phi_w, \phi_v )</td>
</tr>
<tr>
<td>( K_{u}, K_{w}, K_{v} )</td>
</tr>
<tr>
<td>( K_{u\phi}, K_{w\phi}, K_{v\phi} )</td>
</tr>
</tbody>
</table>

**Fig. 1.** Schematic of space launch system cylindrical shell real boundary conditions and testing methods.
perfectly free edge induced by uniformly axial stress were validated against Sander’s shell formulations in which all displacements are independent. Later on in 1984, Yamaki [39] summarised his 17 years of research in an excellent textbook presenting the outcomes of buckling and post-buckling behaviours of isotropic cylindrical shells taking into account different loading combinations. To the best of the authors’ knowledge, no such investigation of boundary effects on linear buckling load and mode shapes of composite cylindrical shells is given in the literature. The present paper aims at investigating the effect of each translational and rotational displacements and their combinations on the the linear buckling response of quasi-isotropic composite shells. The physics behind the formation of individual mode shapes corresponding the the linear buckling behaviour is discussed.

In the current study, the cylinder is assumed to be axially supported \((u = 0)\) and the effect of translational boundaries, such as radial \((w = 0)\) and tangential \((v = 0)\) are studied. In addition, the effect of the angles of rotation \((\phi_u, \phi_v, \phi_w)\) individually and in combination are also investigated. This study is done to mimic the effect of potting/cascade along a cylinder that is used to avoid crippling of edges during testing. In other words, to avoid crippling when a cascade is added to the circumferential unintentionally, the effective local boundary conditions change from \(u = 0\) to \(u = w = v = 0\) to fully clamped conditions which alters critical buckling loads and mode shapes considerably. In contrast, if test boundaries deviate significantly from real boundaries, then this can be a major source for over prediction or under prediction of buckling load in a test. In addition to this, these boundary imperfections may come from uneven surfaces at the ends of the cylinder which causes non-uniform load introduction. Therefore, to understand the transitions between two classical boundaries, spring models are used in Abaqus FE models that can further be translated into the real structures.

In this paper, Section 2 describes the effect of boundary conditions and ply orientations for different configurations of cylinders taken from the literature. In section 3, numerical results on the linear buckling behaviour of quasi-isotropic (QI) laminated cylindrical shells under different boundary conditions are presented. Then, seven distinct mode shapes are identified that cover an entire design domain, and detailed analysis is presented. Subsequently, various ways of attaining maximum linear buckling loads with corresponding mode shapes are discussed. Finally, conclusions are drawn in Section 4.

2. Sensitivity of linear buckling load to boundary conditions

Buckling of thin-walled cylindrical shells is sensitive to boundary conditions and could reduce the critical load by 50% [40]. In reality, these cylinders are usually stacked together in series over other cylinders or cones in space structures and neighbouring structural stiffness depends on the height, radius, thickness, and material type. In other words, these structures are surrounded by finite stiffness structures, which can significantly affect the buckling load and mode shape. Besides, these cylinders are often either welded or riveted/boiled together at the flange to develop the full structure. Hence, if the appropriate value of stiffness of the surrounding structure is not appropriately accounted for, the critical buckling load can be significantly underestimated or overestimated.

For a perfect cylinder (geometry and boundary), the critical buckling load is the same as the classical buckling load and can be calculated by \(P_{CR} = 2.22 E t^2\) [6], where \(E\) is the elastic modulus and \(t\) is the thickness of the cylinder. However, the presence of boundary imperfections can reduce the buckling load substantially. Hence, from a design point of view, this buckling load is overpredicted by a designer and could lead to an under-designed structure. Therefore, accurate prediction of critical buckling load is important to take an appropriate margin of safety for robust design subject to buckling.

### Table 1

Knockdown factors for different cylindrical configurations (two values reported where available [41]).

<table>
<thead>
<tr>
<th>Specimen [41]</th>
<th>Test (kN) [41]</th>
<th>KDF (Eq. (1))</th>
<th>KDF (elastic) [40]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z11%/Z22</td>
<td>228.00/221.70</td>
<td>0.47</td>
<td>0.45</td>
</tr>
<tr>
<td>Z12%/Z24</td>
<td>93.50/90.20</td>
<td>0.47</td>
<td>0.76</td>
</tr>
<tr>
<td>Z14</td>
<td>82.80</td>
<td>0.39</td>
<td>0.57</td>
</tr>
<tr>
<td>Z17%/Z25</td>
<td>278.50/227.90</td>
<td>0.47</td>
<td>0.45</td>
</tr>
<tr>
<td>Z18</td>
<td>212.60</td>
<td>0.47</td>
<td>0.57</td>
</tr>
<tr>
<td>Z21</td>
<td>69.30</td>
<td>0.39</td>
<td>0.51</td>
</tr>
<tr>
<td>Z22</td>
<td>34.40</td>
<td>0.39</td>
<td>0.79</td>
</tr>
</tbody>
</table>

In the 1960s, several buckling tests were conducted and a lower bound curve was given by NASA SP-8007 due to scatter in test data to address the effect of geometrical imperfections. The empirical relation of the KDF lower bound curve for cylindrical shells under axial compressive loading is given by [24]

\[
KDF = 1 - 0.901(1 - e^{-4})
\]

where

\[
\psi = \frac{1}{16} \sqrt{\frac{R}{t}} \quad \text{(isotropic)}
\]

\[
\phi = \frac{1}{29.8} \sqrt{\frac{R}{A_{11}A_{22}}} \quad \text{(orthotropic)}
\]

where \(R\) is the radius of the cylinder, \(t\) is the thickness, \(A_{11}\) and \(A_{22}\) are in-plane stiffness terms, while \(D_{11}\) and \(D_{22}\) are bending stiffness terms, respectively. It is noted that Eq. (1) was developed for simply supported cylindrical shells based on lumped results of available tests, that include effects of imperfections and non-perfect boundary conditions on the KDF [24]. This study aims to provide a better understanding of sensitivity of linear buckling load to wider variations of boundary conditions with emphasis on the relatively short shells.

Therefore, this study aims to provide a better understanding of the sensitivity of linear buckling load to boundary conditions with an emphasis on relatively short shells. These insights are required before future work can be developed to examine the combined effects of geometrical and boundary imperfections on the buckling load of cylindrical shells.

Table 1 shows the design factors calculated with Eq. (1) for different cylindrical configurations [41]. It is worth noting that the KDF used in this study is only used to assess boundary effects and not geometrical imperfections. In other words, the KDF is the ratio of linear buckling load for an elastic boundary condition to the buckling load for a clamped condition. The KDF representing the boundary imperfection caused by elastic foundations can be as little as 0.45 (see the last column of

![Fig. 2. Cylindrical coordinate system, notations and conventions.](image-url)
Engineering Structures 244 (2021) 112796

Table 1) comparing the two boundary conditions at the base, which are vertically constrained ($u = 0$) and SS1 ($u = w = v = 0$). This leads to a significant drop in KDF as far as the boundary sensitivity is concerned. Furthermore KDF expressions given by Eq. (1) and combined with an elastic boundary KDF could assist in identifying the equivalent or overall KDF for cases where both types of imperfection are present. More importantly, the boundary constraints in real structures can present any finite stiffnesses between these two limiting cases. Therefore, understanding the effect of intermediate boundary conditions on the linear buckling behaviour is important in designing cylindrical shells.

In the following section, the linear buckling behaviour of a quasi-isotropic laminated cylindrical shell is investigated with FE simulations. In particular, the goal of this investigation is to determine the effects of any possible combinations of boundary conditions on the linear buckling load and mode of the cylinder. To study the boundary sensitivity a cylindrical coordinate system is used as shown in Fig. 2, where $u$, $v$ and $w$ denote the displacement in axial, tangential and radial directions respectively and $\phi_u$, $\phi_v$ and $\phi_w$ are rotations about the respective axis.

The transition between two boundary conditions is also investigated to assess the effect of elastic foundations. Moreover, the change of the first linear buckling mode using translational and rotational spring elements in a FE environment is studied. In particular, the boundary conditions are enforced by assigning zero spring stiffness at each node in the corresponding degree of freedom (DoF). For example, the free and fully constrained axial displacement can be obtained with $K_u$ being null or infinite (i.e. 10 GN/m), respectively. Similarly, the axial rotational constraint can be implemented with $K_{\phi_u}$ being null or infinite (10 GNm/rad). Thereby, the stiffness of the elastic foundation is assigned using intermediate values.

3. Numerical results and discussion

In this section, ABAQUS FE models are used to investigate the sensitivity of linear buckling load to boundary conditions of a QI cylinder. The cylinder dimensions are taken from [40], i.e. radius $R = 250$ mm, height $H = 250$ mm and ply thickness $t_{90} = 0.131$ mm (see Fig. 3a). The IM78552 carbon-epoxy prepreg material property [10,42] $E_{11} = 134.3$ GPa, $E_{22} = E_{33} = 9.38$ GPa, $G_{12} = G_{13} = 5.21$ GPa, $\nu_{12} = \nu_{13} = 0.347$, $\nu_{23} = 0.45$ and $G_{23} = 3.98$ GPa is used for the FE simulation. The laminate stacking sequence is $[0/90/45/-45/0/90/45/-45]_s$. To neglect the local loading effects, interaction is created at the top of the cylinder, which connects a reference point at the centre of the top circumference with surrounding points on the wall, as shown Fig. 3b. The axial load is applied at the reference point allowing only axial translations. The bottom nodes of the cylindrical wall are constrained within a cylindrical coordinate system. Different translational and rotational constraints are enforced to investigate the sensitivity of linear buckling load and the corresponding mode shape to boundary conditions of the cylindrical shell. At the bottom of the cylinder, apart from constrained axial displacement, five more DoF including three rotational and two translational displacements are combined to consider the effect of all possible combinations of boundary conditions. Table 2 shows detailed combinations of boundary conditions to be implemented. As the alteration of critical load and mode shapes with respect to gradually enhanced effects of support stiffness is being investigated, it is necessary to assign the corresponding stiffness components for each DoF support. These stiffnesses associated with the boundary conditions are implemented in the FE model using a Python script. The infinite stiffness in this table represents the DoF being fully constrained.

The cylinder is discretised with 54R elements using commercial ABAQUS software. A mesh convergence study indicates that 320 elements along the circumference and 50 elements along the height are needed for accurate evaluations of mode shapes and critical loads. More
Table 3  Linear buckling load under different boundary conditions for quasi-isotropic laminate.

<table>
<thead>
<tr>
<th>Case N</th>
<th>Boundary Condition</th>
<th>Buckling Load (kN)</th>
<th>KDF (w.r.t fully clamped)</th>
<th>Mode Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( u = 0 )</td>
<td>449</td>
<td>0.55</td>
<td>I</td>
</tr>
<tr>
<td>7</td>
<td>( u = \phi_u = 0 )</td>
<td>449</td>
<td>0.55</td>
<td>I</td>
</tr>
<tr>
<td>2</td>
<td>( v = 0 )</td>
<td>472</td>
<td>0.58</td>
<td>II</td>
</tr>
<tr>
<td>21</td>
<td>( u = v = \phi_u = 0 )</td>
<td>472</td>
<td>0.58</td>
<td>II</td>
</tr>
<tr>
<td>5</td>
<td>( u = \phi_u = 0 )</td>
<td>475</td>
<td>0.58</td>
<td>III</td>
</tr>
<tr>
<td>10</td>
<td>( u = \phi_u = 0 )</td>
<td>475</td>
<td>0.58</td>
<td>III</td>
</tr>
<tr>
<td>19</td>
<td>( u = v = \phi_u = 0 )</td>
<td>475</td>
<td>0.58</td>
<td>III</td>
</tr>
<tr>
<td>24</td>
<td>( u = v = \phi_u = 0 )</td>
<td>475</td>
<td>0.58</td>
<td>III</td>
</tr>
<tr>
<td>3</td>
<td>( u = w = 0 )</td>
<td>546</td>
<td>0.67</td>
<td>IV</td>
</tr>
<tr>
<td>12</td>
<td>( u = w = \phi_u = 0 )</td>
<td>546</td>
<td>0.67</td>
<td>IV</td>
</tr>
<tr>
<td>14</td>
<td>( u = w = \phi_w = 0 )</td>
<td>546</td>
<td>0.67</td>
<td>IV</td>
</tr>
<tr>
<td>17</td>
<td>( u = w = \phi_w = 0 )</td>
<td>546</td>
<td>0.67</td>
<td>IV</td>
</tr>
<tr>
<td>6</td>
<td>( u = \phi_u = 0 )</td>
<td>801</td>
<td>0.99</td>
<td>V</td>
</tr>
<tr>
<td>9</td>
<td>( u = \phi_u = 0 )</td>
<td>801</td>
<td>0.99</td>
<td>V</td>
</tr>
<tr>
<td>4</td>
<td>( u = v = 0 )</td>
<td>809</td>
<td>0.99</td>
<td>VI</td>
</tr>
<tr>
<td>13</td>
<td>( u = w = \phi_w = 0 )</td>
<td>809</td>
<td>0.99</td>
<td>VI</td>
</tr>
<tr>
<td>15</td>
<td>( u = w = \phi_w = 0 )</td>
<td>809</td>
<td>0.99</td>
<td>VI</td>
</tr>
<tr>
<td>16</td>
<td>( u = w = \phi_u = 0 )</td>
<td>810</td>
<td>1.00</td>
<td>VII</td>
</tr>
<tr>
<td>18</td>
<td>( u = w = \phi_u = 0 )</td>
<td>810</td>
<td>1.00</td>
<td>VII</td>
</tr>
<tr>
<td>20</td>
<td>( u = v = \phi_u = 0 )</td>
<td>811</td>
<td>1.00</td>
<td>VII</td>
</tr>
<tr>
<td>23</td>
<td>( u = v = \phi_u = 0 )</td>
<td>811</td>
<td>1.00</td>
<td>VII</td>
</tr>
<tr>
<td>22</td>
<td>( u = v = \phi_w = 0 )</td>
<td>812</td>
<td>1.00</td>
<td>VII</td>
</tr>
<tr>
<td>25</td>
<td>( u = v = \phi_w = 0 )</td>
<td>812</td>
<td>1.00</td>
<td>VII</td>
</tr>
<tr>
<td>27</td>
<td>( u = v = w = \phi_u = 0 )</td>
<td>813</td>
<td>1.00</td>
<td>VII</td>
</tr>
<tr>
<td>29</td>
<td>( u = v = w = \phi_w = 0 )</td>
<td>813</td>
<td>1.00</td>
<td>VII</td>
</tr>
<tr>
<td>30</td>
<td>( u = v = w = \phi_w = 0 )</td>
<td>813</td>
<td>1.00</td>
<td>VII</td>
</tr>
<tr>
<td>32</td>
<td>( u = v = w = \phi_w = 0 )</td>
<td>813</td>
<td>1.00</td>
<td>VII</td>
</tr>
</tbody>
</table>

Precisely, 16000 elements are used to mesh the cylinder, while the total number DoF of the single problem is 97926. Finally, the linear buckling behaviour of the cylinder is obtained through the eigenvalue subspace algorithm within ABAQUS.

Table 3 shows the critical buckling loads for all boundary combinations given in Table 2. Table 3 shows a significant sensitivity of the critical load to variations in boundary conditions. More specifically, compared to the fully constrained boundary, the KDF values range from 0.55 to 1. Based on the KDF/critical load, the boundary conditions under consideration are categorised into seven groups as shown in Table 3. The maximum critical load is observed for clamped edges, hence, the KDF in this Table is the ratio of buckling load for elastic boundaries compared to the buckling load with fully clamped condition. This KDF examines the reduction of linear buckling load due to the presence of elastic boundary conditions.

It is also observed that all combinations of boundary conditions belonging to the same group have the same critical mode shape. Therefore, understanding the formation of mode shape by each individual constraint is important. In the following Section, the change of the critical mode shapes with respect to the step change in buckling load is discussed. In particular, seven groups of critical mode shapes and their corresponding values of buckling loads are presented.

3.0.1. Mode I: cases 1 and 7

Fig. 4 shows the critical buckling mode of the first group, which is obtained under two distinct boundary conditions, i.e. cases 1 and 7. In mode I, the cylinder is axially constrained and free to move in the radial and tangential directions at the bottom. As such, a local wavy mode shape exists at the bottom circumference of the cylinder. The negligible effect of radial rotation \( (\phi_u) \) in case 7 can be explained by the fact that as the cylinder is simply supported in an axial direction \( (u = 0) \), the relative axial displacement of the bottom edge is zero. In other words, the radial rotation constraint is naturally satisfied by the constrained axial translation. The smallest linear buckling load of 449 kN is obtained in this case.

3.0.2. Mode II: cases 2 and 21

When the axial and tangential translations are constrained \( (u = v = 0) \) at the bottom circumferential nodes, the buckling mode extends lengthwise in comparison to mode I, as shown in Fig. 5. Interestingly, the cylinder is free in the radial direction \( (K_u = 0) \), which creates a shape with a sharp lip (resembling an octopus) at the bottom. In comparison with cases 1 and 7, a further constraint in the tangential displacement reduces the number of waves and increases the critical load to 472 kN.

3.0.3. Mode III: cases 5, 10, 19 and 24

In addition to the boundary conditions causing mode II, the main
Fig. 5. Buckling mode II.

Fig. 6. Buckling mode III.

Fig. 7. Buckling mode IV.

Fig. 8. Buckling mode V.
stiffening factor in these cases is the constrained axial rotational boundary $\phi_u$, which drives the mode shape to have a uniform expansion around the circumference as revealed in Fig. 6. This response can be explained by the relationship between the axial rotation and the variation of radial displacement. Since $\phi_u$ is persistently zero along the circumference, the radial displacement is constant forming a circular shape at the bottom of the shell. Although the mode shape changes significantly, the critical load increases to 475 kN.

3.0.4. Mode IV: Cases 3, 12, 14 and 17

The primary boundary constraints, in this case, are the axial and radial displacements ($u = w = 0$ or $K_u = K_w = \infty$), which restricts the wavy shape at the bottom as shown in Fig. 7. In other words, the bottom circumference remains circular with the initial radius and the peak displacement slightly pushed upwards from the bottom. As both axial and radial displacements are zero, they do not vary along the circumference. Hence, the additional axial rotations $\phi_u$ and radial rotations $\phi_w$ are automatically constrained and do not contribute to the mode shape or affect the critical load. The critical buckling load achieved in each of these boundary conditions is identically 546 kN.

3.0.5. Mode V: Cases 6 and 9

It is observed from the mode shape in Fig. 8 that, if axial displacement and tangential rotation ($u = \phi_v = 0$ or $K_u = K_v = \infty$) are constrained, a significantly higher buckling load is achieved for the QI cylindrical shell. The linear buckling load changes from 449 kN in the case of axially constrained ($u = 0$) to 601 kN, equivalent to a 78% increase, when the tangential rotation is fully fixed along with axial constraint ($u = \phi_v = 0$). It shows that the torsional stiffness of the boundary plays an important role in achieving a higher buckling load. By constraining the tangential rotation, the mode shape expands from being local at the bottom to being global across the entire surface. The wavy shape at the bottom boundary indicates that it is free to translate in radial and tangential directions. Besides, longitudinal waves are inclined that indicates bend-torsion coupling and appears due to $D_{16}$ and $D_{26}$ bending stiffness components [28].

3.0.6. Mode VI: cases 4, 26, 28 and 31

If all three translations are constrained ($u = v = w = 0$ or $K_u = K_v = K_w = 0$) at the boundary, creating fully pinned boundary conditions, the mode shape develops as shown in Fig. 9. In this case, the bottom circumference is not allowed to translate in any directions but all rotations are free, which makes each node’s kinematics equivalent to the ball and socket joint. Hence, the top and bottom circumferential nodal responses are similar. In other words, the deformed shape is symmetric to the middle circumference of the shell. Compared to mode V, a slight improvement in buckling load is obtained due to the enhancement of local stiffness at the based circumference. The critical buckling load for each of these boundary cases is 809 kN.

3.0.7. Mode VII: Cases 8, 11, 13, 15, 16, 14, 20, 23, 22, 20, 27, 29, 30 and 32

When axial translation is constrained along with axial and tangential rotation the buckling mode becomes mode VII as illustrated in Fig. 10. The bottom of the shell is fully encastred. In fact, the combination of locking the axial and tangential rotations together with constrained axial displacement is the simplest way to obtain the maximum critical load. Due to the pinned conditions between the reference node and cylindrical walls at the top circumference (see Fig. 3), the bottom is now more constrained than that at the top and the peak deformation shifts to the middle circumference. From these boundary combinations, the maximum linear buckling load of a cylindrical shell is achieved, i.e. 813 kN.
3.1. Critical buckling behaviour changes due to constrained translational boundary

In this section, the change of the buckling mode shape and its corresponding critical load considering only translational boundaries is discussed. Fig. 11 summarises the buckling load and mode shape changes due to the three translations \( u \), \( v \), and \( w \). The combinations of translational constraints produce four cases and four corresponding distinct mode shapes I, II, IV, and VI. The minimum buckling load of 449 kN is represented by mode I and maximum by mode VI with the corresponding buckling load of 809 kN. In addition, cases of mode II and III represent the buckling behaviour under the boundary combinations of axial and tangential and axial and radial, respectively.

The gradual change of mode shape across different classical translational boundaries can be traced by changing corresponding elastic foundations as shown in Fig. 12 and Fig. 13. These elastic foundations are represented by spring elements in Abaqus, where the spring stiffness changes from infinitesimal (equivalent to free condition) to infinite value (10 GN/m, equivalent to fixed condition).

In Fig. 12, three different boundary conditions are considered. In the first, the boundary condition has \( u = 0 \), while the radial translational spring stiffness \( K_w \) varies (see green curve shown in Fig. 12). With the second boundary condition both translations \( u \) and \( w \) are constrained, whilst the tangential translational stiffness spring \( K_v \) varies (see red curve shown in Fig. 12). Finally, for the third boundary conditions then \( u = 0 \), whilst both the radial and tangential translational spring stiffnesses \( K_w \) and \( K_v \), respectively vary. In particular, the green curve shows that when radial stiffness is applied the linear buckling load increases from 449 kN to 546 kN. In this case, the mode shape changes

---

**Fig. 11.** Change in mode shape under axial, radial and tangential boundary conditions [40].

**Fig. 12.** Transition between Mode I, IV and VI.

**Fig. 13.** Transition between Mode I, II, and VI.

**Fig. 14.** Effects of translational boundary conditions on the linear buckling load.
from Mode I to Mode IV as radial stiffness approaches infinity. However, when tangential stiffness is added, the linear buckling load changes from 546 kN to 809 kN as shown by the red curve and the associated mode shape changes from Mode IV to Mode VI. Finally, the change of critical load due to an equal increment of the radial and tangential stiffnesses is also shown in this figure (the black dashed line), where the mode shape changes from Mode I to Mode VI.

Fig. 13 presents another possible transition between different classical boundaries such as $u = 0$ to $u = v = 0$ to $u = v = w = 0$. In particular, the blue curve shows how the linear buckling load varies when $u = 0$, whilst $K_v$ varies. The blue curve shows that when radial and tangential translations are free and axial translation constrained the structure buckles into Mode I. Whereas, if the tangential translation is fixed (or tangential stiffness is infinite), the mode shape changes from Mode I to Mode II and the linear buckling load reaches 472 kN from 449 kN. The red curve shows the linear buckling behaviour of the cylinder when it is subjected to $u = v = 0$, while $K_w$ varies. For this combination of boundary conditions, the buckling load increases from 472 kN to 809 kN, and mode shape VI is formed when additional radial stiffness is applied. Mode II also represents a case when the axial and tangential translations are fixed, and the radial translational DoF is free and therefore, the end of the blue curve matches well with the initiation of the red curve. The above discussion reveals that contributions of each translational stiffness to the fully constrained critical load can be different depending on the particular boundary combination under consideration.

Fig. 14 shows the change of the linear buckling load when using different boundary conditions. It is observed that when $u = 0$ the linear buckling load is 449 kN. When the tangential constraint is added, the buckling load increase is 23 kN (see the second bar of Fig. 14). However, when the boundary condition is $u = w = 0$ the linear buckling load increases from 449 kN to 546 kN (see the third bar of Fig. 14). Finally, if the boundary conditions are $u = v = w = 0$ the linear buckling load reaches 809 kN. It is worth observing that the increase of the linear buckling load of the cylindrical shell is strongly affected by the boundary conditions to which it is subjected. In fact, when $v = 0$ and $w = 0$ the increments of the buckling load are 23 kN (see the second bar of Fig. 14) and 97 kN (see the third bar of Fig. 14), respectively. However, when both $v$ and $w$ are constrained the increment is 360 kN (see the fourth bar of Fig. 14). This observation also indicates a strong coupling between the radial and tangential boundary stiffnesses in the buckling response of this cylindrical shell.

3.2. Critical buckling behaviour changes due to constrained rotational boundary

The possibility of improving the critical behaviour of cylindrical shells by enforcing rotational stiffness is discussed in this section. Fig. 15 illustrates the effect of each rotational constraint on the critical buckling load and the corresponding mode shape. The structure under these conditions
boundary conditions can exhibit three different mode shapes out of the seven possible modes. In the case of axial rotation, the top view looks like a flattened ring, which indicates no rotation about the longitudinal axis of cylinder, and Mode III is obtained. When tangential rotation is fixed, a mode shape of a twisted-like cylinder is formed. Furthermore, the combination of these two rotational constraints together with axial support completes a fully clamped condition giving the critical load its maximum value.

Fig. 16 shows the ongoing variations in the linear buckling load among these four modes due to the contribution of rotational constraints. Firstly, the green curve indicates the linear buckling load change due to different radial rotational stiffnesses. As discussed previously, the additional constraint of radial rotation does not affect the critical load. Secondly, a small contribution due to the axial rotational stiffness is observed in the blue curve (from 449 kN to 475 kN) and the highest contribution is due to the tangential rotational stiffness (from 475 kN to 801 kN, the red curve). Finally, the black curve shows the combined effect of axial and tangential rotational stiffnesses, which is close to a fully fixed condition and the linear buckling load increases from 801 kN to 810 kN.

An examination of boundary condition effect, i.e. different combinations of radial and tangential rotation stiffnesses, is shown in Fig. 17. From this graph it is observed that $\phi_g$ gives a larger contribution to the linear buckling load than $\phi_u$, hence, there is no significant coupling in axial and tangential rotations.

Fig. 18 presents the effect of rotational stiffnesses ($K_\phi$) on the linear buckling load of the cylindrical shell subjected to two different combinations of constrained translational DoFs. It is observed that only the tangential rotation stiffness contributes to the critical load as either radial or tangential displacement is fixed together with axial displacement. However, the range of contribution from tangential rotation is different for these two cases. More specifically, the enforcement of tangential rotational constraint increases the critical load up to the maximum value of 813 kN from 546 kN of constrained radial displacement from 472 kN of constrained tangential displacement.

4. Conclusions

In this study, an intensive virtual testing investigation on the contribution of each translational and rotational component of boundary conditions on the linear buckling behaviour of a quasi-isotropic cylindrical shell has been conducted. The shell is axially constrained at the bottom and loaded at the top circumference. Using Python code in Abaqus finite element simulations, the constraints at the boundary are enforced by spring elements for finite stiffness or fully fixed conditions.

The virtual testing investigation revealed that, among all combinations of boundary conditions under consideration, seven mode shapes occur with seven corresponding similar linear buckling loads. Based on this finding, a categorisation of various combinations of classical boundary conditions into seven groups has been devised. A strong coupling effect of radial and tangential translational constraints on the linear buckling load was identified. In other words, the contribution of each translational constraint depends on the particular combination of boundary conditions under consideration. Among the rotational constraints, the fixed condition of radial rotation is naturally satisfied by the constrained axial displacement, hence does not contribute to the linear buckling behaviour of the cylinder. Meanwhile, the constraint of tangential rotation provides a significant enhancement to linear buckling behaviour, i.e. it can shift the buckling behaviour from being local at the base to being global throughout the structure. Also, axial rotational constraint gives only slight improvement to the linear buckling load. Several means to obtain the highest critical load by different combinations of edge stiffening are also discussed, that can help inform future theoretical or empirical models for cascade, stiffeners, flange or welded designs of real structural applications.

CRediT authorship contribution statement

Quaiyum M. Ansari: Conceptualization, Methodology, Formal analysis, Investigation, Validation, Writing - original draft. Luan C. Trinh: Conceptualization, Methodology, Visualization, Investigation, Writing - review & editing. Giovanni Zucco: Conceptualization, Methodology, Validation, Writing - review & editing. Paul M. Weaver: Conceptualization, Supervision, Funding acquisition, Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The authors would like to thank Science Foundation Ireland (SFI) for funding Spatially and Temporally VARIABLE COMPosite Structures (VARICOMP) Grant N. (15/RP/2773) under its Research Professor program.
References


